ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 3, Fall 1995

© 1995 by the Board of Trustees of the University of Illinois
Manufactured in the United States of America

# ON THE SET OF TOPOLOGICALLY INVARIANT MEANS ON THE VON NEUMANN ALGEBRA $VN(G)$

ZHIGUO HU[1]

## 1. Introduction

The study of the cardinality of the set of invariant means on a group was initiated by Day [3] and Granirer [8]. In 1976, Chou [1] showed that for a discrete infinite amenable group $G$ the cardinality of the set $ML(G)$ of all left invariant means on $l^\infty(G)$ is $2^{2^{|G|}}$. Later, Lau and Paterson [20] proved that if $G$ is a noncompact amenable locally compact group, then the set $MTL(G)$ of all topologically left invariant means on $L^\infty(G)$ has cardinality $2^{2^{d(G)}}$, where $d(G)$ is the smallest cardinality of a covering of $G$ by compact sets. (Of course, when $G$ is compact, $MTL(G)$ is the singleton containing only the normalized Haar measure of $G$). For results on the size of the set $ML(G) \setminus MTL(G)$, see Granirer [9], Rudin [29], and Rosenblatt [26]. See also Yang [32] and Miao [21] for some recent developments in certain related aspects. We refer the readers to the books of Pier [23] and Paterson [22] for more details on the study of the size and the structure of the set of invariant means on groups and semigroups.

Let $G$ be a locally compact group, $A(G)$ the Fourier algebra of $G$, $VN(G)$ the von Neumann algebra defined by the left regular representation $\{\rho, L^2(G)\}$ and $TIM(\hat{G})$ the set of all topologically invariance means on $VN(G)$. The set $TIM(\hat{G})$ was first studied by Dunkl and Ramirez for compact groups. They showed [4] that if $G$ is an infinite compact group, then $|TIM(\hat{G})| \geq 2$. Renaud [25, Theorem 1] proved that there exists a unique topologically invariant mean on $VN(G)$ when $G$ is discrete. In Theorem 1 of [10], Granirer showed the following: if $G$ is non-discrete and second countable (i.e., there is a countable basis for open sets in $G$), then $TIM(\hat{G})$ is not norm separable. A stronger results was obtained by Chou in [2, Theorem 3.3]: if $G$ is non-discrete and metrizable, then there exists a linear isometry of $(l^\infty)^*$ into $VN(G)^*$ which embeds a "big subset" (having cardinality $2^c$) of $(l^\infty)^*$ into $TIM(\hat{G})$. See also Granirer [13, p.172–173] for the discussion on the set $TIM_p(\hat{G})$ of topologically invariant means on $A_p(G)^*$, where $A_p(G)$ is

Received May 25, 1993.
1991 Mathematics Subject Classification. Primary 22D25, 43A07, 43A30; Secondary 22D15, 47D35.
[1]This research was supported in part by the Killam Memorial Scholarship at the University of Alberta.

the Figà-Talamanca-Herz space ($1 < p < \infty$) and $A_p(G) = A(G)$ if $p = 2$. In particular, he proved that $|TIM_p(\hat{G})| \geq 2^c$ in case $G$ is second countable and non-discrete. Recently, Lau and Losert showed, among many other results, that if $VN(G)$ has a unique topologically invariant mean, then $G$ must be discrete (see [19, Theorem 4.10 and Corollary 4.11]). They actually remedied Renaud's result by using a totally different machinery (as noticed by a number of mathematicians, there is a gap in the proof of Proposition 8 in [25]; see [19, p. 21]).

The main purpose of this paper is to prove that if $G$ is a non-discrete locally compact group, then $TIM(\hat{G})$ has cardinality $2^{2^{b(G)}}$, where $b(G)$ is the smallest cardinality of an open basis at the unit element $e$ of $G$.

Section 2 consists of some notations and preliminary results.

For an initial ordinal $\mu$, let $X$ be the set of all ordinals less than $\mu$ with its natural order. We introduce a subset of $l^\infty(X)^*$:

$$\mathscr{F}(X) = \Big\{\phi \in l^\infty(X)^*;\ \|\phi\| = \phi(\mathbf{1}) = 1 \text{ and } \phi(f) = 0 \text{ if } f \in l^\infty(X) \text{ and } \lim_{\alpha \in X} f(\alpha) = 0\Big\},$$

where $\mathbf{1}$ is the constant function of value one. This set with $X = \mathbf{N}$ was first considered by Chou [2]. We shall prove in Section 3 that $|\mathscr{F}(X)| = 2^{2^{|X|}}$ (Proposition 3.3).

The main idea of Section 4 was inspired by Lau-Losert [19, Lemma 4.8]. If $G$ is a $\sigma$-compact non-metrizable locally compact group, let $\mu$ be the initial ordinal satisfying $|\mu| = b(G)$. We shall show that there exists a decreasing family $(N_\alpha)_{\alpha \leq \mu}$ of normal subgroups of $G$ such that $N_0 = G$, $N_\mu = \{e\}$ and $b(N_\alpha) = b(G)$ for all $\alpha < \mu$; $N_\alpha$ is compact if $\alpha > 0$; $N_\alpha/N_{\alpha+1}$ is metrizable but $N_{\alpha+1} \neq N_\alpha$ for $\alpha < \mu$; and $N_\gamma = \bigcap_{\alpha < \gamma} N_\alpha$ for each limit ordinal $\gamma \leq \mu$ (Proposition 4.3). This interesting property concerning the local structure of $G$ at $e$ plays a key role in proving the main results. The proof of Proposition 4.3 will constitute the major technical part of this paper.

In Section 5, we shall present a weaker version of Chou's result [2, Theorem 3.3] and obtain the exact cardinality of $TIM(\hat{G})$ for a non-discrete locally compact group $G$. Let $\mu$ be the initial ordinal with $|\mu| = b(G)$ and $X = \{\alpha;\ \alpha < \mu\}$. In case $G$ is $\sigma$-compact and non-metrizable, by modifying the technique of Chou [2], we construct a family of linear isometries $(\pi_j^*)_j$ of $l^\infty(X)^*$ into $VN(G)^*$. For each $\phi \in l^\infty(X)^*$, let

$$W_\phi = \Big\{\text{all } w^*\text{-cluster points of } \big(\pi_j^*\phi\big)_j \text{ in } VN(G)^*\Big\}.$$

It is shown that $\{W_\phi;\ \phi \in l^\infty(X)^*\}$ is a family of pairwise disjoint non-empty subsets of $VN(G)^*$ and $W_\phi \subseteq TIM(\hat{G})$ if $\phi \in \mathscr{F}(X)$ (Theorem 5.4). Consequently, if $G$ is non-discrete, then there exists a one-one map $W: l^\infty(X)^* \to$

$2^{VN(G)^*}\backslash\{\emptyset\}$ such that $W(\mathscr{F}(X)) \subseteq 2^{TIM(\hat{G})}$ (Corollary 5.5). Finally, the equality $|TIM(\hat{G})| = 2^{2^{b(G)}}$ is proved when $G$ is a non-discrete locally compact group (Theorem 5.9). If $G$ is abelian and $\hat{G}$ is its dual group, then $A(G)$ can be identified with $L^1(\hat{G})$ and $VN(G)$ with $L^\infty(\hat{G})$; now each $m \in VN(G)^*$ belongs to $TIM(\hat{G})$ if and only if the corresponding mean in $L^\infty(\hat{G})$ is a topologically left invariant mean. Since $b(G) = d(\hat{G})$ (see [17, (24.48)]), our Theorem 5.9 coincides with Lau-Paterson's result [20, Theorem 1] when $G$ is abelian.

Some applications of the main theorems are given in Section 6. We show that if $G$ is non-discrete, then $TIM(\hat{G})$ contains a subset $E$ with $|E| = |TIM(\hat{G})| = 2^{2^{b(G)}}$ such that $\|m_1 - m_2\| = 2$ for $m_1, m_2 \in E$ and $m_1 \neq m_2$. Let $UCB(\hat{G})$ be the space of all uniformly continuous functionals on $A(G)$ and $F(\hat{G})$ be the space of topological almost convergent elements in $VN(G)$. Note that each $m \in TIM(\hat{G})$ is determined by its value on $UCB(\hat{G})$. Hence, by Theorem 5.9, any norm dense subset of $UCB(\hat{G})/F(\hat{G}) \cap UCB(\hat{G})$ has cardinality greater than $b(G)$ when $G$ is non-discrete; in particular, $UCB(\hat{G})/F(\hat{G}) \cap UCB(\hat{G})$ is not norm separable. Our Theorem 5.9 also implies Theorem 12 of [11].

## 2. Preliminaries and some notations

Let $\mathbf{C}$ be the complex field. If $E$ is a Banach space over $\mathbf{C}$, let $E^*$ denote the Banach space of all bounded linear functionals $E \to \mathbf{C}$. If $\phi \in E^*$, then the value of $\phi$ at an element $x$ in $E$ will be written as $\phi(x)$ or $\langle \phi, x \rangle$.

Throughout this paper, $G$ denotes a locally compact group with unit element $e$ and a fixed Haar measure $\lambda$. Let $L^2(G)$ be the Banach space of square $\lambda$-integrable complex-valued function $f$ on $G$ with norm $\|f\|_2 = (\int_G |f|^2 \, d\lambda)^{1/2}$. Let $VN(G)$ be the von Neumann algebra defined by the left regular representation $\{\rho, L^2(G)\}$, i.e., the closure of the linear span of $\{\rho(a);\ a \in G\}$ in the weak operator topology, where $\rho(a)f(x) = f(a^{-1}x)$, $x \in G$, $f \in L^2(G)$. Let $A(G)$ be the Fourier algebra of $G$, consisting of all functions of the form $f * \tilde{g}$, where $f, g \in L^2(G)$, $\tilde{g}(x) = \overline{g(x^{-1})}$, and

$$f * \tilde{g}(x) = \int_G f(t)\tilde{g}(t^{-1}x)\,dt = \int_G f(t)\overline{g(x^{-1}t)}\,dt, \quad x \in G.$$

Then each $\phi = f * \tilde{g}$ in $A(G)$ can be regarded as an ultraweakly continuous functional on $VN(G)$ defined by

$$\phi(T) = \langle Tf, g \rangle \quad \text{for } T \in VN(G),$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(G)$. Furthermore, as shown by P. Eymard in [6, p. 210 and p. 218], each ultraweakly continuous functional on

$VN(G)$ is of this form. Therefore, $A(G)$ is the predual of $VN(G)$, i.e., $A(G)^* = VN(G)$. In particular, the $w^*$- and weak operator topologies on $VN(G)$ coincide. Also, $A(G)$ with pointwise multiplication and the norm

$$\|\phi\| = \sup\{|\phi(T)|;\, T \in VN(G) \text{ and } \|T\| \leq 1\}$$

forms a commutative Banach algebra. There is a natural action of $A(G)$ on $VN(G)$ given by

$$\langle u \cdot T, v\rangle = \langle T, uv\rangle \quad \text{for } u, v \in A(G), T \in VN(G).$$

For more details on the algebras $VN(G)$ and $A(G)$, see Eymard [6].

An $m \in VN(G)^*$ is called a *topologically invariant mean* on $VN(G)$, if

(i) $\|m\| = \langle m, I\rangle = 1$, where $I = \rho(e)$ denotes the identity operator,

(ii) $\langle m, u \cdot T\rangle = \langle m, T\rangle$ for $T \in VN(G)$ and $u \in A(G)$ with $u(e) = 1$.

Let $TIM(\hat{G})$ be the set of all topologically invariant means on $VN(G)$. It is known that $TIM(\hat{G})$ is a non-empty $w^*$-compact convex subset of $VN(G)^*$ (see Renaud [25] for a further discussion). Let $C(G)$ denote the Banach space of bounded continuous complex-valued functions on $G$ with the supremum norm and $C_{00}(G)$ denote all functions in $C(G)$ with compact support, where the support of a continuous function $u$ on $G$ is the closure of the set $\{x \in G;\, u(x) \neq 0\}$. The support of an element $f \in L^2(G)$ is defined by saying that $x \notin \operatorname{supp} f$ if and only if there exists a neighborhood $V$ of $x$ such that $\langle f, v\rangle = 0$ for all $v \in C_{00}(G)$ with $\operatorname{supp} v \subseteq V$. The support of an operator $T \in VN(G)$ is defined by saying that $x \notin \operatorname{supp} T$ if and only if there exists a neighborhood $U$ of $e$ such that $x \notin \operatorname{supp}(Tu)$ for all $u \in C_{00}(G)$ with $\operatorname{supp} u \subseteq U$ (see [15, p. 117]). An equivalent definition for $\operatorname{supp} T$ is that $x \in \operatorname{supp} T$ if and only if $u \cdot T = \mathbf{0}$ implies $u(x) = 0$ for all $u \in A(G)$ (see [6, Proposition 4.4] or [14, p. 119]).

Let $UCB(\hat{G})$ denote the norm closure of $A(G) \cdot VN(G)$. Then $UCB(\hat{G})$ is a $C^*$-subalgebra and an $A(G)$-submodule of $VN(G)$ (see [12]) which coincides with the norm closure of $\{T \in VN(G);\, \operatorname{supp} T \text{ is compact}\}$. In case $G$ is abelian, $UCB(\hat{G})$ is isometrically algebra isomorphic to the algebra of bounded uniformly continuous functions on the dual group $\hat{G}$ of $G$. For this reason, operators in $UCB(\hat{G})$ are called uniformly continuous functionals on $A(G)$ (see [11]). The $C^*$-algebra $UCB(\hat{G})$ and its relationship with other $C^*$-subalgebras of $VN(G)$ have been studied by Granier in [11] and [12] and by Lau in [18]. By the definitions of $TIM(\hat{G})$ and $UCB(\hat{G})$, each element $m$ in $TIM(\hat{G})$ is determined by its value on $UCB(\hat{G})$.

Dunkl-Ramirez [5] called $\{T \in VN(G);\, u \mapsto u \cdot T$ is a weakly compact operator of $A(G)$ into $VN(G)\}$ the space of weakly almost periodic functionals of $A(G)$ and denoted it by $W(\hat{G})$. It turns out that $W(\hat{G})$ is a self-adjoint closed $A(G)$-submodule of $VN(G)$ which coincides with the space of weakly almost periodic functions in $L^\infty(\hat{G})$ when $G$ is abelian (see [5] for more details). Chou [2] used $F(\hat{G})$ to denote the space of all $T \in VN(G)$ such that $m(T)$ equals a fixed constant $d(T)$ as $m$ runs through $TIM(\hat{G})$ and called $F(\hat{G})$ the space of topological almost convergent elements in $VN(G)$. We can easily check that $F(\hat{G})$ is a norm closed self-adjoint $A(G)$-submodule of $VN(G)$. It is known what $W(\hat{G})$ has a unique topologically invariant mean (see [5] and [11]). In particular, this gives that $W(\hat{G}) \subseteq F(\hat{G})$. The above inclusion is also obtained by Chou using his results on characterizations of $F(\hat{G})$. See Chou [2] for more information on $F(\hat{G})$.

Let $\phi_1$ and $\phi_2$ be two positive definite functions in $A(G)$. We say that $\phi_1$ is orthogonal to $\phi_2$ if $\|\phi_1 - \phi_2\| = \|\phi_1\| + \|\phi_2\|$ (see [30, p. 31]).

If $M$ is a locally compact group with unit element $e$, we use $b(M)$ to denote the smallest cardinality of an open basis at $e$. When $M$ is abelian and $\hat{M}$ is the dual group of $M$, Hewitt and Stromberg showed that $b(M) = d(\hat{M})$, the smallest cardinality of a covering of $\hat{M}$ by compact sets (see [16] and [17, (24.48)]).

For any two sets $A$ and $B$, $A \setminus B$ denotes their difference, $1_A$ denotes the characteristic function of $A$ as a subset of the underlying set or locally compact group, $2^A$ is the set of all functions from $A$ to $\{0, 1\}$, an $|A|$ is the cardinality of $A$. Then $|2^A| = 2^{|A|}$, the cardinality of the set of all subsets of $A$. So we also use $2^A$ to denote the set of all subsets of $A$. When $\alpha$ is an ordinal number, $|\alpha|$ means the cardinality of the set $\{\beta;\ \beta$ is an ordinal and $\beta < \alpha\}$. An ordinal $\alpha$ is called an *initial ordinal* if $|\alpha|$ is infinite and $\beta < \alpha$ implies $|\beta| < |\alpha|$ (see [27, p. 271]).

LEMMA 2.1. *Let $\alpha$ be an initial ordinal. If $\beta$ and $\gamma > 0$ are ordinals such that $\beta + \gamma = \alpha$, then $\gamma = \alpha$.*

*Proof.* Since $\gamma > 0$, $\beta < \beta + \gamma = \alpha$ (see [27, p. 193]). Then $|\beta| < |\alpha|$ because $\alpha$ is an initial ordinal. But $|\beta| + |\gamma| = |\alpha|$. It follows that $|\gamma| = |\alpha|$. Also, $\gamma \le \beta + \gamma = \alpha$ (see [27, p. 193]). Therefore, $\gamma = \alpha$. □

If $X$ is a set, let $l^\infty(X)$ be the Banach space of all bounded complex-valued functions on $X$ with the supremum norm. It is well known that if $\phi \in l^\infty(X)^*$, then any two of the following three conditions implies the remaining one:

(i) $\|\phi\| = 1$,

(ii) $\phi(\mathbf{1}) = 1$,

(iii) $\phi \ge 0$, that is, $\phi(f) \ge 0$ for all non-negative $f \in l^\infty(X)$,

where $\mathbf{1}$ is the constant function of value one. When $\phi \in l^\infty(X)^*$ has any two of the above properties, we call $\phi$ a *mean* on $l^\infty(X)$. If $X$ is a directed set, we define

$$\mathscr{F}(X) = \Big\{\phi \in l^\infty(X)^*;\ \|\phi\| = \phi(\mathbf{1}) = 1,\ \phi(f) = 0 \\ \text{if } f \in l^\infty(X) \text{ and } \lim_{\alpha \in X} f(\alpha) = 0\Big\}.$$

This set with $X = \mathbf{N}$, the set of all positive integers, was first considered by Chou when he introduced the technique to embed a large set in $TIM(\hat{G})$ (see [2]). Yang in [32] studied the case $X = \Lambda(Y)$, the set of all non-empty finite subsets of a infinite set $Y$ directed by inclusion. When $X$ is a directed set, a tail in $X$ is defined by

$$T_\alpha = \{ \beta \in X;\ \beta \geq \alpha \}, \quad \alpha \in X.$$

Therefore, $\phi \in \mathscr{F}(X)$ if and only if $\phi$ is a mean on $l^\infty(X)$ and $\phi(1_{T_\alpha}) = 1$ for all $\alpha \in X$.

If $X$ is a set (with the discrete topology), $\beta X$ denotes the Stone-Čech compactification of $X$. Then $l^\infty(X)$ is isometrically isomorphic to $C(\beta X)$. Thus $\beta X$ can be identified with the spectrum of $l^\infty(X)$, i.e., the set of all nonzero multiplicative linear functionals on $l^\infty(X)$ with the Gelfand topology (see, say, [31, Proposition 4.5, p. 18]). In this way, each $x \in X$ is identified with the evaluation $\hat{x}$ on $l^\infty(X)$ at $x$, i.e., $\hat{x}(f) = f(x)$ for $f \in l^\infty(X)$. On the other hand, $\beta X$ can also be obtained by "fixing" the free ultrafilters on $X$, that is, $\beta X = \{$all ultrafilters on $X\}$ with $\{Z^*;\ Z \subseteq X\}$ as a base for closed subsets of $\beta X$, where $Z^* = \{\phi \in \beta X;\ Z \in \phi\}$ (see [7, pp. 86–87]). Now, every $x \in X$ corresponds to the fixed ultrafilter $\phi_x$ on $X$ containing $\{x\}$, i.e., $\phi_x = \{E;\ x \in E \subseteq X\}$. Either way of the above embeddigs will be used later.

When $X = \mathbf{N}$, Chou in [2] pointed out that $\beta\mathbf{N}\backslash\mathbf{N} \subseteq \mathscr{F}(\mathbf{N})$. For the general case, we have:

LEMMA 2.2. *Let $X$ be a directed set. If $\phi \in \beta X$ and $\phi$ contains $\{T_\alpha;\ \alpha \in X\}$, then $\phi \in \mathscr{F}(X)$.*

*Proof.* Let $\phi \in \beta X$ and $\phi$ contain $\{T_\alpha;\ \alpha \in X\}$. Since $\phi$ is in the spectrum of $l^\infty(X)$, $\phi$ is a mean on $l^\infty(X)$. It is known that $E \in \phi$ if and only if $\phi(1_E) = 1$. Now, for each $\alpha \in X$, $T_\alpha \in \phi$ and hence $\phi(1_{T_\alpha}) = 1$. Therefore, $\phi \in \mathscr{F}(X)$. □

## 3. The cardinality of $\mathscr{F}(X)$

For a directed set $X$, let $\mathscr{F}(X)$ be the subset of $l^\infty(X)^*$ defined as in §2. If $X = \mathbf{N}$, then $|\mathscr{F}(\mathbf{N})| = 2^c = 2^{2^{|X|}}$, since $\beta\mathbf{N}\backslash\mathbf{N} \subseteq \mathscr{F}(\mathbf{N})$ (see Chou [2, p. 208]), where $c$ is the cardinality of the continuum. When $X = \Lambda(Y)$, the set of all non-empty finite subsets of an infinite set $Y$ directed by inclusion, Yang proved in [32, Lemma 2.1] that $|\mathscr{F}(X)| = 2^{2^{|X|}}$ if $l^\infty(X)$ is the real Banach space. Throughout this section, $\mu$ will be an initial ordinal and $X$ denote the set $\{\beta;\ \beta$ is an ordinal and $\beta < \mu\}$ with its natural order. We shall show that $|\mathscr{F}(X)| = 2^{2^{|X|}}$. For this purpose, we begin with a technical lemma which provides us a family $\mathscr{Y}$ of functions in $2^X$ such that $|\mathscr{Y}| = 2^{|X|}$ and any two functions in $\mathscr{Y}$ are not cofinal.

LEMMA 3.1. *There exists a family* $\{f_i;\ i \in I\} \subseteq 2^X$ *such that* $|I| = 2^{|X|}$ *and*

$$f_i|_{T_\alpha} \neq f_j|_{T_\alpha} \quad \textit{for } i, j \in I \textit{ with } i \neq j,\ \alpha \in X,$$

*where* $f|_A$ *is the restriction of function* $f$ *to the set* $A$.

*Proof.* *Case* (*i*). Assume that $2^{|\alpha|} < 2^{|X|}$ for all $\alpha \in X$.

For each pair $f, g \in 2^X$, we define $f \sim g$ if there exists an element $\alpha \in X$ such that $f|_{T_\alpha} = g|_{T_\alpha}$. Then " $\sim$ " is an equivalent relation on $2^X$. Let $f \in 2^X$. We put $[f] = \{g \in 2^X;\ g \sim f\}$, the equivalence class containing $f$. Let $I$ be the set of all such equivalence classes. Then $2^X = \bigcup\{[f];\ [f] \in I\}$.

Fix an $f \in 2^X$. We have $[f] = \bigcup_{\alpha \in X} F_\alpha$, where $F_\alpha = \{g \in 2^X;\ g|_{T\alpha} = f|_{T_\alpha}\}$. Since $|F_\alpha| = 2^{|\alpha|}$, $|[f]| \leq \Sigma_{\alpha \in X} 2^{|\alpha|}$. This is true for every $f \in 2^X$. Hence,

$$\begin{aligned} 2^{|X|} = |2^X| &= \sum_{[f] \in I} |[f]| \leq \sum_{[f] \in I} \left( \sum_{\alpha \in X} 2^{|\alpha|} \right) \\ &= |I| \left( \sum_{\alpha \in X} 2^{|\alpha|} \right) = \max\left( |I|, \sum_{\alpha \in X} 2^{|\alpha|} \right). \end{aligned}$$

By König-Zermelo's inequality (see [27, p. 313]), we have

$$\sum_{\alpha \in X} 2^{|\alpha|} < \prod_{\alpha \in X} 2^{|X|} = \left(2^{|X|}\right)^{|X|} = 2^{|X|}. \tag{3.1}$$

Obviously, $|I| \leq 2^{|X|}$. Consequently,

$$2^{|X|} \leq \max\left( |I|, \sum_{\alpha \in X} 2^{|\alpha|} \right) \leq 2^{|X|}. \tag{3.2}$$

Now (3.1) and (3.2) combined give $|I| = 2^{|X|}$. For each equivalence class $i \in I$, we choose an $f_i \in i$. Then the family $\{f_i;\ i \in I\}$ satisfies the requirement.

*Case* (ii). Assume that $2^{|\alpha|} = 2^{|X|}$ for some $\alpha \in X$.

Let $\alpha_0 = \min\{\beta;\ \beta \in X \text{ and } 2^{|\beta|} = 2^{|X|}\}$. Then $\alpha_0$ is a limit ordinal. By the generalized division algorithm (see [24, p. 177]), there exists a unique pair of ordinals $\eta$ and $\varepsilon$ such that $\mu = \alpha_0\eta + \varepsilon$ and $\varepsilon < \alpha_0$. Note that $\varepsilon < \alpha_0 < \mu$ and $\alpha_0(\zeta + 1) = \alpha_0\zeta + \alpha_0$ for any ordinal $\zeta$. By Lemma 2.1, $\varepsilon = 0$ and $\eta$ has to be a limit ordinal.

Let $\omega$ be the initial ordinal satisfying $|\omega| = 2^{|X|}$. Let $I = \{i;\ i$ is an ordinal and $i < \omega\}$. Then $|I| = |\omega| = 2^{|X|}$. In the following we inductively construct a family $(X_i)_{i<\omega}$ of subsets of $X$ such that

$$X_i \cap T_\alpha \neq X_j \cap T_\alpha \quad \text{for } i, j < \omega \text{ with } i \neq j,\ \alpha \in X. \tag{3.3}$$

Let $i_0 < \omega$. Assume that we have chosen a family $(X_i)_{i<i_0}$ of subsets of $X$ satisfying (3.3). Recall that $\mu = \alpha_0\eta$. For every $\xi < \eta$, let $S_\xi$ be the segment of ordinals between $\alpha_0\xi$ and $\alpha_0(\xi+1)$, i.e.,

$$S_\xi = \{\alpha;\ \alpha_0\xi \leq \alpha < \alpha_0(\xi+1)\}.$$

Since $\alpha_0\xi_1 < \alpha_0\xi_2$ if and only if $\xi_1 < \xi_2$ (see [27, p. 200]), $\{S_\xi;\ \xi < \eta\}$ is pairwise disjoint. Furthermore, $X = \bigcup_{\xi<\eta} S_\xi$ and, for every $\alpha \in X$, there exists a $\xi < \eta$ such that $S_\xi \subseteq T_\alpha$. Now, $|S_\xi| = |\alpha_0|$ and hence $2^{|S_\xi|} = 2^{|\alpha_0|} = 2^{|X|}$ for all $\xi < \eta$. But we have

$$\left|\{X_i \cap S_\xi;\ i < i_0\}\right| \leq |i_0| < 2^{|X|} \text{ for } \xi < \eta.$$

Consequently, for each $\xi < \eta$, there exists a set $B_\xi \subseteq S_\xi$ such that $B_\xi \notin \{X_i \cap S_\xi;\ i < i_0\}$. Let $X_{i_0} = \bigcup_{\xi<\eta} B_\xi$. Then $X_{i_0} \cap S_\xi \neq X_i \cap S_\xi$ if $i < i_0$ and $\xi < \eta$. Hence $X_{i_0} \cap T_\alpha \neq X_i \cap T_\alpha$ for all $i < i_0$ and $\alpha \in X$. Therefore, the family $\{X_i;\ i \leq i_0\}$ has property (3.3). By transfinite induction, we obtain a family $(X_i)_{i<\omega}$ of subsets of $X$ satisfying (3.3).

Finally, for each $i \in I$, let $f_i : X \to \{0, 1\}$ be the characteristic function of $X_i$. Then $\{f_i;\ i \in I\}$ has the required property. □

*Remark* 3.2. Under the generalized continuum hypothesis (GCH, for short), $a < b$ implies that $2^a < 2^b$, where $a$ and $b$ are any two cardinal numbers. In Lemma 3.1, $|\alpha| < |\mu| = |X|$ for all $\alpha \in X$, since $\mu$ is an initial ordinal. Thus, if the GCH is assumed, we always have $2^{|\alpha|} < 2^{|X|}$ for all $\alpha \in X$; this is the case (i) in the above proof. To avoid using the GCH, we have to consider case (ii) in our proof as well.

Now we are ready to prove that $\mathscr{F}(X)$ is a "big subset" of $l^\infty(X)^*$.

PROPOSITION 3.3. $|\mathscr{F}(X)| = 2^{2^{|X|}}$.

*Proof.* Obviously, $|\mathscr{F}(X)| \leq |l^\infty(X)^*| = 2^{2^{|X|}}$. By Lemma 2.2, it suffices to show that there are $2^{2^{|X|}}$ many ultrafilters on $X$ containing $\{T_\alpha;\ \alpha \in X\}$. We now follow an argument of Rudin [28, Theorem 1.3] (see also the proof of [32, Lemma 2.1]).

Let $\Lambda = \Lambda(X)$ be the set of all non-empty finite subsets of $X$. In the following, we shall construct a family $\{A_\tau;\ \tau \in \Lambda\}$ of subsets of $X$ satisfying:

(i) $|A_\tau| = 2^{2^{|\tau|}}$;
(ii) if $\tau \neq \tau'$, then $A_\tau \cap A_{\tau'} = \varnothing$;
(iii) if $\alpha \in A_\tau$, then $\alpha \geq \max(\tau)$, where $\max(\tau) = \max\{\beta;\ \beta \in \tau\}$.

Since $|\Lambda| = |X| = |\mu|$, we can write $\Lambda = \{\tau_i;\ i < \mu\}$. Let $i_0 < \mu$. Assume that we have defined a family $\{A_{\tau_i};\ i < i_0\}$ of subsets of $X$ satisfying (i)–(iii). Let $B = \bigcup_{i<i_0} A_{\tau_i}$. Then $|B| < |\mu|$, since each $A_{\tau_i}$ is finite and $|i_0| < |\mu|$. Let $\alpha$ be the unique ordinal satisfying $\max(\tau_{i_0}) + \alpha = \mu$ (see [27, p. 194]). Since $\alpha \neq 0$; $\alpha = \mu$, by Lemma 2.1. In particular, $|\alpha| = |\mu| > |B|$. But $|\mu|$ is infinite. So, we can choose a finite set $A_{\tau_{i_0}} \subseteq \{\beta;\ \max(\tau_{i_0}) \leq \beta < \mu\} \setminus B$ with $|A_{\tau_{i_0}}| = 2^{2^{|\tau_{i_0}|}}$. Clearly, the family $\{A_{\tau_i};\ i \leq i_0\}$ has properties (i)–(iii). By transfinite induction, we have constructed a family $\{A_{\tau_i};\ i < \mu\} = \{A_\tau;\ \tau \in \Lambda\}$ of subsets of $X$ satisfying (i)–(iii).

For each $\tau \in \Lambda$, label the elements of $A_\tau$ by ordered $2^{|\tau|}$-tuples $(x_1, x_2, \ldots, x_{2^{|\tau|}})$ with $x_i \in \{0, 1\}$. Let $E_i$ be the subset of $A_\tau$ consisting of the $2^{|\tau|}$-tuples which have $x_i = 0$. If we let $E_i^0 = E_i$ and $E_i^1 = A_\tau \setminus E_i$, then $\bigcap_{i=1}^{2^{|\tau|}} E_i^{\varepsilon_i}$ is not empty for any choice of $\varepsilon_i \in \{0, 1\}$, since $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2^{|\tau|}}) \in \bigcup_{i=1}^{2^{|\tau|}} E_i^{\varepsilon_i}$. Denote the sets $E_i$, $i = 1, 2, \ldots, 2^{|\tau|}$, by $E(h)$, where $h$ is a map from $\tau$ to $\{0, 1\}$.

Let $\mathscr{Y} = \{f_j;\ j \in I\} \subseteq 2^X$ be the same family of functions as in Lemma 3.1. For each $f \in \mathscr{Y}$, we define

$$B(f) = \bigcup \{E(f|_\tau);\ \tau \in \Lambda\},$$

where $f|_\tau$ is the restriction of $f$ to the set $\tau$.

Suppose that $f_1, \ldots, f_n, f_{n+1}, \ldots, f_m$ are distinct functions in $\mathscr{Y}$ and $\alpha \in X$. Since $f_1|_{T_\alpha}, \ldots, f_n|_{T_\alpha}, f_{n+1}|_{T_\alpha}, \ldots, f_m|_{T_\alpha}$ are different (by Lemma 3.1), there exists an element $\tau \in \Lambda$ such that $\tau \subseteq T_\alpha$ and $f_1|_\tau, \ldots, f_n|_\tau, f_{n+1}|_\tau, \ldots, f_m|_\tau$ are different. Hence the above argument gives

$$E(f_1|_\tau) \cap \cdots \cap E(f_n|_\tau) \cap (A_\tau \setminus E(f_{n+1}|_\tau)) \cap \cdots \cap (A_\tau \setminus E(f_m|_\tau)) \neq \varnothing.$$

Using $\tau \subseteq T_\alpha$ and property (iii), we have $A_\tau \subseteq T_\alpha$. Therefore,

$$E(f_1|_\tau) \cap \cdots \cap E(f_n|_\tau) \cap (A_\tau \setminus E(f_{n+1}|_\tau)) \cap \cdots \cap (A_\tau \setminus E(f_m|_\tau)) \\ \cap T_\alpha \neq \varnothing.$$

Note that $\{A_\tau;\ \tau \in \Lambda\}$ is pairwise disjoint. It follows that

$$B(f_1) \cap \cdots \cap B(f_n) \cap (X \setminus B(f_{n+1})) \cap \cdots \cap (X \setminus B(f_m)) \cap T_\alpha \neq \varnothing.$$

Hence for any map $F : \mathscr{Y} \to \{0, 1\}$, the collection

$$\left\{B(f)^{F(f)};\ f \in \mathscr{Y}\right\} \cup \{T_\alpha;\ \alpha \in X\},$$

where $B(f)^0 = B(f)$ and $B(f)^1 = X \setminus B(f)$, generates a filter base. Consequently, we have $2^{|\mathscr{Y}|} = 2^{2^{|X|}}$ different ultrafilters on $X$ containing $\{T_\alpha;\ \alpha \in X\}$. This completes the proof of the proposition. $\square$

## 4. The local structure of $\sigma$-compact non-metrizable groups

Let $G$ be a locally compact group with unit element $e$ and $b(G)$ be the smallest cardinality of an open basis at $e$ defined as in §2. In this section, we shall present an important property of a $\sigma$-compact non-metrizable locally compact group $G$ concerning its local structure at $e$. This property is very crucial for our main results and is interesting in itself. We begin with two lemmas. The first one is similar to [19, Lemma 4.7]. The second one deals with the relation between $b(N)$ and $b(G)$, where $N$ is a closed subgroup of $G$.

This section is motivated by Lau-Losert [19, Lemma 4.8].

LEMMA 4.1. *Let $G$ be a $\sigma$-compact locally compact group. Let $N$ be a closed normal subgroup of $G$ and $U$ an open neighborhood of $e$. Then there exists a compact normal subgroup $M$ of $G$ such that $M \subseteq N \cap U$ and $N/M$ is metrizable.*

*Proof.* By the Kakutani-Kodaira Theorem (see [17, (8.7)]), there exists a compact normal subgroup $K$ of $G$ such that $G/K$ is metrizable and $K \subseteq U$. Let $M = K \cap N$. Then $M$ is a compact normal subgroup of $G$ and $M \subseteq N \cap U$. Note that $N/M \cong NK/K \subseteq G/K$. Therefore, $N/M$ is metrizable. □

LEMMA 4.2. *Let $G$ be a locally compact group and $N$ be a closed subgroup of $G$. Let $\aleph$ be a cardinal number. If $N$ is an intersection of no more than $\aleph$ open subsets of $G$, then $b(G) \leq \aleph b(N)$.*

*Proof.* Choose a set $I$ with $|I| = \aleph$. Since $G$ is a normal topological space (i.e., any two disjoint closed subsets of $G$ can be separated by two disjoint open subsets of $G$), by the assumption, we can write $N = \bigcap_{i \in I} A_i$, where each $A_i$ is a closed subset of $G$ and $N \subseteq \mathring{A}_i$ (the interior of $A_i$ in $G$). Also we choose a set $J$ with $|J| = b(N)$ such that $\{B_j \cap N;\ j \in J\}$ is a neighborhood basis at $e$ in $N$, where each $B_j$ is a compact neighborhood of $e$ in $G$. We can assume that $B_j \subseteq K$ for all $j \in J$, where $K$ is a fixed compact subset of $G$. Let $\Lambda(I)$ (resp. $\Lambda(J)$) be the set of all non-empty finite subsets of $I$ (resp. $J$). For any $\xi \in \Lambda(I)$ and $\eta \in \Lambda(J)$, denote $A_\xi = \bigcap_{i \in \xi} A_i$ and $B_\eta = \bigcap_{j \in \eta} B_j$. Then $A_\xi$ and $B_\eta$ are neighborhoods of $e$ in $G$.

We claim that $\{A_\xi \cap B_\eta;\ \xi \in \Lambda(I),\ \eta \in \Lambda(J)\}$ is a neighborhood basis at $e$ in $G$. Assume that there exists a neighborhood $U$ of $e$ in $G$ such that $A_\xi \cap B_\eta \subsetneq U$ for all $\xi \in \Lambda(I)$ and $\eta \in \Lambda(J)$. Choose an element $x_{\xi,\eta} \in (A_\xi \cap B_\eta)\setminus U$ for each pair $(\xi, \eta) \in \Lambda(I) \times \Lambda(J)$. We direct $\Lambda(I)$ and $\Lambda(J)$ by counter inclusion (i.e., $\zeta_1 \leq \zeta_2$ if and only if $\zeta_2 \subseteq \zeta_1$), and direct $\Lambda(I) \times \Lambda(J)$ by $(\xi_1, \eta_1) \leq (\xi_2, \eta_2)$ if and only if $\xi_1 \leq \xi_2$ and $\eta_1 \leq \eta_2$. Then the net $(x_{\xi,\eta})_{(\xi,\eta) \in \Lambda(I) \times \Lambda(J)}$ in $K$ has a cluster point, say, $x \in K$. By the direction on

$\Lambda(I) \times \Lambda(J)$ and the compactness of $A_\xi \cap B_\eta$, we have $x \in A_\xi \cap B_\eta$ for all $(\xi, \eta) \in \Lambda(I) \times \Lambda(J)$. Consequently,

$$x \in \bigcap_{(\xi,\eta)\in\Lambda(I)\times\Lambda(J)} (A_\xi \cap B_\eta) = \bigcap_{\eta\in\Lambda(J)} (N \cap B_\eta) = \bigcap_{j\in J} (N \cap B_j) = \{e\};$$

i.e., $x = e$. But $U$ is a neighborhood of $e$ in $G$ and $x_{\xi,\eta} \notin U$ for all $(\xi, \eta) \in \Lambda(I) \times \Lambda(J)$. This contradicts the fact that $x = e$ is a cluster point of $(x_{\xi,\eta})_{(\xi,\eta)\in\Lambda(I)\times\Lambda(J)}$. It follows that

$$\{A_\xi \cap B_\eta;\ \xi \in \Lambda(I),\ \eta \in \Lambda(J)\}$$

is a neighborhood basis at $e$ in $G$.

Since $|J| = b(N)$, $|J| = 1$ or $|J|$ is infinite. In any case, we have that $|\Lambda(J)| = |J| = b(N)$. So,

$$b(G) \leq |\Lambda(I) \times \Lambda(J)| = |\Lambda(I)|\,|\Lambda(J)| = |\Lambda(I)|b(N).$$

If $\aleph$ is infinite, then $|\Lambda(I)| = |I| = \aleph$ and hence $b(G) \leq \aleph b(N)$ by the above inequality. If $\aleph$ is finite, then $N$ is an open subgroup of $G$ and now $b(N) = b(G)$. Therefore, we always have that $b(G) \leq \aleph b(N)$. □

The main result of this section is contained in the following proposition.

PROPOSITION 4.3. *Let $G$ be a $\sigma$-compact non-metrizable locally compact group with unit element $e$. Then there exists a limit ordinal $\mu$ and a decreasing family $(N_\alpha)_{\alpha \leq \mu}$ of normal subgroups of $G$ (i.e., $\alpha \leq \beta$ implies $N_\alpha \supseteq N_\beta$) such that*:

(i) *$N_0 = G$ and $N_\mu = \{e\}$;*
(ii) *$N_\alpha$ is compact for each $\alpha > 0$;*
(iii) *$N_\alpha/N_{\alpha+1}$ is metrizable but $N_{\alpha+1} \neq N_\alpha$ for all $\alpha < \mu$;*
(iv) *$N_\gamma = \bigcap_{\alpha<\gamma} N_\alpha$ for every limit ordinal $\gamma \leq \mu$;*
(v) *$b(N_\alpha) = b(G)$ for all $\alpha < \mu$.*

*Furthermore, $\mu$ is minimal among all such families and $\mu$ is the initial ordinal satisfying $|\mu| = b(G)$.*

*Proof.* Let $d$ be the initial ordinal satisfying $|d| = b(G)$. Then $d$ is a limit ordinal. Let $\{O_\alpha;\ \alpha < d\}$ be an open basis at $e$ in $G$. Let $N_0 = G$. By Lemma 4.1, there exists a compact normal subgroup $N_1$ of $G$ such that $N_1 \subseteq N_0 \cap O_0$ and $N_0/N_1$ is metrizable. Let $d_0 < d$. Assume that we have chosen a decreasing family $(N_\alpha)_{\alpha<d_0}$ of normal subgroups of $G$ such that $N_\alpha$ is compact for each $0 < \alpha < d_0$, $N_{\alpha+1} \subseteq N_\alpha \cap O_\alpha$ and $N_\alpha/N_{\alpha+1}$ is metrizable if $\alpha + 1 < d_0$, and $N_\gamma = \bigcap_{\alpha<\gamma} N_\alpha$ for every limit ordinal $\gamma < d_0$. If $d_0$ is a limit ordinal, then we put $N_{d_0} = \bigcap_{\alpha<d_0} N_\alpha$. If $d_0 = \beta + 1$ (such $\beta$ is

unique), then, by Lemma 4.1, we choose $N_{d_0}$ to be the compact normal subgroup of $G$ such that $N_{d_0} \subseteq N_\beta \cap O_\beta$ and $N_\beta / N_{d_0}$ is metrizable. By transfinite induction, we get a decreasing family $(N_\alpha)_{\alpha < d}$ of normal subgroups of $G$ such that $N_0 = G$, $N_\alpha$ is compact for all $0 < \alpha < d$, $N_{\alpha+1} \subseteq N_\alpha \cap O_\alpha$ and $N_\alpha / N_{\alpha+1}$ is metrizable for $\alpha < d$, and $N_\gamma = \bigcap_{\alpha < \gamma} N_\alpha$ for every limit ordinal $\gamma < d$. Now

$$\bigcap_{\alpha<d} N_\alpha \subseteq \bigcap_{\alpha<d} O_\alpha = \{e\},$$

so, $\bigcap_{\alpha < d} N_\alpha = \{e\}$. Let $N_d = \{e\}$. Then $N_d = \bigcap_{\alpha < d} N_\alpha$.

We claim that for each $0 < \alpha < d$, $N_\alpha$ is an intersection of no more than $|\alpha| \aleph_0$ open subsets of $G$, where $\aleph_0$ is the first infinite cardinal number. This is true for $\alpha = 1$ because $N_1$ is a $G_\delta$-set in $G$ (i.e., $N_1$ is an intersection of countably many open subsets of $G$, since $G/N_1$ is metrizable). Let $d_0 < d$. Assume that the above statement is true for all $0 < \alpha < d_0$. If $d_0$ is a limit ordinal, then $N_{d_0} = \bigcap_{\alpha < d_0} N_\alpha$ and hence, by the inductive assumption, $N_{d_0}$ is an intersection of no more than $|d_0|^2 \aleph_0 = |d_0| \aleph_0$ open subsets of $G$. If $d_0 = \beta + 1$ for some $\beta < d$, then $N_{d_0}$ is a $G_\delta$-set in $N_\beta$, since $N_\beta / N_{d_0}$ is metrizable. By the assumption that $N_\beta$ is an intersection of no more than $|\beta| \aleph_0$ open subsets of $G$, $N_{d_0}$ is an intersection of no more than $|\beta| \aleph_0^2 = |\beta| \aleph_0 = |d_0| \aleph_0$ open subsets of $G$. By transfinite induction, our assertion follows.

Let $0 < \alpha < d$. By the above claim and Lemma 4.2, we have

$$b(G) \leq (|\alpha| \aleph_0) b(N_\alpha). \tag{4.1}$$

Since $d$ is the initial ordinal satisfying $|d| = b(G) > \aleph_0$, then $|\alpha| < b(G)$ and hence

$$|\alpha| \aleph_0 = \max(|\alpha|, \aleph_0) < b(G). \tag{4.2}$$

Now (4.1) and (4.2) combined give

$$\begin{aligned} b(G) &\leq (|\alpha| \aleph_0) b(N_\alpha) \\ &= \max(|\alpha| \aleph_0, b(N_\alpha)) = b(N_\alpha); \end{aligned}$$

i.e., $b(G) \leq b(N_\alpha)$. Conversely, $b(N_\alpha) \leq b(G)$, since $N_\alpha$ is a subgroup of $G$. Therefore, $b(N_\alpha) = b(G)$ for all $\alpha < d$. We conclude that $(N_\alpha)_{\alpha \leq d}$ is a decreasing family of normal subgroups of $G$ satisfying:

(i) $N_0 = G$ and $N_d = \{e\}$;
(ii) $N_\alpha$ is compact for each $\alpha > 0$;
(iii) $N_\alpha / N_{\alpha+1}$ is metrizable for all $\alpha < d$;
(iv) $N_\gamma = \bigcap_{\alpha < \gamma} N_\alpha$ for every limit ordinal $\gamma \leq d$;
(v) $b(N_\alpha) = b(G)$ for all $\alpha < d$.

Let $\mu$ be the minimal ordinal among all such families. We see that $\mu$ has to be a limit ordinal. In fact, assume that $\mu = \nu + 1$ ($\nu$ is infinite, since $G$ is non-metrizable). Then $N_\nu = N_\nu/N_\mu$ is metrizable, contradicting the fact that $b(N_\nu) = b(G) > \aleph_0$. It follows that $\mu$ is a limit ordinal. By passing to an appropriate subfamily, we can achieve that $N_{\alpha+1} \neq N_\alpha$ for all $\alpha$. The ordinal type of this subfamily will be still $\mu$, by minimality. Note that (i)–(v) implies (i), (ii), (iii)′, (iv), and (v). Consequently, $\mu$ is minimal among all families satisfying (i)–(v).

By the same procedure as above, we can prove that for each $0 < \alpha < \mu$, $N_\alpha$ is an intersection of no more than $|\alpha|\aleph_0$ open subsets of $G$. Since $\bigcap_{\alpha<\mu} N_\alpha = N_\mu = \{e\}$, $\{e\}$ is an intersection of no more than $|\mu|^2\aleph_0 = |\mu|$ open subsets of $G$. Applying Lemma 4.2 to $N = \{e\}$, we get that $b(G) \leq |\mu| b(N) = |\mu|$. But $\mu \leq d$, by the minimality of $\mu$, and $|d| = b(G)$. Therefore, $|\mu| = b(G)$ and hence $\mu = d$; i.e., $\mu$ is the initial ordinal satisfying $|\mu| = b(G)$. This completes the proof of the proposition. □

*Remark* 4.4. The basic idea used in constructing $(N_\alpha)_{\alpha \leq \mu}$ is essentially the same as that used in Lau-Losert [19, Lemma 4.8]. The net $(N_\alpha)_{\alpha \leq \lambda}$ there possesses property (i)–(iv). Here, for our purpose, we begin by showing the existence of the family of subgroups of $G$ satisfying (i)–(v). Hence the result is strengthened in the following two related aspects, which are important in the sequel.

(1) The limit ordinal $\mu$ is totally determined by the local structure of the $\sigma$-compact non-metrizable group $G$ ($\mu$ is actually the first ordinal satisfying $|\mu| = b(G)$).

(2) $b(N_\alpha) = b(G)$ for all $\alpha < \mu$ (this property reflects, in some sense, that each compact normal subgroup $N_\alpha$ in this net has the same "non-metrizability" as $G$ does).

## 5. Main results

Let $G$ be a $\sigma$-compact non-metrizable locally compact group. Let $(N_\alpha)_{\alpha \leq \mu}$ be the decreasing family of normal subgroups of $G$ as in Proposition 4.3. By the properties of $(N_\alpha)_{\alpha \leq \mu}$, we can define a family $(P_\alpha)_{\alpha < \mu}$ of projections in $VN(G)$ as in the proof of Lau-Losert [19, Theorem 4.10]. Let $P_0 = \mathbf{0} \in VN(G)$. For $0 < \alpha < \mu$, let $P_\alpha \in VN(G)$ be the central projection defined by convolution with the normalized Haar measure $\lambda_\alpha$ of $N_\alpha$. More exactly,

$$P_\alpha : L^2(G) \to L^2(G/N_\alpha)(\subseteq L^2(G))$$

is given by

$$(P_\alpha f)(x) = \int_{N_\alpha} f(t^{-1}x)\, d\lambda_\alpha(t), \quad f \in L^2(G), 0 < \alpha < \mu,$$

where $L^2(G/N_\alpha)$ is the subspace of $L^2(G)$ consisting of all functions in $L^2(G)$ which are constant on the cosets of $N_\alpha$ (see [6, (3.23)]). Now $(P_\alpha)_{\alpha<\mu}$ is an increasing net of projections in $VN(G)$; i.e., $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for $\alpha < \beta < \mu$. Define

$$Q_\alpha = P_{\alpha+1} - P_\alpha, \quad \alpha < \mu.$$

Then $(Q_\alpha)_{\alpha<\mu}$ is an orthogonal net of projections in $VN(G)$; that is,

$$Q_\alpha Q_\beta = \begin{cases} Q_\alpha & \text{if } \alpha = \beta, \\ \mathbf{0} & \text{if } \alpha \neq \beta. \end{cases}$$

We begin with a technical lemma.

LEMMA 5.1. *Let $G$ be a $\sigma$-compact non-metrizable locally compact group and $(N_\alpha)_{\alpha\leq\mu}$ be the decreasing family of normal subgroups of $G$ as in Proposition 4.3. Let $(Q_\alpha)_{\alpha<\mu}$ be the orthogonal net of projections in $VN(G)$ defined as above. If $U$ is a neighborhood of the unit element $e$ of $G$ and $\alpha < \mu$, then there exists an $f \in L^2(G)$ such that $\|f\|_2 = 1$, $\operatorname{supp} f \subseteq UN_\alpha$, and $Q_\alpha f = f$.*

*Proof.* Since $N_{\alpha+1} \subsetneqq N_\alpha$, there exists an $x_0 \in N_\alpha$ such that $x_0 N_{\alpha+1} \cap N_{\alpha+1} = \varnothing$. By the compactness of $N_{\alpha+1}$, there exists a neighborhood $V$ of $e$ in $G$ such that

$$x_0 N_{\alpha+1} V \cap N_{\alpha+1} V = \varnothing. \tag{5.1}$$

We can assume that $V \subseteq U$ and $V$ is compact.

Let $g = 1_{N_{\alpha+1}V}$. Then $g \in L^2(G)$, $\operatorname{supp} g \subseteq VN_{\alpha+1} \subseteq UN_{\alpha+1}$, and $g$ is constant on the cosets of $N_{\alpha+1}$; i.e., $g \in L^2(G/N_{\alpha+1})$. If $\alpha = 0$, then $Q_0 g = P_1 g = g$, since $g \in L^2(G/N_1)$. Now $g/\|g\|_2$ satisfies the requirements. In the following we assume that $\alpha > 0$ and we shall show that $g \notin L^2(G/N_\alpha)$.

Assume that $g \in L^2(G/N_\alpha)$. Then there exists an $h \in L^2(G)$ such that $h$ is constant on the cosets of $N_\alpha$ and $g = h$ a.e. Now $g = 1$ on $N_{\alpha+1}V$ and $g = 0$ on $N_\alpha V \setminus N_{\alpha+1}V$. Hence, there exist measurable subsets $W_1 \subseteq N_{\alpha+1}V$ and $W_2 \subseteq N_\alpha V \setminus N_{\alpha+1}V$ such that

$$\lambda(W_1) = \lambda(N_{\alpha+1}V), \tag{5.2}$$

$$\lambda(W_2) = \lambda(N_\alpha V \setminus N_{\alpha+1}V), \tag{5.3}$$

and $h = 1$ on $W_1$, $h = 0$ on $W_2$. Therefore, $h = 1$ on $N_\alpha W_1$ and $h = 0$ on

$N_\alpha W_2$, since $h$ is constant on the cosets of $N_\alpha$. It follows that $N_\alpha W_1 \cap N_\alpha W_2 = \varnothing$. But

$$\begin{aligned}
\lambda(x_0 W_1 \cap W_2) &= \lambda(x_0 N_{\alpha+1} V \cap W_2) && \text{(by (5.2))}\\
&= \lambda(x_0 N_{\alpha+1} V \cap (N_\alpha V \setminus N_{\alpha+1} V)) && \text{(by (5.3))}\\
&= \lambda(x_0 N_{\alpha+1} V) && \text{(by (5.1))}\\
&= \lambda(N_{\alpha+1} V) \geq \lambda(V) > 0.
\end{aligned}$$

In particular, $x_0 W_1 \cap W_2 \neq \varnothing$ and hence $N_\alpha W_1 \cap N_\alpha W_2 \neq \varnothing$, a contradiction. We conclude that $g \notin L^2(G/N_\alpha)$.

Let $f = Q_\alpha g$ $(= (P_{\alpha+1} - P_\alpha) g = g - P_\alpha g$, since $g \in L^2(G/N_{\alpha+1})$). Then $f \in L^2(G)$ and $f \neq 0$ in $L^2(G)$. Now

$$Q_\alpha f = Q_\alpha^2 g = Q_\alpha g = f;$$

i.e., $Q_\alpha f = f$. Also,

$$\begin{aligned}
(P_\alpha g)(x) &= \int_{N_\alpha} g(t^{-1}x)\, d\lambda_\alpha(t)\\
&= \int_{N_\alpha} 1_{N_{\alpha+1}V}(t^{-1}x)\, d\lambda_\alpha(t)\\
&= \lambda_\alpha\left(N_\alpha \cap x(N_{\alpha+1}V)^{-1}\right), \quad x \in G.
\end{aligned}$$

Then $(P_\alpha g)(x) = 0$ if $x \notin N_\alpha N_{\alpha+1} V = N_\alpha V$. This gives $\operatorname{supp}(P_\alpha g) \subseteq N_\alpha V$. But $\operatorname{supp} g \subseteq N_{\alpha+1} V \subseteq N_\alpha V$. Consequently,

$$\operatorname{supp} f = \operatorname{supp}(g - P_\alpha g) \subseteq N_\alpha V = VN_\alpha \subseteq UN_\alpha.$$

Replacing $f$ by $f/\|f\|_2$, we complete the proof of the lemma. □

Let $G$, $(N_\alpha)_{\alpha \leq \mu}$ and $(Q_\alpha)_{\alpha < \mu}$ be the same as in Lemma 5.1. Let $J$ be a set with $|J| = b(G)$, where $b(G)$ is the smallest cardinality of an open basis at $e \in G$ defined as in §2. Let $\{U_j;\ j \in J\}$ be an open basis at $e$. For each $j \in J$, we choose a symmetric neighborhood $V_j$ of $e$ such that $V_j^2 \subseteq U_j$. If $\alpha < \mu$ and $j \in J$, then, by Lemma 5.1, there exists an $f_\alpha^j \in L^2(G)$ such that $\|f_\alpha^j\|_2 = 1$, $\operatorname{supp} f_\alpha^j \subseteq V_j N_\alpha$, and $Q_\alpha f_\alpha^j = f_\alpha^j$. Let

$$u_\alpha^j = f_\alpha^j * \tilde{f}_\alpha^j, \quad \alpha < \mu,\ j \in J.$$

Then $u_\alpha^j \in A(G)$, $\|u_\alpha^j\| = u_\alpha^j(e) = 1$, $\operatorname{supp} u_\alpha^j \subseteq (V_j N_\alpha)(V_j N_\alpha)^{-1} = V_j N_\alpha N_\alpha^{-1} V_j^{-1} = V_j^2 N_\alpha \subseteq U_j N_\alpha$; i.e., $\operatorname{supp} U_\alpha^j \subseteq U_j N_\alpha$.

Fix $j \in J$. We have $\|u_\alpha^j - u_\beta^j\| \leq \|u_\alpha^j\| + \|u_\beta^j\| = 2$ for $\alpha, \beta < \mu$. Note that $(Q_\gamma)_{\gamma < \mu}$ is an orthogonal net of projections in $VN(G)$, $u_\gamma^j = f_\gamma^j * \tilde{f}_\gamma^j$ and $Q_\gamma f_\gamma^j = f_\gamma^j$. It follows that

$$u_\alpha^j(Q_\beta) = \langle Q_\beta f_\alpha^j, f_\alpha^j \rangle = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \tag{5.4}$$

But $\|Q_\alpha - Q_\beta\| = 1$ if $\alpha, \beta < \mu$ and $\alpha \neq \beta$, since $(Q_\alpha)_{\alpha < \mu}$ is orthogonal. So we get

$$\|u_\alpha^j - u_\beta^j\| \geq \left|(u_\alpha^j - u_\beta^j)(Q_\alpha - Q_\beta)\right| = \left|u_\alpha^j(Q_\alpha - Q_\beta) - u_\beta^j(Q_\alpha - Q_\beta)\right| = 2.$$

Consequently, $\|u_\alpha^j - u_\beta^j\| = \|u_\alpha^j\| + \|u_\beta^j\| = 2$ for all $\alpha, \beta < \mu$ with $\alpha \neq \beta$, that is, $(u_\alpha^j)_{\alpha < \mu}$ is an orthogonal net in $A(G)$.

Let $X = \{\alpha;\ \alpha$ is an ordinal and $\alpha < \mu\}$ directed by its natural order. Direct $J$ by $i < j$ if and only if $U_j \subseteq U_i$, and $J \times X$ by $(i, \alpha) \leq (j, \beta)$ if and only if $i \leq j$ and $\alpha \leq \beta$. A few properties of the net $(u_\alpha^j)_{(j, \alpha) \in J \times X}$ is summarized in the following lemma.

LEMMA 5.2. *Under the same assumptions as above, the net* $(u_\alpha^j)_{(j, \alpha) \in J \times X}$ *has the following properties.*

(i) $u_\alpha^j \in A(G)$, $\|u_\alpha^j\| = u_\alpha^j(e) = 1$ *and* $\operatorname{supp} u_\alpha^j \subseteq U_j N_\alpha$ *for all* $(j, \alpha) \in J \times X$.

(ii) *For each fixed* $j \in J$, $(u_\alpha^j)_{\alpha \in X}$ *is an orthogonal net in* $A(G)$.

(iii) $(u_\alpha^j)_{(j, \alpha) \in J \times X}$ *is topologically convergent to invariance; that is, if* $v \in A(G)$ *and* $v(e) = 1$, *then*

$$\lim_{(j, \alpha) \in J \times X} \|v u_\alpha^j - u_\alpha^j\| = 0.$$

*Proof.* By the above argument, we only need to show that the net $(u_\alpha^j)_{(j, \alpha) \in J \times X}$ possesses property (iii). Our proof follows Renaud [25, Proposition 3]. Let $\varepsilon > 0$ and $K$ be a compact neighborhood of $e$ in $G$. Then there exists a $u \in A(G)$ such that $u = 1$ on $K$. Now, $(v - u)(e) = 0$. Since points are synthesis for $A(G)$ (see [6, (4.11) Corollary 2]), there exists a $w \in A(G) \cap C_{00}(G)$ such that $\|(v - u) - w\| < \varepsilon$ *and* $w = 0$ *on some neighborhood* $U$ *of* $e$.

Note that $\operatorname{supp} u_\alpha^j \subseteq U_j N_\alpha$. If $U_j N_\alpha \subseteq U \cap K$, then $u u_\alpha^j = u_\alpha^j$ and $w u_\alpha^j = 0$. Hence for such $u_\alpha^j$, we have

$$\begin{aligned} \|v u_\alpha^j - u_\alpha^j\| &= \|(v - u - w) u_\alpha^j\| \\ &= \|v - u - w\|\, \|u_\alpha^j\| \\ &= \|v - u - w\| < \varepsilon. \end{aligned}$$

Therefore, by the direction on $J \times X$, we only have to show that there exists an element $(j_0, \alpha_0) \in J \times X$ such that $U_{j_0} N_{\alpha_0} \subseteq U \cap K$. Choose a neighborhood $V$ of $e$ such that $V^2 \subseteq U \cap K$. Since $(N_\alpha)_{0<\alpha<\mu}$ is a decreasing net of compact subgroups of $G$ and $\bigcap_{0<\alpha<\mu} N_\alpha = \{e\}$, there exits an $\alpha_0 < \mu$ such that $N_{\alpha_0} \subseteq V$. Let $j_0 \in J$ be such that $U_{j_0} \subseteq V$. Then $U_{j_0} N_{\alpha_0} \subseteq V^2 \subseteq U \cap K$. □

*Remark* 5.3. Recall that if $N$ is a compact normal subgroup of a locally compact group $G$, then $A(G/N)$ embeds into $A(G)$ (corresponding to the subspace of all $N$-periodic functions in $A(G)$, see [6, Proposition (3.25)]). In our case, now $\bigcup_{0<\alpha<\mu} A(G/N_\alpha)$ is norm dense in $A(G)$ (since $\bigcup_{0<\alpha<\mu} L^2(G/N_\alpha)$ is norm dense in $L^2(G)$). For a fixed $0 < \alpha < \mu$, $(u_\alpha^j)_{j\in J}$ may not be topologically convergent to invariance. However, since $N_\alpha$ is synthesis for $A(G)$ (see [15, p. 94]), we still can show that $(u_\alpha^j)_{j\in J}$ is topologically convergent to invariance "for $A(G/N_\alpha)$"; that is,

$$\lim_{j\in J} \|vu_\alpha^j - u_\alpha^j\| = 0$$

for all $v \in A(G)$ with $v = 1$ on $N_\alpha$. But this fact will not be needed in the sequel.

For a directed set $X$, let $\mathscr{F}(X)$ be the subset of $l^\infty(X)^*$ defined as in §2. Chou in [2] showed that if $G$ is a non-discrete metrizable locally compact group, then there exists an orthogonal sequence in $A(G)$ which is topologically convergent to invariance. Using such a sequence, he constructed a linear isometry of $(l^\infty)^*$ into $VN(G)^*$ which embeds the large set $\mathscr{F}(\mathbf{N})$ into $TIM(\hat{G})$ (see [2, Theorem 3.3]). Recall that the net $(u_\alpha^j)_{(j,\alpha)\in J\times X}$ in Lemma 5.2 is topologically convergent to invariance and $(u_\alpha^j)_{\alpha\in X}$ is orthogonal for each fixed $j \in J$. Thus in case $G$ is non-metrizable, although we can not set up one linear isometry embedding a big set into $TIM(\hat{G})$, we still have the following weaker version of Chou's results obtained by modifying his technique.

THEOREM 5.4. *Let $G$ be a $\sigma$-compact non-metrizable locally compact group and $(N_\alpha)_{\alpha\le\mu}$ be the decreasing family of normal subgroups of $G$ as in Proposition 4.3. Let $X = \{\alpha;\ \alpha < \mu\}$ with its natural order and $(u_\alpha^j)_{(j,\alpha)\in J\times X}$ be the same net in $A(G)$ as in Lemma 5.2. For every element $j$ in $J$, define $\pi_j : VN(G) \to l^\infty(X)$ by*

$$\pi_j(T)(\alpha) = \langle T, u_\alpha^j\rangle, \quad T \in VN(G),\ \alpha \in X.$$

*Then*:

(a) *For each $j \in J$, $\pi_j$ is a positive linear mapping of $VN(G)$ onto $l^\infty(X)$ with $\|\pi_j\| = 1$ and the conjugate $\pi_j^*$ is a linear isometry of $l^\infty(X)^*$ into $VN(G)^*$.*

(b) *For each* $\phi \in l^\infty(X)^*$, *if we let*

$$W_\phi = \left\{ \textit{all } w^*\textit{-cluster points of } \left(\pi_j^*\phi\right)_{j\in J} \textit{ in } VN(G)^* \right\},$$

*then* $W_\phi \neq \varnothing$, $W_\phi \subseteq TIM(\hat{G})$ *if* $\phi \in \mathscr{F}(X)$, *and the family* $\{W_\phi;\ \phi \in l^\infty(X)^*\}$ *is pairwise disjoint.*

*Proof.* (a) Fix $j \in J$. Clearly, $\pi_j$ is linear, $\pi_j(I)$ is the constant function of value one, and $\pi_j(T) \geq \mathbf{0}$ if $T \geq \mathbf{0}$. If $T \in VN(G)$ and $\alpha \in X$, then $|\pi_j(T)(\alpha)| = |\langle T, u_\alpha^j\rangle| \leq \|T\|\,\|u_\alpha^j\| = \|T\|$. Therefore, $\|\pi_j\| = 1$. To see that $\pi_j$ is onto and $\pi_j^*$ is an isometry, we only have to show that for each $f \in l^\infty(X)$, there exists a $T \in VN(G)$ such that $\pi_j(T) = f$ and $\|T\| = \|f\|_\infty$.

Let $\Lambda = \Lambda(X)$ be the set of all non-empty finite subsets of $X$ directed by inclusion. Let $f \in l^\infty(X)$. For each $\tau \in \Lambda$, let $S_\tau = \sum_{\alpha\in\tau} f(\alpha) Q_\alpha$. Since $(Q_\alpha)_{\alpha<\mu}$ is an orthogonal net of projections in $VN(G)$ and $f \in l^\infty(X)$, then

$$\|S_\tau\| \leq \|f\|_\infty \quad \text{for all } \tau \in \Lambda,$$

and the net $(S_\tau)_{\tau\in\Lambda}$ is convergent in the weak operator topology to an operator $T \in VN(G)$ with $\|T\| \leq \|f\|_\infty$. Recall that on $VN(G)$ the weak operator topology coincides with the $\sigma(VN(G), A(G))$-topology. Consequently, by $u_\alpha^j \in A(G)$ and formula (5.4), we get

$$\begin{aligned}
\pi_j(T)(\alpha) &= \langle T, u_\alpha^j\rangle = \lim_{\tau\in\Lambda} \langle S_\tau, u_\alpha^j\rangle \\
&= \lim_{\tau\in\Lambda} \sum_{\beta\in\tau} f(\beta)\langle Q_\beta, u_\alpha^j\rangle \\
&= f(\alpha) \quad \text{for all } \alpha \in X;
\end{aligned}$$

i.e., $\pi_j(T) = f$. In particular, $\|f\|_\infty \leq \|\pi_j\|\,\|T\| = \|T\|$, and hence $\|T\| = \|f\|_\infty$. This completes the proof of (a).

An interesting fact here is that the above operator $T$ is independent of the choice of $j$ in $J$, that is, given $f \in l^\infty(X)$, there exists a "common" $T \in VN(G)$ such that

$$\|T\| = \|f\|_\infty \text{ and } \pi_j(T) = f \text{ for all } j \in J.$$

We need this fact later.

(b) Let $\phi \in l^\infty(X)^*$. Since $\|\pi_j^*\phi\| = \|\phi\|$ for all $j \in J$ and the unit ball in $VN(G)^*$ is $w^*$-compact, then the net $(\pi_j^*\phi)_{j\in J}$ must have a $w^*$-cluster point in $VN(G)^*$. So, $W_\phi \neq \varnothing$.

Let $\phi \in \mathscr{F}(X)$ and $F \in W_\phi$. Then there exists a subnet $(\pi_{j'}^*\phi)_{j'}$ of $(\pi_j^*\phi)_{j\in J}$ such that $\pi_{j'}^*\phi \to F$ in the $\sigma(VN(G)^*, VN(G))$-topology. Now,

$$\|F\| \leq \liminf_{j'} \|\pi_{j'}^*\phi\| = \|\phi\| = 1,$$

and

$$\langle F, I \rangle = \lim_{j'} \langle \pi_{j'}^* \phi, I \rangle = \lim_{j'} \big\langle \phi, \pi_{j'}(I) \big\rangle = \phi(\mathbf{1}) = 1,$$

where $\mathbf{1}$ is the constant function of value one. Therefore, $\|F\| = \langle F, I \rangle = 1$. Let $T \in VN(G)$ and $v \in \hat{A}(G)$ with $v(e) = 1$. Then

$$\langle F, v \cdot T - T \rangle = \lim_{j'} \langle \pi_{j'}^* \phi, v \cdot T - T \rangle = \lim_{j'} \big\langle \phi, \pi_{j'}(v \cdot T - T) \big\rangle. \quad (5.5)$$

By Lemma 5.2, $\lim_{j', \alpha} \|v u_\alpha^{j'} - u_\alpha^{j'}\| = 0$. Thus, we get

$$\begin{aligned} \lim_{j', \alpha} \pi_{j'}(v \cdot T - T)(\alpha) &= \lim_{j', \alpha} \langle v \cdot T - T, u_\alpha^{j'} \rangle \\ &= \lim_{j', \alpha} \langle T, v u_\alpha^{j'} - u_\alpha^{j'} \rangle = 0. \end{aligned}$$

So, given $\varepsilon > 0$, there exists $j_0'$ and $\alpha_0$ such that

$$\big| \pi_{j'}(v \cdot T - T)(\alpha) \big| < \varepsilon \quad \text{for all } (j', \alpha) \geq (j_0', \alpha_0). \quad (5.6)$$

Since $\phi \in \mathscr{F}(X)$, then (5.6) implies that

$$\Big| \big\langle \phi, \pi_{j'}(v \cdot T - T) \big\rangle \Big| \leq \varepsilon \quad \text{for all } j' \geq j_0'. \quad (5.7)$$

Consequently, (5.5) and (5.7) combined give

$$\langle F, v \cdot T - T \rangle = \lim_{j'} \big\langle \phi, \pi_{j'}(v \cdot T - T) \big\rangle = 0;$$

i.e., $\langle F, v \cdot T \rangle = \langle F, T \rangle$ for all $T \in VN(G)$, $v \in A(G)$ with $v(e) = 1$. We conclude that $W_\phi \subseteq TIM(\hat{G})$ for all $\phi \in \mathscr{F}(X)$.

Let $\phi_1, \phi_2 \in l^\infty(X)^*$ be two different elements. Assume that $F \in W_{\phi_1} \cap W_{\phi_2}$. Let $f \in l^\infty(X)$. By the fact mentioned after the proof of (a), there exists a "common" $T \in VN(G)$ such that

$$\pi_j(T) = f \quad \text{for all } j \in J.$$

Then $\langle \phi_1, f \rangle = \langle \phi_1, \pi_j(T) \rangle = \langle \pi_j^* \phi_1, T \rangle$ for all $j \in J$. Similarly, we have $\langle \phi_2, f \rangle = \langle \pi_j^* \phi_2, T \rangle$ for all $j \in J$. By taking limits on subnets, we thus get $\langle \phi_1, f \rangle = \langle F, T \rangle$ and $\langle \phi_2, f \rangle = \langle F, T \rangle$; i.e., $\langle \phi_1, f \rangle = \langle \phi_2, f \rangle$. This is true for all $f \in l^\infty(X)$. It follows that $\phi_1 = \phi_2$, contradicting the fact that $\phi_1 \neq \phi_2$. Therefore, $W_{\phi_1} \cap W_{\phi_2} = \varnothing$ for all $\phi_1, \phi_2 \in l^\infty(X)^*$ with $\phi_1 \neq \phi_2$. This completes the proof of the theorem. $\square$

Recall that if $A$ is a set, then $2^A$ denotes the set of all subsets of $A$. The above theorem together with the embedding results for $VN(G)^*$ will yield the following result.

COROLLARY 5.5. *Let $G$ be a non-discrete locally compact group. Let $\mu$ be the initial ordinal satisfying $|\mu| = b(G)$ and $X = \{\alpha;\ \alpha < \mu\}$ with its natural order. Then there exists a one-one map $W : l^\infty(X)^* \to 2^{VN(G)^*}$ such that:*

(i) *$W(\phi) \neq \varnothing$ for all $\phi \in l^\infty(X)^*$;*
(ii) *$W(\phi_1) \cap W(\phi_2) = \varnothing$ if $\phi_1, \phi_2 \in l^\infty(X)^*$ and $\phi_1 \neq \phi_2$;*
(iii) *$W(a\phi) = aW(\phi)$ and $W(\phi_1 + \phi_2) \subseteq W(\phi_1) + W(\phi_2)$ for all $\phi, \phi_1, \phi_2 \in l^\infty(X)^*$ and $a \in \mathbf{C}$;*
(iv) *$W(\phi) \subseteq TIM(\hat{G})$ if $\phi \in \mathscr{F}(X)$.*

*Proof.* When $G$ is metrizable, this corollary is a consequence of Chon [2, Theorem 3.3]. In the following we assume that $G$ is non-metrizable.

If $G$ is $\sigma$-compact, let $W : l^\infty(X)^* \to 2^{VN(G)^*}$ be defined by $W(\phi) = W_\phi$, where $W_\phi \subseteq VN(G)^*$ is the same as in Theorem 5.4(b). Then $W$ satisfies (i), (ii) and (iv). It is easy to check that $W$ also satisfies (iii).

In the general case ($G$ not necessarily $\sigma$-compact), let $G_0$ be a compactly generated open subgroup of $G$. Let $t : A(G_0) \to A(G)$ be the extension map defined by $tv = \mathring{v}$, where $\mathring{v} = v$ on $G_0$ and 0 outside $G_0$. Then, by Granirer [10, Theorem 3], $t^{**}$ is a linear isometry of $VN(G_0)^*$ into $VN(G)^*$ and $t^{**}(TIM(\hat{G}_0)) = TIM(\hat{G})$. Note that now $G_0$ is $\sigma$-compact and non-metrizable and $b(G_0) = b(G)$. Let $W_1 : l^\infty(X)^* \to 2^{VN(G_0)^*}$ be the map given in the previous paragraph. Define $W = \widehat{t^{**}} \circ W_1$, where $\widehat{t^{**}} : 2^{VN(G_0)^*} \to 2^{VN(G)^*}$ is the map generated by $t^{**}$; i.e., $\widehat{t^{**}}(\mathscr{E}) = \{t^{**}F;\ F \in \mathscr{E}\}$ for all $\mathscr{E} \subseteq VN(G_0)^*$. Then $W : l^\infty(X)^* \to 2^{VN(G)^*}$ has properties (i)–(iv). □

COROLLARY 5.6. *Let $G$ be a non-discrete locally compact group. Then*

$$|TIM(\hat{G})| \geq 2^{2^{b(G)}}.$$

*Proof.* Let $\mu$ be the initial ordinal with $|\mu| = b(G)$ and $X = \{\alpha;\ \alpha < \mu\}$. Let $W : l^\infty(X)^* \to 2^{VN(G)^*}$ be the one-one map in Corollary 5.5. Then, by properties (i), (ii), and (iv) and Proposition 3.3, we have

$$|TIM(\hat{G})| \geq |\mathscr{F}(X)| = 2^{2^{|X|}} = 2^{2^{b(G)}}. \ \square$$

To show that the equality in Corollary 5.6 holds, we need two more technical lemmas.

LEMMA 5.7. *Let $G$ be a non-discrete locally compact group and $K$ be a compact subset of $G$. Let*

$$C_K(G) = \{f; f \in C_{00}(G) \text{ and } \operatorname{supp} f \subseteq K\}.$$

*Then there exists a subset $\mathscr{L}$ of $L^2(G)$ such that $|\mathscr{L}| \leq b(G)$ and $\mathscr{L}$ is $\|\cdot\|_2$-dense in $C_K(G)$.*

*Proof.* Choose a set $J$ with $|J| = b(G)$. Let $\{U_j; j \in J\}$ be an open basis at the unit element $e$ of $G$. Since $K$ is compact, for each fixed $j \in J$, there exist $x_1^j, \ldots, x_{n_j}^j \in K$ such that $K \subseteq \bigcup_{k=1}^{n_j} x_k^j U_j$. Let $\mathscr{E}_0$ be the set of all such sets $x_k^j U_j \cap K$, $j \in J$ and $k = 1, \ldots, n_j$. Then $|\mathscr{E}_0| \leq |J| = b(G)$ (since $b(G)$ is infinite), and $\mathscr{E}_0$ is a basis for open sets in $K$ (with the relative topology). Let

$$\mathscr{E} = \left\{E; E = \bigcup_{k=1}^{n} H_k \text{ for some } H_1, \ldots, H_n \in \mathscr{E}_0\right\}.$$

Then we still have $|\mathscr{E}| \leq b(G)$. Define

$$\mathscr{L} = \left\{\sum_{k=1}^{n} a_k 1_{E_k}; a_k \in \mathbf{Q}_c, E_k \in \mathscr{E}, k = 1, \ldots, n\right\},$$

where $\mathbf{Q}_c = \{a + ib \in \mathbf{C};\ a, b \text{ are rationals}\}$. Then $\mathscr{L} \subseteq L^2(G)$ and $|\mathscr{L}| \leq b(G)$, since $\mathbf{Q}_c$ is countable and $|\mathscr{E}| \leq b(G)$ (with $b(G)$ infinite).

We claim that $\mathscr{L}$ is $\|\cdot\|_2$-dense in $C_K(G)$. We can assume that $\lambda(K) > 0$. Let $f \in C_K(G)$ with $\|f\|_\infty > 0$ and let $\varepsilon > 0$. Then there exists a partition $\{F_k; k = 1, \ldots, n\}$ of $\operatorname{supp} f$ ($\subseteq K$) such that each $F_k$ is measurable and

$$|f(x) - f(y)| < \delta_1 \quad \text{for } x, y \in F_k,\ k = 1, \ldots, n,$$

where $\delta_1 = \varepsilon(4\lambda(K)^{1/2})^{-1}$. By the density of $\mathbf{Q}_c$ in $\mathbf{C}$, for each $k$, we can choose an $a_k \in \mathbf{Q}_c$ such that

$$|a_k| \leq \|f\|_\infty \text{ and } |f(x) - a_k| < 2\delta_1 \text{ for } x \in F_k. \tag{5.8}$$

Fix $1 \leq k \leq n$. By the regularity of the left Haar measure $\lambda$ of $G$, there exist an open set $O_k$ and a compact set $M_k$ such that $M_k \subseteq F_k \subseteq O_k$ and

$$\lambda(O_k \setminus M_k) < \delta_2, \tag{5.9}$$

where $\delta_2 = \varepsilon^2(2n\|f\|_\infty)^{-2}$. Note that $\mathscr{E}_0$ is a basis for open sets in $K$, $O_k \cap K$ is open in $K$ and $M_k$ is compact. Then there exist $H_1^k, \ldots, H_m^k \in \mathscr{E}_0$ such that $M_k \subseteq \bigcup_{l=1}^m H_l^k \subseteq O_k \cap K$. Let $E_k = \bigcup_{l=1}^m H_l^k$. Then $E_k \in \mathscr{E}$. Now $M_k \subseteq E_k \subseteq O_k$ and $M_k \subseteq F_k \subseteq O_k$. Hence (5.9) implies

$$\lambda(E_k \Delta F_k) \leq \lambda(O_k \setminus M_k) < \delta_2, \tag{5.10}$$

where $E_k \Delta F_k$ is the symmetric difference of $E_k$ and $F_k$. Let

$$g = \sum_{k=1}^{n} a_k 1_{E_k}.$$

Then $g \in \mathscr{S}$. Recall that $f = \sum_{k=1}^{n} f 1_{F_k}$. Hence,

$$\begin{aligned}
|f - g| &= \left| \sum_{K=1}^{n} (f 1_{F_k} - a_k 1_{E_k}) \right| \leq \sum_{k=1}^{n} |f 1_{F_k} - a_k 1_{E_k}| \\
&= \sum_{k=1}^{n} |f - a_k| 1_{F_k \cap E_k} + \sum_{k=1}^{n} |f| 1_{F_k \setminus E_k} + \sum_{K=1}^{n} |a_k| 1_{E_k \setminus F_k} \\
&\leq 2\delta_1 \sum_{k=1}^{n} 1_{F_k \cap E_k} + \|f\|_\infty \sum_{k=1}^{n} 1_{F_k \Delta E_k} \qquad \text{(by (5.8))} \\
&\leq 2\delta_1 1_K + \|f\|_\infty \sum_{k=1}^{n} 1_{F_k \Delta E_k}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\|f - g\|_2 &\leq 2\delta_1 \|1_K\|_2 + \|f\|_\infty \sum_{k=1}^{n} \|1_{F_k \Delta E_k}\|_2 \\
&= 2\delta_1 \lambda(K)^{1/2} + \|f\|_\infty \sum_{k=1}^{n} \lambda(F_k \Delta E_k)^{1/2} \\
&< \frac{\varepsilon}{2} + \|f\|_\infty n \delta_2^{1/2} \qquad \text{(by (5.10))} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon;
\end{aligned}$$

i.e.,, $\|f - g\|_2 < \varepsilon$. It follows that $\mathscr{S}$ is $\|\cdot\|_2$-dense in $C_K(G)$. □

LEMMA 5.8. *Let $G$ be a non-discrete locally compact group and $V$ a compact subset of $G$. Let*

$$A_V(G) = \{v;\ v \in A(G) \text{ and } \operatorname{supp} v \subseteq V\}.$$

*Then there exists a subset $\mathscr{S}$ of $A(G)$ such that $|\mathscr{S}| \leq b(G)$ and $\mathscr{S}$ is $\|\cdot\|_{A(G)}$-dense in $A_V(G)$.*

*Proof.* Choose a compact neighborhood $K$ of $e$ such that $V \subseteq K$. Define

$$A_1 = \operatorname{span}\{f * \tilde{g},\ f, g \in C_{00}(G),\ \operatorname{supp} f \subseteq K,\ \operatorname{supp} g \subseteq K\},$$

where span $E$ means the linear span of $E$.

Assume that there exists a $v \in A_V(G)$ such that $v \notin \overline{A_1}$ (the norm closure of $A_1$ in $A(G)$). Then by the Hahn-Banach Theorem, there exists a $T \in VN(G) = A(G)^*$ such that $\langle T, v \rangle \neq 0$ but $\langle T, f * \tilde{g} \rangle = \langle Tf, g \rangle = 0$ for all $f, g \in C_{00}(G)$ with $\operatorname{supp} f \subseteq K$ and $\operatorname{supp} g \subseteq K$. By the definition of $\operatorname{supp} T$, we have that $K \subseteq G \backslash \operatorname{supp} T$; i.e., $\operatorname{supp} T \subseteq G \backslash K \subseteq G \backslash V$. Note that $v \in A_V(G)$ and hence $\operatorname{supp} v \subseteq V$. It follows that $\operatorname{supp} v \cap \operatorname{supp} T = \varnothing$. By [6, Proposition (4.6) and (4.8)], $v \cdot T = \mathbf{0}$. We choose a $u \in A(G)$ such that $u = 1$ on $V$. Then $vu = v$, and hence

$$0 = \langle v \cdot T, u \rangle = \langle T, uv \rangle = \langle T, v \rangle,$$

contradicting the fact that $\langle T, v \rangle \neq 0$. We conclude that $A_V(G) \subseteq \overline{A_1}$.

Let

$$C_K(G) = \{f; f \in C_{00}(G) \text{ and } \operatorname{supp} f \subseteq K\}.$$

By Lemma 5.7, there exists an $\mathscr{L} \subseteq L^2(G)$ such that $|\mathscr{L}| \leq b(G)$ and $\mathscr{L}$ is $\|\cdot\|_2$-dense in $C_K(G)$. Define

$$\mathscr{S} = \left\{ \sum_{i=1}^{n} a_i f_i * \tilde{g}_i;\ a_i \in \mathbf{Q}_c, f_i, g_i \in \mathscr{L}, i = 1, \dots, n \right\},$$

where $\mathbf{Q}_c$ is the same dense subset of $\mathbf{C}$ as in the proof of Lemma 5.7. Then $|\mathscr{S}| \leq b(G)$, since $\mathbf{Q}_c$ is countable and $|\mathscr{L}| \leq b(G)$. Since $\mathscr{L}$ is $\|\cdot\|_2$-dense in $C_K(G)$ and $\mathbf{Q}_c$ is dense in $\mathbf{C}$, by the definition of $A(G)$, $\mathscr{S}$ is $\|\cdot\|_{A(G)}$-dense in $A_1$. Recall that $A_V(G) \subseteq \overline{A_1}$. Therefore, $\mathscr{S}$ is $\|\cdot\|_{A(G)}$-dense in $A_V(G)$. □

We are now ready to find out the precise cardinality of $TIM(\hat{G})$ for any non-discrete locally compact group $G$.

THEOREM 5.9. *Let $G$ be a non-discrete locally compact group. Let $b(G)$ be the smallest cardinality of an open basis at the unit element $e$ of $G$. Then*

$$|TIM(\hat{G})| = 2^{2^{b(G)}}.$$

*Proof.* By Corollary 5.6, we only have to show that $|TIM(\hat{G})| \leq 2^{2^{b(G)}}$.

Let $U$ and $V$ be two compact neighborhoods of $e$ in $G$ such that $U \subsetneqq V$. We choose two functions $u_0$ and $v_0$ in $A(G)$ such that $u_0(e) = 1$, $v_0 = 1$ on $U$, $\operatorname{supp} u_0 \subseteq U$ and $\operatorname{supp} v_0 \subseteq V$. Then $u_0 = u_0 v_0$. Let

$$\mathscr{B} = \{u_0 \cdot T; T \in VN(G)\}.$$

Then $\mathscr{B}$ is a subspace of $VN(G)$, and each $m \in TIM(\hat{G})$ is determined by its value on $\mathscr{B}$, by the definition of $TIM(\hat{G})$. Hence we have

$$\left|TIM(\hat{G})\right| \leq c^{|\mathscr{B}|}, \tag{5.11}$$

where $c$ is the cardinality of the continuum.

In the following we shall prove that $|\mathscr{B}| \leq c^{b(G)}$. Let $T \in VN(G)$ and $v \in A(G)$. Then

$$\langle u_0 \cdot T, v\rangle = \langle T, u_0 v\rangle = \langle T, u_0 v_0 v\rangle = \langle u_0 \cdot T, v_0 v\rangle. \tag{5.12}$$

Now $v_0 v \in A(G)$ with support contained in $V$. Define

$$A_V(G) = \{v \in A(G); \operatorname{supp} v \subseteq V\}.$$

Then, by (5.12), each $u_0 \cdot T \in \mathscr{B}$ is determined by its value on $A_V(G)$. By Lemma 5.8, there exists an $\mathscr{S} \subseteq A(G)$ such that $|\mathscr{S}| \leq b(G)$ and $\mathscr{S}$ is $\|\cdot\|_{A(G)}$-dense in $A_V(G)$. Hence each $u_0 \cdot T \in \mathscr{B}$ is determined by its value on $\mathscr{S}$. Consequently,

$$|\mathscr{B}| \leq c^{|\mathscr{S}|} \leq c^{b(G)}. \tag{5.13}$$

Finally, (5.11) and (5.13) combined give

$$\left|TIM(\hat{G})\right| \leq c^{|\mathscr{B}|} \leq c^{c^{b(G)}} = 2^{2^{b(G)}},$$

since $b(G)$ is infinite. □

*Remark* 5.10. Lau and Paterson showed that if $G$ is a non-compact amenable locally compact group, then $|MTL(G)| = 2^{2^{d(G)}}$, where $MTL(G)$ is the set of all topologically left invariant means on $L^\infty(G)$ and $d(G)$ is the smallest cardinality of a covering of $G$ by compact sets (see [20, Theorem 1]). When $G$ is abelian and $\hat{G}$ is the dual group of $G$, $A(G)$ can be identified with $L^1(\hat{G})$ (by Fourier transform) and $VN(G)$ with $L^\infty(\hat{G})$; each $f \in L^\infty(\hat{G})$ can be regarded as a multiplication operator on $L^2(\hat{G})$ which is isomorphic to $L^2(G)$ by Plancherel's theorem. Under these identifications, the module action of $L^1(\hat{G})$ on $L^\infty(\hat{G})$ is just the usual convolution. Consequently, $m \in VN(G)^*$ belongs to $TIM(\hat{G})$ if and only if the corresponding mean on $L^\infty(\hat{G})$ is a topologically left invariant mean. In particular, $|TIM(\hat{G})| = |MTL(\hat{G})|$. Now $b(G) = d(\hat{G})$ (see [17, (24.48)]). Therefore, when $G$ is abelian, our Theorem 5.9 coincides with Lau-Paterson's result.

## 6. Some applications

For a locally compact group $G$, let $b(G)$ be the smallest cardinality of an open basis at the unit element $e$ of $G$ defined as before. The format of the following proposition and corollary is due to Chou [2]. He discussed the case when $G$ is metrizable.

PROPOSITION 6.1. *If $G$ is a non-discrete locally compact group, then $TIM(\hat{G})$ contains a subset $E$ such that $|E| = |TIM(\hat{G})| = 2^{2^{b(G)}}$ and if $m_1, m_2 \in E$ and $m_1 \neq m_2$, then $\|m_1 - m_2\| = 2$. In particular, $TIM(\hat{G})$ is not norm separable.*

*Proof.* When $G$ is metrizable, this is shown by Chou (see [2, Corollary 3.5]).

In the following we assume that $G$ is non-metrizable. By Granirer [10, Theorem 3], we may assume that $G$ is $\sigma$-compact. Let $\mu$ be the limit ordinal associated with $G$ as in Proposition 4.3, $X = \{\alpha;\ \alpha < \mu\}$ with its natural order and $\mathscr{F}(X)$ the subset of $l^\infty(X)^*$ define as in §2. Let

$$\mathscr{A} = \{\phi \in \beta X;\ \phi \text{ contains } \{T_\alpha;\ \alpha \in X\}\},$$

where $\beta X$ is the Stone-Čech compactification of the discrete set $X$ and $T_\alpha$ is a tail in $X$ as in §2. Then, by Lemma 2.2 and the proof of Proposition 3.3, $\mathscr{A} \subseteq \mathscr{F}(X)$ and $|\mathscr{A}| = 2^{2^{|X|}} = 2^{2^{b(G)}}$.

Let $\phi_1, \phi_2 \in \mathscr{A}$ with $\phi_1 \neq \phi_2$. Then $\|\phi_1 - \phi_2\| = 2$, since $\phi_1, \phi_2 \in \beta X$. Let $\psi_1 \in W_{\phi_1}$ and $\psi_2 \in W_{\phi_2}$, where $W_\phi$ is the non-empty subset of $TIM(\hat{G})$ defined for each $\phi \in \mathscr{F}(X)$ as in Theorem 5.4. Then, there exist subnets $(\pi_{j_1}^*)_{j_1}$ and $(\pi_{j_2}^*)_{j_2}$ of $(\pi_j^*)_{j \in J}$, where $(\pi_j^*)_{j \in J}$, is the net of linear maps associated with $G$ as in Theorem 5.4, such that

$$\pi_{j_1}^*\phi_1 \to \psi_1 \quad \text{and} \quad \pi_{j_2}^*\phi_2 \to \psi_2$$

in the $\sigma(VN(G)^*, VN(G))$-topology. Since $\|\psi_1\| = \|\psi_2\| = 1$, $\|\psi_1 - \psi_2\| \leq 2$. On the other hand, if $f \in l^\infty(X)$ with $\|f\|_\infty = 1$, then, by the fact mentioned in the proof of Theorem 5.4, there exists a "common" $T \in VN(G)$ such that $\|T\| = \|f\|_\infty = 1$ and

$$\pi_j(T) = f, \quad \text{for all } j \in J.$$

Hence, we get

$$\begin{aligned}
\|\psi_1 - \psi_2\| &\geq |\langle \psi_1 - \psi_2, T\rangle| \\
&= \lim_{j_1, j_2} |\langle \pi_{j_1}^*\phi - \pi_{j_2}^*\phi_2, T\rangle| \\
&= \lim_{j_1, j_2} |\langle \phi_1, \pi_{j_1}(T)\rangle - \langle \phi_2, \pi_{j_2}(T)\rangle| \\
&= |\langle \phi_1, f\rangle - \langle \phi_1, f\rangle| \\
&= |\langle \phi_1 - \phi_2, f\rangle|,
\end{aligned}$$

that is,

$$\|\psi_1 - \psi_2\| \geq |\langle \phi_1 - \phi_2, f\rangle| \quad \text{for all } f \in l^\infty(X) \text{ with } \|f\|_\infty = 1.$$

It follows that $\|\psi_1 - \psi_2\| \geq \|\phi_1 - \phi_2\| = 2$. Consequently, $\|\psi_1 - \psi_2\| = 2$ for $\psi_1 \in W_{\phi_1}$ and $\psi_2 \in W_{\phi_2}$.

For each $\phi \in \mathscr{A}$, choose a $\psi \in W_\phi$. Let $E$ be the set of all such $\psi$. Then $|E| = |\mathscr{A}| = 2^{2^{b(G)}}$ and $\|m_1 - m_2\| = 2$ for all $m_1, m_2 \in E$ with $m_1 \neq m_2$. □

Recall that $F(\hat{G})$ is the space of all $T \in VN(G)$ such that $m(T)$ equals a fixed constant $d(T)$ as $m$ runs through $TIM(\hat{G})$. Also, each $m \in TIM(\hat{G})$ is determined by its value on $UCB(\hat{G})$ and $W(\hat{G}) \subseteq F(\hat{G})$. Thus Theorem 5.9 will yield the following result.

Corollary 6.2. *Let $G$ be a non-discrete locally compact group. If $\mathscr{B}$ is a norm dense subset of the quotient Banach space $VN(G)/F(\hat{G})$ (or $UCB(\hat{G})/F(\hat{G}) \cap UCB(\hat{G})$), then*

$$|\mathscr{B}| > b(G).$$

*In particular, $VN(G)/F(\hat{G})$, $UCB(\hat{G})/F(\hat{G}) \cap UCB(\hat{G})$ and $UCB(\hat{G})/W(\hat{G}) \cap UCB(\hat{G})$ are not norm separable.*

*Proof.* Assume that $\mathscr{B}$ is norm dense in $VN(G)/F(\hat{G})$. Then there exists a subset $\mathscr{D}$ of $VN(G)$ such that $|\mathscr{D}| = |\mathscr{B}|$ and the set

$$\mathscr{E} = \{T + S;\ T \in \mathscr{D} \text{ and } S \in F(\hat{G})\}$$

in norm dense in $VN(G)$. Thus each $m \in TIM(\hat{G})$ is determined by its value on $\mathscr{E}$. Fix an $m_0 \in TIM(\hat{G})$. We have

$$m(T+S) = m(T) + m(S) = m(T) + m_0(S),$$

for all $m \in TIM(\hat{G})$, $T \in \mathscr{D}$ and $S \in F(\hat{G})$. Therefore, each $m \in TIM(\hat{G})$ is determined by its value on $\mathscr{D}$. Consequently, we have

$$|TIM(\hat{G})| \leq c^{|\mathscr{D}|} = c^{|\mathscr{B}|} = 2^{\aleph_0|\mathscr{B}|}, \tag{6.1}$$

where $c$ is the cardinality of the continuum and $\aleph_0$ is the first infinite cardinal number. On the other hand, by Theorem 5.9,

$$|TIM(\hat{G})| = 2^{2^{b(G)}} > 2^{b(G)}. \tag{6.2}$$

Now (6.1) and (6.2) combined give

$$\aleph_0|\mathscr{B}| > b(G).$$

But $b(G) \geq \aleph_0$, since $G$ is non-discrete. Therefore, $|\mathscr{B}| > b(G)$.

Similarly, we can prove the $UCB(\hat{G})/F(\hat{G}) \cap UCB(\hat{G})$ case, since each $m \in TIM(\hat{G})$ is determined by its value on $UCB(\hat{G})$, by the definitions of $TIM(\hat{G})$ and $UCB(\hat{G})$. □

If $u \in A(G)$ with $u(e) = 1$, let

$$u^{\perp} = \{T \in VN(G); u \cdot T = \mathbf{0}\}.$$

If $T \in u^{\perp}$ and $m \in TIM(\hat{G})$, then $m(T) = m(u \cdot T) = m(\mathbf{0}) = 0$. Hence $u^{\perp} \subseteq F(\hat{G})$. Note that $W(\hat{G}) \subseteq F(\hat{G})$. By the same procedure as in the proof of Corollary 6.2, we can also prove the following.

COROLLARY 6.3 (Granirer [11, Theorem 12]). *If $G$ is a locally compact group such that there exists a $u \in A(G)$ with $u(e) = 1$ and an $X$, a norm separable subspace of $VN(G)$, such that $UCB(\hat{G})$ is contained in the norm closure of $W(\hat{G}) + u^{\perp} + X$, then $G$ is discrete.*

*Acknowledgments.* This paper will form part of the author's Ph.D. thesis under the guidance of Professor Anthony T. Lau, to whom she wishes to express her deepest gratitude for the valuable suggestions and constant encouragement. The author is also indebted to Professors Lau and Losert for providing her a preprint of their paper [19].

REFERENCES

1. C. CHOU, *The exact cardinality of the set of invariant means on a group*, Proc. Amer. Math. Soc. **55** (1976), 103–106.
2. ______, *Topological invariant means on the von Neumann algebra VN(G)*, Trans. Amer. Math. Soc. **273** (1982), 207–229.
3. M.M. DAY, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.
4. C. DUNKL and D. RAMIREZ, *Existence and nonuniqueness of invariant mean on $\mathscr{L}^{\infty}(\hat{G})$*, Proc. Amer. Math. Soc. **32** (1972), 525–530.
5. C. DUNKL and D. RAMIREZ, *Weakly almost periodic functionals on the Fourier algebra*, Trans. Amer. Math. Soc. **185** (1973), 501–514.
6. P. EYMARD, *L'algèbra de Fourier d'un groupe locallement compact*, Bull. Soc. Math. France **92** (1964), 181–236.
7. L. GILLMAN and M. JERISON, *Rings of continuous functions*, Van Nostrand, Princeton, New Jersey, 1960.
8. E.E. GRANIRER, *On amenable semigroups with a finite-dimensional set of invariant means I; II*, Illinois J. Math. **7** (1963), 32–58.
9. ______, *Criteria for compactness and for discreteness of locally compact amenable groups*, Proc. Amer. Math. Soc. **40** (1973), 615–624.

10. ______, *Properties of the set of topological invariant means on P. Eymard's $W^*$-algebra $VN(G)$*, Indag. Math. **36** (1974), 116–121.
11. ______, *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **189** (1974), 371–382.
12. ______, *Density theorems for some linear subspaces and some $C^*$-subalgebras of $VN(G)$*, Istituto Nazionale di Alta Mathematica, Symposia Mathematica **22** (1977), 61–70.
13. ______, *Geometric and topological properties of certain $w^*$ compact convex subsets of double duals of Banach spaces, which arise from the study of invariant means*, Illinois J. Math. **30** (1986), 148–174.
14. F.P. GREENLEAF, *Invariant means on topological groups*, Van Nostrand, New York, 1969.
15. C. HERZ, *Harmonic synthesis for subgroups*, Annales de l'Institut Fourier (Grenoble) **23** (1973), 91–123.
16. E. HEWITT and K. R. STROMBERG, *A remark on Fourier-Stieltjes transforms*, An Acad. Brasil. Ciênc. **34** (1962), 175–180.
17. E. HEWITT and K.A. ROSS, *Abstract Harmonic Analysis I*, Springer-Verlag, New York, 1979.
18. A.T. LAU, *Uniformly continuous functionals on the Fourier algebra of any locally compact group*, Trans. Amer. Math. Soc. **251** (1979), 39–59.
19. A.T. LAU and V. LOSERT, *The $C^*$-algebra generated by operators with compact support on a locally compact group*, J. Funct. Anal., **112** (1993), 1–30.
20. A.T. LAU and A.L.T. PATERSON, *The exact cardinality of the set of topological left invariant means on an amenable locally compact group*, Proc. Amer. Math. Soc. **98** (1986), 75–80.
21. T. MIAO, *On the sizes of the sets of invariant means*, Illinois J. Math. **36** (1992), 53–72.
22. A.L.T. PATERSON, *Amenability*, Amer. Math. Soc., Providence, Rhode Island, 1988.
23. J.P. PIER, *Amenable locally compact groups*, Wiley, New York, 1984.
24. C.C. PRINTER, *Set theory*, Addison-Wesley, Reading, Mass., 1971.
25. P.E. RENAUD, *Invariant means on a class of von Neumann algebras*, Trans. Amer. Math. Soc. **170** (1972), 285–291.
26. J.M. ROSENBLATT, *Invariant means for the bounded measurable functions on a non-discrete locally compact group*, Math. Ann. **22** (1976), 219–228.
27. J.E. RUBIN, *Set theory for the mathematician*, Holden-Day, San Francisco, 1967.
28. W. RUDIN, *Homogeneity problems in the theory of Čech compactification*, Duke Math. J. **23** (1956), 409–419.
29. ______, *Invariant means on $L^\infty$*, Studia Math. **44** (1972), 219–227.
30. S. SAKAI, *$C^*$-algebras and $W^*$-algebras*, Springer-Verlag, New York, 1971.
31. M. TAKESAKI, *Theory of operator algebras I*, Springer-Verlag, New York, 1979.
32. Z. YANG, *On the set of invariant means*, J. London Math. Soc. (2) **37** (1988), 317–330.

UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA, CANADA

UNIVERSITY OF WINDSOR
WINDSOR, ONTARIO, CANADA