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PARAMETRIZING THE SOLUTIONS OF AN ANALYTIC DIFFERENTIAL EQUATION

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Dedicated to the memory of Lee A. Rubel (1928–1995)

Introduction

One of Rubel's many research problems about algebraic differential equations (cf. [5, problem 21] and [6, problem 28]) is as follows:

Given a sequence (z_n) of distinct complex numbers tending to infinity, and any sequence (w_n) of complex numbers, does there exist a differentially algebraic entire function f such that $f(z_n) = w_n$ for all $n \in \mathbf{N}$?

It is classical that the answer is "yes" if the "differentially algebraic" requirement is dropped. Below, in (2.4), we show the answer is "no" in our case:

Given any such sequence (z_n), the set of sequences $(w_n) \in \mathbf{C}^{\mathbf{N}}$ for which there is a differentially algebraic entire function f with $f(z_n) = w_n$ for all n, is meagre in the sequence space $\mathbf{C}^{\mathbf{N}}$ equipped with the product topology. (Rubel had already shown in [6] that if one prescribes not only the values of $f(z_n)$ but also those of $f^{(j)}(z_n)$ for $1 \leq j \leq n$, then the interpolation problem is in general not solvable.)

The goal of this paper is to reproduce Malgrange's local parametrization of the solutions of an analytic differential equation, which is perhaps not as widely known as it should be, and to indicate its role in answering questions of this sort.

Let us now define some of our terms precisely. By *region in* $\mathbf{C}$ we mean a nonempty connected open subset of the complex plane $\mathbf{C}$, and a holomorphic function f on such a region is said to be *differentially algebraic* (or DA, for short) if there is a nonzero polynomial $P(Z, W, W', \dots, W^{(m)})$ in the variables $Z, W, W', \dots, W^{(m)}$ over $\mathbf{C}$ such that $P(z, f(z), f'(z), \dots, f^{(m)}(z)) = 0$ for all z in the region.

We then also say that f is a *solution* of the algebraic differential equation (ADE)

$$P(z, w, w', \dots, w^{(m)}) = 0.$$

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One relevant observation (cf. [4]) is that then P can be taken to have rational coefficients; so only countably many ADE's need to be considered. (Quick proof. Let $K \subseteq \mathbf{C}$ be the field generated by the coefficientsof the polynomial $P(Z, W, W', \ldots, W^{(m)})$. Consider $K(z)$ as a field of meromorphic functions on the region $D = \mathrm{domain}(f)$ and note that $K(z)$ has finite transcendence degree over $\mathbf{Q}$. The field $K(z, f, f^{(1)}, \ldots)$ of meromorphic functions on D generated over $K(z)$ by f and its successive derivatives is already generated over $K(z)$ by $f, f^{(1)}, f^{(2)}, \ldots, f^{(m)}$ alone, so $K(z, f, f^{(1)}, \ldots)$ has transcendence degree $\leq m$ over $K(z)$, hence $K(z, f, f^{(1)}, \ldots)$ has finite transcendence degree over $\mathbf{Q}$. This gives a nontrivial polynomial relation with rational coefficientsrelating $f, f^{(1)}, \ldots,$ as desired.)

Now remember the somewhat vague but useful slogan:

The solutions of a differential equation $\Phi(z, w, w', \ldots, w^{(m)}) = 0$ *near a given point* $z = z_0$ *form an* m*-parameter family.*

(The idea is that a solution $w(z)$ is largely determined near z_0 by the values $w(z_0), w'(z_0), \ldots, w^{(m-1)}(z_0)$.) It is understood here that the not identically vanishing function Φ is holomorphic in its variables $z, w, w', \ldots, w^{(m)}$ (or real analytic, if one is interested in the real case), and that only holomorphic (respectively, real analytic) solutions are considered.

This slogan, together with the countability observation above, certainly suggests that the answer to Rubel's question should be negative. The problem is to give a precise sense to this slogan and then to prove it. The following result of Boshernitzan [1] shows that some caution is needed here:

There is an ADE of order 19 *with real coefficients whose real polynomial solutions lie dense in the space of all real valued continuous functions on* $\mathbf{R}$, *with the topology of uniform convergence on compacta.*

This seems incompatible with the "slogan" view that the solutions of a given ADE form a finite dimensional family. We still have an escape route, namely to view Boshernitzan's example as analogous to $\mathbf{Q}$ being dense in $\mathbf{R}$ and nevertheless very small inside $\mathbf{R}$ (countable, hence meagre, of measure 0, etc.). This view turns out to be correct: roughly speaking, *the solutions of a given ADE of order* m *form a union of countably many analytic families of dimension* $\leq m$.

Such a result is deduced in Section 1 from Malgrange [3], who refers for part of his argument to Douady [2]. This part is replaced here by a more elementary argument, since I was unable to extract the desired result from Douady's paper. Actually, Malgrange works with analytic rather than algebraic differential equations, and most of the results below are stated also in that more general context.

Another of Rubel's problems (cf. [5, problem 20]) can be stated as follows:

Given a sequence (z_n) that tends to infinity in the complex plane, must there exist a differentially algebraic entire function whose zeros are exactly the z_n's?

We answer this negatively, in (3.3), by showing that for many such sequences (z_n) there is no DA entire function vanishing at all z_n. Similar ideas are involved.

I thank Lee Rubel for stimulating conversations on this topic.

1. Analytic parametrization according to Malgrange

The ring of holomorphic functions on an open set U in $\mathbf{C}^M$ is denoted by $H(U)$. A *region* in $\mathbf{C}^M$ is a non-empty connected open subset of $\mathbf{C}^M$.

Let a holomorphic function Φ on a region E in $\mathbf{C}^{2+m}$ be given. In connection with differential equations we find it convenient to let $z, w, w', \dots, w^{(m)}$ denote the usual coordinate functions on E (rather than, say, $z_1, \dots, z_{2+m}$). The partial derivative $\partial\Phi/\partial w^{(m)}$ of Φ with respect to the last variable is of course also holomorphic on E, and is called in this connection the *separant* of Φ. We now consider the following differential equation of order $\leq m$:

$$(*) \qquad \Phi(z, w, w', \dots, w^{(m)}) = 0.$$

A *solution* of $(*)$ in this paper is always a *holomorphic* function f on a region $D \subseteq \mathbf{C}$ such that $(z, f(z), f'(z), \dots, f^{(m)}(z)) \in E$ and $\Phi(z, f(z), f'(z), \dots, f^{(m)}(z)) = 0$, for all $z \in D$. If moreover f is not a solution of the separant equation

$$(\partial\Phi/\partial w^{(m)})(z, w, w', \dots, w^{(m)}) = 0,$$

then f is called a *non-singular* solution of $(*)$.

Next, let D be an open disc in the complex plane, centered, say, at 0, with closure $\overline{D}$ and boundary $\partial D := \overline{D} \setminus D$, and define

$$B^m(D) := \left\{ f \in H(D) : f^{(k)} \text{ extends continuously to } \overline{D}, \text{ for } k = 0, \dots, m \right\}.$$

$B^m(D)$ is clearly a $\mathbf{C}$-linear subspace of $H(D)$, and we equip $B^m(D)$ with the norm

$$\|f\| := \|f\|_{\sup} + \|f'\|_{\sup} + \cdots + \|f^{(m)}\|_{\sup},$$

which is easily seen to make $B^m(D)$ a complex Banach space.

Suppose now that $f_0 \in B^m(D)$ *is a solution of* $(*)$ *such that in addition*:

(i) f_0 *extends holomorphically to a solution of* $(*)$ *on a region containing* $\overline{D}$,

(ii) $\partial\Phi/\partial w^{(m)}(z, f_0(z), \ldots, f_0^{(m)}(z)) \neq 0$ *for all* $z \in \partial D$. *(Here we use* f_0 *also to denote its holomorphic extension to a region containing* $\overline{D}$.*)*

Under these assumptions we have:

(1.1) THEOREM (Malgrange [3, §4]). *There is a holomorphic function* F: $U \times D \to \mathbf{C}$, *for some open set* U *in* $\mathbf{C}^k$, $k \leq m$, *and an open neighborhood* V_0 *of* f_0 *in* $B^m(D)$ *such that for every solution* $f \in V_0$ *of* $(*)$ *there is a unique point* $a \in U$ *with*

$$f(z) = F(a, z) \text{ for all } z \in D.$$

Remark. The result is not stated in this form by Malgrange, but this comes out of the proof below. Later we show how to arrange that the set of $a \in U$ for which the function $F(a, z)$ on D is a solution of $(*)$ is an analytical subset of U.

Proof. First take an open neighborhood V of f_0 in $B^m(D)$ such that if $f \in V$, then

$$\left(z, f(z), f'(z), \ldots, f^{(m)}(z)\right) \in E(= \text{domain}(\Phi)) \quad \text{for all } z \in \overline{D}.$$

Define the map Ψ: $V \to B^0(D)$ by

$$\Psi(f) := \text{the function } \Phi\left(z, f(z), \ldots, f^{(m)}(z)\right) \text{ on } D.$$

Note that for small $\varepsilon \in \in B^m(D)$ and all $z \in D$ we have, by Taylor expansion,

$$\begin{aligned}
\Psi(f_0 + \varepsilon)(z) &= \Phi\left(z, f_0(z) + \varepsilon(z), f_0'(z) + \varepsilon'(z), \ldots, f_0^{(m)}(z) + \varepsilon^{(m)}(z)\right) \\
&= \Phi\left(z, f_0(z), f_0'(z), \ldots, f_0^{(m)}(z)\right) \\
&\quad + (\partial\Phi/\partial w)\left(z, f_0(z), f_0'(z), \ldots, f_0^{(m)}(z)\right) \cdot \varepsilon(z) \\
&\quad + (\partial\Phi/\partial w')\left(z, f_0(z), f_0'(z), \ldots, f_0^{(m)}(z)\right) \cdot \varepsilon'(z) \\
&\quad\ \vdots \\
&\quad + (\partial\Phi/\partial w^{(m)})\left(z, f_0(z), f_0'(z), \ldots, f_0^{(m)}(z)\right)\varepsilon^{(m)}(z) \\
&\quad + \text{terms of degree} \geq 2 \text{ in } \varepsilon(z), \varepsilon'(z), \ldots, \varepsilon^{(m)}(z).
\end{aligned}$$

In other words,

$$\Psi(f_0 + \varepsilon) = \Psi(f_0) + a_0 \cdot \varepsilon + a_1 \cdot \varepsilon' + \cdots + a_m \cdot \varepsilon^{(m)} \\ + \text{terms of degree } \geq 2 \text{ in } \varepsilon, \varepsilon', \ldots, \varepsilon^{(m)},$$

where a_i is the holomorphic function $(\partial\Phi/\partial w^{(i)})(z, f_0(z), f_0'(z), \ldots, f_0^{(m)}(z))$ on D.

This shows that Ψ is differentiable at f_0 with derivative $\Psi'(f_0)$: $B^m(D) \to B^0(D)$ given by $\Psi'(f_0)(\varepsilon) = a_0 \cdot \varepsilon + a_1 \cdot \varepsilon' + \cdots + a_m \cdot \varepsilon^{(m)}$. The same argument shows Ψ is differentiable at each $f \in V$, and even continuously differentiable on V.

Now $\Psi'(f_0)$ is a linear differential operator of order m with coefficients a_i that extend holomorphically to a region containing $\overline{D}$, and whose "leading" coefficient a_m has no zero on ∂D, by assumption.

In this situation Malgrange shows by a beautiful argument (§1 of [3]) that the image $L := \Psi'(f_0)(B^m(D))$ is closed in $B^0(D)$ of finite codimension. It is classical that the kernel $K := \Psi'(f_0)^{-1}(0)$ is a closed subspace of $B^m(D)$ of dimension $k \leq m$. Next Malgrange appeals to "raisonnements connus" and Douady [2] to conclude that "l'espace analytique banachique $\Psi^{-1}(0)$ est, au voisinage de f_0, de dimension finie". This probably amounts to the argument we give below: we change Ψ so as to make the Inverse Mapping Theorem applicable.

Take continuous linear projection maps $p : B^m(D) \to K$ and $q : B^0(D) \to L$, so p and q are the identity on K and L. Now modify Ψ to a map $\Psi^{\#} : V \to K \times L$ given by $\Psi^{\#}(f) = (p(f), q(\Psi f))$. Clearly $\Psi^{\#}$ is also continuously differentiable on V, with derivative $(\Psi^{\#})'(f_0) : B^m(D) \to K \times L$ at f_0, given by $(\Psi^{\#})'(f_0)(\varepsilon) = (p(\varepsilon), \Psi'(f_0)(\varepsilon))$. Hence $(\Psi^{\#})'(f_0)$ is a bijection, and since we are dealing with Banach spaces, the inverse of $(\Psi^{\#})'(f_0)$ is continuous.

Then by the Inverse Mapping Theorem there is an open neighborhood $V_0 \subseteq V$ of f_0 in $B^m(D)$ and open neighborhoods V_K of $p(f_0)$ in K and V_L of $0 = q(0) = q(\Psi(f_0))$ in L such that $\Psi^{\#}$ maps V_0 homeomorphically onto $V_K \times V_L$ with continuously differentiable inverse $\theta : V_K \times V_L \to V_0 \subseteq B^m(D)$. Since $\Psi^{\#}(V_0 \cap \Psi^{-1}(0)) \subseteq V_K \times \{0\}$ we have

$$(\#) \qquad V_0 \cap \Psi^{-1}(0) \subseteq \theta(V_K \times \{0\}).$$

Take a linear isomorphism $e : \mathbf{C}^k \cong K$ and let $U := e^{-1}(V_K)$, an open set in $\mathbf{C}^k$. Then the map $F : U \times D \to \mathbf{C}$ given by $F(a, z) := \theta(e(a), 0)(z)$ is easily seen to be continuously differentiable, hence holomorphic. Moreover, if $f \in V_0$ is a solution of $(*)$, then by $(\#)$ there is $a \in U$ such that $f = \theta(e(a), 0)$, that is, $f(z) = F(a, z)$ for all $z \in D$. $\square$

(1.2) *Remarks.* (1) In the theorem above we can take F, U, V_0 and an analytic set

$$A = \{a \in U: g_0(a) = \cdots = g_N(a) = 0\}$$

with $g_0, \dots, g_N \in H(U)$ such that the map

$$a \mapsto (\text{the function } F(a, z) \text{ on } D)$$

is a bijection from A onto $V_0 \cap \Psi^{-1}(0)$ (the set of solutions of $(*)$ in V_0). (In this way we obtain a parametrization of the set of solutions of $(*)$ near f_0 in the space $B^m(D)$ by an analytic set A of dimension $\leq m$.)

To get A, note first that our construction of F shows that, given any $a \in U$, the function $F(a, z)$ on D actually belongs to V_0. It will be a solution to $(*)$ if and only if

$$\Phi(z, F(a, z), (\partial F/\partial z)(a, z), \dots, (\partial^m F/\partial z^m)(a, z)) = 0 \text{ for all } z \in D.$$

The left hand side here is a holomorphic function $G(a, z)$ of $(a, z) \in U \times D$. Hence there are $g_n \in H(U)$, $n = 0, 1, \dots,$ such that $G(a, z) = \Sigma g_n(a) z^n$ for $(a, z) \in U \times D$, the infinite sum converging absolutely, and uniformly on compact subsets of $U \times D$. Let $a_0 \in U$ be the point for which $F(a_0, z) = f_0(z)$ on D. By a well known result on several complex variables, finitely many of the g_n's, say $g_0, \dots, g_N$, generate the ideal of $H(U)$ generated by all the g_n's, provided U is first suitably decreased to a smaller domain containing a_0. Then we can change V_0 accordingly such that the result stated above holds for our new F, U and V_0, with $g_0, \dots, g_N$ generating the ideal of $H(U)$ generated by all g_n's.

(2) The assumption in the theorem and the previous remark that D is an open disc centered at 0 can be replaced by the assumption that D is the image of the open unit disc in $\mathbf{C} = \mathbf{R}^2$ under a C^1-diffeomorphism of an open neighborhood of the closed unit disc with an open set in $\mathbf{C} = \mathbf{R}^2$. (Same definition of $B^m(D)$.) Exactly the same proof works.

Next we use the theorem to parametrize solutions of $(*)$ in spaces $H(D)$. We equip the $\mathbf{C}$-linear space $H(D)$, for any region D in the complex plane, with the topology of uniform convergence on compact subsets of D. We will need the fact that $H(D)$ is then a Fréchet space (complete metrizable locally convex space), with a countable basis for the open sets. Let $D(r)$ denote be the open disc $|z| < r$ of radius r centered at 0.

(13) Corollary. *Let $0 < r < R$, and let $f_0 \in H(D(R))$ be a non-singular solution of the differential equation $(*)$. Then there is a holomorphic function F on $U \times D(r)$, for some open set $U \subseteq \mathbf{C}^k$, $k \leq m$, such that for every solution f*

of $(*)$ *in some neighborhood* $V(f_0)$ *of* f_0 *in the space* $H(D(R))$ *there is* $a \in U$ *with*

$$f(z) = F(a, z) \quad \textit{for all } z \in D(r).$$

Proof. By increasing r we may assume that $(\partial\Phi/\partial w^{(m)})(z, f_0(z), \dots, f_0^{(m)}(z)) \neq 0$ for all z with $|z| = r$. Then we can apply the theorem to $D = D(r)$. This gives a neighborhood V_0 of $f_0|D$ in $B^m(D)$ and a holomorphic function F on $U \times D$ for some open $U \subseteq \mathbf{C}^k$, $k \leq m$, such that if $f \in V_0$ is a solution of $(*)$, then $f(z) = F(a, z)$ for some $a \in U$ and all $z \in D$. We may assume $V_0 = \{f \in B^m(D)\colon \|f - (f_0|D)\| < \varepsilon\}$, $0 < \varepsilon$, where $\|\cdot\|$ is the norm on $B^m(D)$ given in (1.3). Let now $r < r' < R$. Then it follows from the formula

$$g^{(j)}(z) = (j!/2\pi i)\int_{|x| = r'} \left(g(x)/(x - z)^{j+1}\right) dx \quad (g \in H(D(R)), j \in \mathbf{N})$$

that for some $\varepsilon' > 0$, if $f \in H(D(R))$ and $|f(x) - f_0(x)| < \varepsilon'$ on $D(r')$, then $\|(f - f_0)|D\| < \varepsilon$. (Apply the formula to $g = f - f_0$, $j = 0, \dots, m$.) Hence

$$\{f \in H(D(R))\colon |f(x) - f_0(x)| < \varepsilon' \text{ on } D(r')\}$$

is a neighborhood of f_0 in $H(D(R))$ with the desired property. □

Next we consider all solutions of $(*)$ in $H(D(R))$, not just those near a given non-singular solution.

(1.4) COROLLARY. *Let* Φ *not be identically* 0, *and* $0 < r < R$. *Then there are holomorphic functions* $F_n\colon U_n \times D(r) \to \mathbf{C}$, *with* U_n *open in* $\mathbf{C}^{k(n)}$, $0 \leq k(n) \leq m$, *for* $n = 1, 2, 3, \dots,$ *such that for each solution* $f \in H(D(R))$ *of* $(*)$ *there are* n *and* $a \in U_n$ *with*

$$f(z) = F_n(a, z) \quad \textit{for all } z \textit{ in } D(r).$$

Proof. For each non-singular solution $f_0 \in H(D(R))$ of $(*)$ we choose a function F and an open neighborhood $V(f_0)$ in $H(D(R))$ with the properties of Corollary (1.3). Since the space $H(D(R))$ has a countable base for the topology, countably many of the $V(f_0)$'s will cover the union of all $V(f_0)$'s (Lindelöf property), and so we can find countably many functions F as above such that each non-singular solution in $H(D(R))$ is of the desired form on $D(r)$ for one of those countably many F's. This takes care of the non-singular solutions. The singular solutions are solutions of the separant equation, and hence we apply the same process with Φ replaced by its separant $\partial\Phi/\partial w^{(m)}$. This takes care of the non-singular solutions of the separant equation.

Continuing this way we treat all solutions of $(*)$ in $H(D(R))$ such that for some $k \in \mathbf{N}$ and $z \in D(R)$ we have $(\partial^k \Phi / \partial (w^{(m)})^k)(z, f(z), \ldots, f^{(m)}(z)) \neq 0$.

Next suppose that $f \in H(D(R))$ and $(\partial^k \Phi / \partial (w^{(m)})^k)(z, f(z), \ldots, f^{(m-1)}(z), f^{(m)}(z)) = 0$ for all $k \in \mathbf{N}$ and all $z \in D(R)$. Take any $R^{\#}$ with $r < R^{\#} < R$. Then Taylor expansion of Φ with respect to $w^{(m)}$ around each point $(z, f(z), \ldots, f^{(m-1)}(z), f^{(m)}(z))$ with $|z| \leq R^{\#}$ shows that for some $\varepsilon > 0$ we have

$$\left(z, f(z), \ldots, f^{(m-1)}(z), u\right) \in E = \operatorname{domain}(\Phi)$$

and

$$\Phi\left(z, f(z), \ldots, f^{(m-1)}(z), u\right) = 0$$

for all z and u with $|z| \leq R^{\#}$ and $|u - f^{(m)}(z)| < \varepsilon$. Take a polynomial $u(Z) \in \mathbf{Q}(i)[z]$ such that $|u(z) - f^{(m)}(z)| < \varepsilon$ for $|z| \leq R^{\#}$, and a region $E^{\#}$ in $\mathbf{C}^{2+(m-1)}$ such that $E^{\#}$ contains all points $(z, f(z), \ldots, f^{(m-1)}(z))$ with $|z| \leq R^{\#}$ and such that if

$$\left(z, w, w', \ldots, w^{(m-1)}\right) \in E^{\#},$$

then

$$\left(z, w, w', \ldots, w^{(m-1)}, u(z)\right) \in E.$$

Next define the holomorphic function $\Phi^{\#} : E^{\#} \to \mathbf{C}$ by

$$\Phi^{\#}\left(z, w, w', \ldots, w^{(m-1)}\right) := \Phi\left(z, w, w', \ldots, w^{(m-1)}, u(z)\right).$$

Note that then $f^{\#} := f|D(R^{\#})$ is a solution of the differential equation $\Phi^{\#}(z, w, w', \ldots, w^{(m-1)}) = 0$, which is only of order $\leq m - 1$, while the original differential equation $(*)$ was of order $\leq m$. We may also assume that $\Phi^{\#}$ is not identically 0 (choosing $u(Z)$ suitably), and that $E^{\#}$ is a union of finitely many open balls in $\mathbf{C}^{2+(m-1)}$, each with rational radius and centered at a point of $\mathbf{Q}(i)^{2+(m-1)}$. This leaves only countably many possibilities for $\Phi^{\#}$, depending on the original Φ. Hence an obvious inductive hypothesis gives the desired result. □

(1.5) *Remark.* Actually Corollary (1.4) extends as follows:

Let Φ not be identically 0, and let D' and D be simply connected bounded regions in $\mathbf{C}$ such that $\overline{D'} \subseteq D$. Then there are holomorphic functions $F_n : U_n \times D' \to \mathbf{C}$, with U_n open in $\mathbf{C}^{k(n)}$, $0 \leq k(n) \leq m$, for $n = 1, 2, 3, \ldots,$ such that for each solution $f \in H(D)$ of $(*)$ there are n and $a \in U_n$ with $f(z) = F_n(a, z)$ for all z in D'.

To see this one first proves the corresponding extension of Corollary (1.3), and then adapts the proof of (1.4) in a straightforward way.

2. Interpolation for differentially algebraic functions

We now apply the results of the previous section to answer Rubel's interpolation problem for differentially algebraic holomorphic functions.

(2.1) Given natural numbers m and M, let us say a set $S \subseteq \mathbf{C}^M$ has *complex-analytic dimension* $\leq m$ if S is contained in the union of countably many complex-analytic (embedded) submanifolds of $\mathbf{C}^M$ of dimension $\leq m$. Note that then every subset of S has complex-analytic dimension $\leq m$, and that the union of countably many subsets of $\mathbf{C}^M$ of complex-analytic dimension $\leq m$ has complex-analytic dimension $\leq m$. If $m < M$, then a set $S \subseteq \mathbf{C}^M$ of complex-analytic dimension $\leq m$ is very small in $\mathbf{C}^M$, in several ways: S is meagre in $\mathbf{C}^M$, and of Lebesgue measure 0.

If $G : U \to \mathbf{C}^M$ is a complex-analytic map with $U \subseteq \mathbf{C}^m$ open, then $G(U)$ is actually a union of countably many complex-analytic submanifolds of dimension $\leq m$, in particular $G(U)$ is of complex-analytic dimension $\leq m$. This can be seen by partitioning U into countably many complex-analytic submanifolds M_i (of various dimensions) such that each $G|M_i : M_i \to \mathbf{C}^M$ has constant rank r_i, and then using the rank theorem and the fact that M_i has a countable basis to conclude that $G(M_i)$ is a countable union of complex-analytic submanifolds of $\mathbf{C}^M$ of dimension r_i.

(2.2) As in the previous section, we consider the differential equation

$$(*) \qquad \Phi(z, w, w', \ldots, w^{(m)}) = 0$$

where Φ is a holomorphic function on a region in $\mathbf{C}^{2+m}$. *We also assume now that* Φ *is not identically* 0. Let points $z_1, \ldots, z_M \in \mathbf{C}$ be given and define $I(\Phi, z_1, \ldots, z_M)$ to be the set of all $(f(z_1), \ldots, f(z_M)) \in \mathbf{C}^M$ with f a solution of $(*)$ defined on a region in the complex plane containing the points $z_1, \ldots, z_M$. (The region may vary with f.)

(2.3) PROPOSITION. *The set* $I(\Phi, z_1, \ldots, z_M)$ *has complex-analytic dimension* $\leq m$ *in* $\mathbf{C}^M$. *In particular, if* $m < M$, $I(\Phi, z_1, \ldots, z_M)$ *is meagre and of Lebesgue measure* 0 *in* $\mathbf{C}^M$.

Proof. Let f be a solution of $(*)$ on a region D such that $z_1, \ldots, z_M \in D$. Then there is a connected set $P \subseteq D$ such that:

(i) $z_1, \ldots, z_M \in P$;

(ii) P is the union of finitely many (compact) line segments inside D;

(iii) the endpoints of these line segments are among $\{z_1, \ldots, z_M\} \cup \{a + bi \colon a, b \in \mathbf{Q}\}$;

(iv) no two distinct segments intersect except at endpoints.

By deleting superfluous segments we get a "spanning tree", so we may even assume P is simply connected. Hence, for the purpose of the proposition we may as well assume D is the ε-neighborhood of such a polygonal path P for some positive rational $\varepsilon > 0$. Then D is also simply connected and bounded.

Let $0 < \varepsilon' < \varepsilon$ with ε' also rational and let D' be the ε'-neighborhood of P. Note that, given the points $z_1, \dots, z_M$, only countably many pairs (D, D') of this form exist. Now (1.5) gives us countably many holomorphic functions $F_{n,D,D'} : U_{n,D,D'} \to \mathbf{C}$ with $U_{n,D,D'}$ open in $\mathbf{C}^{k(n,D,D')}$, $0 \le k(n, D, D') \le m$, such that for each solution f of $(*)$ in $H(D)$ there are n and $a \in U_{n,D,D'}$ with $f(z) = F_{n,D,D'}(a, z)$ for all $z \in D'$. Define $G_{n,D,D'} : U_{n,D,D'} \to \mathbf{C}^M$ by

$$G_{n,D,D'}(a) = \left(F_{n,D,D'}(a, z_1), \dots, F_{n,D,D'}(a, z_M)\right).$$

Then $G_{n,D,D'}$ is a complex-analytic map, and the considerations above show that $I(\Phi, z_1, \dots, z_M)$ is contained in the union of the sets $G_{n,D,D'}(U_{n,D,D'}) \subseteq \mathbf{C}^M$, with (D, D') ranging over countably many possible pairs of regions and n over $\mathbf{N}$. The desired result is now immediate from the remarks in (2.1). □

(2.4) COROLLARY. *Let $(z_n)_{n\in\mathbf{N}}$ be a sequence of complex numbers. Then the set*

$$\{(f(z_n)) \in \mathbf{C}^{\mathbf{N}} : f \text{ is a differentially algebraic holomorphic function on a region containing the points } z_0, z_1, z_2, \dots\}$$

is meagre in the sequence space $\mathbf{C}^{\mathbf{N}}$. (Here we equip $\mathbf{C}^{\mathbf{N}}$ with the product topology making it a Polish space.)

Proof. This is almost immediate from the previous result and the fact quoted in the introduction that each differentially algebraic holomorphic function on a region in $\mathbf{C}$ is a solution of an algebraic differential equation with rational coefficients.

3. Zeros of differentially algebraic functions

Here we answer Rubel's question on zeros of differentially algebraic holomorphic functions. We need one more technical lemma.

(3.1) LEMMA. *Let $U \subseteq \mathbf{C}^k$ be open, D a region in the complex plane, and $F : U \times D \to \mathbf{C}$ a holomorphic function. Let $n > k$ and put*

$$A := \{(z_1, \dots, z_n) \in D^n : \text{there is } a \in U \text{ such that the function } F(a, z) \text{ on } D \text{ does not vanish identically but vanishes at the points } z_1, \dots, z_n\}.$$

Then A has complex-analytic dimension $\le k$ in $\mathbf{C}^n$, hence is meagre in $\mathbf{C}^n$.

Proof. By removing from U the closed subset $\{a \in U\colon F(a, z) = 0$ for all $z \in D\}$ we may assume that for all $a \in U$ there is $z \in D$ with $F(a, z) \neq 0$. For each n-tuple $i = (i_1, \dots, i_n)$ of natural numbers, put

$$A_i := \{(z_1, \dots, z_n) \in D^n\colon \text{there is } a \in U \text{ such that}$$

$$\begin{aligned}&(\partial^j F/\partial z^j)(a, z_1) = 0 \text{ for } 0 \le j \le i_1, (\partial^{i_1+1} F/\partial z^{i_1+1})(a, z_1) \neq 0,\\ &\vdots\\ &(\partial^j F/\partial z^j)(a, z_n) = 0 \text{ for } 0 \le j \le i_n, (\partial^{i_n+1} F/\partial z^{i_n+1})(a, z_n) \neq 0\}.\end{aligned}$$

Then A is the union of the A_i's, so it sufficesto show that each A_i is of complex-analytic dimension $\le k$. For given i as above we have $A_i \subseteq \pi(B_i)$ where $\pi : U \times D^n \times \mathbf{C}^n \to D^n$ is the projection map onto the middle factor and B_i is the set of all points

$$(a, z_1, \dots, z_n, w_1, \dots, w_n) \in U \times D^n \times \mathbf{C}^n$$

such that

$$(\partial^{i_1} F/\partial z^{i_1})(a, z_1) = \cdots = (\partial^{i_n} F/\partial z^{i_n})(a, z_n) = 0$$

and

$$(\partial^{i_1+1} F/\partial z^{i_1+1})(a, z_1) \cdot w_1 = 1, \dots, (\partial^{i_n+1} F/\partial z^{i_n+1})(a, z_n) \cdot w_n = 1.$$

So $B_i = G^{-1}(0, \dots, 0, 1, \dots, 1)$, where $G : U \times D^n \times \mathbf{C}^n \to \mathbf{C}^n \times \mathbf{C}^n$ is the complex-analytic map given by

$$\begin{aligned}G(a, z_1, \dots, z_n, w_1, \dots, w_n) = \big(&(\partial^{i_1} F/\partial z^{i_1})(a, z_1), \dots, (\partial^{i_n} F/\partial z^{i} n)(a, z_n),\\ &(\partial^{i_1+1} F/\partial z^{i_1-1})(a, z_1) \cdot w_1, \dots, (\partial^{i_n+1} F/\partial z^{i_n 1})(a, z_n) \cdot w_n\big).\end{aligned}$$

One easily checks that the Jacobian matrix of G has full rank $2n$ at each point of B_i; hence B_i is a complex-analytic submanifold of $U \times D^n \times \mathbf{C}^n$ of dimension $k + 2n - 2n = k$. Now use the fact that $A_i \subseteq \pi(B_i)$. □

As before we consider a differential equation of order $\le m$:

$$(*) \qquad \Phi(z, w, w', \dots, w^{(m)}) = 0$$

(Φ a holomorphic function on a region in $\mathbf{C}^{2+m}$).

(3.2) COROLLARY. *Assume Φ is not identically zero. Then for each $n > m$, the set*

$$\{(z_1, \dots, z_n) \in \mathbf{C}^n: \text{ there is a non-zero solution } f \in H(D) \text{ of } (*) \text{ on a region } D \subseteq \mathbf{C} \text{ such that } z_1, \dots, z_n \in D \text{ and } f(z_1) = \cdots = f(z_n) = 0\}$$

has complex-analytic dimension $\leq m$ in $\mathbf{C}^n$, in particular, it is meagre in $\mathbf{C}^n$.

Proof. Combine the arguments in the proof of (2.3) with Lemma (3.1). □

For each infinite sequence (z_n) of points in $\mathbf{C}$ such that $z_n \to \infty$ there is a nonzero entire function f such that $f(z_n) = 0$ for all n. But in general we cannot take f here differentially algebraic as follows easily from (3.2):

(3.3) COROLLARY. *Let $\Omega := \{(z_0, z_1, \dots) \in \mathbf{C}^{\mathbf{N}}: |z_n| \geq n$ for all $n\}$, a closed subset of $\mathbf{C}^{\mathbf{N}}$. Then $\{(z_n) \in \Omega$: there is a nonzero entire DA function f such that $f(z_n) = 0$ for all $n\}$ is meagre in Ω.*

4. The real analytic case

Let Φ be a real analytic function defined on a non-empty connected open subset E of $\mathbf{R}^{2+m}$. Consider the real differential equation of order $\leq m$

$$(*)_{\mathbf{R}} \qquad \Phi(x, y, y', \dots, y^{(m)}) = 0.$$

A *solution of* $(*)_{\mathbf{R}}$ is now a *real analytic* function f on an open interval $I \subseteq \mathbf{R}$ such that

$$(x, f(x), f'(x), \dots, f^{(m)}(x)) \in E$$

and

$$\Phi(x, f(x), f'(x), \dots, f^{(m)}(x)) = 0 \quad \text{for all } x \in I.$$

If moreover f is not a solution of the corresponding separant equation, then we call f a non-singular solution of $(*)_{\mathbf{R}}$. Note that we can always extend Φ analytically to a holomorphic function on a region in $\mathbf{C}^{2+m}$ and a solution of $(*)_{\mathbf{R}}$ to a holomorphic solution on a region in $\mathbf{C}$ of the corresponding complex-analytic differential equation. This is essentially what makes it possible to apply the results in the previous sections to the present real analytic case. We leave the details to the reader and only state the outcome. Define "having real-analytic dimension $\leq m$" (for subsets of $\mathbf{R}^M$) in the same way as "having complex-analytic dimension $\leq m$" (for subsets of $\mathbf{C}^M$), with real-analytic instead of complex-analytic submanifolds.

In the rest of this section assume that Φ is not identically 0.

Let points $x_1, \ldots, x_M \in \mathbf{R}$ be given and let $I_{\mathbf{R}}(\Phi, x_1, \ldots, x_M)$ be the set of all $(f(x_1), \ldots, f(x_M)) \in \mathbf{R}^M$ with f a solution of $(*)_{\mathbf{R}}$ defined on an open interval in $\mathbf{R}$ containing the points $x_1, \ldots, x_M$. (The interval may vary with the solution.)

(4.1) PROPOSITION. *The set $I_{\mathbf{R}}(\Phi, x_1, \ldots, x_M)$ has real-analytic dimension $\leq m$ in $\mathbf{R}^M$. Hence, if $M > m$, then $I_{\mathbf{R}}(\Phi, x_1, \ldots, x_M)$ is meagre and of Lebesgue measure 0 in $\mathbf{R}^M$.*

This may be compared with Boshernitzan [1], which gives a real algebraic differential equation $(*)_{\mathbf{R}}$ of order $m = 19$, such that for all distinct real numbers $x_1, \ldots, x_M$ the set $I_{\mathbf{R}}(\Phi, x_1, \ldots, x_M)$ is dense in $\mathbf{R}^M$.

(4.2) PROPOSITION. *For each $M > m$, the set*

$$\{(x_1, \ldots, x_M) \in \mathbf{R}^M : \text{there is a non-zero solution } f \text{ of } (*)_{\mathbf{R}} \text{ on an open interval } I \subseteq \mathbf{R} \text{ such that } x_1, \ldots, x_M \in I \text{ and } f(x_1) = \cdots = f(x_M) = 0\}$$

has real-analytic dimension $\leq m$ in $\mathbf{R}^M$; in particular, it is meagre in $\mathbf{R}^M$.

Again, Boshernitzan [1], gives a real algebraic differential equation $(*)_{\mathbf{R}}$ of order $m = 19$, such that for each $M > 19$ the set in (4.2) is dense in $\mathbf{R}^M$.

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