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AN EXTREMAL PROPERTY OF CONTRACTION SEMIGROUPS IN BANACH SPACES

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1. Introduction and the main result

Let the closed operator B generate a (C_0) contraction semigroup T on a complex Banach space X, and let f be a unit vector in X. It is evident that if f is an eigenvector of B corresponding to a purely imaginary eigenvalue, then $|\langle T_t f, x^* \rangle| = |\langle f, x^* \rangle|$ for every functional x^* in the dual space X^*. In the converse direction Goldstein [3] proved that if X is a Hilbert space and f is a unit vector in X satisfying

$$\lim_{t \to \infty} |\langle T_t f, f \rangle| = 1,$$

then f is an eigenvector of B belonging to a purely imaginary eigenvalue. He also gave an example in [3] showing that the corresponding natural generalization of his assumption in the case of a general Banach space X need not imply the desired conclusion: in the space $X = C[0, 1]$ there is a unit vector f and a unit vector x^* in X^* satisfying

$$\langle f, x^* \rangle = 1, \qquad |\langle T_t f, x^* \rangle| = 1$$

for every $t \geq 0$ without f being an eigenvector of the generator operator B.

Trying to find an extension of the main result of [3] to the case of a general Banach space X we first observe (see Lemma 2 below) that the "single functional assumption" of [3] in a Hilbert space immediately implies a corresponding "every functional statement": for every g in the Hilbert space X we then have

$$\lim_{t \to \infty} |\langle T_t f, g \rangle| = |\langle f, g \rangle|.$$

Since we have to postulate more than the "single functional assumption" in an arbitrary Banach space, we shall suppose what we can call the "every

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functional assumption". In other words, we shall prove the following main result of this note:

THEOREM. *Let X be a complex Banach space, T a (C_0) contraction semigroup of bounded linear operators on X generated by the closed linear operator B, and $f \in X$ a nonzero vector satisfying*

$$\lim_{t\to\infty} |\langle T_t f, x^* \rangle| = |\langle f, x^* \rangle|$$

for every x^* in the dual space X^*. Then f is an eigenvector of B corresponding to a purely imaginary eigenvalue.

The proofs of the assertions above will be given in six lemmas in Section 2. Lemma 2 will deal with the mentioned equivalence in Hilbert space, whereas Lemmas 1 and 3 through 6 will give parts of the proof of the main result in the Banach space case. The main tool will be the reduction of the investigation to the smallest T-invariant subspace containing f, where the restriction of the semigroup will be shown to be weakly almost periodic in the sense of deLeeuw and Glicksberg [1].

Notations will be standard or explained in the text. Note that $|\cdot|$ will denote any norm, and $\langle x, x^* \rangle$ will often denote the value $x^*(x)$ of the functional x^* at x. The operators $T(t)$ in the semigroup will also be denoted by T_t.

2. Lemmas and proofs

It is well-known that in any Banach space X for any unit vector f there exist *orthoprojectors* P_f onto f, i.e. bounded linear idempotent operators P_f with norm 1 and range $L(f)$, the (one-dimensional) subspace spanned by f, satisfying $P_f f = f$. Any such othoprojector has the form $P_f = y^* \otimes f$, i.e. $P_f(\cdot) = y^*(\cdot) f$, where y^* has norm 1 in X^* and satisfies $y^*(f) = 1$; the converse is also valid. Using these notations we start with the following simple result.

LEMMA 1. *Under the conditions of the preceding paragraph, for any $x^* \in X^*$ we have*

$$|\langle f, x^* \rangle| = |P_f^* x^*|.$$

Further, if T is a contraction semigroup, the following are equivalent:

(a) $|\langle P_f T_t f, x^* \rangle| \to |\langle f, x^* \rangle|$ *as $t \to \infty$ for every $x^* \in X^*$;*

(b) $|\langle T_t f, y^* \rangle| \to 1$ *as $t \to \infty$ for the functional y^* of P_f.*

Proof. Clearly,

$$|P_f^* x^*| = \sup_{|x|=1} \left|\langle x, P_f^* x^* \rangle\right| \geq |\langle f, x^* \rangle|.$$

On the other hand,

$$|P_f^* x^*| = \sup_{|x|=1} |\langle y^*(x) f, x^* \rangle| \leq |\langle f, x^* \rangle|.$$

The equivalence of (a) and (b) is immediate from the considerations above.

If X is a Hilbert space, the "single functional assumption" of the introduction clearly implies the "every functional assumption" via the main result of Goldstein [3]. We show here that this can be proved immediately (without recourse to [3]), which in the end will yield a completely different proof of the main result of [3].

LEMMA 2. *Let X be a Hilbert space, f a unit vector in X and T a contraction semigroup in X. $|\langle T_t f, f\rangle| \to 1$ as $t \to \infty$ if and only if for every $g \in X$;*

$$\lim_{t\to\infty} |\langle T_t f, g\rangle| = |P_f g| = |\langle f, g\rangle|,$$

where P_f denotes the selfadjoint projection onto $L(f)$.

Proof. We shall prove the only if statement. By assumption, if $|P_f g| \neq 0$, then

$$\lim_{t\to\infty} \left|\langle T_f f, P_f g / |P_f g|\rangle\right| = 1.$$

Hence for every $g \in X$ we obtain

$$\lim_{t\to\infty} \left|\langle T_t f, P_f g\rangle\right| = |P_f g|.$$

Assuming nonzero denominators, Bessel's inequality gives, with $P_f^c = I - P_f$,

$$\left|\left\langle T_t f, \frac{P_f g}{|P_f g|}\right\rangle\right|^2 + \left|\left\langle T_t f, \frac{P_f^c g}{|P_f^c g|}\right\rangle\right|^2 \leq |T_f f|^2.$$

By assumption, the first term and the right-hand side tend to 1; hence the second term converges to 0 as $t \to \infty$. Since

$$\langle T_t f, g\rangle = \langle T_t f, P_f g\rangle + \langle T_t f, P_f^c g\rangle,$$

we obtain

$$\lim_{t\to\infty} |\langle T_t f, g\rangle| = \lim_{t\to\infty} |\langle T_t f, P_f g\rangle| = |P_f g|.$$

If $P_f^c g = 0$, then $P_f g = g$, and our assertion is valid by what has been proved above. If $P_f g = 0$, hence $g = P_f^c g$, take the vector f in place of $P_f g$ and apply Bessel's inequality again. It yields

$$\lim_{t\to\infty} |\langle T_t f, g\rangle| = 0 = |P_f g|,$$

and the proof is complete.

From now on we shall consider only the general case where X is a complex Banach space. Also, in what follows T will denote a contraction semigroup, and f a fixed unit vector in X.

LEMMA 3. *With the notations above assume that for every* $x^* \in X^*$,

$$\lim_{t\to\infty} |\langle T_t f, x^*\rangle| = |\langle f, x^*\rangle|.$$

Then the orbit $\{T_t f : t \geq 0\}$ *is a relatively compact set in the weak topology of* X.

Proof. By assumption and by Lemma 1,

$$\lim_{t\to\infty} |\langle (I - P_f)T_t f, x^*\rangle| = |P_f^*(I^* - P_f^*)x^*| = 0$$

for every $x^* \in X^*$. Hence

$$\lim_{t\to\infty} [T_t f - y^*(T_t f) f] = 0$$

in the weak topology of X (here, as above, y^* denotes the linear functional corresponding to the orthoprojector P_f). With the notation $c_t = y^*(T_t f)$ we have

$$\lim_{t\to\infty} |c_t| = 1.$$

Therefore for every real sequence converging to infinity there is a subsequence $\{t(n)\}$ such that $\lim_{n\to\infty} c(t(n)) = c$, where the complex number c has modulus 1. Hence

$$\lim_{n\to\infty} T_{t(n)} f = cf$$

in the weak topology of X. For any nonnegative real sequence with a finite limit point the strong continuity of the orbit $\{T_t f : t \geq 0\}$ yields the existence of a subsequence for which $T_{t(n)} f$ converges in the strong topology of X. Hence the orbit is relatively weakly compact.

Definition. Any vector $x \in X$ whose orbit satisfies the conclusion of Lemma 3 will be called a weakly almost periodic vector (with respect to the semigroup T), and we shall write $x \in WAP(T)$.

LEMMA 4. *The closure L in the norm topology of the linear span of the orbit of the vector f is contained in $WAP(T)$.*

L is a T-invariant subspace, and T restricted to L is a weakly almost periodic semigroup in the sense of deLeeuw and Glicksberg [1]. *Hence L is the topological direct sum of the closed subspaces L_0 and L_1 of the flight vectors and of the reversible vectors* (*cf.* [1, *Theorem* 4.11]).

Proof. The vectors $\{T_s f : s \geq 0\}$ and their linear combinations evidently have relatively weakly compact orbits. The fact that the set $WAP(T)$ of weakly almost periodic vectors is norm-closed can be proved for the semigroup case exactly as for the group case by Eberlein [2, Theorem 4.2]. The T-invariance of L is again clear. For the rest see deLeeuw and Glicksberg [1, Section 4].

As an alternate reference for the deLeeuw-Glicksberg theory, see Krengel's book [6].

LEMMA 5. *In the notation of Lemma* 4 *let $I = P_0 + P_1$ be the sum of the projectors corresponding to the direct sum decomposition $L = L_0 \oplus L_1$. Then $f = P_1 f \in L_1$.*

Proof. The vector $P_0 f$ is in L_0, which means, by definition, that the vector 0 is in the weak closure of the orbit $\{T_t P_0 f : t \geq 0\}$ (see [1, pp. 73–74]). Hence for some generalized sequence $\{t(\alpha) : \alpha \in A\}$ of nonnegative reals we have $\lim_A T_{t(\alpha)} P_0 f = 0$ in the weak topology of L. The range of the generalized sequence above has at least one limit point p in the extended set of the nonnegative reals. If $p \in R$, then there is a subsequence $\{t(n) : n \in \mathbf{N}\}$ of the generalized sequence such that $\lim_{n \to \infty} t(n) = p$. By the strong continuity of the semigroup and by the preceding remarks then

$$T_p P_0 f = \lim_{n \to \infty} T_{t(n)} P_0 f = \lim_A T_{t(\alpha)} P_0 f = 0,$$

where the limits are taken in the weak topology of L. Hence

$$T_t P_0 f = 0 \quad (t \geq p),$$

and the proof can be finished exactly as in the case of the other logical possibility. This is $p = \infty$, which implies the existence of a subsequence $\{t(n) : n \in \mathbf{N}\}$ tending to ∞ such that

$$\lim_{n\to\infty} \left\langle T_{t(n)} f, P_0^* z^* \right\rangle = \lim_{n\to\infty} \left\langle T_{t(n)} P_0 f, z^* \right\rangle = 0$$

for every $z^* \in L^*$, since the projector P_0 clearly commutes with the restriction $T|L$. On the other hand, by assumption,

$$|\langle P_0 f, z^* \rangle| = |\langle f, P_0^* z^* \rangle| = \lim_{n\to\infty} |\langle T_{t(n)} f, P_0^* z^* \rangle| = 0$$

for every $z^* \in L^*$. Hence $P_0 f = 0$, and $f = P_1 f$ is in L_1.

LEMMA 6. *The restriction $S = \{S_t : t \geq 0\}$ of T to L_1 can be extended to a group $G = \{G_t : t \in \mathbf{R}\}$, almost periodic in the sense that for any $x \in L_1$ the orbit $\{G_t x : t \in \mathbf{R}\}$ is relatively compact in the norm topology of L_1. There is a real number λ such that, if B denotes the generator operator of the semigroup T, then*

$$Bf = i\lambda f, \quad T_t f = e^{i\lambda t} f.$$

Proof. By [1, Lemma 4.6], L_1 is the closed linear subspace of L spanned by the common eigenvectors of T having eigenvalues of modulus 1, i.e. by those $x \in L$ that satisfy

$$T_t x = S_t x = e^{i\mu t} x \quad (t \geq 0)$$

for some $\mu \in \mathbf{R}$. Linear combinations of such vectors have finite dimensional, hence relatively norm-compact orbits with respect to T. Eberlein [2, Theorem 4.2] shows again that the set of all vectors with relatively norm-compact orbits is closed in the norm topology, hence $S = T|L_1$ is almost periodic. By Lyubich and Lyubich [7, pp. 80–81] the semigroup S extends to an almost periodic group G; further for every $\lambda \in \mathbf{R}$ there is an orthoprojector

$$Q(\lambda) = \lim_{t\to\infty} 1/t \int_0^t S_r e^{-i\lambda r}\, dr,$$

where the integral exists in the strong operator topology. These orthoprojectors have the following properties:

$$\begin{aligned} Q(\lambda)Q(\mu) &= \delta_{\lambda\mu}Q(\lambda), \\ \overline{\operatorname{span}}\{Q(\lambda)L_1 : \lambda \in \mathbf{R}\} &= L_1, \\ \cap\{\ker Q(\lambda) : \lambda \in \mathbf{R}\} &= \{0\}, \\ Q(\lambda)L_1 &= \{x \in L_1 : Bx = i\lambda x\} \\ &= \{x \in L_1 : T_t x = e^{i\lambda t}x \text{ for } t \in \mathbf{R}\}, \end{aligned}$$

and the operators S_t and $Q(\lambda)$ clearly commute for every t, $\lambda \in \mathbf{R}$.

Should there exist no $\lambda \in \mathbf{R}$ with the property stated in the lemma, there would be λ, $\mu \in \mathbf{R}$ such that

$$Q(\lambda)f \neq 0, \quad Q(\mu)f \neq 0, \quad \lambda \neq \mu.$$

Assuming this, for every $z^* \in L_1^*$ we have

$$\begin{aligned} |\langle e^{i\lambda t}Q(\lambda)f + e^{i\mu t}Q(\mu)f, z^*\rangle| &= |\langle S_t[Q(\lambda) + Q(\mu)]f, z^*\rangle| \\ &= |\langle T_t f, [Q(\lambda)^* + Q(\mu)^*]z^*\rangle| \\ &\to |\langle f, [Q(\lambda) + Q(\mu)]^* z^*\rangle| \\ &= |\langle [Q(\lambda) + Q(\mu)]f, z^*\rangle| \end{aligned}$$

as $t \to \infty$. There exists a linear functional $x^* \in L_1^*$ such that

$$c(\lambda) = \langle Q(\lambda)f, x^*\rangle \neq 0, \quad c(\mu) = \langle Q(\mu)f, x^*\rangle \neq 0.$$

Taking this x^* in place of z^* in the formula above we obtain

$$\lim_{t\to\infty} |e^{i\lambda t}c(\lambda) + e^{i\mu t}c(\mu)| = |c(\lambda) + c(\mu)|.$$

Hence we obtain

$$\lim_{t\to\infty} \left|\frac{c(\lambda)}{c(\mu)} + e^{i(\mu-\lambda)t}\right| = \left|\frac{c(\lambda)}{c(\mu)} + 1\right|,$$

which is clearly absurd. The proof is complete.

3. The Hilbert space case again

In this section we shall give a different (third) proof of the Hilbert space special case of the main theorem by using the following extension of a result of Norbert Wiener (for the extension and for historical remarks see, e.g., Goldstein [4]):

WIENER'S THEOREM. *Let B generate a (C_0) contraction semigroup T on the Hilbert space X. Then for all $f_1, f_2 \in X$,*

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s |\langle T(t)f_2, f_1\rangle|^2\, dt = \sum_{\lambda\in\Lambda} |\langle P_\lambda f_2, f_1\rangle|^2,$$

where Λ is the set of all purely imaginary eigenvalues of B, and for $\lambda \in \Lambda$, P_λ is the orthogonal projection onto the kernel of $B - \lambda I$.

Proof of the Hilbert space theorem. Let f be a unit vector in the Hilbert space X satisfying

$$\lim_{t\to\infty} |\langle T_t f, f\rangle| = 1.$$

Applying Wiener's theorem above gives

$$\sum_{\lambda\in\Lambda} |P_\lambda f|^4 = \lim_{s\to\infty} \frac{1}{s}\int_0^s |\langle T_t f, f\rangle|^2\, dt = 1.$$

From this we obtain

$$1 = \sum_{\lambda\in\Lambda} |P_\lambda f|^4 \le \sum_{\lambda\in\Lambda} |P_\lambda f|^2 \le |f|^2 = 1,$$

since the distinct orthogonal projections of f are pairwise orthogonal (cf. Jacobs [5]). This line of inequalities shows that there is exactly one $\lambda \in \Lambda$ for which

$$0 \ne P_\lambda f = f,$$

which is the assertion of the theorem.

REFERENCES

1. K. DELEEUW and I. GLICKSBERG, *Applications of almost periodic compactifications*, Acta Math. **105** (1961), 63–97.
2. W.F. EBERLEIN, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67** (1949), 217–240.
3. J.A. GOLDSTEIN,, *External properties of contraction semigroups on Hilbert and Banach spaces*, Bull. London Math. Soc., **25** (1993), 369–376.
4. ______, "Asymptotics for bounded semigroups on Hilbert space" in *Aspects of positivity in functional analysis*, ed. R. Nagel et al., Elsevier (North Holland), Amsterdam, 1986, pp. 49–62.

5. K. JACOBS, *Ergodentheorie und fastperiodische Funktionen auf Halbgruppen*, Math. Zeitschr. **64** (1956), 298–338.
6. V. KRENGEL, *Ergodic theorems*, W. de Gruyter, Berlin, 1985.
7. M. YU. LYUBICH and YU. I. LYUBICH, *The splitting of the boundary spectrum for almost periodic operators and representations of semigroups*, Theory of functions, functional analysis and their applications (Harkov) **42** (1986), 69–84 (in Russian).

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