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TRACE IDEAL CRITERIA FOR OPERATORS OF HANKEL TYPE

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In this paper we obtain trace ideal criteria for commutators of multiplication operators with integral operators having kernels of a critical homogeneity. The prototype for the class of operators considered is the Hankel operator associated with the Bergman projection. Specifically, let D be a domain in $\mathbf{C}^n$, and let P be the orthogonal projection of $L^2(D)$ onto its holomorphic subspace $A^2(D)$. When f is a function on D, the *Hankel operator* with *symbol* f is defined formally by

$$H_f = (I - P)M_f P = [M_f, P]P,$$

where $[M_f, P]$ is the *commutator* defined by

$$[M_f, P] = M_f P - PM_f$$

and M_f is the multiplication operator defined by $M_f g = fg$.

Trace ideal criteria for Hankel operators with conjugate holomorphic symbols have been obtained in various settings by several authors [2], [8], [12], [14], [19]. For general symbols, trace ideal criteria for the commutators $[M_f, P]$, and hence also for H_f, were first obtained in the unit ball by K. Zhu [21], and later in bounded symmetric domains by D. Zheng [20]. More recently, D. Luecking [15] has given a direct characterization of the symbol classes for trace ideals of Hankel operators in the unit disk, without the mediation of the commutator operators that was used in earlier work. (Of course, the distinction between H_f and $[M_f, P]$ evaporates when the symbol f is conjugate holomorphic, as in the case of classical Hankel operators.)

The purpose of this paper is to extend the results of [21] and [20] to a general class of operators which are loosely modeled on the operators H_f. As special cases, we will obtain trace ideal criteria for Hankel operators on strictly pseudoconvex domains in $\mathbf{C}^n$ and on finite type domains in $\mathbf{C}^2$ (see

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Corollaries 4.3 and 4.8). In our work here, we view the commutator operator $[M_f, P]$, rather than H_f, as the fundamental object of interest. In fact, sufficient trace ideal criteria will be obtained for commutators of M_f with a more general class of integral operators, requiring only that the kernel have the same homogeneity as the Bergman kernel. We do not require the integral operator to be a projection, nor do we impose any analyticity condition on the kernel. In fact, our sufficiency results use only real variable techniques, and we have found it convenient to give an axiomatic treatment, using the same general framework we have used previously in [4] (see also [1], [10], [17]) to establish boundedness and compactness criteria. Specifically, we work with kernels defined on sets which are bounded by a space of homogeneous type, and homogeneity of the kernel is expressed in terms of the homogeneous structure on the boundary. A detailed description of this setup is given in Section 1, along with a formulation of a sufficient condition for the commutator to be in the trace ideal $\mathcal{S}_p(L^2)$ when $p \geq 2$ (Theorem 1.5). Sections 2 and 3 are devoted to the proof of this result. In Section 4, using an idea of J. Burbea [7], we show that, under mild regularity assumptions, our sufficient conditions are also necessary when the integral operator is the Bergman projection.

1. Preliminaries

We begin by defining a homogeneous structure. Let X be a locally compact Hausdorff space. A *homogeneous structure* on X consists of a positive, regular Borel measure μ on X, and a family $\{B(x, r) : x \in X, r > 0\}$ of basic open subsets of X such that for some constants $c > 1$ and $K > 1$ we have:

(1) $x \in B(x, r)$ for every $x \in X$ and every $r > 0$;
(2) If $x \in X$ and $0 < r_1 \leq r_2$, then $B(x, r_1) \subset B(x, r_2)$;
(3) If $B(x_1, r_1) \cap B(x_2, r_2) \neq \varnothing$ and $r_1 \geq r_2$, then $B(x_1, cr_1) \supseteq B(x_2, r_2)$;
(4) $X = \bigcup_{r>0} B(x, r)$ for some (and hence every) $x \in X$;
(5) $0 < \inf_{x \in X} \mu(B(x, 1)) \leq \sup_{x \in X} \mu(B(x, 1)) < \infty$.
(6) $\mu(B(x, cr)) \leq K\mu(B(x, r))$ for all $x \in X$ and all $r > 0$.

The constants c and K are called the *homogeneous structure constants*. For the remainder of this section, we work with some fixed homogeneous structure on the space X.

Let $\tilde{X} = X \times (0, \infty)$, and for any $z = (x, s) \in \tilde{X}$, write $z^* = x$ and $\delta(z) = s$. The *Carleson region* $C(z)$ is defined by

$$C(z) = C(x, s) = B(x, s) \times (0, s).$$

Let μ_0 be the measure on $\tilde{X}$ defined by

$$d\mu_0(x, s) = d\mu(x)\, ds$$

and for any real number β, define

$$d\mu_\beta(z) = \mu_0(C(z))^{-\beta}\, d\mu_0(z).$$[1]

For any $x, y \in X$, let

$$\rho(x, y) = \inf\{s > 0 : y \in B(x, s) \text{ and } x \in B(y, s)\}.$$

Then ρ is a quasimetric on X. For any two points $z = (x, s)$ and $w = (y, t)$ in $\tilde{X}$, let

$$r(z, w) = r((x, s), (y, t)) = s + t + \rho(x, y).$$

From the defining properties of homogeneous structures, one easily checks that $r(z, w)$ is comparable with

$$r_1(z, w) = r_1((x, s), (y, t)) = \inf\{\sigma \geq s : C(x, \sigma) \supseteq C(w)\}.$$

Let

$$C(z, w) = C(z^*, r(z, w)).$$

The *kernel* of a homogeneous structure is the function on $\tilde{X} \times \tilde{X}$ defined by

$$\mathscr{K}(z, w) = \frac{1}{\mu_0(C(z, w))}.$$

From the defining properties of a homogeneous structure, one checks that

$$\mathscr{K}(z, w) \approx \frac{1}{\mu_0(C(z^*, r_1(z, w)))} \approx \frac{1}{\mu_0(C(w^*, r_1(z, w)))} \approx \mathscr{K}(w, z).$$

The following estimate for the growth of the measure of Carleson regions is a special case of Lemma 1.2 of [4].

LEMMA 1.1. *Let c and K be the constants of the homogeneous structure, and let $a = 1 + \log K/\log c$. Then for any $(x, t) \in \tilde{X}$ and any $0 < s \leq 1$, we have*

$$\mu_0(C(x, st)) \geq K^{-2} s^a \mu_0(C(x, t)).$$

[1]When $\beta < 0$, this is at odds with the notation of [4].

The following integrability result for $\mathscr{K}(z, \cdot)$ is a slight variation on Lemma 1.4 of [4].

LEMMA 1.2. *Let $ab < 1$ with $a > 1$ as in Lemma 1.1, and let $\alpha + b > 1$. Then*

$$\int_{\tilde{X}} \mathscr{K}(z, w)^{\alpha}\, d\mu_b(w) \leq C_{\alpha, b}\, \mu_0(C(z))^{1-\alpha-b}.$$

Proof. The case $b < 0$ is a straightforward consequence of the case $b = 0$, so we may assume that $b \geq 0$. Let c and K be the homogeneous structure constants, and for each non-negative integer j let $Q_j = C(x, c^j s)$ where $z = (x, s)$. Letting I denote the integral we wish to estimate, we have

$$I = \sum I_j$$

where

$$I_0 = \int_{Q_0}$$

and

$$I_j = \int_{Q_j \setminus Q_{j-1}}.$$

For $(y, t) \in Q_0$, Lemma 1.1 gives the estimate

$$\mu_0(C(y, t)) \geq C\left(\frac{t}{s}\right)^{a} \mu_0(C(x, s))$$

so

$$\begin{aligned} I_0 &\leq \mu(Q_0)^{-\alpha-b} \int_{Q_0} \left(\frac{t}{s}\right)^{-ab} dt \\ &= C\mu_0(Q_0)^{-\alpha-b} s\mu(B(x, s)) \\ &= C\mu_0(Q_0)^{1-\alpha-b}. \end{aligned}$$

Similarly, since $\mu_0(C(x, s+t+\rho(x,y))) \approx \mu_0(Q_j)$ for $(y,t) \in Q_j \setminus Q_{j-1}$, we have

$$\begin{aligned} I_j &\le C\mu_0(Q_j)^{-\alpha-b} \int_{Q_j} \left(\frac{t}{s}\right)^{-ab} dt \\ &= C\mu_0(Q_j)^{1-\alpha-b} \\ &\le Cc^{(1-\alpha-b)j}\mu_0(Q_0)^{(1-\alpha-b)}, \end{aligned}$$

and summing over j gives the lemma.

A homogeneous structure is called *smooth* if each of the functions $s \mapsto \mu(B(x,s))$ is differentiable on $(0,\infty)$, and

$$\frac{\partial}{\partial s}\mu(B(x,s)) \le C\frac{\mu(B(x,s))}{s}.$$

The most immediate example of a smooth homogeneous structure is $\mathbf{R}^n$ with the Lebesgue measure and Euclidean balls. Additional examples include boundaries of bounded strictly pseudoconvex domains in $\mathbf{C}^n$ with the Euclidean surface measure and the Korányi-Stein pseudo-balls, and boundaries of bounded finite type domains in $\mathbf{C}^2$ with the Euclidean surface measure and the balls defined in [3]. It is immediate from the definition that the kernel of a smooth homogeneous structure satisfies

$$\max\{D_z\mathscr{K}(z,w), D_w\mathscr{K}(z,w)\} \le C\frac{\mathscr{K}(z,w)}{r(z,w)} \tag{1}$$

where D denotes differentiation in the half-line direction in $\tilde{X}$.

For each $z \in \tilde{X}$, define a positive function κ_z on $\tilde{X}$ by

$$\kappa_z(w) = \left(\int_{\tilde{X}} \mathscr{K}(z,\cdot)^2\, d\mu_0\right)^{-1} \mathscr{K}(z,w)^2.$$

The following lemma summarizes some elementary properties of the functions κ_z.

LEMMA 1.3. (1) $\kappa_z \approx \mu_0(C(z))\mu_0(C(z,w))^{-2} = \mathscr{K}(z,z)^{-1}\mathscr{K}(z,w)^2$;
(2) $|D_z\kappa_z(w)| \le C\kappa_z(w)/\delta(z)$;
(3) $\int \kappa_z\, d\mu_0 = 1$.

Proof. (1) and (2) are immediate from Lemma 1.2, and (3) is trivial.

For any measurable function f on $\tilde{X}$, we define

$$\hat{f}(z) = \int_{\tilde{X}} f\kappa_z \, d\mu_0$$

for any $z \in \tilde{X}$ such that the integral converges absolutely.

LEMMA 1.4. *If $f \in L^p(\tilde{X}, d\mu_0)$ with $1 \le p \le \infty$, then $\hat{f}$ is defined on $\tilde{X}$, and*

$$|\hat{f}(z)| \le C\|f\|_{L^p}\mu_0(C(z))^{-1/p}.$$

Proof. By Jensen's Inequality,

$$\begin{aligned}|\hat{f}(z)|^p &\le \int |f|^p\kappa_z \, d\mu_0 \\ &\le C\mu_0(C(z))^{-1}\|f\|_{L^p}^p.\end{aligned}$$

We now turn to the formulation of our main theorem. For $1 \le p \le \infty$, we define Y^p to be the space of all functions $f \in L^p(\tilde{X}, d\mu_0)$ such that

$$\|f\|_{Y^p} = \|f\|_{L^2(d\mu_0)} + \|S_2 f\|_{L^p(d\mu_1)}$$

where

$$(S_p f)(z) = \left(\int_{\tilde{X}} |f - \hat{f}(z)|^p \kappa_z \, d\mu_0\right)^{1/p}. \tag{2}$$

When f is a function on $\tilde{X}$, the *multiplication operator* M_f is defined formally by

$$M_f g = fg.$$

When G is a function on $\tilde{X} \times \tilde{X}$, the *integral operator* $\mathcal{I}_G$ is defined formally by

$$\mathcal{I}_G f(z) = \int_{\tilde{X}} f(w)G(z,w) \, d\mu_0(w).$$

The *commutator operator* with *symbol* f and *kernel* G is defined formally by

$$[M_f, \mathcal{I}_G] = M_f\mathcal{I}_G - \mathcal{I}_G M_f.$$

For $1 \le p < \infty$, we let $\mathcal{S}_p$ denote the usual trace ideal of operators on $L^2(\tilde{X}, d\mu_0)$. (Recall that an operator T on a Hilbert space $\mathcal{H}$ is in the trace ideal $\mathcal{S}_p$ if and only if $|T|_p = (\mathrm{tr}(T^*T)^{p/2})^{1/p} < \infty$.)

THEOREM 1.5. *Let G be a measurable function on $\tilde{X} \times \tilde{X}$ such that $|G(z,w)| \leq \mathscr{K}(z,w)$ for all $(z,w) \in \tilde{X} \times \tilde{X}$, and let $f \in Y^p$ with $2 \leq p < \infty$. Then the commutator operator $[M_f, \mathscr{I}_G]$ is in the trace ideal $\mathscr{S}_p$, and $\|[M_f, \mathscr{I}_G]\|_p \leq C\|f\|_{Y^p}$.*

2. Estimates for the smooth part

Trace ideal criteria for commutators will be obtained by splitting the symbol into a smooth part and a non-smooth remainder, and estimating the corresponding commutator operators separately. In this section, we will obtain estimates for the commutators associated with the smooth part of the symbol. We will prove:

THEOREM 2.1. *Let $G(z,w)$ be a measurable function on $\tilde{X} \times \tilde{X}$ satisfying $|G(z,w)| \leq \mathscr{K}(z,w)$, and let $f \in Y^p$ with $2 \leq p < \infty$. Then the commutator $[M_{\hat{f}}, \mathscr{I}_G]$ is in $\mathscr{S}_p$, and $\|[M_{\hat{f}}, \mathscr{I}_G]\|_p \leq C\|f\|_{Y^p}$.*

We shall use a result of B. Russo [18] which provides a sufficient condition for a general integral operator to be in a trace ideal. Russo's condition may be formulated as follows.

THEOREM 2.2. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space, and let $k(x,y)$ be a measurable function on $X \times X$ satisfying*

$$\|k\|_{p',p} = \left(\int\left(\int|k(x,y)|^{p'}\,d\mu(x)\right)^{p/p'} d\mu(y)\right)^{1/p} < \infty$$

and

$$\|k^*\|_{p',p} = \left(\int\left(\int|k(y,x)|^{p'}\,d\mu(x)\right)^{p/p'} d\mu(y)\right)^{1/p} < \infty$$

for some $2 \leq p < \infty$. Then the associated integral operator $\mathscr{I}_k$, defined on a dense subspace of $L^2(\mu)$ by

$$\mathscr{I}_k f(x) = \int f(y)k(x,y)\,d\mu(y),$$

is in the trace ideal $\mathscr{S}_p$, and moreover

$$|\mathscr{I}_k|_p \leq \left(\|k\|_{p',p}\|k^*\|_{p',p}\right)^{1/2}. \tag{3}$$

In our case, the operator of interest, namely $[M_f, \mathscr{I}_G]$, is an integral operator with kernel

$$G_f(z,w) = (f(z) - f(w))G(z,w).$$

Since G_f is dominated pointwise by

$$\mathscr{K}_f(z,w) = (f(z) - f(w))\mathscr{K}(z,w),$$

it is clear that $[M_f, \mathscr{I}_G] \in \mathscr{S}_p$ whenever $\mathscr{K}_f$ satisfies the conditions of Theorem 2.2. Moreover, since $\mathscr{K}_f(w,z) \approx \mathscr{K}_f(z,w)$, when $k = \mathscr{K}_f$, the righthand side of (3) is comparable with $\|\mathscr{K}_f\|_{p',p}$. Thus, we have:

COROLLARY 2.3. *Let G be a measurable function on $\tilde{X} \times \tilde{X}$ satisfying $|G(z,w)| \le \mathscr{K}(z,w)$, and let f be a measurable function on $\tilde{X}$ such that $\|\mathscr{K}_f\|_{p',p} < \infty$ for some $p \le 2 < \infty$. Then $[M_f, \mathscr{I}_G] \in \mathscr{S}_p$ with $\mathscr{S}_p$-norm at most a constant multiple of $\|\mathscr{K}_f\|_{p',p}$.*

We will need the following variant of Hardy's Inequality.

LEMMA 2.4. *Let g be a non-negative, measurable function on $(0,\infty)$ such that $g(st) \le bs^{-a}g(t)$ for every $t > 0$ and $0 < s < 1$, with $a < 1$. Then for any $1 \le p < \infty$ and any non-negative, measurable function f on $(0,\infty)$ we have*

$$\left(\int_0^\infty \left(\int_t^\infty f(s)\,\frac{ds}{s}\right)^p g(t)\,dt\right)^{1/p} \le \frac{pb^{1/p}}{1-a}\left(\int_0^\infty f(t)^p g(t)\,dt\right)^{1/p}.$$

Proof. Letting I denote the left hand side of the above inequality, changing variables in the inner integral and applying Minkowski's Inequality gives

$$\begin{aligned} I &= \left(\int_0^\infty \left(\int_1^\infty f(st)\,\frac{ds}{s}\right)^p g(t)\,dt\right)^{1/p} \\ &\le \int_1^\infty \left(\int_0^\infty f(st)^p g(t)\,dt\right)^{1/p} \frac{ds}{s}. \end{aligned}$$

Changing variables in the inner integral once again and using the estimate $g(t/s) \le bs^a g(t)$ when $s \ge 1$ gives

$$\begin{aligned} I &\le \int_1^\infty \left(\int_0^\infty f(t)^p s^{-1} g(t/s)\,dt\right)^{1/p} \frac{ds}{s} \\ &\le b^{1/p} \int_1^\infty s^{(a-1)/p}\,\frac{ds}{s}\left(\int_0^\infty f(t)^p g(t)\,dt\right)^{1/p} \end{aligned}$$

and the proof is complete.

We will apply Lemma 2.4 with $g(t)$ a negative power of $\mu_0(C(x,t))$ with $x \in X$.

COROLLARY 2.5. *Let $a > 1$ be as in Lemma 1.1, let $1 \le p < \infty$, and let b be a real number with $ab < 1$. Then there is a constant C, depending on c, K, b, and p, such that for any non-negative function f on $(0,\infty)$ we have*

$$\int_0^\infty \left(\int_t^\infty f(s) \frac{ds}{s} \right)^p \mu_0(C(y,t))^{-b}\, dt \le C \int_0^\infty f(t)^p \mu_0(C(y,t))^{-b}\, dt.$$

We next establish some estimates for the functions $\hat{f}$.

LEMMA 2.6. *There is a constant C such that for every $(x,s) \in \tilde{X}$,*

$$\left| \frac{\partial}{\partial s} \hat{f}(x,s) \right| \le C s^{-1} \inf_{a \in \mathbf{C}} \int |f - a| \kappa_{(x,s)}\, d\mu_0.$$

Proof. For any complex number a we have

$$\frac{\partial}{\partial s} \hat{f}(x,s) = \frac{\partial}{\partial s} \big(\hat{f}(x,s) - a \big) = \int_{\tilde{X}} (f - a) \frac{\partial}{\partial s} \kappa_{(x,s)}\, d\mu_0$$

so (2) of Lemma 1.3 gives the result.

LEMMA 2.7. *Let b be a real number with $ab < 1$, where $a > 1$ is as in Lemma 1.1. If $f \in L^q(\tilde{X}, d\mu_0)$ with $1 \le q \le \infty$, then for any $1 \le p < \infty$ we have*

$$\int |\hat{f}|^p\, d\mu_b \le C_p \int_{\tilde{X}} \left(\int_{\tilde{X}} |f(w) - \hat{f}(z)| \kappa_z(w)\, d\mu_0(w) \right)^p d\mu_b(z)$$

Proof. Since, by Lemma 1.4, $\hat{f}(x,\cdot\,)$ vanishes at infinity, we have

$$\int_{\tilde{X}} |\hat{f}|^p\, d\mu_b = \int_{\tilde{X}} \left| \int_s^\infty \frac{\partial}{\partial s} \hat{f}(x,t)\, dt \right|^p d\mu_b(x,s)$$

and so Lemma 2.6 gives

$$\int_{\tilde{X}} |\hat{f}|^p\, d\mu_b \le C \int_{\tilde{X}} \left(\int_s^\infty (S_1 f)(x,t) \frac{dt}{t} \right)^p d\mu_b(x,s),$$

where $S_1 f$ is defined by (2), and the result follows from Lemma 2.4.

COROLLARY 2.8. *Let b be a positive number with $ab < 1$, and let $f \in Y^p$ with $2(1-b) < p < \infty$. Then*

$$\int_{\tilde{X}} |\hat{f}|^p \, d\mu_b \leq C_{p,b} \|f\|_{Y^p}^p.$$

Proof. By Lemma 2.7, it suffices to show that

$$\int_{\tilde{X}} (S_1 f)^p \, d\mu_b \leq \|f\|_{Y^p}^p,$$

where $S_1 f$ is defined by (2). Letting $I(f)$ denote the left side of the above inequality, write

$$I(f) = I_1(f) + I_2(f)$$

where $I_1(f)$ is the integral over $X \times (0,1]$ and $I_2(f)$ is the integral over $X \times (1,\infty)$. Since $\mu_0(C(x,s))^{-b} \leq C\mu_0(C(x,s))^{-1}$ on $X \times (0,1]$, the estimate

$$I_1(f) \leq C\|f\|_{Y^p}^p$$

is elementary. Moreover

$$I_2(f) \leq 2^{p-1} \int_{X \times (1,\infty)} \left(\left(|f|\hat{\ } \right)^p + |\hat{f}|^p \right) d\mu_b$$

so Lemma 1.4 gives

$$I_2(f) \leq 2\|f\|_{L^2}^p \int_{X \times (1,\infty)} d\mu_{b+p/2} = C_{p,b} \|f\|_{L^2}^p$$

and the proof is complete.

From the defining properties of a homogeneous structure and (1) of Lemma 1.3, one easily obtains:

LEMMA 2.9. *Let $M \geq 1$. If $(x,s) \in \tilde{X}$ and $x' \in B(x, Ms)$, then*

$$\kappa_{(x', Ms)}(y,t) \leq C\kappa_{(x,s)}(y,t).$$

LEMMA 2.10. *Let $a > 1$ be as in Lemma 1.1, let b be a real number with $ab < 1$, and let $2(1 - 1/a) < p < \infty$. Then there is a constant $C = C_{b,p}$ such* *that for any* $f \in Y^p$,

$$(4)\qquad \int_{\tilde X}\int_{\tilde X} \left|\hat f(w) - \hat f(z)\right|^p \mathscr{K}(z,w)^{2-b}\, d\mu_b(w)\, d\mu_0(z) \le C\|f\|_{Y^p}^p.$$

Proof. Since the left hand side of (4) increases with b, it suffices to consider the case $1 - p/2 < b < 1/a$ (so that the hypothesis of Corollary 2.8 is satisfied). Letting $\alpha = (2-b)/p$, we have

$$\begin{aligned} I_1(z) &= \int_{\tilde X} \left|\hat f(w) - \hat f(z)\right|^p \mathscr{K}(z,w)^{2-b}\, d\mu_b(w) \\ &= \int_X \int_0^\infty \left| \int_t^\infty \frac{\partial}{\partial r}(\hat f(y,r) - \hat f(z))\mathscr{K}(z,(y,r))^\alpha\, dr\right|^p \mu_0(C(y,t))^{-b}\, dt\, d\mu(y) \end{aligned}$$

and Lemma 2.4 gives

$$\begin{aligned} (5)\quad & I_1(z) \\ &\le \int_X\int_0^\infty \left|\frac{\partial}{\partial t}(\hat f(y,t) - \hat f(z))\mathscr{K}(z,(y,t))^\alpha\right|^p t^p \mu_0(C(y,t))^{-b}\, dt\, d\mu(y) \\ &\le 2^{p-1}\int_{\tilde X}\left|\frac{\partial \hat f(y,t)}{\partial t}\right|^p \mathscr{K}(z,(y,t))^{\alpha p} t^p \mu_0(C(y,t))^{-b}\, d\mu_0(y,t) \\ &\quad + 2^{p-1}\int_{\tilde X}\left|\hat f(y,t) - \hat f(z)\right|^p \left|\frac{\partial}{\partial t}\mathscr{K}(z,(y,t))^\alpha\right|^p t^p\, d\mu_b(y,t) \\ &= I_{11}(z) + I_{12}(z). \end{aligned}$$

Lemma 2.6 and Jensen's Inequality give

$$\begin{aligned} I_{11}(z) &\le C\int (S_1 f)(w)^p \mathscr{K}(z,w)^{\alpha p}\, d\mu_b(w) \\ &\le C\int (S_2 f)(w)^p \mathscr{K}(z,w)^{\alpha p}\, d\mu_b(w), \end{aligned}$$

where $S_p f$ is defined by (2). Integrating with respect to z and applying Lemma 1.2 gives

$$(6)\qquad \int_{\tilde X} I_{11}(z)\, d\mu_0(z) \le C\|f\|_{Y^p}^p$$

since $p\alpha > 1$. (Recall that $a > 1$.)

To complete the proof, we must estimate $\int I_{12}$. The estimate (1) gives

$$\begin{aligned}(7)\quad &\int_{\tilde{X}_\sigma} I_{12}\, d\mu_0 \\ &\le C\int_{\tilde{X}}\int_{\tilde{X}_\sigma} |\hat{f}(w)-\hat{f}(z)|^p \mathscr{K}(z,w)^{\alpha p}\left(\frac{\delta(w)}{r(z,w)}\right)^p d\mu_0(z)\, d\mu_b(w) \\ &\le C\int_{\tilde{X}}\int_{C(w_M)} + C\int_{\tilde{X}}\int_{\tilde{X}_\sigma\setminus C(w_M)} \\ &= I^1(M) + I^2(M)\end{aligned}$$

for any $M > 0$, where $\tilde{X}_\sigma = X \times (\sigma, \infty)$ and $w_M = (w^*, M\delta(w))$. Combining the estimates (5), (6), and (7) gives

$$(8)\qquad \int_{\tilde{X}_\sigma} I_1(z)\, d\mu_0(z) \le C\|f\|_{Y^p}^p + I^1(M) + I^2(M).$$

Since the kernel $\mathscr{K}$ is bounded on $\tilde{X}_\sigma$, with bound depending on σ, it follows from Lemma 2.8 that I^2 is finite. Moreover, since $\delta(w)/r(z,w) \le 1/M$ when $z \in \tilde{X} \setminus C(y, Mt)$, it follows that $I^2(M)$ is less than or equal to half the left hand side of (8) when M is sufficiently large, independent of σ. Thus, if M is sufficiently large, the left hand side of (8) is at most a constant multiple of $\|f\|_{Y^p}^p + I^1(M)$, where the constant does not depend on σ. Letting $\sigma \to 0$ gives

$$I \le C\left(\|f\|_{Y^p}^p + I^1\right)$$

where I denotes the left hand side of (4). We will complete the proof by showing that $I^1 \le C\|f\|_{Y^p}^p$. Since $\mathscr{K}(z,w) \le C\mu_0(C(w))^{-1}$, we have

$$I^1 \le C\int_{\tilde{X}}\int_{C(w_M)} |\hat{f}(w)-\hat{f}(z)|^p\, d\mu_0(z)\, d\mu_2(w).$$

For the inner integral, writing $w = (y, t)$, we have

$$\begin{aligned}&\int_{C(w_M)} |\hat{f}(w)-\hat{f}(z)|^p\, d\mu_0(z) \\ &\quad\le 2^{p-1}\int_{C(w_M)}\left|\int_s^{Mt}\frac{\partial}{\partial\sigma}\hat{f}(x,\sigma)\, d\sigma\right|^p d\mu_0(x,s) \\ &\qquad + 2^{p+1}\int_{C(w_M)} |\hat{f}(w)-\hat{f}(x,Mt)|^p\, d\mu_0(x,s) \\ &\quad= J_1(w) + J_2(w).\end{aligned}$$

For the integrand of J_2, we have the estimate

$$\begin{aligned}|\hat{f}(w)-\hat{f}(x,Mt)|^p &= \left|\int_{\tilde{X}}(f-\hat{f}(w))\kappa_{(x,Mt)}\,d\mu_0\right|^p\\ &\le \left(\int_{\tilde{X}}|f-\hat{f}(w)|^2\kappa_{(x,Mt)}\,d\mu_0\right)^{p/2},\end{aligned}$$

so it follows from Lemma 2.9 that

$$\begin{aligned}|\hat{f}(w)-\hat{f}(x,Mt)|^p &\le C\left(\int_{\tilde{X}}|f-\hat{f}(w)|^2\kappa_w\,d\mu_0\right)^{p/2}\\ &= (S_2f)(w)^p\end{aligned}$$

and therefore

$$J_2(w)\le C\mu_0(C(w_M))(S_2f)(w)^p\le C\mu_0(C(w))(S_2f)(w)^p.$$

Integrating both sides over $\tilde{X}$ gives

$$\int_{\tilde{X}}J_2(w)\,d\mu_2(w)\le C\|f\|_{Y^p}^p,$$

and it remains to estimate $\int J_1$. By Hardy's Inequality,

$$J_1(w)=J_1(y,t)\le C\int_{B(y,Mt)}\int_0^{Mt}\frac{\partial}{\partial s}\hat{f}(x,s)^p s^p\,ds\,d\mu(x)$$

so by Lemma 2.6, we have

$$\begin{aligned}J_1(w) &\le C\int_{C(w_M)}\left(\int_{\tilde{X}}|f-\hat{f}(x,s)|\kappa_{(x,s)}\,d\mu_0\right)^p d\mu_0(x,s)\\ &\le C\int_{C(w_M)}(S_2f)^p\,d\mu_0.\end{aligned}$$

Integrating over $\tilde{X}$ and using the fact that $\mathscr{K}(z,w)\approx\mu_0(C(w))^{-1}$ when $z\in C(w_M)$ gives

$$\int_{\tilde{X}}J_1\,d\mu_2\le C\int\int(S_2f)(z)^p\mathscr{K}(z,w)^2\,d\mu_0(w)\,d\mu_0(z).$$

Finally, applying Fubini's Theorem and Lemma 1.2 gives

$$\int_{\tilde{X}} J_1 \, d\mu_2 \leq \|f\|_{Y^p}^p$$

and the proof is complete.

We now turn to the proof of Theorem 2.1. By Corollary 2.3, it suffices to prove that $\|\mathscr{K}_f\|_{p',p} \leq C\|f\|_{Y^p}$. In the case $2 < p < \infty$, applying Hölder's Inequality with the pair of conjugate exponents p/p' and $(p-1)/(p-2)$, we obtain

$$\begin{aligned}
&\left(\int_{\tilde{X}} |\hat{f}(w) - \hat{f}(z)|^{p'} \mathscr{K}(z,w)^{p'} \, d\mu_0(w)\right)^{p/p'} \\
&\quad \leq \int_{\tilde{X}} |\hat{f}(w) - \hat{f}(z)|^{p} \mathscr{K}(z,w)^{\alpha p} \, d\mu_0(w) \\
&\quad\quad \times \left(\int_{\tilde{X}} \mathscr{K}(z,w)^{(1-\alpha)p/(p-2)} \, d\mu_0(w)\right)^{p-2}.
\end{aligned}$$

Choosing $\alpha < 2/p$, the second integral on the right can be estimated by Lemma 1.2 to obtain

$$\begin{aligned}
&\left(\int_{\tilde{X}} |\hat{f}(w) - \hat{f}(z)|^{p'} \mathscr{K}(z,w)^{p'} \, d\mu_0(w)\right)^{p/p'} \\
&\quad \leq C\mu_0(C(z))^{\alpha p - 2} \int_{\tilde{X}} |\hat{f}(w) - \hat{f}(z)|^{p} \mathscr{K}(z,w)^{\alpha p} \, d\mu_0(w).
\end{aligned}$$

On the other hand, this last inequality holds trivially if $p = 2$, so it is in fact valid for $2 \leq p < \infty$ and $\alpha < 2/p$. Choosing α so that $2 - \alpha p$ is small and positive, integrating both sides, and applying Lemma 2.10, we obtain

$$\|\mathscr{K}_f\|_{p',p} \leq C\|f\|_{Y^p}^p,$$

which completes the proof of Theorem 2.1.

3. Estimates for the remainder

In this section we complete the proof of Theorem 1.5 by showing that for $p \geq 2$ the commutator operator with symbol $f - \hat{f}$ is in the trace ideal $\mathscr{S}_p$ whenever $f \in Y^p$. In fact, we prove a somewhat stronger result.

THEOREM 3.1. *Let G be a measurable function on $\tilde{X} \times \tilde{X}$ satisfying $|G(z,w)| \le \mathscr{K}(z,w)$ for all $z, w \in X$. Then for any $f \in Y^p$ with $2 \le p \le \infty$, the operator $M_{f-\hat{f}}\mathscr{I}_G$ is in the trace ideal $\mathscr{S}_p$, and $|M_{f-\hat{f}}\mathscr{I}_G|_p \le C\|f\|_{Y^p}$.*

Before proceeding, let us note that, since $\mathscr{S}_p$ is closed under the adjoint operation, it follows that, under the conditions of the theorem, the operator $\mathscr{I}_G M_{f-\hat{f}}$ is also in $\mathscr{S}_p$, with norm at most a constant multiple of $\|f\|_{Y^p}$. Thus, we have:

COROLLARY 3.2. *Let G and f be as above. Then the commutator operator $[M_f, \mathscr{I}_G]$ is in $\mathscr{S}_p$, and $\|[M_f, \mathscr{I}_G]\|_p \le C\|f\|_{Y^p}$.*

Theorem 1.5 is an immediate consequence of Theorem 2.1 and Corollary 3.2.

The proof of Theorem 3.1 requires an interpolation result for a new scale of function spaces, which we now introduce. For any measurable function f on $\tilde{X}$, we define

$$T(f)(z) = \left(\int |f(w)|^2 \kappa_z(w)\, d\mu_0(w) \right)^{1/2}$$

and for $1 \le p < \infty$ we define $\tilde{L}^p$ to be the space of measurable functions on $\tilde{X}$ such that $T(f) \in L^p(d\mu_1)$, with

$$\|f\|_{\tilde{L}^p} = \|f\|_{L^2(d\mu_0)} + \|T(f)\|_{L^p(d\mu_1)}.$$

In order to establish the basic properties of the spaces $\tilde{L}^p$, we introduce some machinery. For any $z = (x,s) \in \tilde{X}$ and any $0 < \varepsilon < 1$, let

$$\mathscr{B}_\varepsilon(z) = \left\{ (y,t) \in \tilde{X} : y \in B(x, \varepsilon s) \text{ and } |s-t| < \varepsilon s \right\}.$$

The following properties of the "balls" $\mathscr{B}_\varepsilon(z)$ are immediate.

LEMMA 3.3. *For any fixed $0 < \varepsilon < 1$ we have:*

(1) *$\kappa_w(\zeta) \approx \kappa_z(\zeta)$ uniformly for $z, \zeta \in \tilde{X}$ and $w \in \mathscr{B}_\varepsilon(z)$.*
(2) *$\mathscr{K}(w,w) \approx \mathscr{K}(z,z)$ uniformly for $z \in \tilde{X}$ and $w \in \mathscr{B}_\varepsilon(z)$.*
(3) *$\mathscr{K}(w,\zeta) \approx \mathscr{K}(z,\zeta)$ uniformly for $z, \zeta \in \tilde{X}$ and $w \in \mathscr{B}_\varepsilon(z)$.*
(4) *$\mu_0(\mathscr{B}_\varepsilon(z)) \approx \mathscr{K}(z,z)^{-1}$.*
(5) *There is a constant M and a sequence $\{z_j\}$ in $\tilde{X}$ such that the family $\{\mathscr{B}_\varepsilon(z)$ covers $\tilde{X}$, and no point of $\tilde{X}$ is covered more than M times.*

The relevant properties of the spaces $\tilde{L}^p$ are summarized in the next result.

LEMMA 3.4. *Let a be as in Lemma* 1.1.

(1) $\tilde{L}^p$ *is a (possibly trivial) Banach space for* $1 \le p \le \infty$.

(2) *For* $2 \le p \le \infty$, *we have* $L^p(d\mu_1) \subset \tilde{L}^p$, *and the embedding is continuous.*

(3) $\tilde{L}^{p_1} \subset \tilde{L}^{p_2}$ *for* $1 \le p_1 \le p_2 \le \infty$, *and the embedding is continuous.*

(4) *For* $p > 2(1 - 1/a)$, *every bounded, measurable, compactly supported function on* $\tilde{X}$ *is in* $\tilde{L}^p$.

(5) *For* $p > 2(1 - 1/a)$, *the simple functions with compact support on* $\tilde{X}$ *form a dense subspace of* $\tilde{L}^p$.

Proof. The proof of (1) is straightforward, and will be omitted.

The case $p = \infty$ of (2) is trivial. On the other hand, for $f \in L^p(d\mu_1)$ with $2 \le p < \infty$, Jensen's Inequality gives

$$T(f)(z)^p \le \int |f|^p \kappa_z \, d\mu_0,$$

so Fubini's Theorem and Lemmas 1.3 and 1.2 give

$$\begin{aligned}\|f\|_{\tilde{L}^p}^p &\le C \int |f(w)|^p \int \frac{d\mu_0(z)}{\mu_0(C(z,w))^2} \, d\mu_0(w) \\ &\le C \int |f(w)|^p \frac{d\mu_0(w)}{\mu_0(C(w))} = \|f\|_{L^p(d\mu_1)}^p\end{aligned}$$

and (2) is proved.

For (3), let $f \in \tilde{L}^{p_1}$. By Lemma 3.3, for any fixed $0 < \varepsilon < 1$ we have

$$\begin{aligned}T(f)^{p_1}(z) &= \frac{1}{\mu_0(\mathscr{B}_\varepsilon(z))} \int_{\mathscr{B}_\varepsilon(z)} T(f)^{p_1}(z) \, d\mu_0(w) \\ &\le C \int_{\mathscr{B}_\varepsilon(z)} T(f)^{p_1}(w) \frac{d\mu_0(w)}{\mu_0(C(w))} \le C\|f\|_{\tilde{L}^{p_1}}^{p_1},\end{aligned}$$

which establishes (3) when $p_2 = \infty$. When $p_1 \le p_2 < \infty$, we have

$$\begin{aligned}\|f\|_{\tilde{L}^{p_2}}^{p_2} &= \int T(f)^{p_2} \, d\mu_1 \\ &\le \|T(f)\|_\infty^{p_2 - p_1} \int T(f)^{p_1} \, d\mu_1 = \|f\|_{\tilde{L}^\infty}^{p_2 - p_1} \|f\|_{\tilde{L}^p}^{p_1},\end{aligned}$$

and from the case $p_2 = \infty$, we obtain $\|f\|_{\tilde{L}^{p_2}} \le C\|f\|_{\tilde{L}^{p_1}}$, so (3) is proved.

To prove (4), it suffices to show that every point in $\tilde{X}$ has a neighborhood whose characteristic function is in $\tilde{L}^p$. For fixed $0 < \varepsilon < 1$, if f is the characteristic function of $\mathscr{B}_\varepsilon(z_0)$,

$$T(f)(z) \leq C_{\varepsilon, z_0} \frac{\mu_0(C(z))^{1/2}}{\mu_0(C(z, r(z, z_0)))}$$

and by Lemma 1.2,

$$\begin{aligned}\|f\|_{\tilde{L}^p}^p &\leq C_{\varepsilon, z_0} \int \frac{\mu_0(C(z))^{p/2-1}}{\mu_0(C(z, r(z, z_0)))^p} \, d\mu_0(z) \\ &\leq C_{\varepsilon, z_0, p} \, \mu_0(C(z_0))^{-p/2}\end{aligned}$$

if $(p/2 - 1)a < 1$, and (4) is proved.

Finally, we prove (5). Let $f \in \tilde{L}^p$ and let $\varepsilon > 0$ be arbitrary. Fix $x_0 \in X$, and let $z_0 = (x_0, 1)$. Since $T(f) \in L^p(d\mu_1)$, by the Monotone Convergence Theorem, there is an $M > 1$ such that

$$\int_{\tilde{X} \setminus Q_M} T(f)^p \, d\mu_1 < \varepsilon^p \tag{9}$$

where

$$Q_M = B(x_0, M) \times [1/M, M].$$

Then

$$\begin{aligned}\int_{\tilde{X}} |f(w)|^2 \frac{d\mu_0(w)}{\mu_0(C(z_0, w))} &\leq C \int_{\tilde{X}} |f(w)|^2 \kappa_{z_0} \, d\mu_1(w) \\ &= C T(f)(z_0)^2 < \infty.\end{aligned}$$

It follows that there is a simple function g with compact support in $\tilde{X}$ such that $|g| \leq |f|$ pointwise and

$$\int_{\tilde{X}} | f(w) - g(w)|^2 \frac{d\mu_0(w)}{\mu_0(C(z_0, w))} < \delta,$$

where δ is a small positive number to be chosen below. From the defining properties of homogeneous structures, for any $z \in Q_M$ we have

$$\begin{aligned} T(f-g)(z)^2 &\le C\int_{\tilde{X}} |f(w) - g(w)|^2 \frac{d\mu_0(w)}{\mu_0(C(z,w))} \\ &\le C_M \int_{\tilde{X}} |f(w) - g(w)|^2 \frac{d\mu_0(w)}{\mu_0(C(z_0,w))} \\ &\le C_M \delta, \end{aligned}$$

where C_M is a constant depending only on z_0 and M. Integrating over Q_M gives

$$\int_{Q_M} T(f-g)^p \, d\mu_1 \le (C_M\delta)^{p/2} \mu_1(Q_M),$$

so choosing δ sufficiently small gives

$$\int_{Q_M} T(f-g)^p \, d\mu_1 < \varepsilon^p. \tag{10}$$

On the other hand, since $|g| \le |f|$, Minkowski's Inequality gives

$$T(f-g) \le T(f) + T(g) \le 2T(f),$$

so (9) gives

$$\int_{\tilde{X}\setminus Q_M} d\mu_1 T(f-g)^p \le (2\varepsilon)^p.$$

Combining this with (10) gives

$$\|f-g\|_{\tilde{L}^p} \le 3\varepsilon$$

and the lemma is proved.

We next show that the spaces $\tilde{L}^p$ interpolate properly by the complex method when $p \ge 2$. For a compatible pair of Banach spaces $(\mathscr{B}_0, \mathscr{B}_1)$, and for $0 < \theta < 1$, we denote by $[\mathscr{B}_0, \mathscr{B}_1]_\theta$ the usual complex method interpolating space.

LEMMA 3.5. *Let $0 < \theta < 1$ and $1/p = (1-\theta)/2$. Then $\tilde{L}^p \subset [\tilde{L}^2, \tilde{L}^\infty]_\theta$, with a continuous embedding.*

Proof. First, note that since, by Lemma 3.4, $\tilde{L}^2 \subset \tilde{L}^\infty$, we have $[\tilde{L}^2, \tilde{L}^\infty]_\theta \subset \tilde{L}^\infty$. For any $f \in \tilde{L}^\infty$, we denote by $|f|_\theta$ the $[\tilde{L}^2, \tilde{L}^\infty]_\theta$ norm of f (which may, of course, be infinite). By (5) of Lemma 3.4, it suffices to prove that for any simple function f with compact support in $\tilde{X}$ and $\|f\|_{\tilde{L}^p} = 1$ we have $|f|_\theta \leq C$, where C is a constant independent of f. We first note that, with f as above, $T(f)$ is a bounded, positive function on $\tilde{X}$. Thus, for any $\lambda \in \mathbf{C}$, the function

$$G_\lambda(z) = T(f)(z)^{p(1-\lambda)/(1-\theta)-1} f(z)$$

is a bounded function with compact support on $\tilde{X}$, and so, by (3) and (5) of Lemma 3.4, $G_\lambda \in \tilde{L}^2 \subset \tilde{L}^\infty$, and the map $\lambda \mapsto G_\lambda$ is a holomorphic mapping from $\mathbf{C}$ into $\tilde{L}^\infty$. One easily checks that $\|G_\lambda\|_{\tilde{L}^\infty} \leq C_1 c_2^{1-\operatorname{Re}\lambda}$, where C_1 and C_2 are positive constants depending on f, so the map $\lambda \mapsto G_\lambda$ is bounded (in $\tilde{L}^\infty$ norm) in a neighborhood of the strip $\mathscr{S} = \{0 \leq \operatorname{Re}\lambda \leq 1\}$. Letting

$$F_\lambda = e^{\lambda^2 - \theta^2} G_\lambda,$$

it follows that the map $\lambda \mapsto F_\lambda$ is holomorphic and bounded in a neighborhood of $\mathscr{S}$, and that $\|F_\lambda\|_{\tilde{L}^\infty} \to 0$ as $|\operatorname{Im}\lambda| \to \infty$ in $\mathscr{S}$. Since $F_\theta = G_\theta = f$, the lemma will be proved by showing that $\|F_{it}\|_{\tilde{L}^2}$ and $\|F_{1+it}\|_{\tilde{L}^\infty}$ are both bounded uniformly in $t \in \mathbf{R}$ by a constant independent of f. (Recall that $\|f\|_{\tilde{L}^p} = 1$.) Moreover, since $e^{\lambda^2-\theta^2}$ is bounded on each of the lines $\{\operatorname{Re}\lambda = 0\}$ and $\{\operatorname{Re}\lambda = 1\}$, it suffices to bound $\|G_{it}\|_{\tilde{L}^2}$ and $\|G_{1+it}\|_{\tilde{L}^\infty}$. But

$$|G_{it}| = |G_0| = T(f)^{p/2-1}|f|$$

and

$$|G_{1+it}| = |G_1| = \frac{|f|}{T(f)},$$

so the proof will be finished by establishing the two inequalities

$$\left\|T(f)^{p/2-1} f\right\|_{\tilde{L}^2} \leq C \tag{11}$$

and

$$\left\|\frac{f}{T(f)}\right\|_{\tilde{L}^\infty} \leq C \tag{12}$$

with constants independent of f.

Fix $\varepsilon \in (0,1)$, and let $\{z_j\}$ be the sequence from item 5 of Lemma 3.3. Appealing to Lemma 3.3, the left hand side of (12) can now be estimated as follows:

$$\begin{aligned}
T\left(\frac{f}{T(f)}\right)^2(z) &= \int \frac{|f(w)|^2}{T(f)^2(w)} \kappa_z(w)\, d\mu_0(w) \\
&\le \sum_j \int_{\mathscr{B}_j} \frac{|f(w)|^2}{T(f)^2(w)} \kappa_z(w)\, d\mu_0(w) \\
&\le C \sum_j \frac{|\kappa_z(z_j)|^2}{T(f)^2(z_j)} \int_{\mathscr{B}_j} |f(w)|^2\, d\mu_0(w) \\
&\le C \sum_j \frac{\kappa_z(z_j)}{T(f)^2(z_j)} \mu_0(\mathscr{B}_j) \int_{\mathscr{B}_j} |f(w)|^2 \kappa_{z_j}(w)\, d\mu_0(w) \\
&\le C \sum_j \kappa_z(z_j)\, \mu_0(\mathscr{B}_j) \\
&\le C \sum_j \int_{\mathscr{B}_j} \kappa_z(w)\, d\mu_0(w) \\
&\le C \int_{\tilde{X}} \kappa_z(w)\, d\mu_0(w) = C,
\end{aligned}$$

so (12) is proved.

To prove (11), we first establish a pointwise estimate. Arguing as above, we have

$$\begin{aligned}
&T\big(T(f)^{p/2-1} f\big)^2(z) \\
&\quad = \int T(f)(w)^{p-2} |f(w)|^2 \kappa_z(w)\, d\mu_0(w) \\
&\quad \le C \sum_j T(f)(z_j)^{p-2} \kappa_z(z_j)\, \mu_0(\mathscr{B}_j) \int_{\mathscr{B}_j} |f(w)|^2 \kappa_{z_j}(w)\, d\mu_0(w) \\
&\quad \le C \sum_j T(f)(z_j)^p \kappa_z(z_j)\, \mu_0(\mathscr{B}_j) \\
&\quad \le \int_{\tilde{X}} T(f)(w)^p \kappa_z(w)\, d\mu_0(w).
\end{aligned}$$

Finally, integrating with respect to z and applying Fubini's Theorem gives

$$\|T(f)^{p/2-1}f\|_{\tilde{L}^2}^2 \le C\iint T(f)(w)^p \mu_0(C(z,w))^{-2}\, d\mu_0(z)\, d\mu_0(w)$$

$$\le C\int T(f)(w)^p \mu_0(C(w))\, d\mu_0(w) = C\|f\|_{\tilde{L}^p}^p = C$$

and the proof is complete.

The following lemma, and its proof, are modeled on an analogous result on the Bergman kernel in bounded symmetric domains which is proved in [5].

LEMMA 3.6. *Let G be a measurable function on $\tilde{X}\times\tilde{X}$ such that $|G(z,w)| \le \mathscr{K}(z,w)$ for almost every $(z,w)\in\tilde{X}\times\tilde{X}$. Then $M_f\mathscr{I}_G$ is a bounded operator on L^2 whenever $f\in\tilde{L}^\infty$, with operator norm at most a constant multiple of $\|f\|_{\tilde{L}^\infty}$.*

Proof. Fix $\varepsilon\in(0,1)$, and let $\{z_j\}$ be the sequence in item 5 of Lemma 3.3 and $\mathscr{B}_j = \mathscr{B}_\varepsilon(z_j)$. Let $\varphi\in L^2$, and let

$$g(z) = \int |\varphi(w)|\mathscr{K}(z,w)\, d\mu_0(w).$$

It's clear that

$$|(\mathscr{I}_G\varphi)(z)| \le g(z) \tag{13}$$

for almost every $z\in\tilde{X}$, and moreover, by Lemma 1.4 of [4], we have $\|g\|_2 \le C\|f\|_2$. Also, it follows from (3) of Lemma 3.3 that $g(w)\approx g(z)$ for $w\in\mathscr{B}_\varepsilon(z)$, so for $w\in\mathscr{B}_j$, we have

$$g(w)^2 \le Cg(z_j)^2 = C\frac{1}{\mathscr{B}_j}\int_{\mathscr{B}_j} g(z_j)^2\, d\mu_0 \le C\frac{1}{\mathscr{B}_j}\int_{\mathscr{B}_j} g^2\, d\mu_0,$$

so

$$\sup_{\mathscr{B}_j} g^2 \le \frac{C}{\mu_0(\mathscr{B}_j)}\int_{\mathscr{B}_j} g^2\, d\mu_0.$$

Thus, by (13), we obtain

$$\begin{aligned}\|(M_f\mathscr{I}_G)(\varphi)\|_2^2 &\le \int |f|^2 g^2\, d\mu_0\\ &\le \sum_j \int_{\mathscr{B}_j} |f|^2 g^2\, d\mu_0\\ &\le \sum_j \frac{C}{\mu_0(\mathscr{B}_j)} \int_{\mathscr{B}_j} g^2\, d\mu_0 \int_{\mathscr{B}_j} |f|^2\, d\mu_0.\end{aligned}$$

But by Lemma 3.3, we have $\mu_0(\mathscr{B}_j)^{-1} \approx k_{z_j}(z)$ uniformly for $z \in \mathscr{B}_j$, so we obtain

$$\begin{aligned}\|(M_f\mathscr{I}_G)\varphi\|_2^2 &\le C\sum_j \int_{\mathscr{B}_j} g^2\, d\mu_0 \int_{\mathscr{B}_j} |f|^2 k_{z_j}\, d\mu_0\\ &\le C\|T(f)\|_\infty^2 \sum_j \int_{\mathscr{B}_j} g^2\, d\mu_0\\ &\le C\|T(f)\|_\infty^2 \int_{\tilde{X}} g^2\, d\mu_0\\ &= C\|T(f)\|_\infty^2 \|g\|_2^2 \le C\|T(f)\|_\infty^2 \|\varphi\|_2^2\end{aligned}$$

and the lemma is proved.

LEMMA 3.7. *Let G be a measurable function on $\tilde{X}\times\tilde{X}$ such that $|G(z,w)| \le \mathscr{K}(z,w)$ for almost every $(z,w) \in \tilde{X}\times\tilde{X}$. Then for any $f \in \tilde{L}^p$, with $2 \le p < \infty$, the operator $M_f\mathscr{I}_G$ is in the trace ideal $\mathscr{S}_p$, and $|M_f\mathscr{I}_G|_p \le C_p\|f\|_{\tilde{L}^p}$.*

Proof. For $f \in \tilde{L}^2 = L^2(d\mu_0)$, since $M_f\mathscr{I}_G$ is the integral operator with kernel $G_f(z,w) = f(z)G(z,w)$, the Hilbert Schmidt criterion gives

$$\begin{aligned}|M_f\mathscr{I}_G|_2^2 &= \iint |f(z)|^2 |G(z,w)|^2\, d\mu_0(w)\, d\mu_0(z)\\ &\le \iint |f(z)|^2 \mathscr{K}(z,w)^2\, d\mu_0(w)\, d\mu_0(z)\\ &\le C\int |f(z)|^2\, d\mu_1(z) = C\|f\|_{\tilde{L}^2}^2,\end{aligned}$$

which proves the lemma when $p = 2$.

To complete the proof, we use an interpolation argument. For $0 < \theta < 1$ we have $[\mathscr{S}_2, \mathscr{L}]_\theta = \mathscr{S}_p$, with $p = 2/(1-\theta)$, where $\mathscr{L}$ is the space of all bounded operators on L^2. The proof of the inclusion $[\mathscr{S}_2, \mathscr{L}]_\theta \subset \mathscr{S}_p$ can be found on pages 137–138 of [11],[2] while the reverse inclusion is an elementary consequence of the canonical Schmidt representation of a compact operator (see for example, [11], page 28). On the other hand, Lemma 3.5 asserts that $\tilde{L}^p \subset [\tilde{L}^2, \tilde{L}^\infty]_\theta$. We showed in the preceding paragraph that $|M_f \mathscr{I}_G|_2 \le C\|f\|_{\tilde{L}^2}$, and Lemma 13 gives $\|M_f \mathscr{I}_G\|_{\mathscr{L}} \le C\|f\|_{\tilde{L}^\infty}$, so interpolation gives $|M_f \mathscr{I}_G|_p \le C_p\|f\|_{\tilde{L}^p}$ for $2 < p < \infty$.

In view of the preceding lemma, Theorem 3.1 will be established as soon as we prove:

LEMMA 3.8. *Let a be as in Lemma 1.1. If $f \in Y^p$ with $2(1 - 1/a) < p < \infty$, then $f - \hat{f} \in \tilde{L}^p$, and $\|f - \hat{f}\|_{\tilde{L}^p} \le C\|f\|_{Y^p}$.*

Proof. We have

$$\begin{aligned}\|f - \hat{f}\|_{\tilde{L}^p}^p &= \int\left(\int|f(w) - \hat{f}(w)|^2\kappa_z(w)\right)^{p/2} d\mu_1(z)\\ &\le 2^{p-1}\int\left(\int|f(w) - \hat{f}(z)|^2\kappa_z(w)\right)^{p/2} d\mu_1(z)\\ &\quad + 2^{p-1}\int\left(\int|\hat{f}(z) - \hat{f}(w)|^2\kappa_z(w)\right)^{p/2} d\mu_1(z).\end{aligned}$$

The first integral on the right is clearly dominated by $\|f\|_{Y^p}^p$, and, by Lemma 2.10 with $b = 0$, the second integral on the right is also dominated by $\|f\|_{Y^p}^p$, so the proof is complete.

4. Hankel operators

We now turn our attention to Hankel operators on domains in $\mathbf{C}^n$ in the sense of [5]. Let D be a bounded domain in $\mathbf{C}^n$. The *Bergman space* on D is the subspace of $L^2(D)$ consisting of all holomorphic L^2 functions on D. It follows immediately from subharmonicity of $|f|^2$ that $A^2(D)$ is a closed subspace of $L^2(D)$, and that for any $z \in D$, the point evaluation $f \mapsto f(z)$ is a continuous linear functional on $A^2(D)$. By the Riesz Lemma, for each

[2]The theorem in [11] is stated with the space $\mathscr{S}_\infty$ of all compact operators as the right end point, but the proof works without change for the space $\mathscr{L}$ of all bounded operators.

$z \in D$ there is a unique $K_z \in A^2(D)$ such that

$$f(z) = \langle f, K_z \rangle = \int_D f(w)\overline{K_z(w)}\, dw \tag{14}$$

for every $f \in A^2(D)$. The function $K(z,w) = K_w(z)$ is the *Bergman kernel function* for D. Letting $P : L^2(D) \to A^2(D)$ denote the orthogonal projection, it follows that

$$Pf(z) = \langle Pf, K_z \rangle = \langle f, K_z \rangle = \int_D f(w) K(z,w)\, dw \tag{15}$$

for any $z \in D$ and any $f \in L^2(D)$.

When f is a function on D, the Hankel operator with symbol f is the operator H_f defined formally by

$$H_f = (I - P) M_f P = [M_f, P] P$$

where, as in previous sections, $M_f g = fg$. Our purpose in this section is, for suitable domains D, to characterize those symbols which give rise to Hankel operators in the trace ideal $\mathscr{S}_p$ for $2 \le p < \infty$. When D is the unit ball, or a more general bounded symmetric domain, this characterization has been given by K. Zhu [21] and D. Zheng [20].

From (14) we have

$$\|K_z\|_2^2 = \langle K_z, K_z \rangle = K(z, z),$$

so the function

$$k_z = \frac{K_z}{K(z,z)^{1/2}}$$

is a unit vector in $A^2(D)$ for each $z \in D$, i.e.,

$$\int_D |k_z(w)|^2\, dw = 1$$

for every $z \in D$.

We now restrict our attention to domains for which K_z is bounded for each fixed $z \in D$. This class of domains includes any domain for which the $\bar{\partial}$-Neuman problem is globally regular, and in particular includes all finite type domains. For $f \in L^1(D)$, the *Berezin transform* of f is the function

$$\tilde{f}(z) = \int f(w)|k_z(w)|^2 \, dw.$$

Alternatively, by (15),

$$\tilde{f}(z) = K(z,z)^{-1} P(K_z f)(z),$$

so in particular, when f is holomorphic, we have $\tilde{f} = f$.

We now further specialize to smoothly bounded domains D which are either strictly pseudoconvex in $\mathbf{C}^n$ or of finite type in $\mathbf{C}^2$, and we equip the ∂D with the homogeneous structure defined by the Euclidean surface measure σ and the balls determined by the Korányi-Stein pseudo-metric in the strictly pseudoconvex case, or by any of the metrics constructed in [3] in the finite type case. In order to apply our general trace ideal criterion, we identify a relative neighborhood of ∂D in D with a neighborhood of $\partial D \times \{0\}$ in $\widetilde{\partial D} = \partial D \times (0, \infty)$. More precisely, let ρ denote any fixed defining function for D, and for $\delta_0 > 0$ sufficiently small, let π denote the normal projection of S_{δ_0} on ∂D, where

$$S_\delta = \{-\delta < \rho < 0\}.$$

Then for δ_0 sufficiently small, the map $\Phi : S_{\delta_0} \to \widetilde{\partial D}$ defined by

$$\Phi(z) = (\pi(z), -\rho(z))$$

is a diffeomorphism onto $\widetilde{\partial D} \times (0, \delta_0)$. Also, letting $\mathscr{K}$ denote the kernel of the homogeneous structure in the sense of Section 1, it follows from the kernel estimates of [9], [6], [16] that, again for sufficiently small δ_0, we have

$$K(z,\zeta) \approx \mathscr{K}(\Phi(z), \Phi(\zeta))$$

for $z \in S_{\delta_0}$ and $|z - \zeta| < \delta_0$.

For any function f on D, define a function $\Phi_* f$ on $\widetilde{\partial D}$ by $\Phi_* f = f \circ \Phi^{-1}$ on $\partial D \times (0, \delta_0)$ and $\Phi_* f = 0$ on $\partial D \times [\delta_0, \infty)$. From the comparability of the Bergman kernel with the homogeneous structure, one easily deduces:

LEMMA 4.1. *Let D be a smooth bounded domain which is either strictly pseudoconvex, or of finite type in $\mathbf{C}^2$, and let ∂D be equipped with the* *homogeneous structure described above. Let $f \in L^2(D)$. Then for $2 \le p < \infty$, we have*

$$\|\Phi_* f\|_{Y^p}^p \approx \|f\|_{L^2}^p + \int_D \left(\int_D \left| f(w) - \tilde{f}(z) \right|^2 |k_z(w)|^2 \, dw \right)^{p/2} K(z,z)\, dz$$

We will let $Y^p(D))$ denote the space of all functions in $L^2(D)$ such that the right hand side of the above inequality is finite, and we shall denote the pth root of the right hand side by $\|f\|_{Y^p(D)}$.

Now let $L(z, \zeta)$ be a measurable function on $D \times D$ such that $|L(z, \zeta)| \le C|K(z, \zeta)|$ for $z, \zeta \in D$ with $|z - \zeta| < \delta_0$ and $|L(z, \zeta)| \le C$ for $|z - \zeta| \ge \delta_0$, let $\mathscr{I}_L$ denote the integral operator

$$\mathscr{I}_L f(z) = \int_D f(\zeta) L(z, \zeta)\, d\zeta.$$

Our immediate goal is to establish a sufficient condition on $f \in L^2(D)$ for the commutator $[M_f, \mathscr{I}_L]$ to be in the trace ideal $\mathscr{S}_p = \mathscr{S}_p(L^2(D))$ with $p \ge 2$. Write

$$L = L_1 + L_2$$

where

$$L_1 = \chi(z)\chi(\zeta)L(z, \zeta)$$

with χ denoting the characteristic function of S_{δ_0}, so that

$$[M_f, \mathscr{I}_L] = [M_f, \mathscr{I}_{L_1}] + [M_f, \mathscr{I}_{L_2}].$$

Since L_2 is bounded on $D \times D$, it is immediate from the Hilbert-Schmidt criterion that the second term on the right is in $\mathscr{S}_2$, and hence in $\mathscr{S}_p$ for $p \ge 2$, so it suffices to consider the operator $[M_f, \mathscr{I}_{L_1}]$. Using the orthogonal decomposition

$$L^2(D) = L^2(S_{\delta_0}) \oplus L^2(D \setminus S_{\delta_0}),$$

it suffices to view $[M_f, \mathscr{I}_{L_1}]$ as an operator on $L^2(S_{\delta_0})$. Moreover, the map $g \mapsto g \circ \Phi^{-1}$ is an isometry of $L^2(S_{\delta_0})$ onto $L^2(\partial D \times (0, \delta_0), dV)$, where

$$dV(\zeta, t) = \omega(x, t)\, dt\, d\sigma(\zeta),$$

with $1/\omega$ the Jacobian of the map Φ. Thus, $[M_f, \mathscr{I}_{L_1}] \in \mathscr{S}_p$ if and only if $[M_{f\circ\Phi^{-1}}, \mathscr{I}_{L_1\circ(\Phi\times\Phi)^{-1}}] \in \mathscr{S}_p$. Appealing to Theorem 1.5, we obtain:

THEOREM 4.2. *Let D be a smooth bounded domain in $\mathbf{C}^n$ which is either strictly pseudoconvex or of finite type with $n = 2$. Let L be a measurable function on $D \times D$ such that $|L(z, \zeta)| \leq |K(z, \zeta)|$, where $K(z, \zeta)$ is the Bergman kernel function for D. Then for any $f \in Y^p(D)$ with $2 \leq p < \infty$, the commutator $[M_f, \mathscr{I}_L]$ is in the trace ideal $\mathscr{S}_p(L^2(D))$, and $\|[M_f, \mathscr{I}_L]\|_p \leq C\|f\|_{Y^p(D)}$, where C is a constant depending on D and p.*

COROLLARY 4.3. *Let D be a smooth bounded domain in $\mathbf{C}^n$ which is either strictly pseudoconvex or of finite type with $n = 2$. For $2 \leq p < \infty$, the Hankel operator H_f is in $\mathscr{S}_p(L^2(D))$ whenever $f \in Y^p(D)$, and $|H_f|_p \leq C\|f\|_{Y^p}$.*

We now turn to the converse of Theorem 4.2. The basic tool is the following identity of Burbea [7].

LEMMA 4.4. *Let D be a bounded domain in $\mathbf{C}^n$, and let K be its Bergman kernel. Suppose that for each fixed $\zeta \in D$, the function $1/K(\cdot, \zeta)$ is holomorphic and bounded in D, and let $b_z(\zeta) = K(\zeta, z)/K(z, \zeta)$. Then for any $z \in D$,*

$$\left\langle H_{\bar f}(k_z), b_z\overline{H_f(k_z)}\right\rangle = \left\| H_f(k_z)\right\|_2^2 + \left\| H_{\bar f}(k_z)\right\|_2^2 - \|fk_z\|_2^2 + \left|\tilde{f}(z)\right|^2.$$

Proof. We have

$$\left\langle H_f(k_z), b_z\overline{H_{\bar f}(k_z)}\right\rangle = \left\langle H_f(k_z)H_{\bar f}(k_z), b_z\right\rangle \tag{16}$$

and the first factor on the right can be written

$$\begin{aligned} H_f(k_z)H_{\bar f}(k_z) &= (I-P)(fk_z)(I-P)\left(\bar f k_z\right) \\ &= |f|^2k_z^2 - fk_zP\left(\bar f k_z\right) - \bar f k_z P(fk_z) + P(fk_z)P\left(\bar f k_z\right) \end{aligned} \tag{17}$$

We will calculate the inner product of each term on the right with b_z. For the first term, since $k_z^2\bar b_z = |k_z|^2$, we immediately obtain

$$\left\langle |f|^2k_z^2, b_z\right\rangle = \|fk_z\|_2^2. \tag{18}$$

For the second term on the right of (17), we have

(19)
$$\langle fk_z P(\bar{f}k_z), b_z\rangle = \langle P(\bar{f}k_z), \bar{f}k_z b_z\rangle = \langle P(\bar{f}k_z), \bar{f}k_z\rangle = \|P(\bar{f}k_z)\|_2^2$$

and similarly

$$\langle \bar{f}k_z P(fk_z), b_z\rangle = \|P(fk_z)\|_2^2. \tag{20}$$

For the last term on the right of (17), the reproducing property of the Bergman kernel gives

$$\begin{aligned}\langle P(fk_z)P(\bar{f}k_z), b_z\rangle &= \langle P(fk_z)P(\bar{f}k_z)K_z^{-1}, K_z\rangle \\ &= P(fk_z)(z)P(\bar{f}k_z)(z)K(z,z)^{-1} \\ &= P(fK_z)(z)P(\bar{f}K_z)(z)K(z,z)^{-2}.\end{aligned}$$

But direct calculation gives $P(fK_z)(z) = K(z,z)\tilde{f}(z)$, so we obtain

$$\langle P(fk_z)P(\bar{f}k_z), b_z\rangle = \tilde{f}(z)\tilde{\bar{f}}(z) = |\tilde{f}(z)|^2. \tag{21}$$

The identities (17)–(21) give

$$\langle H_f(k_z), b_z\overline{H_{\bar{f}}(k_z)}\rangle = \|fk_z\|_2^2 - \|P(\bar{f}k_z)\|_2^2 - \|P(fk_z)\|_2^2 + |\tilde{f}(z)|^2.$$

Finally noting that, by the Pythagorean Theorem, $\|H_\varphi\psi\|_2^2 = \|\varphi\psi\|_2^2 - \|P(\varphi\psi)\|_2^2$ completes the proof.

We next give an alternative description of $Y^p(D)$.

LEMMA 4.5. *For any $f \in L^2(D)$ and $0 < p \le \infty$ we have*

$$\|f\|_{Y^p(D)} = \|f\|_2 + \left(\int\left(\|fk_z\|_2^2 - |\tilde{f}(z)|^2\right)^{p/2} K(z,z)\,dz\right)^{1/p},$$

with the usual limiting interpretation when $p = \infty$.

Proof. The inner integral in the definition of the $Y^p(D)$ norm is

$$\begin{aligned}&\int|f(\zeta) - \tilde{f}(z)|^2|k_z(\zeta)|^2\,d\zeta \\ &= \int|f(\zeta)|^2|k_z(\zeta)|^2\,d\zeta - 2\hat{f}(z)\int\bar{f}(\zeta)|k_z(\zeta)|^2\,d\zeta + |\tilde{f}(z)|^2 \\ &= \|fk_z\|_2^2 - |\tilde{f}(z)|^2,\end{aligned}$$

and the result follows.

LEMMA 4.6. *Let T be a positive, compact operator on $L^2(D)$ with range contained in $A^2(D)$. Then*

$$\operatorname{tr} T = \int \langle TK_z, K_z \rangle \, dz = \int \langle Tk_z, k_z \rangle K(z, z) \, dz,$$

and, for any $1 \leq p < \infty$,

$$\int \langle Tk_z, k_z \rangle^p K(z, z) \, dz \leq \int \langle T^p k_z, k_z \rangle = \operatorname{tr} T^p.$$

Proof. Write

$$T = \sum \lambda_j \langle \cdot \, , \varphi_j \rangle \varphi_j,$$

where $\{\varphi_j\}$ is an orthonormal basis for $A^2(D)$. Then

$$K_z = \sum \overline{\varphi_j(z)} \varphi_j,$$

and

$$TK_z = \sum \lambda_j \overline{\varphi_j(z)} \varphi_j,$$

so by Parseval's Identity.

$$\int \langle TK_z, K_z \rangle = \sum |\varphi_j(z)|^2 \lambda_j.$$

Integrating both sides and appealing to the Monotone Convergence Theorem to exchange the sum and the integral gives the first assertion.

For the second assertion, Jensen's Inequality gives

$$\begin{aligned} \langle Tk_z, k_z \rangle^p &= \left(\sum \lambda_j \frac{|\varphi_j(z)|^2}{K(z, z)} \right)^p \\ &\leq \sum \lambda_j^p \frac{|\varphi_j(z)|^2}{K(z, z)} \\ &= \langle T^p k_z, k_z \rangle, \end{aligned}$$

and integrating both sides completes the proof.

Lemma 4.4 now gives:

THEOREM 4.7. *Let D be a bounded domain in $\mathbf{C}^n$ with Bergman kernel K, and suppose that for each fixed $\zeta \in D$, the function $1/K(\cdot, \zeta)$ is holomorphic and bounded in D. Let $f \in L^2(D)$.*

(1) *If* $2 \le p \le \infty$, *and if* H_f *ad* $H_{\bar f}$ *are both in* $\mathscr{S}_p$, *then* $f \in Y^p(D)$, *and*

$$\|f\|^2_{Y^p(D)} \le \tfrac{3}{2}\left(|H_f|^2_p + |H_{\bar f}|^2_p\right).$$

(2) *If* $0 < p \le 2$ *and if* $f \in Y^p(D)$, *then* H_f *and* $H_{\bar f}$ *are both in* $\mathscr{S}_p$, *and*

$$|H_f|^p_p + |H_{\bar f}|^p_p \le 2\|f\|^p_{Y^p}.$$

Proof. From Lemma 4.4 and the Schwarz Inequality,

$$\|fk_z\|^2_2 - \left|\hat f(z)\right|^2 \le \tfrac{3}{2}\left(\|H_f k_z\|^2_2 + \|H_{\bar f} k_z\|^2_2\right),$$

from which the case $p = \infty$ is immediate. For $2 \le p < \infty$, Lemma 4.5 and Minkowski's Inequality give

$$\begin{aligned} (22)\quad \|f\|^2_{Y^p(D)} &\le \frac{3}{2}\left(\int\left(\|H_f k_z\|^2_2 + \|H_{\bar f} k_z\|^2_2\right)^{p/2} K(z,z)\,dz\right)^{2/p} \\ &\le \frac{3}{2}\left(\left(\int\|H_f k_z\|^p_2 K(z,z)\,dz\right)^{2/p} + \left(\int\|H_{\bar f} k_z\|^p_2 K(z,z)\,dz\right)^{2/p}\right) \end{aligned}$$

But by Lemma 4.6,

$$\begin{aligned} &\int\|H_f k_z\|^p_2 K(z,z)\,dz + \int\langle H_f^* H_f k_z, k_z\rangle^{p/2} K(z,z)\,dz \\ &\quad\le \operatorname{tr}\left(H_f^* H_f\right)^{p/2} = |H_f|^p_p, \end{aligned}$$

and similarly

$$\int\|H_{\bar f} k_z\|^p_2 K(z,z)\,dz \le |H_{\bar f}|^p_p,$$

so (1) follows from (22).

For (2), we again use Lemma 4.4 and the Schwarz Inequality to obtain

$$\|fk_z\|^2_2 - \left|\hat f(z)\right|^2 \ge \tfrac{1}{2}\left(\|H_f k_z\|^2_2 + \|H_{\bar f} k_z\|^2_2\right).$$

Thus for $0 < p \le 2$, by Lemma 4.6,

$$\begin{aligned}
\|f\|_{Y^p}^p &\ge \frac{1}{2}\int\left(\|H_f k_z\|_2^p + \|H_{\bar f} k_z\|_2^p\right)K(z,z)\,dz \\
&= \frac{1}{2}\int\left(\langle H_f^* H_f k_z, k_z\rangle^{p/2} + \langle H_{\bar f}^* H_{\bar f} k_z, k_z\rangle^{p/2}\right)K(z,z)\,dz \\
&\ge \frac{1}{2}\int\left(\left\langle (H_f^* H_f)^{p/2} k_z, k_z\right\rangle + \left\langle (H_{\bar f}^* H_{\bar f})^{p/2} k_z, k_z\right\rangle\right)K(z,z)\,dz \\
&= \frac{1}{2}\left(|H_f|_p^p + |H_{\bar f}|_p\right)
\end{aligned}$$

and the proof is complete.

Combining this result with Theorem 3.1 gives the following corollary, which extends work of K. H. Zhu [21] in the unit Ball and D. Zheng [20] in the case of bounded symmetric domains.

COROLLARY 4.8. *Let D be either a finite type domain in $\mathbf{C}^n$ or a strictly pseudoconvex domain in $\mathbf{C}^n$ such that the Bergman kernel of D is non-vanishing on $D \times D$, and let $2 \le p < \infty$. Then $f \in Y^p(D)$ if and only if H_f and $H_{\bar f}$ both in $\mathscr{S}_p$, and moreover $\|f\|_{Y^p(D)} \approx |H_f|_p + |H_{\bar f}|_p$.*

We note in passing that with some additional work, one can drop the condition that K is non-vanishing in the strictly pseudoconvex case. The argument is a refinement of the one given above and relies on the asymptotic expansion for the Bergman kernel. We do not know whether the non-vanishing of the Bergman kernel is essential in the finite type case.

We conclude by noting that, in view of the identity

$$[M_f, P] = H_f - (H_{\bar f})^*,$$

Theorem 4.7 can also be formulated in terms of the commutator operator $[M_f, P]$.

COROLLARY 4.9. *Let D and f be as in Theorem 4.7.*

(1) *If $2 \le p \le \infty$, and if $[M_f, P]$ is in $\mathscr{S}_p$, then $f \in Y^p(D)$, and*

$$\|f\|_{Y^p(D)}^2 \le 3\left|[M_f, P]\right|_p^2.$$

(2) *If $0 < p \le 2$ and if $f \in Y^p(D)$, then $[M_f, P] \in \mathscr{S}_p$, and*

$$\left|[M_f, P]\right|_p \le 2\|f\|_{Y^p}.$$

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