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THE DENSITY THEOREM AND HAUSDORFF INEQUALITY FOR PACKING MEASURE IN GENERAL METRIC SPACES

COLLEEN D. CUTLER

1. Introduction

The standard definition of packing ϕ-measure in Euclidean space (Definition 2.2 of this article—see also Taylor and Tricot [20], [21] and Edgar [8]) is based on the diameters of balls. Balls in $\mathbf{R}^N$, obtained from the usual Euclidean norm (or some equivalent metric), possess certain nice regularity properties—the diameter of a ball is twice its radius, and open and closed balls of the same radius have the same diameter. In arbitrary metric spaces, the possible absence of such regularity properties means that the usual measure construction based on diameters can lead to packing measures with undesirable features (see Example 2.3). We note that both Haase [14] and Edgar [9], in extending diameter-based packing measure to a more general metric setting, needed to make certain modifications and assumptions which can be directly attributed to the irregular behaviour of ball diameters (see the discussion following Definition 2.2 as well as Remark 3.17). In earlier work, Haase [12], [13] briefly introduced the notion of radius-based packing measure (Definitions 3.1 and 3.2 below), noting that it produced the usual packing measure in Euclidean space but possessed stronger invariance properties in the general setting. Nonetheless, this radius approach appears to have been largely overlooked by most researchers (an exception is the recent work of Olsen [18]). We will show that, under the radius definition, the fundamental properties of Euclidean packing measure (Theorems 3.7, 3.11, 3.16, and Corollary 3.20) carry over to general metric spaces. We will also exhibit some of the failings of diameter-based packing measure, but prove that it does satisfy the usual inequality with Hausdorff measure (Theorem 2.6) under a very weak restriction on the underlying space.

In the following $\mathscr{X}$ will always denote a metric space with metric d. A continuous nondecreasing function $\phi : [0, 1] \to [0, \infty)$ such that $\phi(0) = 0$ and

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$\phi(t) > 0$ for all $t > 0$ will be called a *measure function*; if ϕ additionally satisfies the regulating condition $\phi(t/2) \geq K\phi(t)$ for some $0 < K < 1$ and all $0 < t \leq 1$, then ϕ will be called a *blanketed* measure function (see [16]). If E is any nonempty subset of $\mathscr{X}$, the *diameter* of E is defined, as usual: $\text{diam}(E) = \sup_{x,y \in E} d(x, y)$. A *closed ball* in $\mathscr{X}$ is any set B which can be described as $B = \{y \in \mathscr{X} \mid d(y, x) \leq r\}$ for some $x \in \mathscr{X}$ and some $r > 0$. The quantities x and r are called, respectively, a *centre* and *radius* of the ball B, and this is indicated by writing $B = B(x, r)$. Similarly, an *open ball* $B = \{y \in \mathscr{X} \mid d(y, x) < r\}$ will be denoted by $B^o(x, r)$. In most "regular" spaces, such as Euclidean space, a ball (open or closed) has one centre and one radius, and typically $r = \text{diam}(B)/2$; however, in general, neither the radius nor centre of a ball need be unique. This is illustrated in Example 1.1 below.

Example 1.1. Let $\mathscr{X} = \{(x, y) \mid y \leq 0\} \cup \{(0, 2), (0, 3)\}$, i.e., $\mathscr{X}$ is the union of the closed lower half plane and the isolated points $(0, 2), (0, 3)$, with the subspace topology inherited from $\mathbf{R}^2$. Let $B = \{(0, 2), (0, 3)\}$. Then B has many representations as a closed ball in $\mathscr{X}$; for convenience, let $a = (0, 2)$ and $b = (0, 3)$. Then we can write $B = B(a, r)$ for any $1 \leq r < 2$, as well as $B = B(b, r)$ for any $1 \leq r < 3$. Note that $\text{diam}(B) = 1$.

We will need to take into account the possibility of various centre-radius representations for a ball in §3. Finally, we note that if r is any radius of a ball B, then we always have $\text{diam}(B) \leq 2r$. In general, however, no similar reverse inequality holds.

2. Diameter-based packing measure

We begin by using the usual diameter method to define a packing measure in the general metric setting. This definition has been used previously by various authors (see, for example, [15], [8], [9], [6]) and represents the obvious extension of Taylor and Tricot's [20] original Euclidean packing measure. (In [14], Haase considers families of diameter-based packing measures of a slightly different nature.) Since ultimately we are going to discard this definition, we use the notation "Q" to denote the resulting measure, reserving the notation "P" for the preferred radius-based construction developed in the next section.

DEFINITION 2.1 (diameter packing ϕ-premeasure). Let $E \subseteq \mathscr{X}$ be nonempty, and let $0 < \delta < 1$. A *δ-packing* of E is a countable collection $\{B_k\}_k$ of disjoint closed balls of $\mathscr{X}$, centred at points of E, such that $\text{diam}(B_k) \leq \delta$ for every k. Given a measure function ϕ, the *diameter packing*

(ϕ, δ)*-premeasure* of E is defined to be

$$(2.1)\quad Q_\delta^\phi(E) = \sup\left\{ \sum_{k=1}^{\infty} \phi(\operatorname{diam}(B_k)) \mid \{B_k\}_k \text{ is a } \delta\text{-packing of } E \right\}.$$

Letting $\delta \to 0$ we obtain the *diameter packing* ϕ*-premeasure*

$$(2.2)\qquad Q_0^\phi(E) = \lim_{\delta \to 0} Q_\delta^\phi(E).$$

We set $Q_\delta^\phi(\emptyset) = Q_0^\phi(\emptyset) = 0$.

It is well known (e.g., [20], [8]) that Q_0^ϕ, while nonnegative and monotone, is generally not countably subadditive, and so does not meet the usual Carathéodory definition of an outer measure on the subsets of $\mathscr{X}$. We can, however, build an outer measure from Q_0^ϕ by applying the Method I construction of Munroe, described in both [17] and [19]. This leads to the following definition.

DEFINITION 2.2 (diameter packing ϕ-measure). The *diameter packing* ϕ*-measure* of $E \subseteq \mathscr{X}$ is defined to be

$$(2.3)\qquad Q^\phi(E) = \inf\left\{ \sum_k Q_0^\phi(E_k) \mid E \subseteq \bigcup_k E_k \right\}.$$

The infimum in (2.3) is taken over all countable coverings $\{E_k\}_k$ of E, and corresponds to the Method I construction ([17], p. 47, or [19], p. 9) of an outer measure from the nonnegative set function Q_0^ϕ. Hence Q^ϕ is an outer measure on the subsets of $\mathscr{X}$.

Both Q_0^ϕ and Q^ϕ possess numerous desirable regularity properties as set functions on $\mathbf{R}^N$ (see Theorems 3.7, 3.11, and 3.16). It appears, however, that to maintain all, or even most, of these properties for Q_0^ϕ and Q^ϕ in a more general setting requires, at the very least, the existence of some sort of smooth relationship between the diameters and radii of balls in the space. This can be gleaned from an examination of the proofs typically employed in the Euclidean setting; for example, to prove that packings by open balls produce the same premeasure as packings by closed balls ((d) of Theorem 3.7), one uses the fact that every closed ball in $\mathbf{R}^N$ contains an open ball of the same diameter. Theorem 3.7 (d) is an important result, because open packings give Theorem 3.7 (c) which in turn yields the regularity properties (c) and (d) of Theorem 3.11, but closed packings are needed in order to apply Lemma 2.5 which gives Theorem 2.6 as well as (h) of Theorem 3.11. Haase [14] notes that his diameter-based constructions do not always yield Theorem

3.7 (c). As another example, Cutler [6] required an assumption on the relationship between ball diameter and ball radius (see (3.10)) in order to obtain the pointwise representation of diameter packing dimension (Corollary 3.20) in spaces more general than $\mathbf{R}^N$. Of course it will be possible to modify the proofs or statements of some of these results to rely less (or not at all) on the existence of an underlying Euclidean-like geometry. However, Example 2.3 below shows that there is a limit to what can be accomplished in this regard. In Example 2.3 we construct a metric space in which a fundamental Euclidean property of diameter packing ϕ-measure is violated. In order to do this, we first need to discuss the notion of *Hausdorff ϕ-measure*. Unlike the case of packing measure, the theory of Hausdorff measures in general metric spaces is a (relatively) old and well-explored topic; see, for example, the treatise by Rogers [19]. For $E \subseteq \mathscr{X}$, the outer measure $H^{\phi}(E)$ is defined by

$$(2.4) \quad H^{\phi}(E) = \lim_{\delta \to 0} \inf\left\{ \sum_{k=1}^{\infty} \phi(\operatorname{diam}(E_k)) \,|\, \{E_k\}_k \text{ is a } \delta\text{-covering of } E \right\}$$

where a δ-covering of E is any countable collection $\{E_k\}_k$ of subsets of $\mathscr{X}$ such that $E \subseteq \bigcup_k E_k$ and $\operatorname{diam}(E_k) \le \delta$ for every k. If no such δ-covering exists (which may occur in a nonseparable space), we obtain $H^{\phi}(E) = \inf \emptyset = \infty$.

In the Euclidean case, we always have $Q^{\phi} \ge H^{\phi}$ for every blanketed measure function ϕ. With the following example we show that this inequality is not universally valid.

Example 2.3. Let $\mathscr{X}$ be an uncountable set with the discrete metric $d(x, y) = 1$ whenever $x \ne y$. For $0 < \delta < 1$, no δ-covering of $\mathscr{X}$ exists, and so $H^{\phi}(\mathscr{X}) = \infty$. Now, if $0 < \delta < 1$ and $\{B_k\}_k$ is any δ-packing of any $E \subseteq \mathscr{X}$, we see that $\operatorname{diam}(B_k) = 0$ for each k and hence $Q_0^{\phi}(E) = 0$. It follows from (2.3) that $Q^{\phi}(\mathscr{X}) = 0$. (We note that the possibility of just such an unfortunate occurrence appears to be suggested by the exercise in Edgar [8], p. 182, Ex. (6.5.8).)

In fact, the relationship between packing measure and Hausdorff measure is not the only standard Euclidean property violated in Example 2.3—the usual relationship between packing premeasure and upper capacity ((g) of Theorem 3.7) fails to hold, as does the relationship between packing dimension and upper capacity ((g) of Theorem 3.11). However, we prefer to leave a full discussion of these additional points until Remarks 3.8 and 3.12. See also Remarks 3.17 and 3.18.

We now show that Example 2.3 is exceptional in that the desired inequality with Hausdorff measure does hold for diameter-based packing measure provided we limit the number of isolated points in the space. We first require the following definition and lemma.

DEFINITION 2.4 (closed Vitali covering). Let $E \subseteq \mathscr{X}$. A collection $\mathscr{V}$ of closed subsets of $\mathscr{X}$ is called a *closed Vitali covering* of E if, for each $x \in E$ and each $\epsilon > 0$, there exists $V \in \mathscr{V}$ such that $x \in V$ and $0 < \operatorname{diam}(V) \leq \epsilon$.

LEMMA 2.5. *Let ϕ be a blanketed measure function, $E \subseteq \mathscr{X}$, and $\mathscr{V}$ a closed Vitali covering of E. Then there exists a sequence $V_1, V_2, \dots$ of disjoint members of $\mathscr{V}$ such that one of the following is true*:
(i) $\sum_{k=1}^{\infty} \phi(\operatorname{diam}(V_k)) = \infty$, *or*
(ii) $H^{\phi}(E \setminus \bigcup_k V_k) = 0$.

Proof. A proof of this lemma, for the case $\mathscr{X} = \mathbf{R}^N$ and $\phi(t) = t^{\alpha}$, is given in Falconer [10], Theorem 1.10. A similar proof works, however, under the more general hypotheses we have given here. The reader should note that the assumption of a blanketed measure function (which is of course automatically satisfied in the case $\phi(t) = t^{\alpha}$) is crucial to the proof.

THEOREM 2.6. *Suppose $\mathscr{X}$ has at most countably many isolated points. Then $Q^{\phi} \geq H^{\phi}$ for every blanketed measure function ϕ.*

Proof. Let $\mathscr{X}_0$ denote the set of isolated points of $\mathscr{X}$. Since $\mathscr{X}_0$ is countable, it follows by a well-known property of Hausdorff measure that $H^{\phi}(\mathscr{X}_0) = 0$, and so $H^{\phi}(E) = H^{\phi}(E \setminus \mathscr{X}_0)$ for every $E \subseteq \mathscr{X}$. Therefore, without loss of generality, we can assume that E has no isolated points. Now it suffices to prove that $Q_0^{\phi}(E) \geq H^{\phi}(E)$ for all such E since, by the countable subadditivity of H^{ϕ}, we then get $\sum_k Q_0^{\phi}(E_k) \geq \sum_k H^{\phi}(E_k) \geq H^{\phi}(E)$ whenever $\bigcup_k E_k = E$. Since E has no isolated points, the collection $\mathscr{V}_{\delta} = \{B(x, r) \mid x \in E,\ 0 < r \leq \delta/2\}$ forms a closed Vitali covering of E for each $\delta > 0$. Applying Lemma 2.5, there exists a sequence $B_1^{(\delta)}, B_2^{(\delta)}, \dots$ of disjoint closed balls from $\mathscr{V}_{\delta}$ such that either (i) $\sum_k \phi(\operatorname{diam}(B_k^{(\delta)})) = \infty$ or (ii) $H^{\phi}(E \setminus \bigcup_k B_k^{(\delta)}) = 0$. If (i) holds for some sequence $\delta_n \downarrow 0$ then clearly $Q_0^{\phi}(E) = \lim_{n \to \infty} Q_{\delta_n}^{\phi}(E) = \infty$, yielding $Q_0^{\phi}(E) = \infty \geq H^{\phi}(E)$. Therefore assume that, for each positive integer n, there exists a sequence $B_1^{(n)}, B_2^{(n)}, \dots$ of disjoint balls from $\mathscr{V}_{1/n}$ such that (ii) holds. Then, since $H^{\phi}(E) = H^{\phi}(E \cap \bigcap_n \bigcup_k B_k^{(n)})$, it follows that

$$H^{\phi}(E) \leq \liminf_{n \to \infty} \sum_k \phi\Big(\operatorname{diam}\big(B_k^{(n)}\big) \leq \lim_{n \to \infty} Q_{1/n}^{\phi}(E) = Q_0^{\phi}(E).$$

This completes the proof.

We remark that we do not know if Theorem 2.6 remains true if open balls, rather than closed balls, are used in the definition of δ-packing.

A close inspection of Example 2.3 suggests that the diameter definition of packing measure does not, in the general setting, always correctly recognize the distance between points in a space. This problem can be rectified by following Haase [12] and making our definitions in terms of ball radii, rather than ball diameters. It is a (perhaps surprising) consequence of this small change in definition that essentially *all* nice properties of Euclidean packing measure can be recovered in the general metric setting. This behaviour suggests to us that the radius-based definition (developed below) is in fact the "right" one to use.

3. Radius-based packing measure

In this section, our definition of a δ-packing will be similar to that of §2, except that to each ball in a packing we attach a (permissible) representation in terms of a centre and radius. We then define the δ bound on the packing in terms of the radii. As noted in §1, neither the radius nor centre of a ball need be unique in general—hence, the particular representations assigned to the balls in a packing become part of the definition of the packing. The same packing (regarded simply as a collection of sets) may appear many times with many distinct centre-radius representations. A representation is considered *almost optimal* if it utilizes close to the maximum possible radii (subject to the ball centres being in E and to the δ bound on the packing). These remarks will become clearer after an examination of Definitions 3.1 and 3.2 below. See also Remark 3.3.

DEFINITION 3.1 (radius packing ϕ-premeasure). Let $E \subseteq \mathscr{X}$ be nonempty, and let $0 < \delta < 1$. A *δ-packing* of E will be any countable collection of disjoint closed balls $\{B(x_k, r_k)\}_k$ with centres $x_k \in E$ and radii satisfying $0 < r_k \leq \delta/2$ for each k. (The centres x_k and radii r_k are considered part of the definition of the packing.) Given a measure function ϕ, the *radius packing (ϕ, δ)-premeasure* of E is then defined to be

$$(3.1) \quad P_\delta^\phi(E) = \sup\left\{ \sum_{k=1}^{\infty} \phi(2r_k) \mid \{B(x_k, r_k)\}_k \text{ is a } \delta\text{-packing of } E \right\}.$$

Thus the supremum in (3.1) takes into consideration all permissible centre-radius representations of a δ-packing of E (regarded as a collection of sets), and the measure function ϕ is applied to the radii, rather than the diameters, of the balls. Letting $\delta \to 0$, we then obtain the *radius packing ϕ-premeasure*

$$(3.2) \quad P_0^\phi(E) = \lim_{\delta \to 0} P_\delta^\phi(E).$$

We then set $P_\delta^\phi(\emptyset) = P_0^\phi(\emptyset) = 0$.

As in the case of Q_0^ϕ, it is easy to see that P_0^ϕ is nonnegative and monotone. Moreover, P_0^ϕ will also generally fail to be countably subadditive, which can be seen by noting that $P_0^\phi = Q_0^\phi$ when $\mathscr{X} = \mathbf{R}^N$. (In fact, equality between P_0^ϕ and Q_0^ϕ holds in any space $\mathscr{X}$ where the radius of a ball B is always given uniquely by $r = \operatorname{diam}(B)/2$.) Hence, to produce an outer measure, we again apply the Method I construction:

DEFINITION 3.2 (radius packing ϕ-measure). The *radius packing ϕ-measure* of $E \subseteq \mathscr{X}$ is defined to be

$$P^\phi(E) = \inf\left\{ \sum_k P_0^\phi(E_k) \mid E \subseteq \bigcup_k E_k \right\}. \tag{3.3}$$

It follows that P^ϕ is an outer measure on the subsets of $\mathscr{X}$.

Remark 3.3. The necessity of considering various possible centre-radius representations of a packing in Definition 3.1 can be eliminated by instead defining a *maximal radius function* over the balls in a packing. That is, we can simply let a *packing* of E be any countable collection of disjoint closed balls $\{B_k\}_k$ which can be centred at points of E, and define $r_\delta^E(B_k) = \sup\{0 < r \leq \delta/2 \mid B_k = B(x, r) \text{ for some } x \in E\}$. Then, due to the continuity and monotonicity of ϕ, (3.1) can be replaced by

$$P_\delta^\phi(E) = \sup\left\{ \sum_{k=1}^{\infty} \phi\left(2r_\delta^E(B_k)\right) \mid \{B_k\}_k \text{ is a packing of } E \right\}. \tag{3.4}$$

However, we have found (3.1) simpler to use in most calculations. Note that a particular representation $\{B(x_k, r_k)\}_k$ of a packing $\{B_k\}_k$ of E will be almost optimal if r_k is very close to $r_\delta^E(B_k)$ for each k. In most cases, it should be possible to actually take $r_k = r_\delta^E(B_k)$.

Before proceeding to a discussion of the general properties of P_0^ϕ and P^ϕ, we show that this radius-based approach eliminates the problem encountered in Example 2.3.

Example 3.4. Let $\mathscr{X}$ be an uncountable set endowed with the discrete topology, and let d be any metric which generates this topology. (A metric produces the discrete topology if and only if each point of $\mathscr{X}$ is isolated under the metric.) Let ϕ be a measure function. We will prove that $P^\phi(\mathscr{X}) = \infty$. Let $\{E_k\}_k$ be any countable partition of $\mathscr{X}$. Then at least one member of this partition, say E_1, is uncountable. It follows that, for some $\delta_0 > 0$, there exists a sequence $\{x_k\}_k$ of distinct points of E_1 such that $B(x_k, \delta_0) = \{x_k\}$ for each k. (Otherwise the uncountability of E_1 is contradicted.) Hence, for every

$0 < \delta < \delta_0$, the collection $\{B(x_k, \delta/2)\}_k$ constitutes a δ-packing of E_1, yielding $P_\delta^\phi(E_1) \geq \sum_k \phi(\delta) = \infty$ and hence $P_0^\phi(E_1) = \infty$. From (3.3) we conclude that $P^\phi(\mathscr{X}) = \infty$.

We now develop the properties of the premeasure P_0^ϕ. Some of these properties have been noted previously in Haase [12]. Of special interest will be the situation where the measure function ϕ takes the form $\phi(t) = t^\alpha$ for some $\alpha > 0$. In this case we use the notation P_0^α, and refer to $P_0^\alpha(E)$ as the *packing α-premeasure* of E. Similarly, we use the notation $P^\alpha(E)$ for the *packing α-measure* of E, and $H^\alpha(E)$ for the *Hausdorff α-measure* of E. We will also need the following definition and lemma.

DEFINITION 3.5 (upper capacity). Let $E \subseteq \mathscr{X}$, and let $N_\delta(E)$ be the minimum number of closed balls of diameter δ (or less) required to cover E. The *upper capacity* of E is then defined to be

$$\Delta^+(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{\log 1/\delta}. \tag{3.5}$$

$\Delta^+(E)$ is also frequently called the *upper box dimension* or *metric entropy* of E.

Note that $N_\delta(E)$ will be finite for every $\delta > 0$ if and only if E is totally bounded in the metric d; for this reason, the definition of upper capacity is often restricted to totally bounded sets. However, we will find it useful to assign a capacity (necessarily infinite) even to sets which are not totally bounded.

While Definition 3.5 above provides the standard definition of upper capacity, we will need the following equivalent form:

LEMMA 3.6. *Let $E \subseteq \mathscr{X}$, and let $M_\delta(E)$ be the maximum number of disjoint closed balls of radius δ which can be centred at points of E. Then*

$$\Delta^+(E) = \limsup_{\delta \to 0} \frac{\log M_\delta(E)}{\log 1/\delta}. \tag{3.6}$$

Proof. This follows from the easy inequalities $N_{4\delta}(E) \leq M_\delta(E) \leq N_\delta(E)$.

THEOREM 3.7. *Let ϕ be any measure function. The packing ϕ-premeasure P_0^ϕ has the following properties*:

(a) $P_0^\phi \geq Q_0^\phi$ *in general, and* $P_0^\phi = Q_0^\phi$ *whenever* $\mathscr{X} = \mathbf{R}^N$ *or any other metric space where the radius of a ball is given uniquely by* $r = \operatorname{diam}(B)/2$.

(b) *P_0^ϕ is nonnegative and monotone; i.e., $A \subseteq C \Rightarrow 0 \le P_0^\phi(A) \le P_0^\phi(C)$.*
(c) *$P_0^\phi(E) = P_0^\phi(\overline{E})$ for every $E \subseteq \mathscr{X}$, where $\overline{E}$ denotes the closure of E.*
(d) *Closed and open packings produce the same premeasure; i.e., if "closed balls" are replaced by "open balls" in Definition 3.1, the same premeasure P_0^ϕ results.*
(e) *P_0^ϕ is finitely subadditive, i.e., $P_0^\phi(A \cup C) \le P_0^\phi(A) + P_0^\phi(C)$, and is finitely additive over positively separated sets, i.e., $d(A, C) > 0 \Rightarrow P_0^\phi(A \cup C) = P_0^\phi(A) + P_0^\phi(C)$.*
(f) *If $P_0^\alpha(E) < \infty$, then $P_0^\beta(E) = 0$ for every $\beta > \alpha$. It follows that either there exists a* **critical point** *$\Delta^*(E)$ such that $P_0^\alpha(E) = \infty$ for $\alpha < \Delta^*(E)$ and $P_0^\alpha(E) = 0$ for $\alpha > \Delta^*(E)$, or that $P_0^\alpha(E) = \infty$ for all $\alpha > 0$. In the latter case, we set $\Delta^*(E) = \infty$. Then, in general,*

$$\Delta^*(E) = \inf\{\alpha > 0 \mid P_0^\alpha(E) = 0\}.$$

(g) *$\Delta^*(E) = \Delta^+(E)$ for every $E \subseteq \mathscr{X}$. It follows that E has finite upper capacity if and only if E has finite packing α-premeasure for some $\alpha > 0$.*

Proof. Part (a) is immediate, and the proofs of (b)–(e) are straightforward arguments similar to those used in Lemma 3.1 of [20]. However, we point out that the proofs of (c) and (d) are made possible in the general setting by the fact that the measure function ϕ is being applied to the radii, rather than the diameters, of the balls. We now prove (f). Suppose $P_0^\alpha(E) < \infty$. Then there exists $\delta_0 > 0$ and $M < \infty$ such that $P_{\delta_0}^\alpha(E) \le M$. Let $\beta > \alpha$ and $0 < \delta < \delta_0$. If $\{B(x_k, r_k)\}_k$ is a δ-packing of E, then $\sum_k (2r_k)^\beta \le \delta^{\beta-\alpha} \sum_k (2r_k)^\alpha \le \delta^{\beta-\alpha} M \to 0$ as $\delta \to 0$. Thus $P_0^\beta(E) = 0$ as claimed, and (f) follows.

We now prove (g). The inequality $\Delta^*(E) \ge \Delta^+(E)$ is easiest. Let $\alpha < \Delta^+(E)$. From Lemma 3.6, it follows that $\limsup_{\delta \to 0} M_\delta(E)\delta^\alpha = \infty$, and hence $P_\delta^\alpha(E) \ge M_\delta(E)\delta^\alpha \to \infty$ as $\delta \to 0$. Thus $P_0^\alpha(E) = \infty$, and so $\Delta^*(E) \ge \alpha$. This gives $\Delta^*(E) \ge \Delta^+(E)$. The opposite inequality requires more work; we follow the proof of Tricot [22] given for the Euclidean case. If $\Delta^*(E) = 0$ there is nothing to prove, so let $0 < \alpha < \Delta^*(E)$. Since $P_0^\alpha(E) = \infty$, it follows that for each $\delta > 0$ there exists a δ-packing $\mathscr{P}_\delta$ of E such that $\sum_{\mathscr{P}_\delta} (2r_k)^\alpha > 1$. For each small $\delta > 0$ and each positive integer n, let $k_\delta(n)$ be the number of balls in $\mathscr{P}_\delta$ with radii satisfying $2^{-(n+1)} \le 2r < 2^{-n}$. Then, given $0 < \beta < \alpha$, it follows that, for each small $\delta > 0$, there exists an integer n_δ such that

$$k_\delta(n_\delta) > 2^{n_\delta \beta}(1 - 2^{-(\alpha-\beta)}).$$

Note that $n_\delta \to \infty$ as $\delta \to 0$. Now for each n_δ there exist $k_\delta(n_\delta)$ disjoint closed balls, centred at points of E, with equal radii r satisfying $2r = 2^{-(n_\delta+1)}$

i.e., $r = 2^{-(n_\delta+2)}$. Hence, from Lemma 3.6,

$$\begin{aligned}\Delta^+(E) &\geq \limsup_{\delta\to 0} \log M_{2^{-(n_\delta+2)}}/\log 2^{(n_\delta+2)}\\ &\geq \lim_{n\to\infty} \log\bigl(2^{n\beta}(1-2^{-(\alpha-\beta)})\bigr)/\log 2^{(n+2)} = \beta.\end{aligned}$$

It follows that $\Delta^+(E) \geq \Delta^*(E)$, and (g) is proved.

Remark 3.8. Let $\mathscr{X}$ be the uncountable space of Example 2.3. Obviously $N_\delta(\mathscr{X}) = \infty$ for each $0 < \delta < 1$, giving $\Delta^+(\mathscr{X}) = \infty$. However, as noted in Example 2.3, $Q_0^\phi(\mathscr{X}) = 0$ for all measure functions ϕ, which shows that the Q critical point $\Delta_Q^*(E) = \inf\{\alpha > 0 \mid Q_0^\alpha(E) = 0\} = 0$. Thus (g) of Theorem 3.7 is entirely false when P_0^α is replaced by Q_0^α.

We also wish to establish, in the case of $P^\alpha(E)$, the existence of a *critical point* as described in (f) of Theorem 3.7:

LEMMA 3.9. *If $P^\alpha(E) < \infty$, then $P^\beta(E) = 0$ for every $\beta > \alpha$.*

Proof. If $P^\alpha(E) < \infty$, then, by (3.3), there must exist a covering $\{E_k\}_k$ of E such that $P_0^\alpha(E_k) < \infty$ for each k. If $\beta > \alpha$, it follows from Theorem 3.7 (f) that $P_0^\beta(E_k) = 0$ for each k. Applying (3.3) again, we conclude $P^\beta(E) = 0$.

DEFINITION 3.10 (packing dimension and Hausdorff dimension). Let $E \subseteq \mathscr{X}$. We define the *packing dimension* of E (denoted $\mathrm{Dim}(E)$) to be the critical point

$$\mathrm{Dim}(E) = \inf\{\alpha > 0 \mid P^\alpha(E) = 0\}. \tag{3.7}$$

It is well known (e.g., [17]) that Lemma 3.9 holds with Hausdorff measure in place of packing measure, and hence the *Hausdorff dimension of E* (denoted $\dim(E)$) is defined by

$$\dim(E) = \inf\{\alpha > 0 \mid H^\alpha(E) = 0\}. \tag{3.8}$$

We note that $\dim(E) = \infty$ and/or $\mathrm{Dim}(E) = \infty$ is possible in some spaces.

It is easily seen that a similar critical point $\mathrm{Dim}_Q(E)$ exists for $Q^\alpha(E)$ (which we call the *diameter packing dimension* of E.) However, in keeping with our belief that P^ϕ is the "correct" definition of packing measure, we regard (3.7) as the "correct" definition of packing dimension.

THEOREM 3.11. *Let ϕ be any measure function. Then the outer measure P^ϕ has the following properties (in addition to countable subadditivity):*

(a) *$P^\phi \geq Q^\phi$ in general, and $P^\phi = Q^\phi$ whenever $\mathscr{X} = \mathbf{R}^N$ or any other metric space where the radius of a ball is given uniquely by $r = \mathrm{diam}(B)/2$.*

(b) P^ϕ *is a metric outer measure (i.e., P^ϕ is finitely additive over positively separated sets) and all Borel sets of $\mathscr{X}$ are P^ϕ-measurable.*

(c) P^ϕ *is Borel regular; i.e., corresponding to each $E \subseteq \mathscr{X}$ is a Borel set $B \supseteq E$ such that $P^\phi(B) = P^\phi(E)$. (B is called a* **Borel cover** *of E.) Consequently, $P^\phi(E_n) \uparrow P^\phi(\bigcup_n E_n)$ for every increasing sequence of sets $\{E_n\}_n$.*

(d) P^ϕ *is inner regular; i.e., if E is P^ϕ-measurable with $P^\phi(E) < \infty$, then, for every $\epsilon > 0$, there exists a closed set $F \subseteq E$ such that $P^\phi(F) > P^\phi(E) - \epsilon$.*

(e) *If E is countable then* $P^\phi(E) = 0$.

(f) $P^\phi(E) = \inf\{\sup_k P_0^\phi(E_k) \mid E_k \uparrow E\}$ *for every* $E \subseteq \mathscr{X}$.

(g) $\mathrm{Dim}(E) = \inf\{\sup_k \Delta^+(E_k) \mid E_k \uparrow E\}$ *for every* $E \subseteq \mathscr{X}$.

(h) *If ϕ is a* **blanketed** *measure function, the inequality $P^\phi \geq H^\phi$ holds. Consequently,* $\mathrm{Dim}(E) \geq \dim(E)$ *for every* $E \subseteq \mathscr{X}$.

Proof. Part (a) is immediate. Consider (b). The finite additivity of P^ϕ over positively separated sets is an easy consequence of (3.3) and (b), (e) of Theorem 3.7. This makes P^ϕ a metric outer measure, from which the rest of (b) follows (see [17], p. 59). Consider (c). From Theorem 3.7 (c), we see that we can replace "all countable coverings of E" in (3.3) with "all countable coverings of E by *closed* sets." From this and Theorem 12.3, p. 53, of [17], it follows that there exists an $\mathscr{F}_{\sigma\delta}$ set B (which is of course Borel) such that $B \supseteq E$ and $P^\phi(B) = P^\phi(E)$. Since the Borel sets are P^ϕ-measurable (from (b)), we see that P^ϕ satisfies Munroe's ([17], p. 50) definition of a *regular* outer measure. The rest of (c) then follows [17, p. 51].

For (d) we follow the argument in Lemma 5.1 of [20]. Suppose E is P^ϕ-measurable and $P^\phi(E) < \infty$. Then, by (c) above, there exists Borel $B_1 \supseteq E$ with $P^\phi(B_1 \setminus E) = 0$, and also Borel $B_2 \supseteq B_1 \setminus E$ with $P^\phi(B_2) = 0$. It follows that $B_1 \setminus B_2 \subseteq E$ and $P^\phi(B_1 \setminus B_2) = P^\phi(E)$. Now the set function $\mu(\cdot)$ defined over the Borel sets of $\mathscr{X}$ by $\mu(B) = P^\phi(B \cap (B_1 \setminus B_2))$ is a finite Borel measure and therefore inner regular (see Theorem 1.1 of [4]); i.e., for each Borel B and each $\varepsilon > 0$, there exists a closed set $F \subseteq B$ such that $\mu(F) > \mu(B) - \epsilon$. Taking $B = B_1 \setminus B_2$, (d) is proved.

Property (e) follows in the usual manner. Now let $P^*(E)$ denote the righthand side of the equation in (f), and suppose $E_k \uparrow E$. Since $P^\phi(E_k) \leq P_0^\phi(E_k)$, it follows from the last part of (c) that $P^\phi(E) = \lim_{k \to \infty} P^\phi(E_k) \leq \sup_k P_0^\phi(E_k)$. This shows that $P^\phi(E) \leq P^*(E)$. To get the reverse inequality, let $\{E_k\}_k$ be any cover of E. Without loss of generality (using (3.3) and Theorem 3.7 (b)) we can assume $E_k \subseteq E$ for each k. Define $E_k^* = \bigcup_{j=1}^k E_j$. Then $E_k^* \uparrow E$ and, for each k, $P_0^\phi(E_k^*) \leq \sum_{j=1}^k P_0^\phi(E_j)$ by finite subadditivity of P_0^ϕ. Hence $\sup_k P_0^\phi(E_k^*) \leq \sum_{k=1}^\infty P_0^\phi(E_k)$, which shows that $P^*(E) \leq P^\phi(E)$. Thus (f) is proved.

Now (g) is a consequence of (f) plus Theorem 3.7 (g). Since $\Delta^+(E) = \Delta^*(E)$, it is enough to prove that $\mathrm{Dim}(E) = D(E)$, where $D(E) = \inf\{\sup_k \Delta^*(E_k) \mid E_k \uparrow E\}$. We first show $\mathrm{Dim}(E) \leq D(E)$. If $D(E) = \infty$ there is nothing to prove, so assume that $D(E)$ is finite and let $\alpha > D(E)$ be arbitrary. Then there exists $E_k \uparrow E$ such that $\sup_k \Delta^*(E_k) < \alpha$. From Theorem 3.7 (f) it follows that $\sup_k P_0^\alpha(E_k) = 0$ and hence, from (f) above, $P^\alpha(E) = 0$. This gives $\mathrm{Dim}(E) \le \alpha$ and proves the desired inequality. The proof of the reverse inequality $D(E) \le \mathrm{Dim}(E)$ is similar. Assume that $\mathrm{Dim}(E)$ is finite and let $\alpha > \mathrm{Dim}(E)$ be arbitrary. Then $P^\alpha(E) = 0$ and there exists $E_k \uparrow E$ such that $\sup_k P_0^\alpha(E_k) \le 1$. Thus $\sup_k \Delta^*(E_k) \le \alpha$, giving $D(E) \le \mathrm{Dim}(E)$. This completes (g).

Now consider (h). If $E \subseteq \mathscr{X}$ contains at most countably many isolated points of $\mathscr{X}$, then from Theorem 2.6 plus (a) of this theorem we conclude that $P^\phi(E) \ge Q^\phi(E) \ge H^\phi(E)$. On the other hand, if E contains uncountably many isolated points of $\mathscr{X}$, the argument of Example 3.4 gives $P^\phi(E) = \infty \ge H^\phi(E)$. As a corollary we get $P^\alpha(E) \ge H^\alpha(E)$ for every $\alpha > 0$ and the inequality $\mathrm{Dim}(E) \ge \dim(E)$ follows. This completes the proof of Theorem 3.11.

Remark 3.12. We note that (g) of Theorem 3.11 is false for diameter packing measure in the case of the discrete uncountable space $\mathscr{X}$ of Example 2.3. Clearly the diameter packing dimension of $\mathscr{X}$ is 0, while $\sup_k \Delta^+(E_k) = \infty$ for every sequence of sets $E_k \uparrow \mathscr{X}$. (If $E_k \uparrow \mathscr{X}$, then there must exist k_0 such that E_k is uncountable for all $k \ge k_0$.)

We now proceed to show that a version of the main density theorem of Taylor and Tricot (Theorem 5.4 of [20] applied to balls) holds true in general metric spaces. We begin with the following definition.

DEFINITION 3.13 (lower ϕ-density). Let μ be a finite Borel measure on $\mathscr{X}$ such that $\mu(\mathscr{X}) > 0$. Let ϕ be a measure function. Then the *lower ϕ-density* of μ is the function $d_\phi : \mathscr{X} \to \mathbf{R}$ defined by

$$d_\phi(x) = \liminf_{\delta \to 0} \frac{\mu(B(x, \delta))}{\phi(2\delta)}. \tag{3.9}$$

Due to the continuity of ϕ, the same function d_ϕ results if closed balls are replaced by open balls in (3.9). However, we will find closed balls more convenient to use.

The main density theorem (Theorem 3.16) links the quantities $\mu(E)$ and $P^\phi(E)$ via the lower ϕ-density. This connection is made through the use of certain Vitali-like coverings. We have the following definition:

DEFINITION 3.14 (centred ball covering). Let $E \subseteq \mathscr{X}$. A collection $\mathscr{C}$ of closed balls of $\mathscr{X}$ is called a *centred ball covering* of E if, for every $x \in E$ and every $\varepsilon > 0$, there exists $B \in \mathscr{C}$ such that $B = B(x, r)$ where $0 < r < \epsilon$.

Note that a centred ball covering of E is also a closed Vitali covering of E if and only if E has no isolated points. We say that a finite Borel measure μ on $\mathscr{X}$ possesses the *centred ball covering property* if, for every Borel set E and every centred ball covering $\mathscr{C}$ of E, there exists a disjoint sequence $B_1, B_2, \ldots$ from $\mathscr{C}$ such that $\mu(E \setminus \bigcup_k B_k) = 0$. The strong form of Theorem 3.16 (as well as the simplest proof) is obtained when μ has the centred ball covering property.

We say that $\mathscr{X}$ itself has the centred ball covering property if every finite Borel measure on $\mathscr{X}$ has the property. It is well known (e.g., [1], [7]) that $\mathbf{R}^N$ possesses the centred ball covering property, as do all finite-dimensional Banach spaces and sufficiently smooth compact Riemannian manifolds (see pp. 145–150 of [11]). Federer [11] also points out that existence of the centred ball covering property on $\mathscr{X}$ is a consequence of a property he calls *directionally limited*, which is closely linked to the notion of finite dimensionality. Thus, to obtain Theorem 3.16 in the general case (where the centred ball covering property need not hold) we need to resort to other methods. Our approach is similar to that taken for the Euclidean case in the original paper of Taylor and Tricot [20]; we extend their method to the general metric setting by developing the following lemma (which is a variation on a component of the original Besicovitch covering lemma in $\mathbf{R}^N$—see Theorem 3.2.1 of [7]). We point out that the verity of this lemma in arbitrary metric spaces is due in part to our use of the radius definition of packing measure.

LEMMA 3.15. *Let $E \subseteq \mathscr{X}$ and suppose $\mathscr{C}$ is a centred ball covering of E. If there exists a measure function ϕ such that $P_0^\phi(E) < \infty$, then, for every $0 < \alpha < 1/2$, there exists a sequence of balls $\{B(x_k, r_k)\}_k$ (centred at points $x_k \in E$) from $\mathscr{C}$ such that*

(i) *$E \subseteq \bigcup_k B(x_k, r_k)$, and*

(ii) *the smaller balls $\{B(x_k, \alpha r_k)\}_k$ are disjoint.*

Proof. Since $P_0^\phi(E) < \infty$, there exists $\delta_0 > 0$ such that $P_{\delta_0}^\phi(E) < \infty$. Let $0 < \alpha < 1/2$ be given, and choose β so that $\alpha(1-\alpha)^{-1} < \beta < 1$. Define $d_1 = \sup\{0 < r \leq \delta_0 \mid B(x,r) \in \mathscr{C}, x \in E\}$. Then we can find $B(x_1, r_1) \in \mathscr{C}$ such that $x_1 \in E$ and $\beta d_1 < r_1 \leq d_1$. For convenience, set $B_1 = B(x_1, r_1)$. If $E \subseteq B_1$ we can stop. If $E \nsubseteq B_1$, define

$$d_2 = \sup\{0 < r \leq \delta_0 \mid B(x,r) \in \mathscr{C},\, x \in E \setminus B_1\}.$$

Obviously $d_2 \leq d_1$. Choose $B(x_2, r_2) \in \mathscr{C}$ such that $x_2 \in E \setminus B_1$ and $\beta d_2 < r_2 \leq d_2$; set $B_2 = B(x_2, r_2)$. If $E \subseteq B_1 \cup B_2$ we can stop. Otherwise, continue this process, defining

$$d_3 = \sup\{0 < r \leq \delta_0 \mid B(x,r) \in \mathscr{C}, x \in E \setminus (B_1 \cup B_2)\}$$

and obtaining B_3, etc. This process either terminates at some finite n (if $E \subseteq \bigcup_{k=1}^{n} B_k$) or generates an infinite sequence of balls $B_1, B_2, \ldots$. We first prove (ii). For each k, let B_k^* denote the smaller ball $B(x_k, \alpha r_k)$. Consider any two of these balls B_i^* and B_j^*, and suppose $i < j$. We know, from the above construction, that $d(x_j, x_i) > r_i$. Let y be any element of B_j^*; then $d(x_j, y) \leq \alpha r_j$. We will show that $y \notin B_i^*$. First note that $r_i > \beta d_i \geq \beta d_j \geq \beta r_j$; i.e., $r_j < \beta^{-1} r_i$. Now, applying the triangle inequality, we get

$$d(y, x_i) \geq d(x_j, x_i) - d(x_j, y) > r_i - \alpha r_j > r_i - \alpha \beta^{-1} r_i > \alpha r_i.$$

Thus $y \notin B_i^*$, which shows that the two balls are disjoint, proving (ii). Now consider (i). If the construction process terminates; i.e., $E \subseteq \bigcup_{k=1}^{n} B_k$ for some n, then we are done. So assume that we have generated an infinite sequence $B_1, B_2, \ldots$ of balls. Note that, by (ii), the smaller balls $B_1^*, B_2^*, \ldots$ form a δ_0-packing of E. It follows that $\sum_k \phi(2\alpha r_k) \leq P_{\delta_0}^{\phi}(E) < \infty$ so we must have $\phi(2\alpha r_k) \to 0$, and hence $r_k \to 0$ as $k \to \infty$. Since $r_k > \beta d_k$, we conclude that also $d_k \to 0$. Now suppose $x \in E$. By Definition 3.14, there exists some $0 < r < \delta_0$ such that $B(x, r) \in \mathscr{C}$. Since $d_k \to 0$, there must exist n such that $d_n < r$, which (by the definition of d_n) implies that $x \in \bigcup_{k=1}^{n-1} B_k$. This proves (i).

THEOREM 3.16. *Let $\mathscr{X}$ be a metric space and let ϕ be any measure function. Let μ be a finite positive Borel measure on $\mathscr{X}$ with lower density d_ϕ. For each Borel subset $E \subseteq \mathscr{X}$, let $\inf d_\phi(E) = \inf_{x \in E} d_\phi(x)$ and $\sup d_\phi(E) = \sup_{x \in E} d_\phi(x)$. Then the following hold:*

(a) *For each Borel set E, we have*

$$\mu(E) \geq P^{\phi}(E) \inf d_\phi(E)$$

where we take the righthand side to be 0 if either $\inf d_\phi(E) = 0$ or $P^{\phi}(E) = 0$.

(b) *If additionally ϕ is blanketed, i.e., there exists $0 < K < 1$ such that ϕ satisfies $\phi(t/2) \geq K\phi(t)$ for all $0 < t \leq 1$, then*

$$\mu(E) \leq K^{-1} P^{\phi}(E) \sup d_\phi(E)$$

for each Borel set E, where we take the righthand side to be ∞ if either $\sup d_\phi(E) = \infty$ or $P^{\phi}(E) = \infty$.

We have the following stronger form of (b) *if μ possesses the centred ball covering property*:

(b′) *If μ has the centred ball covering property, then the restriction that ϕ be blanketed can be removed from* (b), *and we obtain the stronger result*

$$\mu(E) \leq P^{\phi}(E) \sup d_\phi(E)$$

for each Borel set E. As in (b), *the righthand side is taken to be* ∞ *if* $\sup d_\phi(E) = \infty$ *or* $P^\phi(E) = \infty$.

Proof. We begin by showing that $P^\phi(E) < \infty$ whenever $\inf d_\phi(E) > 0$. Let γ be any value satisfying $0 < \gamma < \inf d_\phi(E)$ (γ can be taken arbitrarily large if $\inf d_\phi(E) = \infty$). We will show that $P^\phi(E) \le \gamma^{-1}\mu(\mathscr{X})$. Now $\inf d_\phi(E) > \gamma$ implies that E can be obtained as the limit $E_k \uparrow E$, where

$$E_k = \{x \in E \mid \mu(B(x,r)) \ge \gamma\phi(2r) \text{ for all } 0 < r \le 1/k\}.$$

Thus any closed δ-packing $\mathscr{P}_\delta$ of E_k with $0 < \delta < 2/k$ must satisfy

$$\sum_{\mathscr{P}_\delta} \phi(2r_k) \le \gamma^{-1} \sum_{\mathscr{P}_\delta} \mu(B(x,r_k)) \le \gamma^{-1}\mu(E_k(\delta)),$$

where $E_k(\delta) = \{x \in \mathscr{X} \mid d(x, E_k) \le \delta\}$. Letting $\delta \to 0$, we get

$$P_0^\phi(E_k) \le \gamma^{-1}\mu(\overline{E}_k) \le \gamma^{-1}\mu(\mathscr{X}).$$

Since $E_k \uparrow E$, it follows from Theorem 3.11 (f) that $P^\phi(E) \le \sup_k P_0^\phi(E_k) \le \gamma^{-1}\mu(\mathscr{X})$. Thus $P^\phi(E) < \infty$.

Now consider (a). If $\inf d_\phi(E) = 0$ there is (by definition) nothing to prove, and so we can assume that $\inf d_\phi(E) > 0$. By the argument given above, this implies $P^\phi(E) < \infty$. Hence, by the inner regularity of both P^ϕ and μ, we can find an increasing sequence of closed sets $F_n \subseteq E$ such that $P^\phi(F_n) \uparrow P^\phi(E)$ and $\mu(F_n) \uparrow \mu(E)$. Since $\inf d_\phi(E) \le \lim_{n\to\infty} \inf d_\phi(F_n)$, it follows that it is sufficient to prove (a) for *closed* sets E with $\inf d_\phi(E) > 0$. As before, we let $0 < \gamma < \inf d_\phi(E)$ be arbitrary and write $E_k \uparrow E$, where the E_k are the same as defined earlier. In the identical manner, we again obtain $P_0^\phi(E_k) \le \gamma^{-1}\mu(\overline{E}_k)$, but since E is closed and $E_k \subseteq E$, we have $\overline{E}_k \subseteq E$ and so $P_0^\phi(E_k) \le \gamma^{-1}\mu(E)$. This gives $P^\phi(E) \le \sup_k P_0^\phi(E_k) \le \gamma^{-1}\mu(E)$. Since $0 < \gamma < \inf d_\phi(E)$ was arbitrary, we conclude that $\mu(E) \ge P_0^\phi(E) \inf d_\phi(E)$ as claimed.

Now consider (b). If $\sup d_\phi(E) = \infty$ there is nothing to prove, so assume $\sup d_\phi(E) < \infty$. We will show that $\mu(E) \le \gamma K^{-1} P^\phi(E)$ for every real $\gamma > \sup d_\phi(E)$. It is enough to prove, for every $A \subseteq E$, that $\mu(A) \le \gamma K^{-1} P_0^\phi(A)$, since then

$$\mu(E) \le \inf\left\{ \sum_k \mu(A_k) \;\middle|\; \bigcup_k A_k = E \right\}$$

$$\le \gamma K^{-1} \inf\left\{ \sum_k P_0^\phi(A_k) \;\middle|\; \bigcup_k A_k = E \right\} = \gamma K^{-1} P^\phi(E)$$

as required. So let $A \subseteq E$. If $P_0^\phi(A) = \infty$ there is nothing to prove, so assume $P_0^\phi(A) < \infty$. Now $\sup d_\phi(A) \le \sup d_\phi(E) < \gamma$ and hence, for each $x \in A$, there exists a sequence $r_n = r_n(x) \downarrow 0$ such that $\mu(B(x, r_n)) \le \gamma\phi(2r_n)$. Thus, for each $0 < \delta < 1$, the collection

$$\mathscr{C}_\delta = \{B(x,r) \mid x \in A, \mu(B(x,r)) \le \gamma\phi(2r), 0 < r < \delta/2\}$$

is a centred ball covering of A. Let $\epsilon > 0$ be small and choose $0 < \alpha < 1/2$ sufficiently close to $1/2$ so that, by the uniform continuity of ϕ and the blanketed condition, we get $\phi(2t) \le (K^{-1} + \epsilon)\phi(2\alpha t)$ for all $0 < t \le 1/2$. Since $P_0^\phi(A) < \infty$, we can apply Lemma 3.15 to obtain a sequence of balls $\{B(x_k, r_k)\}_k$ from $\mathscr{C}_\delta$ such that $A \subseteq \bigcup_k B(x_k, r_k)$ and the smaller balls $\{B(x_k, \alpha r_k)\}_k$ are disjoint. This gives

$$\begin{aligned}\mu(A) &\le \sum_k \mu(B(x_k, r_k)) \le \gamma \sum_k \phi(2r_k) \\ &\le \gamma(K^{-1} + \epsilon) \sum_k \phi(2\alpha r_k) \le \gamma(K^{-1} + \epsilon) P_\delta^\phi(A).\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\mu(A) \le \gamma K^{-1} P_\delta^\phi(A)$. Letting $\delta \to 0$ completes the proof of (b).

To see (b′), note that the existence of the centred ball covering property means that we can directly choose a sequence of disjoint balls $\{B(x_k, r_k)\}_k$ from $\mathscr{C}_\delta$ such that $\mu(A \setminus \bigcup_k B(x_k, r_k)) = 0$. This immediately gives

$$\mu(A) \le \sum_k \mu(B(x_k, r_k)) \le \gamma \sum_k \phi(2r_k) \le \gamma P_\delta^\phi(A)$$

and the result follows.

Remark 3.17. It is worth noting that the above proofs of (b) and (b′) do not go through for diameter packing measure Q^ϕ. There are basically two mechanisms for producing similar density theorems in the case of diameter-based packings. One is to place a uniform restriction on the relationship between ball diameter and ball radius; for example,

(3.10) there exist $c > 0$ and $r_0 > 0$ such that $\operatorname{diam}(B(x,r)) \ge cr$ for all x and all $0 < r < r_0$

in which case a version of Theorem 3.16 can be obtained (the constant c will also generally enter into the bounds). The second method involves redefining the lower ϕ-density by replacing the quantity $\phi(2\delta)$ in the denominator of (3.9) by $\phi(\operatorname{diam}(B(x, \delta)))$. Edgar [9] has recently used this approach to obtain a powerful density theorem for diameter-based packing measure under the assumption that μ possesses the centred ball covering property. Haase [14] has also used this method to obtain a weaker density theorem (under similar restrictions) for a different class of diameter packing measure. We note that actual computation of the modified ϕ-density requires knowledge of the local geometry at each point x of the space, a fact which may have important ramifications in practice.

Remark 3.18. Haase [13] has proved a density theorem similar to Theorem 3.16 for radius-based packing measure in the restricted case of separable ultrametric spaces.

Remark 3.19. We also point out that if a result somewhat analogous to Lemma 2.5 can be shown to exist for packing measure, then (b) of Theorem 3.16 can be strengthened to $\mu(E) \leq P^{\phi}(E) \sup d_{\phi}(E)$ for all blanketed measure functions. That is, if it can be proved that $P^{\phi} \mid E$ satisfies the centred ball covering property whenever E is a set of finite packing ϕ-premeasure, we can then eliminate the factor K^{-1} from (b) for blanketed measure functions. Such a result seems too ambitious, however, since it would imply that every finite Borel measure μ possesses the centred ball covering property when restricted to a set of finite packing ϕ-premeasure. We refer the reader to Haase [14] for discussion of a weaker (insufficient for our purposes) centred ball covering property satisfied by certain diameter-based packing measures.

We now consider the pointwise representation of packing dimension with respect to a Borel probability distribution μ on $\mathscr{X}$. In [6], Cutler showed (under the assumption of the centred ball covering property and the relationship (3.10)) that the *upper pointwise dimension map*

$$\alpha^{+}(x) = \limsup_{\delta \to 0} \frac{\log \mu(B(x, \delta))}{\log \delta} \tag{3.11}$$

satisfied the following two properties:

(3.12) (a) $\mathrm{Dim}_Q(\{x \in \mathscr{X} \mid \alpha^{+}(x) \leq \alpha\}) \leq \alpha$ for every $\alpha \geq 0$;

(b) if $E \subseteq \{x \in \mathscr{X} \mid \alpha^{+}(x) \geq \alpha\}$ and $\mu(E) > 0$, then $\mathrm{Dim}_Q(E) \geq \alpha$

where Dim_Q denotes diameter packing dimension. It is easy to see, however, that $\mathrm{Dim}_Q(\cdot) = \mathrm{Dim}(\cdot)$ under assumption (3.10), and it is in fact this equality induced by (3.10) that makes (3.12) work for diameter packing dimension. The properties in (3.12) establish that $\alpha^{+}(\cdot)$ is a version of the *local diameter packing dimension map* (see [6] for terminology and details) and relate the behaviour of $\alpha^{+}(\cdot)$ to the distribution of μ-mass over sets of varying diameter packing dimension. The results in [6] were intended to complement similar results obtained in [5] for Hausdorff dimension in general metric spaces. (See Billingsley [2], [3] for the genesis of the study of pointwise dimension maps and their relation to Hausdorff dimension and sets of positive probability.) Specifically, it is shown in [5] that, for a Borel probability measure μ on a metric space $\mathscr{X}$, the *lower pointwise dimension map*

$$\alpha^{-}(x) = \liminf_{\delta \to 0} \frac{\log \mu(B(x,\delta))}{\log \delta} \tag{3.13}$$

satisfies the following two properties:

(3.14) (a) $\dim(\{x \in \mathscr{X} \mid \alpha^{-}(x) \le \alpha\}) \le \alpha$ for every $\alpha \ge 0$;

(b) if $E \subseteq \{x \in \mathscr{X} \mid \alpha^{-}(x) \ge \alpha\}$ and $\mu(E) > 0$, then $\dim(E) \ge \alpha$.

In particular, (3.14) holds without any restrictions on the metric space $\mathscr{X}$. We now show that (3.12) also holds without any restrictions on $\mathscr{X}$ provided we substitute radius-based packing dimension Dim for Dim_Q. (Hence we obtain the true complement to the results for Hausdorff dimension.) It is possible to prove directly that (3.12) holds, but these properties can be obtained more simply as a corollary to Theorem 3.16:

COROLLARY 3.20. *Let μ be a Borel probability measure on a metric space $\mathscr{X}$, and let $\alpha^{+}(\cdot)$ denote the upper pointwise dimension map of μ. Then the following two properties hold*:

(a) $\mathrm{Dim}(\{x \in \mathscr{X} \mid \alpha^{+}(x) \le \alpha\}) \le \alpha$ *for every* $\alpha \ge 0$;

(b) *if* $E \subseteq \{x \in \mathscr{X} \mid \alpha^{+}(x) \ge \alpha\}$ *and* $\mu(E) > 0$, *then* $\mathrm{Dim}(E) \ge \alpha$.

Hence $\alpha^{+}(\cdot)$ is a version of the local packing dimension map of μ.

Proof. Let $D_\alpha = \{x \in \mathscr{X} \mid \alpha^{+}(x) \le \alpha\}$ and note that (a) is trivially true if $\alpha = \infty$. We therefore assume that $0 \le \alpha < \infty$. To prove (a) it is sufficient to show that $P^{\beta}(D_\alpha) = 0$ for every $\beta > \alpha$. Therefore, let $\beta > \gamma > \alpha$ be arbitrary and let $x \in D_\alpha$. Since $\alpha^{+}(x) \le \alpha$ there exists $\delta_0 > 0$ such that $\mu(B(x,\delta)) \ge \delta^{\gamma}$ for all $0 < \delta < \delta_0$. Hence $\delta^{-\beta}\mu(B(x,\delta)) \ge \delta^{\gamma-\beta} \to \infty$ as $\delta \to 0$. Taking $\phi(t) = t^{\beta}$, this shows that $\inf d_{\phi}(D_\alpha) = \infty$. Noting that D_α is a Borel set and applying Theorem 3.16 (a), we conclude that $P^{\beta}(D_\alpha) = 0$ as required. Now consider the set E given in (b). If $\alpha = 0$, (b) is trivially true, so assume that $\alpha > 0$ and let $0 < \beta < \gamma < \alpha$ be arbitrary. To prove (b), it is sufficient to show that $P^{\beta}(E) = \infty$. Now $x \in E \Rightarrow \alpha^{+}(x) \ge \alpha > \gamma$, and so there exists a sequence $\delta_n \downarrow 0$ such that $\mu(B(x, \delta_n)) \leq \delta_n^\gamma$ for every n. Hence $\liminf_{\delta \to 0} \delta^{-\beta}\mu(B(x, \delta)) = 0$. Taking $\phi(t) = t^\beta$, we conclude that $\sup d_\phi(E) = 0$. Since $\mu(E) > 0$ by assumption, it now follows from Theorem 3.16 (b) that $P^\beta(E) = \infty$.

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