ILLINOIS JOURNAL OF MATHEMATICS
Volume 39, Number 4, Winter 1995

HOW MANY VECTORS ARE NEEDED TO COMPUTE (p, q)-SUMMING NORMS[1]

MARTIN DEFANT AND MARIUS JUNGE

Introduction

In the local theory of Banach spaces the concept of summing operators is of special interest. The presented paper is concerned with the following problem raised up by T. Figiel.

For given p and q, $1 \leq q \leq p \leq \infty$, what is the best rate k_n, such that

$$(*) \qquad \pi_{pq}(T) \leq c \pi_{pq}^{k_n}(T)$$

holds for all operators of rank n and some constant c?

In [DJ] an observation of Figiel and Pelczynski was generalized in showing

$$\pi_{pq}(T) \leq 3 \pi_{pq}^{16^n}(T)$$

for all q, p and all operators T of rank n. This exponential growth cannot be improved in general. Figiel and Pelczynski also showed that there is an operator $T\colon l_\infty^{2^n} \to l_2^n$ (the Rademacher projection) such that for all $k \in \mathbf{N}$

$$\pi_1^k(T) \leq e \sqrt{\frac{1 + \ln k}{n}}\, \pi_1(T).$$

Recently, Johnson and Schechtman [JOS] discovered that for $p = q$ and $q \neq 2$ the rate can not be polynomial. More precisely, every sequence satisfying $(*)$ growth faster than any polynomial, i.e.,

$$\lim_{n \to \infty} k_n n^{-t} = \infty$$

Received July 27, 1993.
1991 Mathematics Subject Classification. Primary 47B10, 47A30, 46B07.

[1]A preliminary version of this paper occurred as *Absolutely summing norms with n vectors* [DJ2].

for all $0 < t < \infty$. This phenomenon is related with the fact that L_p spaces don't have the polynomial approximation property, which was proved by Bourgain. By far the nicest and most important result is Tomczak-Jaegermann's inequality, namely

$$\pi_2(T) \leq \sqrt{2}\,\pi_2^n(T) \qquad \text{for all } T \text{ of rank } n. \tag{1}$$

In [DJ] a certain type of quotient formula was used to generalize Tomczak-Jaegermann's inequality:

$$\pi_{p2}(T) \leq \sqrt{2}\,\pi_{p2}^n(T) \qquad \text{for all } T \text{ of rank } n.$$

König and Tzafriri showed that for all $2 < p < \infty$

$$\pi_{p1}(T) \leq c_p \pi_{p1}^n(T) \qquad \text{for all } T \text{ of rank } n. \tag{2}$$

In contrast to the case $p = q$ we can show that for $q < p$ the (p, q)-summing norm can be well-estimated by a polynomial number of vectors.

THEOREM 1. *Let $1 \leq q \leq p$, $r \leq \infty$ with*

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r}.$$

Then for all operator T of rank n one has

$$\pi_{pq}(T) \leq c_r \begin{cases} \pi_{pq}^n(T) & \text{for } 1 \leq r < 2, \\ \pi_{pq}^{[n(1+\ln n)]}(T) & \text{for } 2 = r, \\ \pi_{pq}^{[n^{r/2}]}(T) & \text{for } 2 < r < \infty. \end{cases}$$

A very helpful tool in the proof of this theorem is again a quotient formula for (p, q)-summing operators which allows a reduction to the (probably worst) case $q = 1$.

THEOREM 2. *Let $1 \leq q \leq p \leq \infty$ and $1 \leq r \leq s \leq q'$ with*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{s}.$$

Then for all operator T: $X \to Y$ and $n \in \mathbf{N}$,

$$\pi_{pq}^n(T) = \sup\{\pi_{r1}^n(TVD_\sigma) \mid V\colon l_{q'} \to X,\, D_\sigma\colon l_\infty \to l_{q'},\, \|\sigma\|_s, \|V\| \leq 1\}.$$

For instance the first case of Theorem 1 is a direct consequence of Theorem 2 and (2). In the other cases a crucial observation of Jameson gives the link between limit orders and number of vectors; see Section 2. We are in debt to W. B. Johnson for showing us Jameson's paper [JAM]. There has always been a quite close connection between the theory of absolutely (p, q)-summing operators and the theory of cotype in Banach spaces. For instances, as a consequence of Tomczak-Jaegermann's inequality and it's generalization the gaussian cotype constant of an n-dimensional Banach space can be calculated with n vectors. This problem is still open in the case of Rademacher cotype. The presented technique allows us to reduce the number of vectors to the order $n(1 + \ln n)^{c_q}$ which indicates a positive solution for the Rademacher cotype. Unfortunately, the constant c_q tends to infinity as q tends to 2.

THEOREM 3. *Let $2 < q < \infty$ and E an n dimensional Banach space. For the Rademacher cotype constant one has*

$$C_q(\mathrm{Id}_E) \leq 2C_q^m(\mathrm{Id}_E),$$

where m satisfies the following estimate for an absolute constant c_0

$$m \leq n\big(c_0(1 + \ln n)\big)^{1/(1-2/q)}.$$

Finally, we want to indicate that a linear growth is only possible if $q = 2$ or $1/q - 1/p \leq \frac{1}{2}$.

THEOREM 4. *Let $1 \leq q \leq p < \infty$, $q \leq r \leq \infty$ with*

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r}.$$

If $q \neq 2$ and $2 < r \leq \infty$. Then there exists $\alpha > 1$ such that for all sequences k_n with

$$\pi_{pq}(T) \leq c\pi_{pq}^{k_n}(T) \qquad \textit{for all } T \textit{ of rank } n,$$

there exists a constant $\bar{c}$ with

$$n^\alpha \leq \bar{c}k_n.$$

The constructed examples are very closely related to limit orders of (p, q)-summing operators. In fact, it is well known that the identity on l_2^n yields an example for the proposition above as long as $q > 2$. In the case $q < 2$ we intensively use the results of Carl, Maurey and Puhl [CMP] about Benett matrices.

Preliminaries

In what follows, $c_0, c_1, \dots$ always denote universal constants. We use standard Banach space notation. In particular, the classical spaces l_q and l_q^n, $1 \le q \le \infty$, $n \in \mathbf{N}$, are defined in the usual way. By ι: $l_q^n \to l_p^n$ we denote the canonical identity. Let $(e_k)_{k \in \mathbf{N}}$ be the sequence of unit vectors in l_∞. For a sequence $\sigma = (\sigma_k)_{k \in \mathbf{N}} \in l_\infty$, $\tau = (\tau_k)_{k \in \mathbf{N}} \in l_\infty$ we define

$$D_\sigma(\tau) := \sum_k \sigma_k \tau_k e_k.$$

The standard reference on operator ideals is the monograph of Pietsch [PIE]. The ideals of linear bounded operators, finite rank operators, integral operators are denoted by $\mathscr{L}$, $\mathscr{F}$, $\mathscr{I}$. Here the integral norm of $T \in \mathscr{I}(X, Y)$ is defined by

$$\iota_1(T) := \sup\{|\operatorname{tr}(ST)| \mid S \in \mathscr{F}(Y, X), \|S\| \le 1\}.$$

Let $1 \le q \le p \le \infty$ and $n \in \mathbf{N}$. For an operator $T \in \mathscr{L}(X, Y)$, the pq-summing norm of T with respect to n vectors is defined by

$$\pi_{pq}^n(T) := \sup\left\{ \left(\sum_1^n \|Tx_k\|^p \right)^{1/p} \middle| \sup_{\|x^*\|_{X^*} \le 1} \left(\sum_1^n |\langle x_k, x^* \rangle|^q \right)^{1/q} \le 1 \right\}.$$

An operator is said to be absolutely pq-summing, short pq-summing, ($T \in \Pi_{pq}(X, Y)$) if

$$\pi_{pq}(T) := \sup_n \pi_{pq}^n(T) < \infty.$$

Then (Π_{pq}, π_{pq}) is a maximal and injective Banach ideal (in the sense of Pietsch). As usual we abbreviate $(\Pi_q, \pi_q) := (\Pi_{qq}, \pi_{qq})$. For further information about absolutely pq-summing operators we refer to the monograph of Tomczak-Jaegermann [TOJ]. In particular, we would like to mention an elementary observation of Kwapien; see [TOJ]. Let $1 \le q \le p \le \infty$, $1 \le \bar{q} \le \bar{p} \le \infty$ with $q \le \bar{q}$, $p \le \bar{p}$ and

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{\bar{q}} - \frac{1}{\bar{p}}.$$

Then for all T, one has

$$\pi_{\bar{p}\bar{q}}^n(T) \le \pi_{pq}^n(T).$$

For $2 \leq q < \infty$, $T \in \mathscr{L}(X,Y)$ and $n \in \mathbf{N}$ the Rademacher (gaussian) Cotype q norm with respect to n-vectors is defined by

$$C_q^n(T)\left(\tilde{C}_q^n(T)\right) := \sup\left\{\left(\sum_1^n \|Tx_k\|^q\right)^{1/q} \,\middle|\, \left(\int_\Omega \left\|\sum_1^n v_k x_k\right\|^2 d\mu\right)^{1/2} \leq 1\right\},$$

where $(v_k)_1^n$ is a sequence of independent Bernoulli (gaussian) variables on a probability space (Ω, μ). An operator is said to be of Rademacher (gaussian) cotype q if the corresponding norm

$$C_q(T) := \sup_{n \in \mathbf{N}} C_q^n(T)\left(\tilde{C}_q(T) := \sup_{n \in \mathbf{N}} \tilde{C}_q^n(T)\right)$$

is finite. For further information and the relation between gaussian cotype and $(q,2)$-summing operators see for example [TOJ].

1. Positive results

Proof of Theorem 2. "$\leq$". Let $x_1, \dots, x_n \in X$ with

$$\sup_{\|\alpha\|_{q'} \leq 1} \left\|\sum_1^n \alpha_k x_k\right\| = \sup_{\|x^*\|_{X^*} \leq 1} \left(\sum_1^n |\langle x_k, x^*\rangle|^q\right)^{1/q} \leq 1.$$

Therefore the operator $V := \sum_1^n e_k \otimes x_k : l_{q'} \to X$ is of norm 1. By the equality case of Hölder's inequality we obtain

$$\begin{aligned}\left(\sum_1^n \|Tx_k\|^p\right)^{1/p} &= \sup_{\|\sigma\|_s \leq 1} \left(\sum_1^n (|\sigma_k|\,\|Tx_k\|)^r\right)^{1/r} \\ &\leq \sup_{\|\sigma\|_s \leq 1} \pi_{r1}^n(TVD_\sigma).\end{aligned}$$

"$\geq$". By the maximality of the norms π_{r1}^n we may assume $D_\sigma : l_\infty^m \to l_{q'}^m$ and $V : l_{q'}^m \to X$ with $\|\sigma\|_s, \|V\| \leq 1$. Now let $U : l_\infty^n \to l_\infty^m$ an operator of norm 1. By an observation of Maurey [MAU] the extreme points of such operators are of the form

$$U = \sum_1^n e_k \otimes g_k,$$

where the g_k's are of norm 1 and have disjoint support. Since we have to estimate the convex expression

$$\left(\sum_1^n \|TVD_\sigma U(e_k)\|^r\right)^{1/r}$$

we can assume that U is of this form. We define τ and $J : l_{q'}^n \to l_{q'}^m$ by

$$\tau_k := \| D_\sigma(g_k) \|_{q'} \quad \text{and} \quad J := \sum_1^n e_k \otimes \frac{D_\sigma(g_k)}{\| D_\sigma(g_k) \|_{q'}}.$$

The operator J is of norm at most 1. Since D_σ is obviously $s1$-summing we have

$$\|\tau\|_s \le \pi_{s1}(D_\sigma)\|U\| \le \|\sigma\|_s \le 1.$$

Therefore by Hölder's inequality we obtain

$$\begin{aligned}
\left(\sum_1^n \| TVD_\sigma U(e_k) \|^r \right)^{1/r} &= \left(\sum_1^n \left\| TV\left(\frac{D_\sigma(g_k)}{\| D_\sigma(g_k) \|_{q'}} \tau_k \right) \right\|^r \right)^{1/r} \\
&= \left(\sum_1^n (\| TVJ(e_k) \| \, |\tau_k|)^r \right)^{1/r} \\
&\le \left(\sum_1^n (\| TVJ(e_k) \|)^p \right)^{1/p} \|\tau\|_s \\
&\le \pi_{pq}^n(T) \|VJ\| \, \|\tau\|_s \le \pi_{pq}^n(T). \qquad \square
\end{aligned}$$

Now we will formulate a generalization of Jameson's lemma [JAM] which he proved in the case $q = 1$, $p = 2$.

LEMMA 1.1 (Jameson). *Let $1 \le q < p \le \infty$ and let $T \in \mathscr{L}(X, Y)$ be a q-summing operator. Then*

$$\pi_{pq}(T) \le 2^{1/p} \pi_{pq}^n(T)$$

where

$$n \le \left(2^{1/p} \frac{\pi_q(T)}{\pi_{pq}(T)} \right)^{1/(1/q - 1/p)}.$$

Proof. Let us assume $\pi_q(T) = 1$. For $\varepsilon > 0$ let $x_1, \dots, x_N$ in X with

$$\sup_{\|x^*\|_{X^*} \le 1} \left(\sum_1^N |\langle x_k, x^* \rangle|^q \right)^{1/q} \le 1 \quad \text{and} \quad (1 - \varepsilon)\pi_{pq}^p(T) \le \sum_1^N \|Tx_k\|^p.$$

Furthermore, we assume $\|Tx_k\|$ nonincreasing. For $0 < \delta$ we choose $n \le N$ minimal such that $\|Tx_k\| \le \delta$ holds for all $k > n$. Then we have $n \le \delta^{-q}$

because

$$n\delta^q \le \pi_q^q(T) \le 1.$$

If $\delta^{p-q} \le \frac{1}{2}(1-\varepsilon)\pi_{pq}^p(T)$ it follows that

$$\begin{aligned}(1-\varepsilon)\pi_{pq}^p(T) &\le \sum_1^N \|Tx_k\|^p \\ &\le \sum_1^n \|Tx_k\|^p + \delta^{p-q}\sum_{n+1}^N \|Tx_k\|^q \\ &\le \sum_1^n \|Tx_k\|^p + \delta^{p-q}\pi_q^q(T) \\ &\le \sum_1^n \|Tx_k\|^p + \frac{1}{2}(1-\varepsilon)\pi_{pq}^p(T).\end{aligned}$$

This means

$$(1-\varepsilon)^{1/p}\pi_{pq}(T) \le 2^{1/p}\pi_{pq}^n(T).$$

Letting ε go to zero we find an $n \in \mathbf{N}$ with

$$n \le \left(\frac{\pi_{pq}^p(T)}{2}\right)^{-q/(p-q)} = \left(2^{1/p}\frac{\pi_q(T)}{\pi_{pq}(T)}\right)^{1/(1/q-1/p)}. \qquad \square$$

Remark 1.2. Exactly the same argument shows that for every operator $T \in \mathscr{L}(X, Y)$ which is of (Rademacher) Cotype 2 one has

$$C_q(T) \le 2^{1/q} C_q^n(T),$$

where $n \in \mathbf{N}$ satisfies

$$n \le \left(2^{1/q}\frac{C_2(T)}{C_q(T)}\right)^{1/(1/q-1/2)}.$$

Proof of Theorem 3. Let E be a n-dimensional Banach space. According to Jameson's lemma we want to compare the cotype 2 norm with the cotype q norm via the gaussian cotype. It is well known [PS, Theorem 3.9.] that the Rademacher cotype 2 can be estimated by the gaussian cotype 2 norm in the following way.

$$C_2(\mathrm{Id}_E) \le c_0\tilde{C}_2(\mathrm{Id}_E)\sqrt{1+\ln\tilde{C}_2(\mathrm{Id}_E)}\,.$$

Using the inequalities $\tilde{C}_2(\mathrm{Id}_E) \le \sqrt{2}n^{1/2-1/q}\tilde{C}_q(\mathrm{Id}_E)$ and $\tilde{C}_q(\mathrm{Id}_E) \le n^{1/q}$, see [TOJ], we obtain

$$\begin{aligned} C_2(\mathrm{Id}_E) &\le c_0\tilde{C}_2(\mathrm{Id}_E)\sqrt{1+\ln \tilde{C}_2(\mathrm{Id}_E)} \\ &\le c_0 n^{1/2-1/q}\tilde{C}_q(\mathrm{Id}_E)\sqrt{2+\ln n}\,. \end{aligned}$$

Combining this estimate with Jameson's lemma, more precisely the remark above, we see that there is a constant $c_1 > 0$ such that

$$C_q(\mathrm{Id}_E) \le 2^{1/q} C_q^m(\mathrm{Id}_E),$$

with

$$m \le n\left(c_1\sqrt{1+\ln n}\right)^{1/(1-2/q)}. \qquad \square$$

In order to apply Jameson's lemma an appropriate estimate of the 1-summing norm by the $r1$-summing norm is needed.

Lemma 1.3. *Let $1 \le r \le \infty$, $n \in \mathbf{N}$ and $T \in \mathscr{L}(X,Y)$ an operator of rank n. Then we have*

$$\pi_1(T) \le c_0\pi_{r1}(T)\begin{cases} \left(\frac{1}{r}-\frac{1}{2}\right)^{-1/2} n^{1/2} & \text{for } 1 \le r < 2, \\ (n(1+\ln n))^{1/2} & \text{for } r = 2, \\ \left(\frac{1}{2}-\frac{1}{r}\right)^{-1/r'} n^{1/r'} & \text{for } 2 < r < \infty. \end{cases}$$

Proof. We may assume $T \in \mathscr{L}(l_\infty, F)$ with $\dim F = n$. The inequality $\pi_2(S: F \to l_\infty) \le \sqrt{n}\,\|S\|$ (see [TOJ]) together with Tomczak-Jagermann's inequality implies

$$\pi_1(T) \le \iota_1(T) \le \sqrt{n}\,\pi_2(T) \le \sqrt{2n}\,\pi_2^n(T).$$

For $2 < r < \infty$ we deduce from Maurey's theorem (see [TOJ]),

$$\pi_2^n(T) \le n^{1/2-1/r}\pi_{r2}^n(T) \le c_0\left(\frac{1}{2}-\frac{1}{r}\right)^{-1/r'} n^{1/2-1/r}\pi_{r1}(T).$$

For $r = 2$ we choose $2 < \bar{r} < \infty$ with

$$\frac{1}{2}-\frac{1}{\bar{r}} = \frac{1}{2+2\ln n}.$$

With $\pi_{\bar{r}1} \le \pi_{21}$ we obtain

$$\pi_2^n(T) \le 2e^2 c_0(1+\ln n)^{1/2}\pi_{21}^n(T).$$

If $1 < r < 2$ we use the other version of Maurey's theorem (again see [TOJ]) to deduce

$$\pi_2(T) \leq c_0\left(\frac{1}{r} - \frac{1}{2}\right)^{-1/2} \pi_{r1}(T).$$

Combining the last three estimates with the first one gives the assertion. □

Proof of Theorem 1. First we prove the theorem in the case $q = 1$, hence $p' = r$. From Jameson's Lemma 1.1 and Lemma 1.3 we deduce that for an operator T of rank n,

$$\pi_{p1}(T) \leq 2^{1/p}\pi_{p1}^{m}(T),$$

where

$$m \leq (2c_0)^r \begin{cases} \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} n & \text{for } 2 < p < \infty, \\ n(1 + \ln n) & \text{for } r = 2, \\ \left(\frac{1}{p} - \frac{1}{2}\right)^{-r/2} n^{r/2} & \text{for } 1 \leq p < 2. \end{cases}$$

An elementary computation shows that for all α, $c \geq 1$ one has

$$\pi_{p1}^{[c\alpha]}(T) \leq (4c)^{1/p}\pi_{p1}^{[\alpha]}(T).$$

Hence we get

$$\pi_{p1}(T) \leq (16c_0)^{r-1} \begin{cases} \left(\frac{1}{r} - \frac{1}{2}\right)^{-1/r'} \pi_{p1}^{n}(T) & \text{for } 1 \leq r < 2 \\ \pi_{21}^{[n(1+\ln n)]}(T) & \text{for } r = 2 \\ \left(\frac{1}{2} - \frac{1}{r}\right)^{-(r-1)/2} \pi_{p1}^{[n^{r/2}]}(T) & \text{for } 2 < r < \infty. \end{cases}$$

For an arbitrary $1 \leq q \leq p \leq \infty$ we define $\bar{p} = r'$. Since we have $1/p = 1/p + 1/q'$ we can deduce from Theorem 2 and the inequalities above that

$$\begin{aligned} \pi_{pq}(T) &= \sup\left\{\pi_{\bar{p}1}(TVD_\sigma) \mid V : l_{q'} \to X, D_\sigma : l_\infty \to l_{q'}, \|\sigma\|_{q'}, \|V\| \leq 1\right\} \\ &\leq c_r \sup\left\{\pi_{\bar{p}1}^{m(r,n)}(TVD_\sigma) \mid V : l_{q'} \to X, D_\sigma : l_\infty \to l_{q'}, \|\sigma\|_{q'}, \|V\| \leq 1\right\} \\ &= c_r \pi_{pq}^{m(r,n)}(T), \end{aligned}$$

where $m(r,n) = n$, $m(r,n) = [n(1 + \ln n)]$, $m(r,n) = [n^{r/2}]$ for $r < 2$, $r = 2$, $2 < r$, respectively.

Remark 1.4. The polynomial order of the vectors needed to compute the pq-summing norm can be improved for several choices of p and q, because they are close enough to the 2. Let $1 \leq q \leq p$, $r \leq \infty$ with $1/q = 1/p + 1/r$. Then for all operators T of rank n one has

$$\pi_{pq}(T) \leq (c_0)^{r/p} \begin{cases} \pi_{pq}^{[n^{1+r(1/q-1/2)}]}(T) & \text{for } 1 \leq q \leq 2 \text{ and } 2 \leq r \leq q', \\ \pi_{pq}^{[n^{r(1/2-1/p)}]}(T) & \text{for } 2 \leq q \leq \infty. \end{cases}$$

Proof. First case. We choose $2 \leq s \leq \infty$ such that $1/q - 1/p = 1/2 - 1/s$. By a result of Carl [CAR], together with Tomczak-Jaegermann's and Kwapien's inequality, we have

$$\begin{aligned} \pi_q(T) &\leq n^{1/q-1/2}\pi_2(T) \leq \sqrt{2}\, n^{1/q-1/2}\pi_2^n(T) \\ &\leq \sqrt{2}\, n^{1/q-1/2} n^{1/2-1/s}\pi_{s2}^n(T) \leq \sqrt{2}\, n^{2/q-1/2-1/p}\pi_{pq}^n(T). \end{aligned}$$

By Jameson's Lemma 1.1 and the elementary estimate in the proof above we obtain

$$\pi_{pq}(T) \leq 8^{1/p} 2^{r/p(1/p+1/2)} \pi_{pq}^{[n^{1+r(1/q-1/2)}]}.$$

Second case. From Kwapien's and Tomczak-Jaegermann's inequality we deduce that

$$\begin{aligned} \pi_q(T) &\leq \pi_2(T) \leq \sqrt{2}\,\pi_2^n(T) \\ &\leq \sqrt{2}\, n^{1/2-1/p}\pi_{p2}^n(T) \leq \sqrt{2}\, n^{1/2-1/p}\pi_{pq}(T). \end{aligned}$$

Again with Jameson's Lemma 1.1 this implies the assertion. □

We would like to note the following.

COROLLARY 1.5. *Let $1 \leq r \leq \infty$, K a compact Hausdorff space and $T \in \mathscr{L}(C(K), Y)$ of rank n. Then we have*

$$\pi_p(T) \leq c_p(1 + \ln n)^{1/p'} \begin{cases} \pi_p^n(T) & \textit{for } 2 < p < \infty, \\ \pi_p^{[n^{p'/2}]}(T) & \textit{for } 1 < p < 2. \end{cases}$$

Proof. Using a result of Carl and Defant, see [CAD], and Theorem 2 we deduce that

$$\begin{aligned}\pi_p(T) &\le c_0(1+\ln n)^{1/p'}\pi_{p1}(T)\\ &\le c_p(1+\ln n)^{1/p'}\pi_{p1}^{[n^{\max(1,p'/2)}]}(T) \le c_p(1+\ln n)^{1/p'}\pi_p^{[n^{\max(1,p'/2)}]}(T).\end{aligned}$$

□

2. Examples

By the positive results of the previous section a polynomial growth can only appear if $1/q - 1/p > 1/2$. Therefore for $1 \le q \le 2$ we define the critical value p_q by $1/p_q = 1/q - 1/2$. In the sequel limit orders of pq-summing operators are of particular interest. We intensively use the results of Carl, Maurey and Puhl (see [CMP]). The next lemma is implicitly contained there but we reproduce the easy interpolation argument.

LEMMA 2.1. *Let* $0 < \theta < 1$, $1 \le q, r \le 2$ *and* $q \le p$ *with*

$$\frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{2} \quad \textit{and} \quad \frac{1}{p} = \frac{1}{q} - \frac{\theta}{2}.$$

Then one has

$$\pi_{pq}\left(\iota : l_{q'}^n \to l_r^n\right) \le n^{1/p}.$$

Proof. Clearly (see [PIE]) one has

$$\pi_q\left(\iota : l_{q'}^n \to l_q^n\right) \le \pi_q\left(\iota : l_\infty^n \to l_q^n\right) \le n^{1/q}.$$

With Kwapien's inequality, $\pi_{qp_q} \le \pi_{21}$, we deduce from the Orlicz property of l_2 that

$$\pi_{qp_q}\left(\iota : l_{q'}^n \to l_2^n\right) \le \pi_{21}(\mathrm{Id}_{l_2^n})\|\iota : l_{q'}^n \to l_2^n\| \le n^{1/q-1/2}.$$

By interpolation, namely $[l_q(l_q), l_{p_q}(l_2)]_\theta = l_p(l_r)$ see [BEL], this means

$$\pi_{pq}\left(\iota : l_{q'}^n \to l_r^n\right) \le n^{(1-\theta)/q}n^{\theta(1/q-1/2)} = n^{1/p}.$$

□

Now we can construct the counterexamples

PROPOSITION 2.2. *Let* $0 < \theta < 1$, $1 \le q \le 2 \le s \le \infty$ *and* $q \le t$ *with*

$$\frac{1}{s} = \frac{1-\theta}{q'} + \frac{\theta}{2} \quad \textit{and} \quad \frac{1}{t} = \frac{1}{q} - \frac{\theta}{2}.$$

For all $n \in \mathbf{N}$ there exists an operator $T \in \mathscr{L}(l_{q'}, l_2^n)$ which satisfies

$$\frac{\pi_{pq}^k(T)}{\pi_{pq}(T)} \leq c_0\sqrt{s}\left(\frac{k}{n^{s/2}}\right)^{1/p-1/t}$$

for all $q \leq p \leq t$ and $k \in \mathbf{N}$.

Proof. Let $m = [n^{s/2}]$ and $A : l_{q'}^m \to l_2^n$ be a random matrix with entries ± 1, a so called Benett matrix. Obviously we have

$$\pi_{pq}(A) \geq m^{1/p}n^{1/2} \geq \tfrac{1}{2}n^{1/2+s/2p} \quad \text{for all } q \leq p.$$

We will see that this estimate is sharp for some indices p. By [CMP, Lemma 5] one has

$$\|A : l_{s'}^m \to l_2^n\| \leq c_0\sqrt{s}\,\max\{n^{1/2}, m^{1/s}\} \leq c_0\sqrt{sn}\,.$$

Since

$$\frac{1}{s'} = \frac{1-\theta}{q} + \frac{\theta}{2}$$

we deduce from Lemma 2.1 that

$$\begin{aligned}\pi_{tq}(A) &\leq \pi_{tq}(\iota : l_{q'}^m \to l_{s'}^m)\|A : l_{s'}^m \to l_2^n\| \\ &\leq c_0\sqrt{s}\,n^{1/2}m^{1/t} \leq c_0\sqrt{s}\,n^{1/2+s/2t}.\end{aligned}$$

Therefore, for arbitrary $q \leq p \leq t$, $k \in \mathbf{N}$, we obtain

$$\begin{aligned}\pi_{pq}^k(A) &\leq k^{1/p-1/t}\pi_{tq}^k(A) \\ &\leq c_0\sqrt{s}\,k^{1/p-1/t}n^{1/2+s/2t} \\ &\leq c_0\sqrt{s}\left(\frac{k}{n^{s/2}}\right)^{1/p-1/t}\pi_{pq}(A).\end{aligned}$$ □

COROLLARY 2.3. *For $1 \leq q < 2$ and $q \leq p < p_q$ let $2 < s \leq \infty$ be defined by*

$$\frac{1}{s} = \frac{1}{q'} + \left(1 - \frac{2}{q'}\right)\left(\frac{1}{q} - \frac{1}{p}\right).$$

For any sequence k_n, $c > 0$ *satisfying*

$$\pi_{pq}(T) \leq C\pi_{pq}^{k_n}(T) \quad \textit{for all } T \textit{ of rank } n$$

there is a constant c_1 *with*

$$n^{s/2} \leq c_1 e^{c_1\sqrt{1+\ln n}} k_n.$$

Proof. We define $\vartheta := 2(1/q - 1/p) < 1$. For $\varepsilon < 1 - \vartheta$ we set $\theta := \vartheta + \varepsilon$ and choose $2 \leq v \leq s$, $p \leq t \leq p_q$ with

$$\frac{1}{v} = \frac{1-\theta}{q'} + \frac{\theta}{2}, \quad \frac{1}{t} = \frac{1}{q} - \frac{\theta}{2}.$$

Now let us consider the quotient $d_n := n^{s/2}k_n^{-1}$. From Proposition 2.2, with an elementary computation, we deduce that

$$\begin{aligned}\frac{1}{C} &\leq c_0\sqrt{s}\left(\frac{k_n}{n^{v/2}}\right)^{1/p-1/t} \leq c_0\sqrt{s}\, d_n^{-\varepsilon/2} n^{(s-v)\varepsilon/4} \\ &\leq c_0\sqrt{s}\, d_n^{-\varepsilon/2} n^{\varepsilon^2 s^2(1/8 - 1/4q')}.\end{aligned}$$

Setting $\varepsilon = (1 - \vartheta)/2$ yields a constant c_2 such that

$$\ln d_n \leq c_2 + \left(s^2(1-\vartheta)\left(\frac{1}{8} - \frac{1}{4q'}\right)\right)\ln n.$$

Therefore there exists an $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ we can choose $\varepsilon_n := \ln d_n/[s^2(1/2 - 1/q')\ln n] < 1 - \vartheta$. An elementary computation gives

$$\frac{1}{C} \leq c_q \exp\left(-\frac{(\ln d_n)^2}{2s^2(1/2 - 1/q')(\ln n)}\right).$$

This is only possible if there exists a constant c_3 depending on s, q and C such that

$$\frac{n^{s/2}}{k_n} = d_n \leq \exp\left(c_3\sqrt{1 + \ln n}\right). \qquad \square$$

Remark 2.4. For $q = 1$ the above results can be slightly improved. For $1 \leq p < 2 \leq s < p' \leq \infty$ there exists an operator $T \in \mathscr{L}(l_\infty^{[n^{s/2}]}, l_2^n)$ such that for all $k \in \mathbf{N}$,

$$\pi_p^k(T) \leq c_0\sqrt{s}\left(\frac{k}{n^{s/2}}\right)^{1/s-1/p'} \pi_{p1}(T).$$

In particular, the inequality

$$\pi_{p1}(T) \le C(1 + \ln n)\pi_p^{k_n}(T) \quad \text{for all } T \text{ of rank } n$$

can only be satisfied, if

$$n^{p'/2} \le \bar{c}e^{\bar{c}\sqrt{1+\ln n}}k_n.$$

This answers a conjecture of Carl and Defant. They suggested

$$\pi_p(T) \le c_p(1 + \ln n)^{1/p'}\pi_{p1}^n(T)$$

for all operators $T \in \mathscr{L}(C(K), Y)$ of rank n, which turns out to be false. Furthermore, we recover the exponential order of vectors for π_1. More precisely, for all $n, k \in \mathbf{N}$ there is an operator $T \in \mathscr{L}(l_\infty^{[n^{1+\ln k}]}, l_2^n)$ with

$$\pi_1^k(T) \le c_0\sqrt{\frac{1 + \ln k}{n}}\,\pi_1(T).$$

Proof. Inspecting the proof of Proposition 2.2 we take a Benett matrix $A : l_\infty^{[n^{s/2}]} \to l_2^n$, whose $p1$-summing norm satisfies

$$\pi_{p1}(T) \ge \tfrac{1}{2}n^{1/2+s/2p}.$$

On the other hand

$$\begin{aligned}
\pi_p^k(A) &\le k^{1/p-1/s'}\pi_{s'}(A) \\
&\le k^{1/p-1/s'}\pi_{s'}\left(\iota : l_\infty^{[n^{s/2}]} \to l_{s'}^{[n^{s/2}]}\right)\|A : l_{s'}^{[n^{s/2}]} \to l_2^n\| \\
&\le c_0\sqrt{s}\,k^{1/p-1/s'}n^{s/2s'}n^{1/2} \\
&\le 2c_0\sqrt{s}\left(\frac{k}{n^{s/2}}\right)^{1/p-1/s'}\pi_{p1}(T).
\end{aligned}$$

The logarithmic factor does not affect the calculation in the proof of Corollary 2.3. For the last assertion we note that $p' = \infty$ and therefore the choice $s = 2(1 + \ln k)$ implies the assertion. □

Proof of Theorem 4. If $1 \le q < 2$ this follows immediately from Corollary 2.3. We only have to note that for all $\varepsilon > 0$ there is a constant C_ε with

$$e^{\bar{c}\sqrt{1+\ln n}} \le C_\varepsilon n^\varepsilon.$$

Now let $2 < q < \infty$. With the help of Benett matrices it was shown in [CMP] that for $2 < q < \infty$,

$$n^{q/2p} \leq c_0\sqrt{q}\,\pi_{pq}\big(\mathrm{id}_{l_2^n}\big).$$

Hence we get

$$\begin{aligned}\pi_{pq}^{k}\big(\mathrm{id}_{l_2^n}\big) &\leq k^{1/p}\|\mathrm{id}_{l_2^n}\| \leq k^{1/p}\\ &\leq c_0\sqrt{q}\left(\frac{k}{n^{q/2}}\right)^{1/p}\pi_{pq}\big(\mathrm{id}_{l_2^n}\big).\end{aligned}$$

Therefore every sequence k_n with $\pi_{pq}(T) \leq C\pi_{pq}^{k_n}(T)$ must satisfy

$$n^{q/2} \leq \big(Cc_0\sqrt{q}\big)^p k_n. \qquad \square$$

Remark 2.5. For operators defined on n-dimensional Banach spaces, the results of [JOS] and [DJ] imply that the pq-summing norm can be calculated with $n^{q/2}(1 + \ln n)$ many vectors. Therefore the order in the proof of the proposition above is quite correct.

REFERENCES

[BEL] J. BERGH AND J. LÖFSTRÖM, *Interpolation spaces*, Springer Verlag, New York, 1976.

[CAD] B. CARL AND A. DEFANT, *An inequality between the p- and $(p, 1)$-summing norm of finite rank operators from $C(K)$ spaces*, Israel J. Math. **74** (1991), 323–325.

[CAR] B. CARL, *Inequalities between absolutely (p, q)-summing norms*, Studia Math. **69** (1980), 143–148.

[CMP] B. CARL, B. MAUREY AND J. PUHL, *Grenzordnungen von absolut (r, p)-summierenden Operatoren*, Math. Nachr. **82** (1978), 205–218.

[DJ] M. DEFANT and M. JUNGE, *On absolutely summing operators with application to the (p, q)-summing norm with few vectors*, J. Funct. Anal. **103** (1992), 62–73.

[DJ2] ______, *Absolutely summing norms with n vectors*, preprint.

[JAM] G. J. O. JAMESON, *The number of elements required to determine $(p, 1)$-summing norms*, Illinois J. of Math **39** (1995), 251–257.

[JOS] W. B. JOHNSON and G. SCHECHTMANN, *Computing p-summing norms with few vectors*, preprint.

[MAU] B. MAUREY, *Type et cotype dans les espaces munis d'un structure localement inconditionelle*, Seminaire Maurey-Schwartz 73-74, Ecole Polytech, Exp. no. 24-25.

[PIE] A. PIETSCH, *Operator Ideals*, Deutscher Verlag Wiss., Berlin 1978 and North Holland, Amsterdam, 1980, Cambridge University Press, Cambridge, 1987.

[PS] G. PISIER, *Factorization of linear operators and geometry of Banach spaces*, CBMS Regional Conference Series $n°$ 60, Amer. Math. Soc., Providence, R.I., 1986.

[TOJ] N. TOMCZAK-JAEGERMANN, *Banach-Mazur distances and finite-dimensional operator ideals*, Longmann and Wiley, Avon, 1989.

UNIVERSITÄT KIEL
KIEL, GERMANY