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NULL HOLOMORPHICALLY FLAT INDEFINITE ALMOST HERMITIAN MANIFOLDS

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1. Introduction

In this paper, we shall study the holomorphic sectional curvatures on indefinite almost Hermitian manifolds, with attention to the behaviour of the Jacobi operator along spacelike, timelike and null geodesics.

The study of sectional curvatures on manifolds with indefinite metrics exhibits significant differences from the positive definite case. In fact, at each point m of a Riemannian manifold M the sectional curvature is a function defined on the Grassmann manifold $G_2(T_m M)$ of planes on the tangent space $T_m M$ at m, and hence bounded. For a semi-Riemannian manifold M, however, $G_2(T_m M)$ at each point m contains degenerate planes, on which the sectional curvature is not defined. That is, the sectional curvature is well-defined only on the noncompact submanifold $G_2^0(T_m M)$ of $G_2(T_m M)$, which consists of all nondegenerate planes in $T_m M$. Thus the sectional curvature is not necessarily bounded.

It is a significant observation by Kulkarni [7] that boundedness of the sectional curvature on a semi-Riemannian manifold implies the constancy of the sectional curvature. It is elementary to recognize that if the sectional curvature is to be a function with definite values over $G_2(T_m M)$, then the curvature tensor R must satisfy the condition

$$R(x, y, x, y) = 0 \tag{1.1}$$

for any degenerate plane $\pi = \{x, y\} \in G_2(T_m M)$, where $x, y \in T_m M$ span the plane π. Dajczer and Nomizu [4] showed that such a condition implies the constancy of the sectional curvature on all nondegenerate planes. See also a work of Thorpe [14] for a Lorentz manifold.

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In the study of an indefinite Kähler manifold M, Barros and Romero [1] showed that the holomorphic sectional curvature on M is constant c if and only if the curvature tensor has the form

$$
\begin{aligned}
R(x,y,z,w) = \frac{c}{4}\{&g(x,z)g(y,w) \\
&-g(x,w)g(y,z) - g(x,Jw)g(y,Jz) \\
&+g(x,Jz)g(y,Jw) - 2g(x,Jy)g(z,Jw)\},
\end{aligned}
\tag{1.2}
$$

for all $x, y, z, w \in T_mM$. It is possible to consider any *degenerate* holomorphic plane and any *degenerate* totally real plane for the curvature tensor R of the above expression. In fact, as an analogue of the necessary condition (1.1), we have three kinds of vanishing conditions for R as follows:

$$R(x, Jx, y, Jy) = 0 \tag{1.3}$$

$$R(x, y, x, y) = 0, \tag{1.4}$$

for all totally real degenerate planes $\pi = \{x, y\}$ ($J\pi \subset \pi^{\perp}$), and

$$R(u, Ju, u, Ju) = 0, \tag{1.5}$$

for all degenerate holomorphic planes $\pi = \{u, Ju\}$ ($J\pi \subset \pi$), where u is a null vector. Note that an indefinite Kähler manifold is called *null holomorphically flat* if the curvature tensor R satisfies the condition (1.5).

In the authors' previous paper [2], it is shown that for a general indefinite almost Hermitian manifold boundedness of the holomorphic sectional curvature leads to spaces of pointwise constant holomorphic sectional curvature, and that for an indefinite Kähler manifold either (1.3) or (1.4) implies the constancy of the holomorphic sectional curvature. Moreover, it is pointed out that the condition (1.5) does not imply the constancy of the holomorphic sectional curvature.

In light of these situations, we shall study in the present paper the problem of the holomorphic sectional curvatures on indefinite almost Hermitian manifolds from a rather general point of view, with a special attention to the behaviour of the curvature tensor, restricted to degenerate holomorphic planes.

The paper is organized as follows. In §2, after a brief review on sectional curvatures on a semi-Riemannian manifold, the definition of *null holomorphically flatness* is given for an indefinite almost Hermitian manifold. Our first main result (Theorem 3.1) is stated in §3 as a generalized version of criteria of Nomizu [10] and Tanno [13] for the pointwise constancy of the holomorphic sectional curvature. In this section, also introduced is a function c_u, which allows us to measure the deviation of null holomorphically flatness from a pointwise constant sectional curvature. There are interesting relations between the boundedness of the holomorphic sectional curvature and the sign of c_u, which will be studied in §4. In §5, some important relations among the function c_u, the Ricci tensor and the *-Ricci tensor are established. Such formulas are applied in §6 to obtain certain general formulas for the curvature tensors of null holomorphically flat manifolds. In the last section (§7), using such expressions of the curvature tensors, we shall state some decomposition theorems for null holomorphically flat indefinite Kähler manifolds.

2. Preliminaries

Let (M, g) be a semi-Riemannian manifold, with an indefinite metric g. Let ∇ denote the metric connection of g, and R its curvature tensor: $R(x, y) = \nabla_{[x,y]} - [\nabla_x, \nabla_y]$, for tangent vectors x, y on M. Put $R(x, y, z, w) = g(R(x, y)z, w)$.

A plane at each point m of M, an element of the Grassmann manifold $G_2(T_m M)$, is denoted by $\pi = \{x, y\}$ if it is spanned by two tangent vectors $x, y \in T_m M$. A plane $\pi = \{x, y\}$ is called *degenerate* if

$$g(x, x)g(y, y) - g(x, y)^2 = 0. \tag{2.1}$$

(Note that if g is a Riemannian metric, then $g(x, x)g(y, y) - g(x, y)^2 > 0$ for any linearly independent x, y.) For each nondegenerate plane $\pi = \{x, y\} \in G_2^0(T_m M)$, the sectional curvature for π is defined by

$$K(\pi) = K(\{x, y\}) = \frac{R(x, y, x, y)}{g(x, x)g(y, y) - g(x, y)^2}.$$

This is clearly not well defined for any degenerate plane $\pi = \{x, y\} \in G_2(T_m M) \setminus G_2^0(T_m M)$.

We now consider a possibility for a sectional curvature be extended for any degenerate plane. Since any degenerate plane π can be considered as a limit plane of a suitably chosen infinite sequence $\{\pi_i\}$ of nondegenerate planes, necessary for $K(\pi)$ to have a finite definite value is the following condition

$$R(x_i, y_i, x_i, y_i) \to 0 \quad (i \to \infty), \tag{2.2}$$

where x_i, y_i span π_i for each i. Therefore (1.1) is a necessary condition for the sectional curvature to be extended over degenerate planes.

We now restrict our attention to an indefinite almost Hermitian manifold (M, g, J). A plane π is called *holomorphic* if it remains invariant under the action of the almost complex structure, $(J\pi \subset \pi)$. This is equivalent to the existence of a vector z such that $\pi = \{z, Jz\}$. If the sectional curvature K is considered only on *holomorphic* planes, we shall write the sectional curvature by H instead of K, and call it the *holomorphic sectional curvature*. Since $g(z,z) = g(Jz,Jz)$ and $g(z,Jz) = 0$, the sectional curvature H for any non-degenerate holomorphic plane $\pi = \{z, Jz\}$ is given by

$$H(\pi) = H(\{z, Jz\}) = \frac{R(z,Jz,z,Jz)}{g(z,z)^2}. \tag{2.3}$$

Note that a holomorphic plane $\pi = \{z, Jz\}$ is *degenerate* iff z is a null vector. A manifold (M, g, J) is called *null holomorphically flat* if the curvature tensor R satisfies for all null vectors z,

$$R(z,Jz,z,Jz) = 0. \tag{2.4}$$

It is interesting to point out the existence of null holomorphically flat indefinite almost Hermitian manifolds which are not of constant holomorphic sectional curvature. Main examples we consider are as follows. The product manifold $M = M_1(c) \times M_2(-c)$ of two indefinite almost Hermitian manifolds of constant holomorphic sectional curvatures c and $-c$. Note that M is Kähler if and only if so are the both factors. Also, it is not difficult to check that previous (2.4) is invariant by conformal changes on the metric. This provides a family of non-Kähler examples; in particular those manifolds locally conformally equivalent to indefinite complex space forms. Finally we note that the tangent bundle TM of a (positive definite) Kähler manifold (M, g, J) endowed with the complete lift, g^C, of the metric g and the complete lift, J^C, of the complex structure J of M is an indefinite Kähler manifold. Moreover, it is shown in [3] that (TM, g^C, J^C) is null holomorphically flat if and only if (M, g, J) is a complex space form.

3. Constancy of the holomorphic sectional curvature

As a key observation of the present issue, we shall establish a theorem on the conditions for the holomorphic sectional curvature to be pointwise constant on an indefinite almost Hermitian manifold. This may be considered as a natural generalization of the expressions obtained by Tanno [13] for the almost Hermitian manifolds and by Barros and Romero [1] for the indefinite Kähler manifolds (see also [8], [9]).

We note that the criteria obtained in [9] and [13] are based on the study of the Jacobi operator, $R(-, Jx)Jx$. The result in the theorem below involves the study of the Jacobi operators $R(-, Jx)Jx$ and $R(-, x)x$. Also, note that the condition $R(x, Jx)Jx + JR(x, Jx)x \sim x$ is reduced to that in [9] and [13] due to curvature identities.

THEOREM 3.1. *Let (M, g, J) be an indefinite almost Hermitian manifold. The holomorphic sectional curvature H on M is pointwise constant if and only if one of the following holds*:

$$R(x, Jx)Jx + JR(x, Jx)x \sim x \quad \text{for all spacelike vectors } x; \tag{3.1}$$

$$R(x, Jx)Jx + JR(x, Jx)x \sim x \quad \text{for all timelike vectors } x; \tag{3.2}$$

$$R(u, Ju)Ju + JR(u, Ju)u = 0 \quad \text{for all null vectors } u, \tag{3.3}$$

where $\sim$ *means is proportional to.*

For the proof of Theorem 3.1, it is convenient to introduce two functions F and L, associated with the curvature tensor R. The functions F and L are defined by

$$F(x, y) = 2R(x, Jx, x, Jy) + 2R(x, Jx, y, Jx) \tag{3.4}$$

$$\begin{aligned} L(x, y) = {} & 2R(x, Jx, y, Jy) + 2R(x, Jy, y, Jx) \\ & + R(x, Jy, x, Jy) + R(y, Jx, y, Jx). \end{aligned} \tag{3.5}$$

In terms of such functions, we have an expression

$$\begin{aligned} & R(\lambda x + \mu y, J(\lambda x + \mu y), \lambda x + \mu y, J(\lambda x + \mu y)) \\ & \quad = \lambda^4 R(x, Jx, x, Jx) + \mu^4 R(y, Jy, y, Jy) \\ & \qquad + \lambda^3\mu F(x, y) + \lambda\mu^3 F(y, x) + \lambda^2\mu^2 L(x, y). \end{aligned} \tag{3.6}$$

There are two more useful formulas similar to the above:

$$\begin{aligned} & R(\lambda x + \mu y, J(\lambda x + \mu y), \lambda x + \mu y, J(\lambda x - \mu y)) \\ & \quad = \lambda^4 R(x, Jx, x, Jx) - \mu^4 R(y, Jy, y, Jy) \\ & \qquad + 2\lambda^3\mu R(x, Jx, y, Jx) - 2\lambda\mu^3 R(y, Jy, x, Jy) \\ & \qquad + \lambda^2\mu^2\{R(y, Jx, y, Jx) - R(x, Jy, x, Jy)\}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} & R(\lambda x + \mu y, J(\lambda x + \mu y), J(\lambda x + \mu y), \lambda x - \mu y) \\ & \quad = \lambda^4 R(x, Jx, Jx, x) - \mu^4 R(y, Jy, Jy, y) \\ & \qquad - 2\lambda^3\mu R(x, Jx, x, Jy) + 2\lambda\mu^3 R(y, Jy, y, Jx) \\ & \qquad + \lambda^2\mu^2\{R(y, Jx, y, Jx) - R(x, Jy, x, Jy)\}. \end{aligned} \tag{3.8}$$

Proof of Theorem 3.1. We shall show first that (3.1) is necessary for H to be pointwise constant. Let x be a spacelike unit vector, i.e., $g(x, x) = 1$.

Take $y \in \langle x \rangle^{\perp}$. Consider a non null vector $\lambda x + \mu y$ so that

$$g(\lambda x + \mu y, \lambda x + \mu y) = \lambda^2 + \epsilon_y \mu^2 \neq 0,$$

where $\epsilon_y = g(y, y)$. Then, $\lambda x + \mu y$ defines a nondegenerate holomorphic plane $\pi = \{\lambda x + \mu y, J(\lambda x + \mu y)\}$. If the holomorphic sectional curvature $H(\pi)$ is pointwise constant, say c, then from (2.3) and (3.6) we have

$$\begin{aligned} c\left(\lambda^2 + \epsilon_y \mu^2\right) = \lambda^4 R(x, Jx, x, Jx) + \mu^4 R(y, Jy, y, Jy) \\ + \lambda^3\mu F(x, y) + \lambda\mu^3 F(y, x) + \lambda^2\mu^2 L(x, y). \end{aligned}$$

Comparing terms in both sides, we get

$$\lambda^3\mu F(x, y) + \lambda^2\mu^2\left(L(x, y) - 2c\epsilon_y\right) + \lambda\mu^3 F(y, x) = 0,$$

and hence $F(x, y) = 0$. From (3.4),

$$g(R(x, Jx)Jx + JR(x, Jx)x, y) = 0.$$

This implies that (3.1) is necessary.

Conversely, suppose that (3.1) holds. Take a unit timelike vector y, which is orthogonal to x, i.e., $y \in \langle x \rangle^{\perp}$, and $g(y, y) = -1$. Consider two mutually orthogonal vectors $\lambda x + \mu y$ and $\mu x + \lambda y$, with

$$g(\lambda x + \mu y, \lambda x + \mu y) = \lambda^2 - \mu^2, \qquad g(\mu x + \lambda y, \mu x + \lambda y) = \mu^2 - \lambda^2.$$

In the case $\lambda^2 - \mu^2 > 0$, replacing x in (3.1) by $\lambda x + \mu y$, we see that

$$\begin{aligned} R(\lambda x + \mu y, \lambda Jx + \mu Jy)(\lambda Jx + \mu Jy) \\ + JR(\lambda x + \mu y, \lambda Jx + \mu Jy)(\lambda x + \mu y) \sim \lambda x + \mu y, \end{aligned}$$

and, hence that it is orthogonal to $\mu x + \lambda y$. Then, we get

$$\begin{aligned} R(\lambda x + \mu y, \lambda Jx + \mu Jy, \lambda Jx + \mu Jy, \mu x + \lambda y) \\ - R(\lambda x + \mu y, \lambda Jx + \mu Jy, \lambda x + \mu y, \mu Jx + \lambda Jy) = 0. \end{aligned}$$

Linearizing previous expression, and comparing the terms with the coefficients $\lambda^3\mu$ and $\lambda\mu^3$, we see that

$$R(x, Jx, x, Jx) = R(y, Jy, y, Jy).$$

Since $\dim\langle x \rangle^{\perp} = \dim M - 1$, any timelike holomorphic plane π intersects $\langle x \rangle^{\perp}$, and hence, the holomorphic sectional curvature of π is obtained by

$H(\pi) = R(y, Jy, y, Jy)$, for any unit vector $y \in \pi \cap \langle x \rangle^{\perp}$. This shows that if (3.1) holds, then H is pointwise constant on timelike holomorphic planes. Analogously, it is shown that the constancy of the holomorphic sectional curvature on spacelike holomorphic planes is obtained.

Finally, we turn our attention to (3.3). If the holomorphic sectional curvature is pointwise constant, say c, then

$$R(x, Jx)Jx + JR(x, Jx)x = cg(x, x)x \tag{3.9}$$

for any non null vector $x \in T_m M$. Since each null vector u can be approximated by a suitable sequence $\{x_n\}$ of non null vectors, taking limits in previous expression when $n \to \infty$, it follows that (3, 3) is a necessary condition for the pointwise constancy of the holomorphic sectional curvature. In order to prove the sufficiency, let us consider orthogonal unit vectors x, y, with $g(x, x) = -g(y, y) = 1$. Then $x + y$ and $x - y$ are null vectors, and hence

$$\begin{aligned} 0 &= R(x+y, J(x+y))J(x+y) + JR(x+y, J(x+y))(x+y) \\ &= R(x-y, J(x-y))J(x-y) + JR(x-y, J(x-y))(x-y). \end{aligned}$$

Now, applying (3.7) and (3.8) to the identity

$$\begin{aligned} 0 = &R(x+y, J(x+y), J(x+y), x-y) \\ &- R(x+y, J(x+y), x+y, J(x-y)) \\ &+ R(x-y, J(x-y), J(x-y), x+y) \\ &- R(x-y, J(x-y), x-y, J(x+y)), \end{aligned}$$

it follows that $R(x, Jx, x, Jx) = R(y, Jy, y, Jy)$, and hence the pointwise constancy of the holomorphic sectional curvature as in above. □

At this point, we note that condition (3.1) is also valid for positive definite almost Hermitian manifolds (just with minor changes in the proof above), and it may be viewed as a natural generalization of the criteria of Nomizu [9] and Tanno [13] for the constancy of the holomorphic sectional curvature.

Note the different behaviour of the operator $\mathfrak{R}_x = R(-, Jx)Jx + JR(x, J-)x$ on spacelike or timelike vectors and null vectors. Equivalent condition for the constancy of the holomorphic sectional curvature is that x to be an eigenvector of $\mathfrak{R}_x$ for all spacelike or timelike vectors. However condition on null vectors requires the associated eigenvalue to be zero. This motivates the study of those spaces with nonvanishing eigenvalue $\mathfrak{R}_u(u) = c(u)u$ as a natural generalization of that of constant holomorphic sectional curvature. The next theorem shows the relation of this property with null holomorphically flatness

THEOREM 3.2. *An indefinite almost Hermitian manifold (M, g, J) is null holomorphically flat if and only if it satisfies*

$$R(u, Ju)Ju + JR(u, Ju)u = c(u)u \tag{3.10}$$

for all null vectors u.

Proof. It is clear that (3.10) leads to (2.4), and hence to null holomorphically flatness.

Conversely, if M is null holomorphically flat, then $R(u, Ju)Ju + JR(u, Ju)u \in \langle u\rangle^{\perp}$. We will show that $g(R(u, Ju)Ju + JR(u, Ju)u, v) = 0$ for all null vector $v \in \langle u\rangle^{\perp}$, and so, it must lie in the direction of u, which proves the result.

For an arbitrary null vector $v \in \langle u\rangle^{\perp}$, $u + \lambda v$ remains null for all $\lambda \in \mathbf{R}$, and hence (2.4) together with (3.6) leads to

$$\begin{aligned} 0 = R(u, Ju, u, Ju) &+ \lambda^4 R(v, Jv, v, Jv) \\ &+ \lambda^3 F(v, u) + \lambda^2 L(u, v) + \lambda F(u, v). \end{aligned}$$

Now, if previous polynomial vanishes, so must be each coefficient, and so we get $F(u, v) = 0$. Hence, the desired result follows from (3.4). □

4. Bounds on the holomorphic sectional curvature

Previous Theorem 3.2 shows that the functions $c(u)$ measure the failure of a null holomorphically flat indefinite almost Hermitian manifold to have pointwise constant holomorphic sectional curvature. In this section we investigate the influence of the sign of $c(u)$ on the values of the holomorphic sectional curvature.

The next identities follow from the definition (2.4) of null holomorphically flatness.

LEMMA 4.1. *Let (M, g, J) be a null holomorphically flat indefinite almost Hermitian manifold. Then*

$$R(x, Jx, x, Jx) + R(y, Jy, y, Jy) = -L(x, y), \tag{4.1}$$

$$F(x, y) + F(y, x) = 0, \tag{4.2}$$

for all orthogonal vectors x, y with $g(x, x) = -g(y, y)$.

Proof. Since x, y are orthogonal vectors with $g(x, x) = -g(y, y)$, $x + y$ and $x - y$ are null vectors. Hence the result follows from (2.4) using the identity (3.6). □

Remark 4.1. For each null vector u, there exist spacelike and timelike orthogonal unit vectors x and y with $g(x, Jy) = 0$, such that $u = \lambda(x + y)$. Note that there exist infinitely many such vectors since the metric g is not Lorentzian. Also note that v defined by $v = (1/2\lambda)(x - y)$ is a null vector, and further $g(u, v) = 1$. We will refer to such v as an *associated* null vector to u.

LEMMA 4.2. *Let (M, g, J) be a null holomorphically flat indefinite almost Hermitian manifold.*

(1) *If the holomorphic sectional curvature H is bounded from below (resp. above) on spacelike holomorphic planes, then $c(u) \leq 0$ (resp. $c(u) \geq 0$).*
(2) *If the holomorphic sectional curvature H is bounded from above (resp. below) on timelike holomorphic planes, then $c(u) \leq 0$ (resp. $c(u) \geq 0$).*

Proof. Let u be a null vector, and consider x, y orthogonal unit vectors with $g(x, x) = 1 = -g(y, y)$, $u = \lambda(x + y)$ for some $\lambda \in \mathbf{R}$. Let $v = (1/2\lambda)(x - y)$ be an associated null vector, so that $g(u, v) = 1$. Since M is null holomorphically flat, due to (3.10) in Theorem 3.2, it follows that

$$\begin{aligned} c(u) &= R(u, Ju, Ju, v) - R(u, Ju, u, Jv) \\ &= \frac{\lambda^2}{2} R(x + y, J(x + y), J(x + y), x - y) \\ &\quad - \frac{\lambda^2}{2} R(x + y, J(x + y), x + y, J(x - y)) \end{aligned}$$

and, from (3.7) and (3.8) with $\lambda = \mu = 1$, we obtain:

$$\frac{1}{\lambda^2} c(u) = R(y, Jy, y, Jy) - R(x, Jx, x, Jx) - F(x, y). \tag{4.3}$$

Next, suppose that H is bounded from below on holomorphic spacelike planes. For all orthonormal vectors x, y as above, $rx + sy$ with $r^2 - s^2 = 1$ is a unit spacelike vector, and hence $k \leq H(rx + sy)$. Using the identity (3.6), it follows that

$$\begin{aligned} k \leq\; & r^4 R(x, Jx, x, Jx) + s^4 R(y, Jy, y, Jy) \\ & + r^2 s^2 L(x, y) + r^3 s F(x, y) + r s^3 F(y, x), \end{aligned}$$

and hence

$$\begin{aligned} k \leq\; & r^2\{R(x, Jx, x, Jx) + s^2 R(x, Jx, x, Jx) + s^2 L(x, y)\} + s^4 R(y, Jy, y, Jy) \\ & + r^3 s F(x, y) + r s^3 F(y, x). \end{aligned}$$

From (4.1) and (4.2) in previous lemma, it follows that

$$\begin{aligned}
k &\leq r^2\big(R(x,Jx,x,Jx) - s^2R(y,Jy,y,Jy)\big)\\
&\quad + s^4R(y,Jy,y,Jy) + (r^3s - rs^3)F(x,y)\\
&= (1+s^2)R(x,Jx,x,Jx) - s^2(1+s^2)R(y,Jy,y,Jy)\\
&\quad + s^4R(y,Jy,y,Jy) + (r^3s - rs^3)F(x,y)\\
&= R(x,Jx,x,Jx) + s^2R(x,Jx,x,Jx) - s^2R(y,Jy,y,Jy) + rsF(x,y).
\end{aligned}$$

Dividing both sides of previous expression by s^2, and taking limits when $s \to \infty$, we obtain

$$0 \leq R(x,Jx,x,Jx) - R(y,Jy,y,Jy) + F(x,y).$$

Hence, it follows from (4.3) that $c(u) \leq 0$.

The result for the holomorphic sectional curvature bounded from above on spacelike planes is obtained in an analogous way. Also, the proof of the second part of the lemma is similar. □

The following theorem shows the mutual relations among the different bounds on the holomorphic sectional curvature and the sign of $c(u)$.

THEOREM 4.1. *Let (M, g, J) be an indefinite almost Hermitian manifold. Then*:

(1) *M is null holomorphically flat and $c(u) \leq 0$ if and only if the holomorphic sectional curvature H is bounded from below on spacelike planes and from above on timelike planes.*
(2) *M is null holomorphically flat and $c(u) \geq 0$ if and only if the holomorphic sectional curvature H is bounded from above on spacelike planes and from below on timelike planes.*

Proof. We will prove the first part of the theorem, and the second one is analogous. Let us show the sufficiency:

Let u be an arbitrary null vector, and consider orthonormal vectors x, y with $g(x,x) = 1 = -g(y,y)$, such that $u = \lambda(x+y)$ for some $\lambda \in \mathbf{R}$. Then $\{rx+y, J(rx+y)\}$ spans a timelike holomorphic plane for $|r| < 1$, and it spans a spacelike holomorphic plane for $|r| > 1$. If the holomorphic sectional curvature is bounded from below on spacelike planes and bounded from above on timelike planes, it follows that

$$\begin{aligned}
k_1 &\leq H(rx+y), \quad |r| > 1\\
k_2 &\geq H(rx+y), \quad |r| < 1,
\end{aligned}$$

and hence

$$(r^2-1)^2 k_1 \le R(rx+y, J(rx+y), rx+y, J(rx+y)), \quad |r|>1$$
$$(r^2-1)^2 k_2 \ge R(rx+y, J(rx+y), rx+y, J(rx+y)), \quad |r|<1.$$

Taking limits in previous expressions when $r \to 1$, it follows that $R(u, Ju, u, Ju) \ge 0$ and also that $R(u, Ju, u, Ju) \le 0$, which shows that M is null holomorphically flat. Moreover, that $c(u) \le 0$ follows from previous lemma.

In order to prove the necessity, we proceed as follows. For each orthonormal vectors x, y satisfying $g(x, x) = 1 = -g(y, y)$, consider the null vector $u = x + y$, and its associate $v = \frac{1}{2}(x - y)$. Then, we have

$$\begin{aligned} c(u) + 4c(v) &= R(u, Ju, Ju, v) - R(u, Ju, u, Jv) \\ &\quad + 4R(v, Jv, Jv, u) - 4R(v, Jv, v, Ju) \\ &= \tfrac{1}{2}R(x+y, J(x+y), J(x+y), x-y) \\ &\quad -\tfrac{1}{2}R(x+y, J(x+y), x+y, J(x-y)) \\ &\quad +\tfrac{1}{2}R(x-y, J(x-y), J(x-y), x+y) \\ &\quad -\tfrac{1}{2}R(x-y, J(x-y), x-y, J(x+y)). \end{aligned}$$

It follows now from the identities (3.7) and (3.8) that

$$c(u) + 4c(v) = 2R(x, Jx, Jx, x) - 2R(y, Jy, Jy, y). \tag{4.4}$$

Now, if $c(w) \le 0$ for each null vector w, equation (4.4) shows that

$$R(x, Jx, x, Jx) \ge R(y, Jy, y, Jy), \tag{4.5}$$

and hence $H(x) \ge H(y)$, for all $\{x, Jx\}$, $\{y, Jy\}$ holomorphic planes of signature $(+,+)$ and $(-,-)$ respectively, with $g(x, y) = 0$.

Let us show now that the holomorphic sectional curvature is bounded from below on holomorphic spacelike planes. Let z be an arbitrary timelike unit vector and consider the subspace $\langle z\rangle^{\perp}$. Since any holomorphic plane intersects $\langle z\rangle^{\perp}$, for any spacelike holomorphic plane π, its holomorphic sectional curvature is given by $H(\pi) = R(x, Jx, x, Jx)$, where x is an unit vector in $\pi \cap \langle z\rangle^{\perp}$. Hence equation (4.5) above shows that $H(\pi) \ge H(z)$, which proves that the holomorphic sectional curvature is bounded from below on spacelike planes. Proceeding in the same way, it is shown that H is bounded from above on timelike planes. □

In [1], Barros and Romero show that the holomorphic sectional curvature of an indefinite Kähler manifold is bounded from above and from below if and only if it is constant. Moreover they exhibit an example showing that boundedness from above (and equivalently from below) does not ensure constant holomorphic sectional curvature (cf. a result of Kulkarni [7] for the sectional curvatures). Küpeli [8] shows that such condition may be relaxed to bounds from above and from below only on spacelike (or timelike) planes. In the author's earlier paper [2], it is shown that such condition holds for arbitrary indefinite almost Hermitian manifolds.

The next theorem shows the possibility of relaxing those conditions for a null holomorphically flat indefinite almost Hermitian manifold.

THEOREM 4.2. *Let (M, g, J) be a null holomorphically flat indefinite almost Hermitian manifold. Then the following are equivalent:*

(1) *The holomorphic sectional curvature is pointwise constant.*
(2) *The holomorphic sectional curvature is bounded from above and from below on spacelike (or timelike) holomorphic planes.*
(3) *The holomorphic sectional curvature is bounded from above or from below.*

Proof. Condition (3) implies that $c_u = 0$ by Theorem 4.1, and hence that the curvature tensor R satisfies (3.3) in Theorem 3.1. Therefore H is pointwise constant. □

5. Ricci tensors and holomorphic sectional curvatures

In this section we investigate the influence of null holomorphically flatness on the Ricci tensors of an almost Hermitian manifold. The results in the present section will be applied in §6 to null holomorphically flat indefinite almost Hermitian manifolds in order to give the curvature tensors explicitly.

Let us recall the definitions of the Ricci, ρ, and *-Ricci tensors, ρ^*, of an (indefinite) almost Hermitian manifold

$$\rho(x, y) = \operatorname{trace}\{z \to R(x, z)y\} \tag{5.1}$$

$$\rho^*(x, y) = \tfrac{1}{2}\operatorname{trace}\{z \to -JR(x, Jy)z\} \tag{5.2}$$

Note that tensors ρ and ρ^* coincide with each other if the manifold is Kähler. Moreover the Ricci tensor ρ is symmetric, but ρ^* is, generally neither symmetric nor antisymmetric. However, it satisfies that $\rho^*(Jx, Jy) = \rho^*(y, x)$.

Next, we will compute the values of both tensors for null vectors. For, let $u \in T_m M$ denote a null vector. According to Remark 4.1, it is possible to choose orthonormal vectors x, y with $g(x, Jy) = 0$, such that $u = \lambda(x + y)$ for some real λ. Further, assume that x to be spacelike, and y to be timelike. At each point m of M, in addition to x and y we can choose $n - 2$

orthonormal vectors $z_1, \dots, z_{n-2}$ so that

$$\{x, y, Jx, Jy, z_1, \dots, z_{n-2}, Jz_1, \dots, Jz_{n-2}\}$$

is an orthonormal basis for $T_m M$; i.e.,

$$T_m M = \langle\{x, y, Jx, Jy\}\rangle \oplus \langle\{z_1, \dots, z_{n-2}, Jz_1, \dots, Jz_{n-2}\}\rangle \tag{5.3}$$

With respect to the basis above, from (5.1) and (5.2), we have

$$\begin{aligned}
&\rho(u,u) + \rho(Ju,Ju) \\
&\quad = R(u,x,u,x) - R(u,y,u,y) + R(Ju,x,Ju,x) \\
&\qquad - R(Ju,y,Ju,y) + R(u,Jx,u,Jx) - R(u,Jy,u,Jy) \\
&\qquad + R(Ju,Jx,Ju,Jx) - R(Ju,Jy,Ju,Jy) \\
&\qquad + \sum_{i=1}^{n-2} \epsilon_{z_i}\{R(u,z_i,u,z_i) + R(u,Jz_i,u,Jz_i) \\
&\qquad\qquad + R(Ju,z_i,Ju,z_i) + R(Ju,Jz_i,Ju,Jz_i)\}.
\end{aligned}$$

Since $u = \lambda(x+y)$, using the identities of the curvature tensor, together with (3.7) and (3.8), it follows that

$$R(u,x,u,x) - R(u,y,u,y) = R(Ju,Jx,Ju,Jx) - R(Ju,Jy,Ju,Jy) = 0,$$

and

$$\begin{aligned}
&R(u,Jx,u,Jx) - R(u,Jy,u,Jy) + R(Ju,x,Ju,x) \\
&\qquad - R(Ju,y,Ju,y) = -2c(u).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\rho(u,u) + \rho^*(Ju,Ju) \\
&\quad = -2c(u) + \sum_{i=1}^{n-2} \epsilon_{z_i}\{R(u,z_i,u,z_i) + R(u,Jz_i,u,Jz_i) \\
&\qquad\qquad + R(Ju,z_i,Ju,z_i) + R(Ju,Jz_i,Ju,Jz_i)\}.
\end{aligned} \tag{5.4}$$

Proceeding in the same way, for the *-Ricci tensor, we have

$$\rho^*(u,u) = R(u,Ju,x,Jx) - R(u,Ju,y,Jy) + \sum_{i=1}^{n-2} \epsilon_{z_i} R(u,Ju,z_i,Jz_i).$$

Using the fact that $u = \lambda(x+y)$, one gets

$$R(u,Ju,x,Jx) - R(u,Ju,y,Jy) = -c(u).$$

Hence

$$\rho^*(u,u) = -c(u) + \sum_{i=1}^{n-2} \epsilon_{z_i} R(u,Ju,z_i,Jz_i). \tag{5.5}$$

To establish a formula relating the Ricci tensors and the function $c(u)$, we need the following:

LEMMA 5.1. *Let (M,g,J) be a null holomorphically flat indefinite almost Hermitian manifold, and let z be a unit vector. Then, for each null vector $u \in \langle z, Jz\rangle^\perp$, we have*

$$\begin{aligned} 2\epsilon_z c(u) = &-R(u,z,u,z) - R(u,Jz,u,Jz) \\ &- R(Ju,z,Ju,z) - R(Ju,Jz,Ju,Jz) - 6R(u,Ju,z,Jz). \end{aligned}$$

Proof. Let v be a null vector, $v \in \langle z, Jz\rangle^\perp$ with $g(u,v) = -1/2$ and, for each $a \in \mathbf{R}$, put

$$w_a = \frac{1}{\sqrt{a}}(u + a\epsilon_z v).$$

We have

$$g(w_a,w_a) = \frac{1}{a} g(u + a\epsilon_z v, u + a\epsilon_z v) = -\epsilon_z,$$

which shows that z, w_a are orthogonal unit vectors, and hence, from Lemma 4.1 and (3.5), we have

$$\begin{aligned} L(z,w_a) &= -R(z,Jz,z,Jz) - R(w_a,Jw_a,w_a,Jw_a) \\ &= 2R(z,Jz,w_a,Jw_a) + 2R(z,Jw_a,w_a,Jz) \\ &\quad + R(z,Jw_a,z,Jw_a) + R(w_a,Jz,w_a,Jz). \end{aligned}$$

Linearizing previous expression, and multiplying both sides by a, we get

$$
\begin{aligned}
&2R(z, Jz, u + a\epsilon_z v, Ju + a\epsilon_z Jv) + 2R(z, Ju + a\epsilon_z Jv, u + a\epsilon_z v, Jz) \\
&\quad + R(z, Ju + a\epsilon_z Jv, z, Ju + a\epsilon_z Jv) + R(u + a\epsilon_z v, Jz, u + a\epsilon_z v, Jz) \\
&\quad = -aR(z, Jz, z, Jz) - \frac{1}{a}R(u, Ju, u, Ju) - a^3\epsilon_z^4 R(v, Jv, v, Jv) \\
&\quad - \epsilon_z F(u, v) - a^2\epsilon_z^3 F(v, u) - a\epsilon_z^2 L(u, v).
\end{aligned}
$$

Taking the limit when $a \to 0$ in the above expression, we have

$$
\begin{aligned}
(5.6) \qquad \epsilon_z F(u, v) = &-2R(u, Ju, z, Jz) - 2R(u, Jz, z, Ju) \\
&- R(u, Jz, u, Jz) - R(Ju, z, Ju, z).
\end{aligned}
$$

If we put Jz instead of z in (5.6), it follows that

$$
\begin{aligned}
(5.7) \qquad \epsilon_z F(u, v) = &\; 2R(u, Ju, Jz, z) + 2R(u, z, Jz, Ju) \\
&- R(u, z, u, z) - R(Ju, Jz, Ju, Jz),
\end{aligned}
$$

and the desired result follows from (5.6) and (5.7) using the first Bianchi identity, together with the expression of $F(u, v)$ in (3.4). □

THEOREM 5.1. *Let (M^{2n}, g, J) be a null holomorphically flat indefinite almost Hermitian manifold. Then, for each null vector u on M,*

$$
(5.8) \qquad c(u) = \frac{-1}{(2n + 4)}\{\rho(u, u) + \rho(Ju, Ju) + 6\rho^*(u, u)\}.
$$

Proof. Considering the result of Lemma 5.1, (5.8) follows directly from (5.4) and (5.5). □

The result of previous theorem will play a fundamental role in the determination of the curvature tensor of a null holomorphically flat indefinite almost Hermitian manifold in the next section. Also note that it provides the following criteria for a null holomorphically flat manifold to be of constant holomorphic sectional curvature

COROLLARY 5.1. *Let (M, g, J) be a null holomorphically flat indefinite almost Hermitian manifold. If M is Einstein and *-Einstein, then it is a space of pointwise constant holomorphic sectional curvature.*

Proof. If M is Einstein and *-Einstein, both Ricci tensors ρ and ρ^* are multiples of the metric, and then $\rho(u, u) = 0$, $\rho(Ju, Ju) = 0$, $\rho^*(u, u) = 0$.

From (5.8), it follows that $c(u) = 0$ for each null vector u, and hence (3.3) in Theorem 3.1 shows that the holomorphic sectional curvature of M is pointwise constant. □

It is a well-known result that the Ricci and *-Ricci tensors of an almost Hermitian manifold of pointwise constant holomorphic sectional curvature are completely determined by the value of the holomorphic sectional curvature [12]. Next, we will show that such result also holds for null holomorphically flat indefinite almost Hermitian manifolds.

LEMMA 5.2. *Let M be a null holomorphically flat indefinite almost Hermitian manifold. If* $\{x, y\}$ *are vectors with* $g(x, x) = 1 = -g(y, y)$, $g(y, x) = g(x, Jy) = 0$, *then*

$$\begin{aligned} 2R(x, Jx, x, Jx) &+ 2R(y, Jy, y, Jy) \\ &= -R(x, y, x, y) - R(x, Jy, x, Jy) \\ &\quad - R(Jx, y, Jx, y) - R(Jx, Jy, Jx, Jy) - 6R(x, Jx, y, Jy). \end{aligned}$$

Proof. Since x, y are orthonormal vectors with $g(x, x) = 1 = -g(y, y)$ and $g(x, Jy) = 0$, then $x \pm y$ and $x \pm Jy$ are null vectors, and the result follows from Lemma 4.1. □

LEMMA 5.3. *Let M be a null holomorphically flat indefinite almost Hermitian manifold. If* $\{x, z\}$ *are vectors with* $g(x, x) = 1 = g(z, z)$, $g(x, z) = g(x, Jz) = 0$, *then*

$$\begin{aligned} 2R(x, Jx, x, Jx) &+ 2R(z, Jz, z, Jz) \\ &= R(x, z, x, z) + R(x, Jz, x, Jz) \\ &\quad + R(Jx, z, Jx, z) + R(Jx, Jz, Jx, Jz) + 6R(x, Jx, z, Jz) \end{aligned}$$

Proof. Let $\{y, Jy\}$ be a timelike holomorphic plane orthogonal to both $\{x, Jx\}$ and $\{z, Jz\}$, and consider the vector

$$\tilde{\omega}_t = \frac{\omega_t}{\sqrt{1 - t^2}},$$

where $\omega_t = tz + y$. It is a timelike unit vector orthogonal to $\{x, Jx\}$, for all $t \in (-1, 1)$ and, according to the result of Lemma 5.2,

$$\begin{aligned} 2R(x, Jx, x, Jx) &+ 2R\big(\tilde{\omega}_t, J\tilde{\omega}_t, \tilde{\omega}_t, J\tilde{\omega}_t\big) \\ &= -R\big(x, \tilde{\omega}_t, x, \tilde{\omega}_t\big) - R\big(x, J\tilde{\omega}_t, x, J\tilde{\omega}_t\big) \\ &\quad - R\big(Jx, \tilde{\omega}_t, Jx, \tilde{\omega}_t\big) - R\big(Jx, J\tilde{\omega}_t, Jx, J\tilde{\omega}_t\big) - 6R\big(x, Jx, \tilde{\omega}_t, J\tilde{\omega}_t\big). \end{aligned}$$

This gives

$$\begin{aligned}2(1-t^2)^2R(x,Jx,x,Jx)&+2R(tz+y,tJz+Jy,tz+y,tJz+Jy)\\ =(t^2-1)\{&R(x,tz+y,x,tz+y)+R(x,tJz+Jy,x,tJz+Jy)\\ &+R(Jx,tz+y,Jx,tz+y)+R(Jx,tJz+Jy,Jx,tJz+Jy)\\ &+6R(x,Jx,tz+y,tJz+Jy)\}.\end{aligned}$$

Finally, the desired result is obtained from the coefficient of t^4 in previous expression. □

As a consequence of Lemmas 5.2 and 5.3, we obtain the following expression for the Ricci tensors of a null holomorphically flat indefinite almost Hermitian manifold.

PROPOSITION 5.1. *Let (M^{2n},g,J) be a null holomorphically flat indefinite almost Hermitian manifold. For each unit vector z,*

$$\rho(z,z)+\rho(Jz,Jz)+6\rho^*(z,z) = 2\epsilon_z\left((n+3)H(\{z,Jz\})+\sum_{i=2}^{n}H(\pi_i)\right) \tag{5.9}$$

where $\{\pi_2,\dots,\pi_n\}$ are orthogonal nondegenerate holomorphic planes in $\langle z\rangle^{\perp}$.

Proof. Let us suppose z is spacelike and consider an orthonormal basis of $\langle z,Jz\rangle^{\perp}$,

$$\{x_2,\dots,x_p,Jx_2,\dots,Jx_p,y_1,\dots,y_q,Jy_1,\dots,Jy_q\},$$

in such a way that $g(x_i,x_i)=1=-g(y_j,y_j)$ where M is of metric signature $(2p,2q)$ $(p+q=n)$. Then, using the results of Lemmas 5.2 and 5.3, it follows that

$$\begin{aligned}\rho(z,z)&+\rho(Jz,Jz)+6\rho^*(z,z)\\ &=8R(z,Jz,z,Jz)+2\sum_{i=2}^{p}\{R(z,Jz,z,Jz)+R(x_i,Jx_i,x_i,Jx_i)\}\\ &\quad+2\sum_{i=1}^{q}\{R(z,Jz,z,Jz)+R(y_i,Jy_i,y_i,Jy_i)\}\\ &=2(n+3)H(\{z,Jz\})+2\sum_{i=2}^{n}H(\pi_i).\end{aligned}$$

Proceeding in an analogous way, we obtain a similar formula for timelike unit vectors. □

Associated with both Ricci tensors, the scalar curvature, τ, and the *-scalar curvature, τ^*, are defined to be the traces of ρ and ρ^* respectively. As a consequence of previous proposition, the scalar curvatures of a null holomorphically flat indefinite almost Hermitian manifold are determined by the holomorphic sectional curvature as follows.

THEOREM 5.2. *The scalar curvatures τ and τ^* of a null holomorphically flat indefinite almost Hermitian manifold (M^{2n}, g, J) satisfy*

$$\tau + 3\tau^* = 4(n+1)\sum_{i=1}^{n} H(\pi_i), \tag{5.10}$$

where $\{\pi_1, \dots, \pi_n\}$ are orthogonal nondegenerate holomorphic planes.

Proof. Let $\{\pi_1, \dots, \pi_n\}$ be nondegenerate holomorphic orthogonal planes, and let $\{e_i, Je_i\}$ be an orthonormal basis of π_i, $i = 1, \dots, n$. Then

$$\begin{aligned} \tau + \tau^* &= \sum_{i=1}^{n} \epsilon_i(\rho(e_i, e_i) + \rho(Je_i, Je_i) + 6\rho^*(e_i, e_i)) \\ &= 2\sum_{i=1}^{n}\left((n+3)H(\pi_i) + \sum_{k=1,\, k\neq i}^{n} H(\pi_k)\right) \\ &= 4(n+1)\sum_{i=1}^{n} H(\pi_i). \end{aligned}$$ □

6. Curvature tensors of null holomorphically flat manifolds

It is a well-known result [1] that the curvature tensor of an indefinite Kähler manifold of constant holomorphic sectional curvature is described in terms of the metric tensor and the complex structure. Such a result is due to the special properties of the curvature of Kähler metrics. Although there is no direct analog for arbitrary almost Hermitian manifolds of constant holomorphic sectional curvature, it is possible to introduce a new curvature function R^* satisfying the curvature identities of the Kähler one to generalize such expression [12]. Also, associated with each curvature function F, the curvature tensor $\tilde{F}$ is defined by $g(\tilde{F}(x,y)z,w) = F(x,y,z,w)$. Let us consider

$$\begin{aligned}16R^*(x,y,z,w)\\ &= 3\{R(x,y,z,w) + R(x,y,Jz,Jw) + R(Jx,Jy,z,w) + R(Jx,Jy,Jz,Jw)\}\\ &\quad + R(x,z,Jy,Jw) + R(Jx,Jz,y,w) - R(x,w,Jy,Jz) - R(Jx,Jw,y,z)\\ &\quad + R(x,Jw,Jy,z) + R(Jx,w,y,Jz) - R(x,Jz,Jy,w) - R(Jx,z,y,Jw).\end{aligned}$$

Note that the curvature tensor R^* satisfies the identities of the curvature tensor of a Kähler manifold. For the purpose of this section, we introduce two curvature functions R_0 and R_1. Define $R_0(x,y,z,w)$ to be

$$\begin{aligned}R_0(x,y,z,w) = \tfrac{1}{4}\{&g(y,z)g(x,w) - g(x,z)g(y,w)\\ &-g(y,Jz)g(Jx,w) + g(x,Jz)g(Jy,w) + 2g(x,Jy)g(Jz,w)\}.\end{aligned}$$

It is to be noted that $R_0(z,Jz,z,Jz) = -1$ for each unit vector z.

An indefinite Kähler manifold is a space of constant holomorphic sectional curvature if and only if the curvature tensor R is a scalar multiple of R_0 [1], and more generally, an (indefinite) almost Hermitian manifold is a space of pointwise constant holomorphic sectional curvature if and only if the curvature tensor R^* is a scalar multiple of R_0 at each point $m \in M$.

Next we introduce a symmetric bilinear form defined in terms of the Ricci tensors ρ and ρ^* as follows.

$$\begin{aligned}(6.1)\qquad \mu(x,y) = \frac{-1}{2(2n+4)}\{&\rho(x,y) + \rho(Jx,Jy)\\ &+3\rho^*(x,y) + 3\rho^*(Jx,Jy)\}.\end{aligned}$$

It is easy to see that μ is Hermitian; i.e., $\mu(Jx,Jy) = \mu(x,y)$.

Define the curvature function $R_1(x,y,z,w)$:

$$\begin{aligned}R_1(x,y,z,w) = \tfrac{1}{4}\{&g(y,z)\mu(x,w) - g(x,z)\mu(y,w) + \mu(y,z)g(x,w)\\ &-\mu(x,z)g(y,w) - g(y,Jz)\mu(Jx,w) + g(x,Jz)\mu(Jy,w)\\ &-\mu(y,Jz)g(Jx,w) + \mu(x,Jz)g(Jy,w)\\ &+2g(x,Jy)\mu(Jz,w) + 2\mu(x,Jy)g(Jz,w)\}\end{aligned}$$

Remark 6.1. Both curvature functions R_0 and R_1 are null holomorphically flat; i.e., for all null vectors u, $R_i(u,Ju,u,Ju) = 0$ ($i = 0,1$). Moreover, the associated curvature tensors satisfy

$$\tilde{R}_i(u,Ju)Ju + J\tilde{R}_i(u,Ju)u = \Phi_i(u)u$$

for each null vector u, ($i = 0,1$). Also note that $\Phi_0(u) = 0$ for each null u.

Next we state the main theorem of this section, which generalizes the expression of the curvature of an almost Hermitian manifold of pointwise constant holomorphic sectional curvature.

THEOREM 6.1. *Let (M, g, J) be a $2n$-dimensional indefinite almost Hermitian manifold. Then, M is null holomorphically flat if and only if the curvature function R^* is written in the form*

$$R^* = \frac{\tau + 3\tau^*}{4(n+1)(n+2)} R_0 + R_1. \tag{6.2}$$

Proof. For each vector $x \in T_m M$, it follows that $R^*(x, Jx, x, Jx) = R(x, Jx, x, Jx)$, and hence M is null holomorphically flat if (6.2) holds. Now prove the converse. Let us consider the curvature function $R - R_1$. Since M is null holomorphically flat, it follows from Remark 6.1 that

$$(R - R_1)(u, Ju, u, Ju) = 0$$

for each null vector u.

Consider the curvature tensor $\tilde{R}_1$. Theorem 5.1 shows that the symmetric bilinear form μ defined in (6.1) satisfies $\mu(u, u) = \frac{1}{2}c(u)$, and hence it follows from the definition of R_1 that $\Phi_1(u) = c(u)$ for each null vector $u \in T_m M$. As a consequence of (3.3) in Theorem 3.1, it follows that $(R - R_1)$ is a curvature function of pointwise constant holomorphic sectional curvature. Hence the associated curvature function $(R - R_1)^*$ is, at each point $m \in M$, a scalar multiple of R_0. Also, since the curvature function R_1 satisfies the identities of the curvature of a Kähler manifold,

$$R_1(x, y, z, w) = R_1(x, y, Jz, Jw) = R_1(Jx, Jy, z, w),$$

it follows that $R_1 = R_1^*$, and hence $(R - R_1)^* = R^* - R_1$. This shows that the curvature tensor R^* satisfies

$$R^* = CR_0 + R_1,$$

for some function C on M.

Next, we will determine such a function C. Since $R^* = CR_0 + R_1$, for each unit vector z, we have

$$R(z, Jz, z, Jz) = -C + \left(\frac{\epsilon_z}{2(n+2)}\right)\big(\rho(z, z) + \rho(Jz, Jz) + 3\rho^*(z, z) + 3\rho^*(Jz, Jz)\big),$$

and hence,

$$\epsilon_z(\rho(z,z) + \rho(Jz,Jz) + 6\rho^*(z,z)) = 2(n+2)(R(z,Jz,z,Jz) + C).$$

If z ranges over an orthonormal frame $\{z_1, Jz_1, \ldots, z_n, Jz_n\}$, we obtain

$$\tau + 3\tau^* = 2(n+2)\sum_{i=1}^{n}(R(z_i,Jz_i,z_i,Jz_i) + C),$$

and from the expression for the scalar curvatures (5.10),

$$\tau + 3\tau^* = (n+2)\left(\frac{\tau + 3\tau^*}{2(n+1)}\right) + n(2n+4)C.$$

This shows that

$$C = \frac{\tau + 3\tau^*}{4(n+1)(n+2)},$$

and the result is obtained. □

As a consequence of Theorem 6.1, we are now able to state a criterion for a null holomorphically flat manifold to have pointwise constant holomorphic sectional curvature.

THEOREM 6.2. *Let (M, g, J) be a null holomorphically flat indefinite almost Hermitian manifold. Then, the holomorphic sectional curvature H is pointwise constant on M if and only if the symmetric tensor μ is proportional to the metric tensor g.*

Remark 6.2. Thanks to Dajczer and Nomizu [5], (see also [10]), it is possible to express the condition of being μ proportional to the metric in terms of boundedness of it on spacelike or timelike unit vectors, as well as in terms of the vanishing of it on null vectors. Note that, the condition of μ being an Einstein tensor occurs, for instance, if M is Einstein and $*$-Einstein. Also, note that the result in Theorem 6.2 may be followed from Theorem 5.1 using the criteria for the pointwise constancy of the holomorphic sectional curvature in Theorem 3.1.

7. The local structures of null holomorphically flat indefinite Kähler manifolds

The Ricci tensors ρ and ρ^* of an indefinite Kähler manifold coincide with each other, and hence $\tau = \tau^*$. This implies that the symmetric tensor μ

defined in (6.1) reduces to

$$\mu(x,y) = \frac{-2}{(n+2)}\rho(x,y),$$

and therefore (6.2) takes the similar form

$$R = \frac{\tau}{(n+1)(n+2)}R_0 + R_1.$$

This fact considerably simplifies the expression (6.2), in which only the scalar curvature appears. However, the main specific feature of the Kähler case is that the curvature function R^* is nothing but the curvature of the semi-Riemannian metric, and hence, it satisfies the second Bianchi identity. This fact has the following consequence

THEOREM 7.1. *Let (M, g, J) be a null holomorphically flat indefinite Kähler manifold. Then the scalar curvature τ of M is constant if and only if it is a locally symmetric space.*

Proof. It is clear that any locally symmetric space has constant scalar curvature. Conversely, let us suppose τ is constant. Using the second Bianchi identity, it follows that

$$\begin{aligned}0 = \sigma(v,x,y)\nabla_v\{&g(y,z)\rho(x,w) - g(x,z)\rho(y,w) + g(y,Jz)\rho(x,Jw)\\ &-g(x,Jz)\rho(y,Jw) - 2g(x,Jy)\rho(z,Jw)\\ &+\rho(y,z)g(x,w) - \rho(x,z)g(y,w) + \rho(y,Jz)g(x,Jw)\\ &-\rho(x,Jz)g(y,Jw) - 2\rho(x,Jy)g(z,Jw)\},\end{aligned}$$

where $\sigma(v, x, y)$ denotes the cyclic sum over v, x, y.

Considering in the expression above $v = w = e_i$, where the $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ range over a local orthonormal frame, multiplying by $\epsilon_i = g(e_i, e_i)$, and taking the sum for $i = 1, \dots, n$, one obtains

$$\begin{aligned}(3n+3)&\big(\nabla_y\rho(x,z) - \nabla_x\rho(y,z)\big) + 2\nabla_{Jz}\rho(x,Jy)\\ &+\nabla_{Jy}\rho(x,Jz) - \nabla_{Jx}\rho(y,Jz)\\ &= g(y,z)\sum_{i=1}^{2n}\epsilon_i\nabla_{e_i}\rho(x,e_i) - g(x,z)\sum_{i=1}^{2n}\epsilon_i\nabla_{e_i}\rho(y,e_i)\\ &\quad+g(y,Jz)\sum_{i=1}^{2n}\epsilon_i\nabla_{e_i}\rho(x,Je_i)\\ &\quad-g(x,Jz)\sum_{i=1}^{2n}\epsilon_i\nabla_{e_i}\rho(y,Je_i) - 2g(x,Jy)\sum_{i=1}^{2n}\epsilon_i\nabla_{e_i}\rho(z,Je_i).\end{aligned}$$

Since the scalar curvature is constant, it follows [11] that the divergence of the Ricci tensor is identically zero, ($2\operatorname{div}\rho = d\tau$), and hence the second term in previous expression vanishes. Consequently

$$\begin{aligned}(7.1)\qquad & 3(n+1)\big(\nabla_y \rho(x,z) - \nabla_x \rho(y,z)\big) + 2\nabla_{Jz}\rho(x,Jy) \\ & + \nabla_{Jy}\rho(x,Jz) - \nabla_{Jx}\rho(y,Jz) = 0.\end{aligned}$$

We already know [15] the identity

$$(7.2)\qquad \nabla_v \rho(x,Jy) + \nabla_x \rho(y,Jv) + \nabla_y \rho(v,Jx) = 0,$$

and considering $x = Jx$, $y = Jy$, $z = Jz$ on it, we obtain

$$(7.3)\qquad \nabla_{Jz}\rho(x,Jy) = -\nabla_{Jx}\rho(y,Jz) + \nabla_{Jy}\rho(x,Jz).$$

Analogously, from (7.2),

$$(7.4)\qquad \nabla_{Jz}\rho(x,Jy) = \nabla_x \rho(y,z) - \nabla_y \rho(x,z).$$

Now, from (7.1), just considering the expressions above, it follows that the Ricci tensor is parallel. □

Next, consider the Ricci operator Q_ρ defined by $g(Q_\rho x, y) = \rho(x,y)$, for each $x, y \in \mathfrak{X}(M)$. Since the Ricci operator of a Kähler manifold is complex ($Q_\rho J = JQ_\rho$) it follows that, for each eigenvector x, the vector Jx is also an eigenvector with the same eigenvalue. Note that the Ricci operator of an indefinite Kähler manifold is not necessarily diagonalizable because of the metric to be indefinite. In fact, in [3] it is shown that the tangent bundle of a non flat complex space form is a null holomorphically flat indefinite Kähler manifold, when one considers the complete lifts of the metric and the complex structure, but the Ricci tensor of the tangent bundle is not diagonalizable.

The next theorem gives a local classification of those null holomorphically flat indefinite Kähler manifolds with constant scalar curvature under the hypothesis of diagonalizable Ricci tensor.

THEOREM 7.2. *Let (M, g, J) be a connected null holomorphically flat indefinite Kähler manifold with constant scalar curvature. Assume the Ricci operator is diagonalizable. Then, the holomorphic sectional curvature is constant, or M is locally isomorphic to a direct product $M = M_1(c) \times M_2(-c)$ of two indefinite Kähler manifolds of constant holomorphic sectional curvatures c and $-c$ respectively.*

Proof. Since ρ is parallel, under the assumption of being diagonalizable, its eigenvalues are constant on M, and have parallel eigenspaces. Hence, they determine complementary totally geodesic foliations on M.

Moreover, the restriction of the Ricci tensor to the leaves are Einstein, and hence, each leaf is an indefinite Kähler manifold of constant holomorphic sectional curvature. In consequence, if Q_ρ has only one eigenvalue, M is an indefinite complex space form according to Theorem 6.2. Let us suppose that Q_ρ has at least two distinct eigenvalues.

Consider a decomposition $M = M_1(c_1) \times M_2(c_2) \times \cdots \times M_k(c_k)$, where each factor is an (indefinite) Kähler manifold of constant holomorphic sectional curvature c_i. Since the product metric $g = g_1 + g_2 + \cdots + g_k$ is strictly semi-Riemannian, some of the g_i must not be positive definite, and hence, suppose that (M_1, g_1, J_1) is a strictly indefinite Kähler manifold.

If the Ricci operator has only one eigenvalue, then M is an Einstein manifold, and the result follows from [8] (see also Theorem 6.3).

If Q_ρ has at least two distinct eigenvalues, we only have to show that the constant curvatures satisfy $c_1 = -c_2$.

Now, the result is obtained by just considering a null vector $u = (x_1, x_2) \in \mathfrak{X}(M)$ such that x_1, x_2 are non null tangent vectors to M_1 and M_2 respectively. Since M is null holomorphically flat, $R(u, Ju, u, Ju) = 0$, and so

$$\begin{aligned} R^1(x_1, J_1x_1, x_1, J_1x_1) &+ R^2(x_2, J_2x_2, x_2, J_2x_2) \\ &= g_1(x_1, x_1)^2(c_1 + c_2) = 0 \end{aligned}$$

which shows that $c_1 = -c_2$.

If the Ricci operator has at least three distinct eigenvalues, then M is locally isometric to a product

$$M = M_1(c_1) \times M_2(c_2) \times M_3(c_3).$$

Assuming that M_1 is not positive definite, and considering null vectors of the form $u = (x_1, x_2, 0)$, $v = (y_1, 0, y_3)$ and $w = (z_1, z_2, z_3)$, if one proceeds as before, it follows that $c_1 = c_2 = c_3 = 0$, which shows that M is locally flat, and hence, a space of constant holomorphic sectional curvature. □

In [2], some examples of null holomorphically flat indefinite Kähler manifolds, as products of strictly positive and negative definite Kähler manifolds of constant holomorphic sectional curvature, are constructed. In that sense, we establish the following decomposition theorem.

THEOREM 7.3. *Let (M, g, J) be a connected null holomorphically flat indefinite Kähler manifold of constant scalar curvature. Assume that at some point $m \in M$ there exists an orthogonal decomposition $T_pM = W_1 \oplus W_2$ where*

W_1 and W_2 are orthogonal J-invariant subspaces such that the restriction of the metric to W_1 and W_2 are both strictly definite.

If the function $c(u)$ is constant for each null vector of the form $u = x + y$, where $x \in W_1$ and $y \in W_2$ are unit vectors, then M has constant holomorphic sectional curvature or it is locally isomorphic to a product $M_1(c) \times M_2(-c)$ of Kähler manifolds of constant holomorphic sectional curvature, where W_i are the tangent spaces at the point m to the integral manifolds M_i, $i = 1, 2$.

Proof. From the hypothesis it follows that the Ricci tensor ρ satisfies $\rho(u,u) = \rho(v, v) = \text{const.}$, for every null $u = x + y$, $v = x - y$, where $x \in W_1$, $y \in W_2$ are unit vectors. Then

$$\rho(x, y) = 0, \quad \rho(x, x) + \rho(y, y) = \text{const.},$$

for every unit vectors $x \in W_1$, $y \in W_2$.

Using the fact that the restriction of the metric tensor to W_i is strictly definite, it follows that the Ricci operator is diagonalizable, having exactly two eigenvalues corresponding to the eigenspaces W_1 and W_2. Since M is locally symmetric, it follows that M has constant holomorphic sectional curvature or it is locally isomorphic to a product as in previous theorem. □

We state the following results, whose proofs are similar to those of previous theorems, just using the expression for the curvature tensor R^* of a null holomorphically flat indefinite almost Hermitian manifold, and for those with pointwise constant holomorphic sectional curvature.

THEOREM 7.4. *Let (M, g, J) be a connected null holomorphically flat indefinite almost Hermitian manifold with parallel bilinear form μ. If the operator Q_μ associated with μ, ($\mu(X,Y) = g(Q_\mu X, Y)$), is diagonalizable, then the holomorphic sectional curvature is constant, or M is (locally) isometric to a product $M = M_1(c) \times M_2(-c)$ of two indefinite almost Hermitian manifolds of constant holomorphic sectional curvature.*

THEOREM 7.5. *Let (M, g, J) be a connected null holomorphically flat indefinite almost Hermitian manifold with parallel bilinear form μ. Assume that at some point $m \in M$ there exists an orthogonal decomposition $T_pM = W_1 \oplus W_2$ where W_1 and W_2 are orthogonal J-invariant subspaces such that the restrictions of the metric to W_1 and W_2 are both strictly definite.*

If the function $c(u)$ is constant for each null vector of the form $u = x + y$, where $x \in W_1$ and $y \in W_2$ are unit vectors, then M has constant holomorphic sectional curvature or it is locally isomorphic to a product $M_1(c) \times M_2(-c)$ of two indefinite almost Hermitian manifolds of constant holomorphic sectional curvature, where W_i are the tangent spaces at the point m to the integral manifolds M_i, $i = 1, 2$.

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