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THE DESCRIPTIVE COMPLEXITY OF HELSON SETS

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Introduction

A closed subset E of the circle group $\mathbf{T}$ is called a *Helson set* if every continuous complex-valued function on E can be extended to a function on $\mathbf{T}$ with absolutely convergent Fourier series. We denote by $\mathscr{H}$ the class of Helson subsets of $\mathbf{T}$.

In this paper we are interested in the descriptive properties of $\mathscr{H}$. We shall need the following definitions: a subset of a compact metric space is called a $\mathbf{G}_{\delta\sigma}$ set if it is the union of countably many $\mathbf{G}_\delta$ sets and an $\mathbf{F}_{\sigma\delta}$ set if its complement is $\mathbf{G}_{\delta\sigma}$. In the sequel we follow the notations of [17]. Thus, the symbols $\mathbf{\Pi}_2^0$, $\mathbf{\Sigma}_3^0$, $\mathbf{\Pi}_3^0$ respectively means $\mathbf{G}_\delta$, $\mathbf{G}_{\delta\sigma}$, $\mathbf{F}_{\sigma\delta}$. However, we sometimes use $\mathbf{G}_\delta$ instead of $\mathbf{\Pi}_2^0$.

Let $\mathscr{K}(\mathbf{T})$ be the space of all compact subsets of $\mathbf{T}$ equipped with its (metric, compact) Hausdorff topology. One natural question (at least for some people) is to find the exact Borel class of $\mathscr{H}$ as a subset of $\mathscr{K}(\mathbf{T})$ (this is what "descriptive properties" meant). It is easy to check (Section 1) that $\mathscr{H}$ is $\mathbf{\Sigma}_3^0$. In this paper we show that $\mathscr{H}$ is a true $\mathbf{\Sigma}_3^0$ set (that is, $\mathbf{\Sigma}_3^0$ but not $\mathbf{\Pi}_3^0$). We do this in two ways. First (Section 2) we prove that even inside the countable sets $\mathscr{H}$ is true $\mathbf{\Sigma}_3^0$. Then (Section 3) we get the same conclusion for *perfect* Helson sets. In fact, our result is slightly more general: we show that for any M_o set E, the perfect Helson sets contained in E form a true $\mathbf{\Sigma}_3^0$ subset of $\mathscr{K}(E) = \{F \in \mathscr{K}(\mathbf{T});\ F \subseteq E\}$ (the definition of an M_o *set* will be given in Section 3). The proof also yields that some other natural classes of thin sets, like the *WTP*, U' or U_o' sets, are true $\mathbf{\Sigma}_3^0$ within any M_o set.

1. Definitions, upper bound for the complexity

Let $\mathbf{M}(\mathbf{T})$ be the space of Borel, complex measures on $\mathbf{T}$ with its natural norm $\|\ \ \|_M$ and $\mathbf{PM}$ be the space of all distributions on $\mathbf{T}$ with bounded

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Fourier coefficients. The norm of an element $S \in \mathbf{PM}$ is defined by $\|S\|_{PM} = \sup_{n \in \mathbb{Z}} |\hat{S}(n)|$. Thus the Fourier transform identifies **PM** with $l^\infty(\mathbb{Z})$.

Evidently $\|\mu\|_M \le \|\mu\|_{PM}$ if $\mu \in \mathbf{M}(\mathbf{T})$, and it is well known (see [4] or [5]) that $E \in \mathscr{K}(\mathbf{T})$ is a Helson set if and only if there is a constant $c \ge 0$ such that

$$\|\mu\|_M \le c\|\mu\|_{PM} \quad \text{for every } \mu \in \mathbf{M}(\mathbf{T}) \text{ supported by } E.$$

From this it is easy to see that $\mathscr{H}$ is a $\mathbf{\Sigma}_3^0$ subset of $\mathscr{K}(\mathbf{T})$. Indeed one can write

$$E \in \mathscr{H} \Leftrightarrow \exists k \in \mathbb{N} \quad \forall \mu \in \mathbf{B}_1(\mathbf{M}(\mathbf{T})) \\ \left(\text{supp}(\mu) \subsetneq E \text{ or } \|\mu\|_M \le \frac{1}{2} \text{ or } \exists n\, |\hat{\mu}(n)| > \frac{1}{k}\right)$$

(here $\mathbf{B}_1(\mathbf{M}(\mathbf{T}))$ is the unit ball of $\mathbf{M}(\mathbf{T})$ with its w^* topology and supp(μ) is the support of the measure μ).

The condition under brackets is clearly $\mathbf{\Pi}_2^0$ in (μ, E). Since $\mathbf{B}_1(\mathbf{M}(\mathbf{T})) \times \mathscr{K}(\mathbf{T})$ is compact, $\mathscr{H}$ is $\mathbf{\Sigma}_3^0$.

From now on we write ω for the set of natural numbers and $\mathbf{2}^\omega$ for the Cantor space of all infinite sequences of 0's and 1's with its usual product topology.

We fix a bijection $(p, q) \mapsto \langle p, q\rangle$ from ω^2 onto ω and denote the inverse map by $n \mapsto ((n)_0, (n)_1)$. For $\alpha \in \mathbf{2}^\omega$, we define $\alpha_p \in \mathbf{2}^\omega$ by $\alpha_p(q) = \alpha(\langle p, q\rangle)$. Finally let **W** be the following subset of $\mathbf{2}^\omega$:

$$\mathbf{W} = \left\{\alpha \in \mathbf{2}^\omega,\ \exists p\ \alpha_p(q) = 1 \text{ for infinitely many } q\text{'s}\right\}$$

It is well known that **W** is a true $\mathbf{\Sigma}_3^0$ subset of $\mathbf{2}^\omega$. Thus, to show that $\mathscr{H}$ is not $\mathbf{\Pi}_3^0$ it is enough to construct a continuous function $\varphi : \mathbf{2}^\omega \to \mathscr{K}(\mathbf{T})$ such that:

$$\text{if } \alpha \in \mathbf{W}, \text{ then } \varphi(\alpha) \in \mathscr{H};$$
$$\text{if } \alpha \notin \mathbf{W}, \text{ then } \varphi(\alpha) \notin \mathscr{H}.$$

This is what we shall do in the next two sections.

2. Countable Helson Sets

In this section we show that $\mathscr{H}$ is true $\mathbf{\Sigma}_3^0$ "inside the countable sets". In other words, we construct a continuous reduction φ such that $\varphi(\alpha)$ is countable for each $\alpha \in \mathbf{2}^\omega$. The advantage of considering only countable sets is that several "arithmetic" conditions are known for a countable set to be Helson.

In the sequel, **T** will be identified with the interval [0, 1[whenever it seems more appropriate.

DEFINITIONS. (a) A subset A of **T** is said to be *independent* if for every $x_1, \dots, x_k \in A$ the equation $\sum_{i=1}^{k} m_i x_i = 0$ has no non-trivial integer solution.

(b) If k is a positive integer, an *arithmetic progression of length k* is a set of the form

$$\{a, a + 1/l, \dots, a + k/l\}$$

for some $a \in \mathbf{T}$ and some positive integer l.

The following facts are well known (see [5]):

(1) *If $E \in \mathcal{K}(\mathbf{T})$ is countable and is the union of finitely many independent closed sets, then E is a Helson set.*

(2) *If $E \in \mathcal{K}(\mathbf{T})$ contains arbitrarily long arithmetic progressions, then E is not a Helson set.*

Remark. It follows from the work of G. Pisier [18] that one can characterize completely the countable Helson sets by means of a very simple arithmetic property. The two preceding facts are of course immediate consequences of this characterization.

THEOREM 1. *There is a continuous map $\varphi : \mathbf{2}^\omega \to \mathcal{K}(\mathbf{T})$ such that $\varphi(\alpha)$ is countable for each $\alpha \in \mathbf{2}^\omega$ and:*
if $\alpha \in \mathbf{W}$, $\varphi(\alpha)$ is a finite union of closed independent sets;
if $\alpha \notin \mathbf{W}$, $\varphi(\alpha)$ contains arbitrarily long arithmetic progressions.

COROLLARY. *There is no $\mathbf{\Pi}_3^0$ subset of $\mathcal{K}(\mathbf{T})$ containing the countable Helson sets and contained in $\mathcal{H}$. In particular $\mathcal{H}$ is a true $\mathbf{\Sigma}_3^0$ set.*

In the proof of Theorem 1 we will need the following Lemma. Recall that a class $\mathcal{C} \subseteq \mathcal{K}(\mathbf{T})$ is said to be *hereditary* if any (closed) subset of an element of $\mathcal{C}$ still belongs to $\mathcal{C}$.

LEMMA 1. *Let $\mathcal{I}$ be the class of independent compact subsets of* **T**. *Then*:
(a) *$\mathcal{I}$ is $\mathbf{G}_\delta$, hereditary, and dense in $\mathcal{K}(\mathbf{T})$;*
(b) *if $E \in \mathcal{I}$, the set $\mathcal{I}_E = \{F \in \mathcal{K}(\mathbf{T});\ E \cup F \in \mathcal{I}\}$ is a dense $\mathbf{G}_\delta$ of $\mathcal{K}(\mathbf{T})$.*

Proof. Part (a) is easy and implies that $\mathcal{I}_E$ is $\mathbf{G}_\delta$ by continuity of the map $(E, F) \mapsto E \cup F$. To prove the density in part (b) it is clearly enough to show that given a non empty open set V there exists a point $x \in V$ such that $E \cup \{x\} \in \mathcal{I}$. So let us fix $E \in \mathcal{I}$, $V \subseteq \mathbf{T}$ open, and consider the subset A of

T defined by

$$x \in A \Leftrightarrow \forall m \neq 0 \, \forall m_1, \ldots, m_k \text{ not all } 0 \, \forall x_1, \ldots, x_k \in E \, \sum_{i=1}^{k} m_i x_i \neq mx$$

$$(m, m_1, \ldots, m_k \text{ are integers}).$$

Since E is independent, A contains all the rational numbers, hence A is dense in **T**. Moreover A is clearly $\mathbf{G}_\delta$ (because E is closed). So, by Baire category theorem, we can find an irrational x in $A \cap V$. Then $E \cup \{x\}$ is independent by definition of A. This proves (b). □

It follows from (b) (by Baire's theorem again) that given independent sets $F_1, \ldots, F_k$ and a non empty open set $V \subseteq \mathbf{T}$ there is a point $x \in V$ such that $\{x\} \cup F_i$ is independent for $i = 1, \ldots, k$.

We now turn to the proof of Theorem 1. Let us first fix some notations. Let $2^{<\omega}$ be the set of all finite sequences of 0's and 1's; for any integer n, $2^{\leq n}$ is the set of sequences of length $\leq n$. If $s \in 2^{<\omega}$, $|s|$ is the length of s, $s_{\lceil n}$ is the restriction of s to $\{0, \ldots, n-1\}$ (for $n \leq |s|$), and for $s, t \in 2^{<\omega}$, $s \precsim t$ means that t is an extension of s (that is, $|s| \leq |t|$ and $t_{\lceil |s|} = s$). If $s \in 2^{<\omega}$, $s \neq \emptyset$, we denote by s' the sequence $s_{\lceil |s|-1}$.

Since $\mathscr{I}$ is $\mathbf{G}_\delta$ and hereditary, we can choose a decreasing sequence $(\mathscr{U}^n)_{n \geq o}$ of open, hereditary subsets of $\mathscr{K}(\mathbf{T})$ such that $\mathscr{I} = \bigcap_{n \geq o} \mathscr{U}^n$. The open sets $\mathscr{U}^n$ are obtained as follows: write $\mathscr{G} = \bigcap_{n \geq o} \mathscr{W}^n$, where the $\mathscr{W}^n$ are open with $\mathscr{W}^{n+1} \subseteq \mathscr{W}^n$. Then let

$$\mathscr{U}^n = \{K \in \mathscr{W}^n; \, L \in \mathscr{W}^n \text{ for every } L \subseteq K\}.$$

$\mathscr{U}^n$ is obviously hereditary and it is easy to check that it is also open. Finally, since $\mathscr{G}$ is hereditary one has $\mathscr{G} \subseteq \mathscr{U}^n \subseteq \mathscr{W}^n$ for all n, hence $\mathscr{G} = \bigcap_{n \geq o} \mathscr{U}^n$.

Finally, we fix a point $x_o \in \mathbf{T}$ such that $\{x_o\} \in \mathscr{I}$ (that is, an irrational x_o).

Now we shall construct for each $s \in 2^{<\omega}$ a closed subset $E(s)$ of **T**. If $s \neq \emptyset$, $E(s)$ will be written as

$$E(s) = \bigcup_{m=o}^{|s|-1} E^m(s)$$

where the $E^m(s)$ are pairwise disjoint and satisfy the following requirements:

(1) $E^m(s) = I_o^m(s) \cup \cdots \cup I_{(m)_o}^m(s)$

where the $I_j^m(s)$ are pairwise disjoint non trivial closed intervals of center $x_j^m(s)$ and of length $\leq 2^{-|s|}$.

(2) $E^m(s) \subseteq]x_o, x_o + 2^{-m}]$.

(3) If $t \precsim s$ then $I_j^m(s) \subseteq I_j^m(t)$ $(m < |t|)$.
(4) If $|s| = n + 1$, $m < n$ and $(m)_o < (n)_o$ then $x_j^m(s) = x_j^m(s')$.
(5) If $|s| = n + 1$ and p is any (nonnegative) integer, then for every $j \leq p$,

$$\{x_o\} \cup \{x_j^m(s), m \leq n, (m)_o = p\} \text{ is independent},$$

$$\{x_o\} \cup \left(\bigcup_{\substack{m \leq n \\ (m)_o = p}} I_j^m(s) \right) \in \mathscr{U}^n.$$

(6) If $|s| = n + 1$ and $s(n) = 0$, then

$$\{x_o^n(s), \dots, x_{(n)_o}^n(s)\} \text{ is an arithmetic progression},$$

$$x_j^m(s) = x_j^m(s') \text{ if } m < n, j \leq (m)_o.$$

(7) If $|s| = n + 1$ and $s(n) = 1$ and if we let $A = \{m \leq n; (m)_o \geq (n)_o\}$, then

$$\{x_o\} \cup \left(\bigcup_{\substack{m \in A \\ j \leq (m)_o}} I_j^m(s) \right) \in \mathscr{U}^n.$$

We first let $E(\emptyset) = \mathbf{T}$ and now describe the inductive step.

Assume the sets $E^m(t)$ have been constructed for each $t \in 2^{\leq n}$ and let s be a sequence of length $n + 1$. We distinguish two cases.

Case 1. $s(n) = 0$. We first define $E^m(s)$ for $m < n$, $(m)_o \neq (n)_o$. So (if there is any) fix $p \in \omega$ with $p \neq (n)_o$ and such that $A_p = \{m < n; (m)_o = p\}$ is non empty. Let also j be an integer $\leq p$.

By induction hypothesis, the set $\{x_o\} \cup \{x_j^m(s'), m \in A_p\}$ is independent, hence belongs to $\mathscr{U}^n$. Since $\mathscr{U}^n$ is open, we can choose intervals $I_j^m(s)$, $m \in A_p$ with center $x_j^m(s')$ and length $\leq 2^{-n}$, such that $\{x_o\} \cup (\bigcup_{m \in A_p} I_j^m(s)) \in \mathscr{U}^n$. Then (1), ..., (5) and one half of (6) are satisfied for $m \in A_p$.

Now we define $E^m(s)$ for those $m \leq n$ with $(m)_o = (n)_o$. Let $A = \{m < n; (m)_o = (n)_o\}$. If $j \leq (n)_o$, the set $F_j = \{x_o\} \cup \{x_j^m(s');\ m \in A\}$ is independent. Thus, by Lemma 1, we can choose $x_o^n(s) \in]x_o, x_o + 2^{-n}[$ such that $\{x_o^n(s)\} \cup F_j$ is independent for all $j \leq (n)_o$. Next let p be a positive integer such that $[x_o^n(s), x_o^n(s) + (n)_o/p] \subseteq]x_o, x_o + 2^{-n}]$ and let $x_j^n(s) = x_o^n(s) + j/p$ for $j \leq (n)_o$. Then obviously $\{x_j^n(s)\} \cup F_j$ is independent for each j. So, letting $x_j^m(s) = x_j^m(s')$ if $m \in A$, we just take for $I_j^m(s)$ some sufficiently small interval around $x_j^m(s)$ to ensure (1), ..., (6).

Case 2. $s(n) = 1$. By (5) we have no freedom in the choice of $E^m(s)$ if $(m)_o < (n)_o$, and we argue as in case 1.

Now let $A = \{m < n;\ (m)_o \geq (n)_o\}$ and $X = \{I_j^m(s');\ m \in A,\ j \leq (m)_o\}$. Using Lemma 1, we find a set $F \in \mathscr{I}$ such that $x_o \in F$ and $F \cap I \neq \emptyset$ for all $I \in X$. Choosing one point x_I in each $I \cap F$ and putting some small interval around it, we get the sets $E_j^m(s)$ for $m \in A$ and $j \leq (m)_o$. If the intervals are well chosen, conditions (1), ..., (7) are then satisfied for $m \neq n$.

Finally we define $E^n(s)$. Actually (in case $s(n) = 1$) $E^n(s)$ is not really essential in the proof: we define it only because it is more convenient to have n blocks at the n'-th step. Nevertheless $E^n(s)$ is easily constructed using Lemma 1 once more.

This concludes the inductive step.

Now we first claim that for each $\alpha \in \mathbf{2}^\omega$ the sets $E(\alpha_{\lceil n})$ converge in $\mathscr{K}(\mathbf{T})$ to some countable (closed) set $E(\alpha)$, and that the map $\alpha \mapsto E(\alpha)$ is continuous. To see this, observe that by (1) and (3) the sequence $(E^m(\alpha_{\lceil n}))_{n>m}$ converges to some finite set $E^m(\alpha)$ (for any $m \in \omega$). By (2), $E^m(\alpha) \subseteq$
$[x_o, x_o + 2^{-m}]$ (in fact $]x_o, x_o + 2^{-m}]$). Hence

$$E(\alpha) = \{x_o\} \cup \left(\bigcup_{n=0}^{\infty} E^m(\alpha)\right)$$

is a countable closed set and clearly $E(\alpha_{|n}) \to E(\alpha)$ as $n \to \infty$.

Next we show that the map $\alpha \to E(\alpha)$ is continuous.

Let V be an open set with $E(\alpha) \cap V \neq \emptyset$. Then $E(\alpha) \cap V \neq \{x_o\}$ as well, so pick $x \in E(\alpha) \cap V$, $x \neq x_o$. Then $x \in E^m(\alpha)$ for some m, hence there is a $j \leq (m)_o$ such that $x \in \bigcap_{n>m} I_j^m(\alpha_{\lceil n})$. If n is big enough, say $n \geq N$, then $I_j^m(\alpha_{\lceil n}) \subseteq V$. Thus, If $\beta_{\lceil N} = \alpha_{\lceil N}$ one has

$$\bigcap_{n>m} I_j^m(\beta_{\lceil n}) \subseteq V \cap E(\beta)$$

by (3), and so $E(\beta) \cap V \neq \emptyset$.

On the other hand, if $E(\alpha) \subseteq V$, then for large m and all $\beta \in \mathbf{2}^\omega$ one has

$$E^m(\beta) \subseteq [x_o, x_o + 2^{-m}] \subseteq V.$$

The diameter condition in (1) now implies that $E(\beta) \subseteq V$ if $\beta_{\lceil n} = \alpha_{\lceil n}$ and n is big enough. This shows that the map $\alpha \mapsto E(\alpha)$ is continuous.

It remains to check that this map satisfies the conclusion of Theorem 1. So we fix $\alpha \in \mathbf{2}^\omega$ and, of course, distinguish two cases.

Case 1. α_{p_o} *is infinite for some* $p_o \in \omega$. We have to show that $E(\alpha)$ is a finite union of independent closed sets. First we note that for any $p \in \omega$ and each $j \leq p$ the set

$$E_{p,j} = \{x_o\} \cup \left(\bigcup_{\substack{m \in \omega \\ (m)_o = p}} E_j^m(\alpha) \right)$$

is (closed and) independent. Indeed, by (3), (5) and the fact that the $\mathscr{U}^n$ are hereditary, every finite subset of $E_{p,j}$ is independent.

Now if α_{p_o} is infinite, then (3) and (7) imply that $E_{p_o} = \{x_o\} \cup (\bigcup_{(m)_o \geq p_o} E^m(\alpha))$ is independent. Then we are done since $E(\alpha) = E_{p_o} \cup (\bigcup_{j \leq p < p_o} E_{p,j})$.

Case 2. α_p *is finite for every* p. Let p_o be a non negative integer. We show that $E(\alpha)$ contains an arithmetic progression of length $p_o + 1$.

Choose q'_o such that $\alpha(\langle p, q \rangle) = 0$ for $p \leq p_o$ and $q > q'_o$ (such a q'_o exists by our hypothesis). Then pick q_o ($> q'_o$) so large that $\langle p_o, q_o \rangle > \mathrm{Max}\{\langle p, q \rangle, p \leq p_o, q \leq q'_o\}$ and let $n_o = \langle p_o, q_o \rangle$.

By the choice of q'_o $\alpha(n_o) = 0$, hence by (6) $E^{n_o}(\alpha_{\lceil n_o+1})$ contains an arithmetic progression of length $p_o + 1$.

Let j be an integer $\leq p_o$. We claim that $x_j^{n_o}(\alpha_{\lceil n+1}) = x_j^{n_o}(\alpha_{\lceil n_o+1})$ for each $n > n_o$. Indeed if $n > n_o$, $n = \langle p, q \rangle$ then:

Either $p > p_o$ and then $x_j^{n_o}(\alpha_{\lceil n+1}) = x_j^{n_o}(\alpha_{\lceil n})$ by (4);

Or else $p \leq p_o$ in which case $q > q'_o$ by the choice of q_o. Then $\alpha(n) = 0$ by the choice of q'_o and $x_j^{n_o}(\alpha_{\lceil n+1}) = x_j^{n_o}(\alpha_{\lceil n})$ by (6).

In any case the claim follows by induction. Condition (1) now implies that $E^{n_o}(\alpha)$ is an arithmetic progression of length $p_o + 1$ (the one already contained in $E^{n_o}(\alpha_{\lceil n_o+1})$). This concludes case 2 and the proof of Theorem 1. □

Remark. A closed set $E \subseteq \mathbf{T}$ is called a *set of analyticity* if the only functions operating on the algebra $\mathbf{A}(E)$ (the restrictions to E of absolutely convergent Fourier series) are the analytic functions. The still open *dichotomy conjecture* (see [4], [5], [6]) asserts that any closed subset of $\mathbf{T}$ is either a Helson set or a set of analyticity (the two cases are of course exclusive). It is known (see [5]) that if $E \in \mathscr{K}(\mathbf{T})$ contains arbitrarily long arithmetic progressions, then E is a set of analyticity. Thus Theorem 1 shows that the class $\mathscr{A}$ of sets of analyticity in $\mathbf{T}$ cannot be $\mathbf{\Sigma}_3^0$ in $\mathscr{K}(\mathbf{T})$. In other words, if the dichotomy conjecture is not true, this is not because $\mathscr{A}$ is "too simple". It can be shown that $\mathscr{A}$ is a $\mathbf{\Pi}_1^1$ (*coanalytic*) set but it does not seem obvious that it should be Borel. This is rather surprising, since if the dichotomy conjecture is true, then the Borel class of $\mathscr{A}$ must be very small ($\mathbf{\Pi}_3^0$).

3. Perfect Helson sets

The preceding result is not really satisfactory because it says nothing about *perfect* Helson sets. In this section, we show that the latter also form a true $\mathbf{\Sigma}_3^0$ subset of $\mathscr{K}(\mathbf{T})$.

First we must introduce some other classes of sets.

If $S \in \mathbf{PM}$ we let $R(S) = \overline{lim}_{n\to\infty} |\hat{S}(n)|$.

For $E \in \mathscr{K}(\mathbf{T})$ define

$$\eta_o(E) = \inf\left\{\frac{R(\mu)}{\|\mu\|_{PM}},\ \mu \in \mathbf{M}_+(E),\ \mu \neq 0\right\}$$

$$\eta_2(E) = \inf\left\{\frac{R(\mu)}{\|\mu\|_{PM}},\ \mu \in \mathbf{M}(E),\ \mu \neq 0\right\}$$

$$\eta_1(E) = \inf\left\{\frac{R(S)}{\|S\|_{PM}},\ S \in \mathbf{N}(E),\ S \neq 0\right\}$$

$$\eta(E) = \inf\left\{\frac{R(S)}{\|S\|_{PM}},\ S \in \mathbf{PM}(E),\ S \neq 0\right\}$$

(here $\mathbf{N}(E)$ denotes the w^* closure of $\mathbf{M}(E)$ in PM; the other notations are self-explanatory).

Then E is called a U_i' *set* if $\eta_i(E) > 0$ and a U' *set* if $\eta(E) > 0$. Evidently $U' \subseteq U_1' \subseteq U_2' \subseteq U_o'$, and it is well known that $\eta_1(E) > 0$ for all Helson sets, that is, $\mathscr{H} \subseteq U_1'$ (see [4], [5]). On the other hand, there are Helson sets which are not sets of uniqueness, hence with $\eta(E) = 0$: this is a deep result, due independently to R. Kaufman and T.W. Körner ([8], [12]). We should also add that $\eta(E) = 0$ for countable sets (which may fail to be Helson): this is a consequence of the fact that pseudomeasures with countable support are almost periodic (Loomis [15]).

E is said to be *without true pseudomeasures* (WTP) if every pseudomeasure supported by E is actually a measure. Equivalently E is WTP if and only if it is a Helson set and a set of synthesis. In particular, $WTP \subseteq \mathscr{H} \cap U'$.

Finally, E is said to be a *Kronecker set* if the characters of $\mathbf{T}$ are uniformly dense in

$$\mathbf{U}(E) = \{f \in \mathbf{C}(E);\ |f(x)| = 1\ \forall x \in E\}.$$

We shall use the following results about Kronecker sets.

(1) Finite unions of Kronecker sets are WTP. This is a consequence of two celebrated results of N. Varopoulos: Kronecker sets are WTP, and Helson

sets (as well as WTP sets) are closed under finite unions. Proofs of these results can be found in [4], [13] and [19].

(2) For any perfect set $P \subseteq \mathbf{T}$, the class of Kronecker subsets of P is $\mathbf{G}_\delta$ hereditary and dense in $\mathscr{K}(P)$ (see [7] or [10] p. 337).

It is easy to check as we did for $\mathscr{H}$, that U', U_o', U_2' and WTP are $\mathbf{\Sigma}_3^0$ subsets of $\mathscr{K}(\mathbf{T})$ (on the other hand, because of the complexity of the notion of spectral synthesis, it seems reasonable to think that U_1' is not even Borel, see [11]). We shall prove below that they are all true $\mathbf{\Sigma}_3^0$ sets. This will follow from a somewhat more general result whose statement unfortunately requires still more definitions.

A measure $\mu \in \mathbf{M}(\mathbf{T})$ is said to be a *Rajchman measure* if $\hat{\mu}(n) \to 0$ as $|n| \to \infty$. For $E \in \mathscr{K}(\mathbf{T})$, we denote by $\mathbf{P}(E)$ the set of all probability measures on E and by $\mathscr{R}(E)$ the set of probability Rajchman measures supported by E (also letting $\mathscr{R} = \mathscr{R}(\mathbf{T})$). $\mathbf{P}(E)$ will always be equipped with the w^* topology induced by $\mathbf{M}(E)$.

A closed set $E \subseteq \mathbf{T}$ is said to be an M_o *set* if it supports a non zero Rajchman measure. By a result of Kechris and Louveau [10, p. 274] also obtained independently by Debs and Saint-Raymond [3] E is an M_o set if and only if it cannot be covered by countably many U_o' sets. E is said to be an M_o^p *set* if for every open set V such that $E \cap V \neq \varnothing$ the set $\overline{E \cap V}$ is in M_o. It is equivalent to say (if $E \neq \varnothing$) that E is the support of a Rajchman probability measure, or that $\mathscr{R}(E)$ is dense in $\mathbf{P}(E)$ (see [2], Lemma 8.3).

The following remark will be useful later: if E is M_o^p, then the set

$$\mathscr{R}'(E) = \{\mu \in \mathscr{R}(E);\, \mathrm{supp}(\mu) = E\}$$

is dense in $\mathbf{P}(E)$. To see this take $\mu_o \in \mathscr{R}$ such that $\mathrm{supp}(\mu) = E$. Then if $\mu \in \mathscr{R}(E)$ and α is any positive number,

$$\mu_\alpha = \frac{1}{1+\alpha}(\mu + \alpha\mu_o)$$

is in $\mathscr{R}'(E)$. Since $\mu_\alpha \to \mu$ as $\alpha \to 0$ we are done by density of $\mathscr{R}(E)$ in $\mathbf{P}(E)$.

We can now state our main result.

THEOREM 2. *Let $E \in \mathscr{K}(\mathbf{T})$ be a non empty M_o^p set and let $\mathscr{G} \subseteq \mathscr{K}(E)$ be G_δ hereditary and dense in $\mathscr{K}(E)$. Then there is a continuous map $\varphi : \mathbf{2}^\omega \to \mathscr{K}(\mathbf{T})$ such that for each $\alpha \in \mathbf{2}^\omega$, $\varphi(\alpha)$ is a perfect subset of E and:*

if $\alpha \in \mathbf{W}$, then $\varphi(\alpha)$ is a finite union of (perfect) $\mathscr{G}$ sets;

if $\alpha \notin \mathbf{W}$, then $\varphi(\alpha) \notin U_o'$.

In particular, there is no $\mathbf{\Pi}_3^0$ subset $\mathscr{A}$ of $\mathscr{K}(E)$ such that $\mathscr{A} \subseteq U_o'$ and $\mathscr{A}$ contains all the finite unions of perfect $\mathscr{G}$ sets.

Of course this result is interesting only if $\mathcal{G} \subseteq U_o'$.

If $\mathcal{G}$ is the class of Kronecker sets (which is dense in $\mathcal{K}(E)$ because M_o^p sets are perfect) we get the following

COROLLARY 1. *Let E be a non empty M_o^p set (e.g., $E = \mathbf{T}$). Then there is no $\mathbf{\Pi}_3^0$ set in $\mathcal{K}(E)$ containing the finite unions of perfect Kronecker subsets of E and contained in U_o'.*

Since every M_o set contains a not empty M_o^p set this implies:

COROLLARY 2. *For any M_o set E, the classes of perfect WTP, $\mathcal{H}$, U', U_o', U_2' sets are true $\mathbf{\Sigma}_3^0$ in $\mathcal{K}(E)$, and U_1' is not $\mathbf{\Pi}_3^0$.*

Remarks. (1) One cannot hope to get the same result as in Theorem 1 for the countable Helson subsets of a given M_o set, because there exist independent M_o sets (they are called *Rudin sets*, see [4] or [13]) and all countable independent sets are Helson.

(2) In [14], T. Linton shows that the so-called *H-sets* (which are not at all the same as the Helson sets) also from a true $\mathbf{\Sigma}_3^0$ set in $\mathcal{K}(\mathbf{T})$. In fact, by results of N. Bary [1, Théorème V], it follows from his proof that the classes U_i' are not $\mathbf{\Pi}_3^0$.

(3) It can be shown (see [4]) that every non U_1' set is a set of analyticity. Thus it follows from Theorem 2 that $\mathcal{A}$ is not $\mathbf{\Sigma}_3^0$ within any M_o set.

To make the proof of Theorem 2 more readable it is better to state first some preliminary results.

LEMMA 2. *Let E be a compact metrizable space and $\mathcal{U} \subseteq \mathcal{K}(E)$ be open and dense in $\mathcal{K}(E)$. Also, let $W_1, \dots, W_k$ be non empty open subsets of E and $\mathcal{V}_1, \dots, \mathcal{V}_k$ be open subsets of $\mathcal{K}(E)$ such that $\overline{W}_i \in \mathcal{V}_i$ for all $i \le k$.*

Then there exists non empty open subsets $V_1, \dots, V_k$ of E such that

$$V_i \subseteq W_i \quad (i \le k)$$
$$\overline{V}_i \in \mathcal{V}_i \quad (i \le k)$$
$$\bigcup_{i \le k} \overline{V}_i \in \mathcal{U}.$$

Proof. For each $i \le k$, choose non empty open subsets of E, say $W_{i1}, \dots, W_{iK_i}$ with $\overline{W}_i \cap W_{ij} \neq \emptyset$ for all j, such that every (compact) subset F of $\overline{W}_i$ with $F \cap W_{ij} \neq \emptyset$ for all $j \le K_i$ belongs to $\mathcal{V}_i$. Now each $W_i \cap W_{ij}$ is a non empty open set in E, so by density we can find a set F in $\mathcal{U}$, $F \subseteq \cup_{i \le k} W_i$, such that $F \cap W_i \cap W_{ij} \neq \emptyset$ for all $i \le k$ and $j \le K_i$. Then, since $\mathcal{U}$ is open, choose an open set $V \subseteq \cup_{i \le k} W_i$ containing F such that $\overline{V} \in \mathcal{U}$ and let $V_i = V \cap W_i$. □

LEMMA 3. *Let E be a non empty M_o^p set and let $\mathscr{U} \subseteq \mathscr{K}(E)$ be open and dense in $\mathscr{K}(E)$. Let $\mathscr{R}_{\mathscr{U}}$ be the subset of $\mathscr{K}(E) \times \mathbf{P}(E)$ defined by*

$$(F, \mu) \in \mathscr{R}_{\mathscr{U}} \Leftrightarrow \mu \in \mathscr{R} \wedge \operatorname{supp}(\mu) = F$$
$$\wedge\ F \in \mathscr{U} \text{ is the closure of an open set in } E.$$

Then $\mathscr{R}_{\mathscr{U}}$ is dense in $\{(F, \mu) \in \mathscr{K}(E) \times \mathbf{P}(E); \operatorname{supp}(\mu) \subseteq F\}$.

Proof. Let us fix (F_o, μ_o) such that $\operatorname{supp}(\mu_o) \subseteq F_o$ and an elementary neighbourhood $\mathscr{U}_o \times N_o$ of (F_o, μ_o) in $\mathscr{K}(E) \times \mathbf{P}(E)$. We may assume that $\mathscr{U}_o = \{F \in \mathscr{K}(E);\ F \subseteq V_o,\ F \cap V_i \neq \varnothing,\ i = 1, \dots, k\}$ where $V_o, V_1, \dots, V_k$ are open in E and $V_i \subseteq V_o$ for i ≥ 1.

Choose a finite set $\{x_1, \dots, x_p\} \subseteq F_o$ and positive numbers $\lambda_1, \dots, \lambda_p$ such that $\sum_{i=1}^p \lambda_i = 1$ and $\mu_1 = \sum_{i=1}^p \lambda_i \delta_{x_i} \in N_o$ (δ_x is the Dirac measure at x). By adding small masses at points of V_i (and normalizing), we can also assume that $p \geq k$ and $x_i \in V_i$ for $1 \leq i \leq k$.

Now choose for each $i \leq p$ an open (in E) neighbourhood W_i of x_i such that:

$$\begin{cases} W_i \subseteq V_i \text{ if } i \leq k; \\ \text{if one takes a point } y_i \text{ in each } W_i, \text{ then } \sum_{i=1}^p \lambda_i \delta_{y_i} \in N_o. \end{cases}$$

Next, by density, take F in $\mathscr{U}$ such that $F \subseteq V_o$ and $F \cap W_i \neq \varnothing$ for all i. Then $F \in \mathscr{U} \cap \mathscr{U}_o$. Since $\mathscr{U} \cap \mathscr{U}_o$ is open, we can find an open set $W \supseteq F$ such that $\overline{W} \in \mathscr{U} \cap \mathscr{U}_o$. Now $\overline{W}$ is an M_o^p set, so the probability Rajchman measures with support $\overline{W}$ are dense in $\mathbf{P}(\overline{W})$. Thus, picking $y_i \in F \cap W_i$ for $1 \leq i \leq p$ and approximating $\sum_{i=1}^p \lambda_i \delta_{y_i}$, we can find a $\mu \in \mathscr{R}$ such that $\mu \in N_o$ and $\operatorname{supp}(\mu) = \overline{W} \in \mathscr{U} \cap \mathscr{U}_o$. This proves the lemma. □

COROLLARY. *Let $E_1, \dots, E_k$ be disjoint non empty M_o^p sets supporting probability measures $\mu_1, \dots, \mu_k$. Let $\mathscr{U}$ be a dense open subset of $\mathscr{K}(E)$, where $E = \bigcup_{i=1}^k E_i$. Let also $\mathscr{V}_1, \dots, \mathscr{V}_k$ be open sets in $\mathscr{K}(E)$ such that $E_i \in \mathscr{V}_i$, $i \leq k$.*

Then for any $\varepsilon > 0$ and any finite set $\mathscr{F} \subseteq$ $\mathbf{C}(\mathbf{T})$ *there exist probability Rajchman measures $\nu_1, \dots, \nu_k$ such that*:

$\operatorname{supp}(\nu_i) \in \mathscr{V}_i$ *and* $\operatorname{supp}(\nu_i)$ *is the closure of an open subset of E_i*;

$|\langle \nu_i, f \rangle - \langle \mu_i, f \rangle| < \varepsilon$ *for every $f \in \mathscr{F}$*;

$\bigcup_{i=1}^k \operatorname{supp}(\nu_i) \in \mathscr{U}$.

Proof. We first choose continuous functions $\varphi_1, \dots, \varphi_k$ with $\varphi_i \geq 0$, $\varphi_i = 1$ on E_i and $\varphi_i = 0$ on E_j if $j \neq i$. We also fix an $\alpha > 0$.

Since E is an M_o^p set, we can apply Lemma 3 to approximate $\mu = \sum_{i=1}^k \mu_i$ and get a positive Rajchman measure ν such that:

$\|\nu\|_M = k$;
$(1-\alpha) < \int \varphi_i \, d\nu < (1+\alpha)$ for $i \leq k$;
supp(ν) is the closure of an open subset of E and belongs to $\mathscr{U}$;
$|\int \varphi_i f \, d\mu - \int \varphi_i f \, d\nu| < \varepsilon$ for $f \in \mathscr{F}$.

Then if we let $\nu_i = \varphi_i \nu / \|\varphi_i \nu\|$, the measures ν_i will work provided α is small enough. □

LEMMA 4. *Let E be a compact metrizable space and $\mathscr{G} \subseteq \mathscr{K}(E)$ be $\mathbf{G}_\delta$. Let $F, F_o, F_1, \ldots$ be closed subsets of E such that:*
$F_n \to F$ as $n \to \infty$;
for every $N \in \omega$, $F \cup (\bigcup_{n \leq N} F_n) \in \mathscr{G}$.
Then $F \cup (\bigcup_{n=o}^{\infty} F_n)$ is the union of two elements of $\mathscr{G}$.

This is a particular case of (the proof of) Lemma 4.1 in [9]. □

DEFINITION. Let N be an integer ≥ 1. A *K-sequence of order N* is a finite sequence

$$((\bar{\mu}^o, \bar{n}^o), \ldots, (\bar{\mu}^p, \bar{n}^p))$$

where $\bar{\mu}^i \in \mathscr{R}^N$, $\bar{n}^i \in \omega^N$, such that:
(i) $|\hat{\mu}_j^{i+1}(r) - \hat{\mu}_j^i(r)| < 2^{-Ni-j-1}$ if $|r| \leq n_{j-1}^{i+1}$ or $|r| \geq n_j^{i+1}$ (we let $n_{-1}^{i+1} = n_{N-1}^i$);
(ii) $0 < n_o^o = n_1^o = \cdots = n_{N-1}^o < n_o^1 < \cdots$.

The letter "K" stands for Kechris because such sequences are used in [9] (see also [3] and [10]). As usual, if S and T are K-sequences $T \precsim S$ means that S is an extension of T. Finally, an *infinite K-sequence (of order N)* is a $\Sigma \in (\mathscr{R}^N \times \omega^N)^\omega$ such that $\Sigma_{\restriction p}$ is a K-sequence for every $p \in \omega$.
The following observations are essential in the proof of Lemma 2.1 in [9].

LEMMA 5. (a) *If*

$$S = ((\bar{\mu}^o, \bar{n}^o), \ldots, (\bar{\mu}^p, \bar{n}^p)) \quad (p \geq 1)$$

is a K-sequence of order N and if we let

$$\mu^o(S) = \frac{1}{N}\left(\sum_{j=o}^{N-1} \mu_j^o\right), \mu^p(S) = \frac{1}{N}\left(\sum_{J=o}^{N-1} \mu_j^p\right),$$

then

$$\| \mu^o(S) - \mu^p(S) \|_{PM} \le \frac{3 - 2^{-Np}}{N}.$$

(b) *If* $\Sigma = ((\overline{\mu}^i, \overline{n}^i))_{i \in \omega}$ *is an infinite K-sequence of order N, then for all* $j \le N - 1$ *the sequence* $(\mu_j^i)_{i \in \omega}$ *converges* ω^* *to a probability measure* μ_j. *If we let*

$$\mu = \frac{1}{N} \left(\sum_{j=o}^{N-1} \mu_j \right),$$

then $R(\mu) \le 3/N$.

Proof. (b) is an immediate consequence of (a). Indeed, it follows from the definition of a K-sequence that $(\mu_j^i)_{i \ge o}$ converges in **P(T)**, and part (a) gives the desired inequality because $\mu^o = \mu^o(\Sigma_{\lceil 1})$ is a Rajchman measure.

To prove (a), let us fix $r \in \mathbb{Z}$. We can write

$$\begin{aligned} \mu^p - \mu^o &= \frac{1}{N} \sum_{j=o}^{N-1} (\mu_j^p - \mu_j^o) \\ &= \frac{1}{N} \sum_{j=o}^{N-1} \sum_{i=o}^{p-1} (\mu_j^{i+1} - \mu_j^i) \end{aligned}$$

Hence

$$| \hat{\mu}^p(r) - \hat{\mu}^o(r) | \le \frac{1}{N} \sum_{j=o}^{N-1} \sum_{i=o}^{p-1} | \hat{\mu}_j^{i+1}(r) - \hat{\mu}_j^i(r) |$$

Now properties (i) and (ii) imply that $|\hat{\mu}_j^{i+1}(r) - \hat{\mu}_j^i(r)|$ is $< 2^{-Ni-j-1}$ except for at most one pair (i, j), and in any case it is bounded by 2. Therefore we obtain

$$| \hat{\mu}^p(r) - \hat{\mu}^o(r) | \le \frac{1}{N} \left(2 + \sum_{j=o}^{N-1} \sum_{i=o}^{p-1} 2^{-Ni-j-1} \right)$$

and we are done because the sum in the right-hand side is exactly $\sum_{k=1}^{Np} 2^{-k}$. □

We can now turn to the proof of Theorem 2. This proof looks very much like that of Theorem 1, but is a little more technical. The arithmetic progressions will be replaced by sets constructed by Kechris in [9], which are finite unions of sets in $\mathscr{G}$ whose η_o is arbitrarily small. To be precise, beginning with a Rajchaman probability measure μ and an integer $N \geq 1$, Kechris constructs an infinite K-sequence of order N, $\Sigma = ((\bar{\mu}^i, \bar{n}^i))_{i \in \omega}$ with $\bar{\mu}^o = (\mu, \dots, \mu)$, such that for all $i \in \omega$, $j \leq N - 1$,

$$\operatorname{supp}\left(\mu_j^{i+1}\right) \subseteq \operatorname{supp}\left(\mu_j^i\right),$$
$$\operatorname{supp}\left(\mu_j^i\right) \in \mathscr{U}^i \quad \left(\text{where } \mathscr{G} = \bigcap \mathscr{U}^i, \mathscr{U}^i \text{ open hereditary}\right).$$

By Lemma 5 the result is then a probability measure

$$\nu = \frac{1}{N} \sum_{j=o}^{N-1} \mu_j$$

where $\operatorname{supp}(\mu_j) \in \mathscr{G}$ and $R(\nu) \leq 3/N$. Thus $F = \operatorname{supp}(\nu)$ is a finite union of $\mathscr{G}$ sets and $\eta_o(F) \leq 3/N$.

This construction plays a key role in the proof below.

Let us fix our notations. E is the given M_o^p set and we let $\mathscr{G} = \bigcap_{n \geq o} \mathscr{U}^n$ where the $\mathscr{U}^n$ are open, hereditary subsets of $\mathscr{K}(E)$ and $\mathscr{U}^{n+1} \subseteq \mathscr{U}^n$ for all n (see the remarks before the proof of Theorem 1).

The class $\mathscr{P}$ of perfect subsets of E is G_δ in $\mathscr{K}(E)$, hence it is a Polish space. Thus we can choose some complete metric δ on $\mathscr{P}$. Of course, δ is not the Hausdorff metric (but it defines the same topology on $\mathscr{P}$).

Finally, if $s \in 2^{<\omega}$, $|s| \geq 1$, recall that s' is the sequence $s_{\lceil |s|-1}$.

Now for each $s \in 2^{<\omega}$ and $m < |x|$ we construct

a closed set $E^m(s) = E_o^m(s) \cup \cdots \cup E_{(m)_o}^m(s)$ where the $E_j^m(s)$ are closed (but not necessarily disjoint),

an integer $p^m(s)$,

a K-sequence $S^m(s)$ of order $(m)_o$ and of length $1 + p^m(s)$,

a non empty open set $V(s) \subseteq E$,

satisfying the following conditions:

(1) $\operatorname{diam}(V(s)) \leq 2^{-|s|}$.

(2) $V(s) \cap (\bigcup_{m<|s|} E^m(s)) = \varnothing$;
The $E^m(s)$ are pairwise disjoint.

(3) Each $E_j^m(s)$ is the closure of a non empty open subset of E.

(4) $\bar{V}(s) \subseteq V(s')$,
$E^n(s) \subseteq V(s')$ if $|s| = n + 1$.

(5) If we denote by $((\mu_o^m(s), \dots, \mu_{(m)_o}^m(s)), \bar{n}^m(s))$ the last coordinate of $S^m(s)$ (i.e., that of index $p^m(s)$) then $E_j^m(s) = \operatorname{supp}(\mu_j^m(s))$.

(6) If $t \precsim s$, $m < |t|$ and $j \leq (m)_o$ then

$$E_j^m(s) \subseteq E_j^m(t),$$
$$\delta\left(E_j^m(s), E_j^m(t)\right) < 2^{-|t|}.$$

(7) If $|s| = n + 1$ and $(m)_o < (n)_o$, then

$$p^m(s) = 1 + p^m(s'),$$
$$S^m(s') \precsim S^m(s).$$

(8) If $|s| = n + 1$ and p is any integer, then

$$\left(\bigcup_{\substack{m \le n \\ (m)_o = p}} E_j^m(s) \right) \cup \overline{V}(s) \in \mathscr{U}^n \quad \text{for any } j \le p.$$

(9) If $|s| = n + 1$ and $s(n) = 0$, then

$$p^n(s) = 0,$$
$$p^m(s) = 1 + p^m(s') \text{ and } S^m(s') \precsim S^m(s) \text{ for } m < n.$$

(10) If $|s| = n + 1$ and $s(n) = 1$, then

$$p^m(s) = 0 \text{ if } (m)_o \ge (n)_o,$$
$$\left(\bigcup_{p \ge (n)_o} \bigcup_{\substack{(m)_o = p \\ j \le p}} E_j^m(s) \right) \cup \overline{V}(s) \in \mathscr{U}^n.$$

We first let $E(\emptyset) = E$. Assume $E^m(t)$, $S^m(t)$ have been constructed for $|t| \le n$, $m < n$, $j \le (m)_o$, and let $s \in 2^{<\omega}$ be a sequence of length $n + 1$. As usual we distinguish two cases.

Case 1. $s(n) = 0$. Let us first modify the $E^m(s')$ for $m < n$ and $(m)_o \neq (n)_o$. So fix $p \neq (n)_o$ such that $(m)_o = p$ for at least one $m \le n$ and let $A_p = \{m < n;\ (m)_o = p\}$.

We will define $p^m(s)$, $S^m(s)$, $E_j^m(s)$ for $m \in A_p$, $j \le p$, and a non empty open set V_p of diameter less than 2^{-n-1} in such a way that

$$E_j^m(s) \subseteq E_j^m(s'),$$
$$\delta\left(E_j^m(s), E_j^m(t)\right) < 2^{-|t|} \quad \text{for each } t \precsim s'$$
$$p^m(s) = 1 + p^m(s')$$
$$S^m(s') \precsim S^m(s)$$
$$\overline{V}_p \subseteq V(s')$$
$$\overline{V}_p \cup \left(\bigcup_{m \in A_p} E_j^m(s) \right) \in \mathscr{U}^n \text{ for every } j \le p.$$

We begin with $j = 0$. Take a non empty open set V such that $\overline{V} \subseteq V(s')$ and with diameter less than 2^{-n-1}. By (2) and (3), the sets $E_o^m(s')$, $m \in A_p$ are pairwise disjoint $\overline{M_o^p}$ sets, disjoint from $\overline{V}$, and $\mathscr{U}^n$ is dense in $\mathscr{K}((\bigcup_{m \in A_p} E_o^m(s')) \cup \overline{V})$. Moreover, by (5), $E_o^m(s') = \operatorname{supp}(\mu_o^m(s'))$ (the notation is that of (5)).

Let $k^m(s')$ be the last integer occurring in $S^m(s')$ (that is, $k^m(s') = n^m_{(m)_o}(s')$). Then, since all the sets involved are perfect, it follows at once from the corollary to Lemma 3 that one can choose probability Rajchman measures $\mu_o^m(s)$, $m \in A_p$ and a non empty open set $V_{p,o}$ such that $E_o^m(s) = \operatorname{supp}(\mu_o^m(s))$ is the closure of an open set and

$\hat{\mu}_o^m(s)$ approximates "closely" $\hat{\mu}_o^m(s')$ on $\{r \in \mathbb{Z};\ |r| \le k^m(s')\}$,

$V_{p,o}$ and $E_o^m(s)$ satisfy the conditions above (with $V_{p,o}$ in place of V_p).

Since $\mu_o^m(s)$ and $\mu_o^m(s')$ are Rajchman measures, we can choose for each $m \in A_p$ an integer $l^m(s)$ such that $|\hat{\mu}_o^m(s)(r)|$ and $|\hat{\mu}_o^m(s')(r)|$ are "small" for $|r| \ge l^m(s)$. Then $|\hat{\mu}_o^m(s)(r) - \hat{\mu}_o^m(s')(r)|$ will be small as well for $|r| \ge l^m(s)$. At this point, we have constructed for $m \in A_p$ the first "coordinate" of $S^m(s)(p^m(s))$, namely $(\mu_o^m(s), l^m(s))$, the sets $E_o^m(s)$ and an auxiliary open set $V_{p,o}$. By repeated applications of Lemma 3 we can now get the K-sequence $S^m(s)$, the sets $E_j^m(s)$, $j \le p$ and open sets $V_{p,o} \supseteq V_{p,1} \supseteq \cdots \supseteq V_{p,p}$ such that for all $j \le p$,

$$\overline{V}_{p,j} \cup \left(\bigcup_{m \in A_p} E_j^m(s) \right) \in \mathscr{U}^n .$$

If we let $V_p = V_{p,p}$ then since $\mathscr{U}^n$ is hereditary, we do have

$$\overline{V}_p \cup \left(\bigcup_{m \in A_p} E_j^m(s) \right) \in \mathscr{U}^n \quad \text{for all } j.$$

Treating in the same way all the $p \neq (n)_o$ such that $A_p \neq \emptyset$, we get the K-sequence $S^m(s)$ and the sets $E_j^m(s)$, $j \le (m)_o$ for each $m < n$ with $(m)_o \neq (n)_o$. Then (3), (5), (6), (7), (8), (9) and one half of (2) are satisfied if $(m)_o \neq (n)_o$. We also obtain a non empty open set U disjoint from the $E_j^m(s)$ such that (8) is true with U for all $p \neq (n)_o$.

Now we define $S^m(s)$, $E^m(s)$ for $m \le n$ and $(m)_o = (n)_o$. We first choose disjoint non empty sets $V, W \subseteq U$. Then $S^m(s)$ and $E^m(s)$ are obtained exactly as before, using $(n)_o + 1$ times the corollary to Lemma 3. $E^n(s)$ is constructed inside $\overline{W}$ and we define

$$S^n(s) = ((\mu, \ldots, \mu), (1, \ldots, 1))$$

where μ is any probability Rajchman measure such that $\operatorname{supp}(\mu) = E^n(s)$; if $m < n$, $E^m(s)$ is constructed inside $E^m(s')$. As before, we also construct open sets $V = V_o \supseteq V_1 \supseteq \cdots \supseteq V_{(m)_o}$ and we let $V(s) = V_{(m)_o}$. Then conditions (1), ..., (9) are satisfied.

Case 2. $s(n) = 1$. We first construct, as in case 1, $E_j^m(s)$, $S^m(s)$ for $(m)_o < (n)_o$ (and $j \leq (m)_o$). Then (7) is true. We also get an auxiliary open set U disjoint from all the $E^m(s)$, $(m)_o < (n)_o$, with $\overline{U} \subseteq V(s')$ and $\operatorname{diam}(U) < 2^{-|s|}$, such that (8) is satisfied for $p < (n)_o$. Finally, we choose disjoint non empty open sets $V, W \subseteq V$ and put $E_j^n(s') = \overline{W}$ for $j \leq (n)_o$.

Now let $A = \{m \leq n; (m)_o \geq (n)_o\}$. Using Lemma 2 and properties (3), (6) for s' we can find closed sets $E_j^m(s)$, $m \in A$, $j \leq (m)_o$ and a non empty open set $V(s) \subseteq V$ such that:

each $E_j^m(s)$ is the closure of an open subset of E;
$E_j^m(s) \subseteq E_j^m(s')$;
$\delta(E_j^m(s), E_j^m(t)) < 2^{-|t|}$ for every $t \precsim s'$;
$\overline{V}(s) \cup (\bigcup_{\substack{m \in A \\ j \leq (m)_o}} E_j^m(s)) \in \mathscr{U}^n$.

Then properties (1), (2), (3), (4), (6), (10) are satisfied, as well as (8) for $p \geq (n)_o$ because $\mathscr{U}^n$ is hereditary.

Finally we define $S^m(s) = ((\mu_o^m(s), \ldots, \mu_{(m)_o}^m(s)), (1, \ldots, 1))$ where the $\mu_j^m(s)$ are Rajchman probability measures such that $\operatorname{supp}(\mu_j^m(s)) = E_j^m(s)$.

This concludes the inductive step.

Now if $\alpha \in \mathbf{2}^\omega$, it follows from (6) that for every $m \in \omega$ and $j \leq (m)_o$, the sequence $(E_j^m(\alpha\lceil n))_{n > m}$ converges in $\mathscr{K}(\mathbf{T})$ to a perfect set $E_j^m(\alpha)$. For $m \in \omega$ we let

$$E^m(\alpha) = \bigcup_{j \leq (m)_o} E_j^m(\alpha).$$

By (1) and (4) there is a unique point $x(\alpha)$ in $\cup_{n \in \omega} \overline{V}(\alpha\lceil n)$ and (4) implies that $E(\alpha) = (\bigcup_{m \in \omega} E^m(\alpha)) \cup \{x(\alpha)\}$ is a closed subset of E. Since the $E^m(\alpha)$ are perfect, $E(\alpha)$ is perfect as well. Furthermore, (1), (4) and (6) together imply that the map $\alpha \mapsto E(\alpha)$ is continuous.

It remains to show that the map just defined is the reduction we are looking for. So we fix $\alpha \in \mathbf{2}^\omega$ and, for the last time, distinguish two cases.

Case 1. α_p is finite for every $p \in \omega$. Let p_o be a non negative integer. We show that $\eta_o(E_\alpha) \leq 3/(p_o + 1)$. Since p_o is arbitrary, this will imply that $E(\alpha) \notin U_o'$. As in the proof of Theorem 1, there is a $q_o > 0$ such that if we let $n_o = \langle p_o, q_o \rangle$ then

$$\forall n > n_o \quad (n)_o \leq p_o \Rightarrow \alpha(n) = 0.$$

Using (7) if $(n)_o > p_o$ and (9) if $\alpha(n) = 0$ we deduce that

$$S^{n_o}(\alpha_{\lceil n+1}) \succ S^{n_o}(\alpha_{\lceil n}) \text{ for every } n > n_o.$$

Thus it follows from Lemma 5 (together with (5)) that $\eta_o(E^{n_o}(\alpha)) \leq 3/(p_o + 1)$. This concludes case 1 since $E(\alpha) \supseteq E^{n_o}(\alpha)$.

Case 2. α_{p_o} is infinite for some $p_o \in \omega$. Let $\mathscr{G}_f \subseteq \mathscr{K}(\mathbf{T})$ be the class of all finite unions of elements of $\mathscr{G}$. First we note that for any integer p and each $j \leq p$

$$E_{j,p} = \{x(\alpha)\} \cup \left(\bigcup_{\substack{m \in \omega \\ (m)_o = p}} E_j^m(\alpha) \right) \in \mathscr{G}_f.$$

Indeed, for any $N \in \omega$,

$$\{x(\alpha)\} \cup \left(\bigcup_{\substack{m \leq N; \\ (m)_o = p}} E_j^m(\alpha) \right)$$

is in $\mathscr{G}$ by (6), (8), the definition of $x(\alpha)$ and the fact that each $\mathscr{U}^n$ is hereditary. Thus we can apply Lemma 4.

It follows that for each $p \in \omega$,

$$E_p = \{x(\alpha)\} \cup \left(\bigcup_{\substack{m \in \omega \\ (m)_o = p}} E^m(\alpha) \right) \in \mathscr{G}_f.$$

Now if α_{p_o} is infinite, we deduce from (10) (using Lemma 4 again) that the set

$$\{x(\alpha)\} \cup \left(\bigcup_{(m)_o \geq p_o} E^m(\alpha) \right)$$

is in $\mathscr{G}_f$. So

$$E(\alpha) = \{x(\alpha)\} \cup \left(\bigcup_{(m)_o \geq p_o} E^m(\alpha) \right) \cup \left(\bigcup_{p < p_o} E_p \right)$$

is indeed a finite union of $\mathscr{G}$ sets.

This concludes the proof of Theorem 2. □

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