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UNIFORM EXTENDIBILITY OF THE BERGMAN KERNEL

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1. Introduction

In this note we simplify the proof of and extend a theorem of S-C Chen on the uniform extendibility of the Bergman kernel function and its relation to global regularity properties of the Bergman projection. Applications are given to Bergman kernel density and finite order vanishing theorems which arise in mapping problems between equidimensional domains.

Suppose Ω is a bounded domain in $\mathbf{C}^n$. The Bergman projection P associated to Ω is the orthogonal projection of $L^2(\Omega)$ onto $H^2(\Omega)$ where $H^2(\Omega)$ denotes the closed subspace of $L^2(\Omega)$ consisting of square integrable holomorphic functions on Ω. The Bergman kernel function $K(z,w)$ is the kernel function for P and satisfies

$$Pu(z) = \int_\Omega K(z,w)u(w)\, dV_w \quad \text{for all } u \in L^2(\Omega).$$

We say that condition Q holds on Ω if P maps $C_0^\infty(\Omega)$ into $\mathscr{O}(\overline{\Omega})$, where $\mathscr{O}(\overline{\Omega})$ denotes the space of holomorphic functions on Ω that can be extended holomorphically to some open set containing $\overline{\Omega}$. Finally, we let $\mathscr{O}(\Omega)$ denote the space of all holomorphic functions on Ω and $A^\infty(\Omega)$ denote the space of holomorphic functions on Ω that are in $C^\infty(\overline{\Omega})$.

Extendibility properties of the Bergman kernel are important in the study of extension of biholomorphic and proper holomorphic mappings between domains in $\mathbf{C}^n$. It has been shown that extendibility of the mappings can be deduced from extendibility properties of the Bergman kernel, which follow from global regularity properties of the Bergman projection (see Bell [1]).

It is known that condition Q holds on a domain Ω whenever the $\bar{\partial}$-Neumann problem is globally real analytic hypoelliptic on Ω; for instance see Bell [2]. This property of the $\bar{\partial}$-problem is known to hold in strictly pseudo-

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convex domains with real analytic boundaries (see Derridj and Tartakoff [11], Komatsu [14], Tartakoff [18–19], Treves [20]) and in certain weakly pseudoconvex domains with real analytic boundaries (see [10], [12] and [13]). Global real analytic hypoellipticity of the $\bar{\partial}$-Neumann problem is also known to hold in domains with lots of symmetries, including Reinhardt domains (see Chen [6]) certain circular domains [7], and other special domains in $\mathbf{C}^2$ [9].

It is easy to see that extendibility properties of the Bergman kernel imply condition Q. The relationship between these two properties is most transparent in a complete Reinhardt domain, where the associated Bergman kernel function $K(z,w)$ extends to be holomorphic in z on a neighborhood of $\overline{\Omega}$ whenever w is restricted to lie in a compact subset of Ω. From this, it follows that condition Q is satisfied on a bounded complete Reinhardt domain in $\mathbf{C}^n$. In this paper, we are concerned with the reverse implication, and with seeing that the two properties that hold on Reinhardt domains are typical.

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2. Main results

We shall prove the following theorem, which generalizes a theorem of Chen [8]. The proof we give also yields an elementary proof of Chen's original result.

THEOREM 2.1. *Let Ω be a bounded domain in $\mathbf{C}^n$. Then the following are equivalent*:

(i) *Condition Q holds on Ω.*

(ii) *For each compact subset $\mathscr{K}$ of Ω, there exists an open set $\mathscr{U}_{\mathscr{K}}$ containing $\overline{\Omega}$ such that*

(a) *for each fixed $w_0 \in \mathscr{K}$, $K(z,w_0)$ is a holomorphic function of z on $\mathscr{U}_{\mathscr{K}}$ and*

(b) $K(z,w) \in C(\mathscr{U}_{\mathscr{K}} \times \mathscr{K})$.

In fact, condition (b) *can be replaced by the following weaker condition*: (b′) $|K(z,w)| < C$, *for some constant $C = C_{\mathscr{K}} > 0$, and $K(z,w)$ is measurable for all $(z,w) \in \mathscr{U}_{\mathscr{K}} \times \mathscr{K}$.*

Next we shall prove the following uniform holomorphic extension theorem for the Bergman kernel function on certain bounded domains in $\mathbf{C}^n$ which are not necessarily smooth. We remark that this theorem contains Chen's main theorem in [8] and that the proof is basically contained in the proof of the previous theorem. We shall see that (ii) ⇒ (i) is rather easy and for (i) ⇒ (ii) we begin by following Chen's argument closely; however, we give a substantial simplification of the proof as a whole. To wit, the need to use the special cut-off functions introduced by Ehrenpreis [8] and an associated estimate, Tartakoff [17], is eliminated. Instead we are able to apply the classical theorems of Morera and Hartogs to come to the same conclusion.

THEOREM 2.2. *Let Ω be a bounded domain in $\mathbf{C}^n$. Then the following are equivalent.*

(i) *Condition Q holds on Ω.*

(ii) *For each compact subset $\mathcal{K}$ of Ω, there exists an open set $\mathcal{U}_{\mathcal{K}}$ containing $\overline{\Omega}$ such that $K(z,w)$ extends holomorphically in z and anti-holomorphically in w to the set $\mathcal{U}_{\mathcal{K}} \times \mathcal{K}$.*

Now we give some applications of Theorem 2.2. S. Bell posed the following question. Given a bounded domain Ω in $\mathbf{C}^n$, a multi-index β and two points, $z_0 \in b\Omega$ and $w_0 \in \Omega$, is there a multi-index α such that $(\partial^{|\alpha|+|\beta|}/\partial z^\alpha\, \partial \overline{w}^\beta)K(z_0, w_0) \neq 0$? If $n = 1$, the answer is yes at every smooth boundary point, and if $n \geq 2$ the answer is yes in the case of smooth domains for which the $\bar{\partial}$-problem satisfies a unique continuation property, Bell [3]. Here we show that the answer is also yes for domains satisfying condition Q.

THEOREM 2.3. *Let Ω be a bounded domain in $\mathbf{C}^n$ which satisfies condition Q. Then, given $z_0 \in b\Omega$, $w_0 \in \Omega$ and a multi-index β there exists a multi-index α such that*

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha\, \partial \overline{w}^\beta} K(z_0, w_0) \neq 0.$$

Let us make the following definition.

DEFINITION 2.4. Let $\mathcal{K}$ be a compact subset of a domain Ω in $\mathbf{C}^n$. We say that $\mathcal{K}$ is H^2-Runge with respect to Ω if any function f which is holomorphic on a neighborhood of $\mathcal{K}$ can be approximated uniformly on $\mathcal{K}$ by functions in $H^2(\Omega)$.

This condition is easy to verify, for example, if $\mathcal{K}$ is a polynomially convex subset of a bounded domain Ω in $\mathbf{C}^n$. Now we can state the following Bergman kernel density theorem for domains satisfying condition Q.

THEOREM 2.5. *Let Ω be a bounded domain in $\mathbf{C}^n$ which satisfies condition Q. Suppose $w_0 \in b\Omega$ and $\mathcal{K}$ is a compact subset of Ω which is H^2-Runge with respect to Ω. Given a holomorphic function f defined on a neighborhood of $\mathcal{K}$ and an $\varepsilon > 0$, there is a function κ in the complex linear span S of*

$$\left\{ \frac{\partial^{|\beta|}}{\partial \overline{w}^\beta} K(z, w_0) : |\beta| \geq 0 \right\}$$

such that $|f - \kappa| < \varepsilon$ on $\mathcal{K}$.

We remark that Bell [3] has proved similar theorems on smoothly bounded domains using an approximation theorem of Catlin [5].

3. Proofs of the theorems

Recall that the Bergman kernel function $K(z,w)$ for Ω is holomorphic in z and anti-holomorphic in w for $z,w \in \Omega$. Also, if for each fixed $a \in \Omega$ we choose a function $\varphi_a \in C_0^\infty(\Omega)$ such that φ_a is radially symmetric about a, $\varphi_a \geq 0$, and $\int_\Omega \varphi_a(z)\, dV_z = 1$, then we can recover the Bergman kernel function by $K(z,a) = (P\varphi_a)(z)$.

DEFINITION 3.1. A subset $\mathscr{A}$ of a domain Ω in $\mathbf{C}^n$ is called a set of uniqueness for Ω if any holomorphic function f on Ω which vanishes on $\mathscr{A}$ is identically zero on Ω.

Note that any subset of Ω with non-empty interior is a set of uniqueness for Ω.

LEMMA 3.2 (Vitali). *Let $\{f_\nu\}$ be a sequence of holomorphic functions on a domain Ω in $\mathbf{C}^n$ and let $\mathscr{A}$ be a set of uniqueness for Ω. Suppose $\{f_\nu\}$ is uniformly bounded on Ω, that is there exists a positive number M such that $|f_\nu(z)| < M$ for all $z \in \Omega$ and for all ν, and suppose that $\{f_\nu(a)\}$ converges for any $a \in \mathscr{A}$. Then $\{f_\nu\}$ converges uniformly on compact subsets of Ω.*

The proof of Lemma 3.2 is a normal family argument and can be found in Narasimhan [15].

Proof of Theorem 2.1. (ii) $\Rightarrow$ (i). Let $\varphi \in C_0^\infty(\Omega)$ and $\mathscr{K} = \operatorname{supp}(\varphi)$, the support of φ. Then there exists an open set $\mathscr{U}_{\mathscr{K}} \supset \overline{\Omega}$ such that (a) and (b′) hold. Now,

$$P\varphi(z) = \int_\Omega K(z,w)\varphi(w)\, dV_w = \int_{\mathscr{K}} K(z,w)\varphi(w)\, dV_w. \tag{3.3}$$

We claim that the integral above provides a holomorphic extension of $P\varphi$ to the set $\mathscr{U}_{\mathscr{K}}$. Indeed, let γ_j, $j \in \{1,\dots,n\}$, be a closed curve in $\mathbf{C}^n$ which lies in $\mathscr{U}_{\mathscr{K}}$ and is parametrized by a function of the form $z(t) = (z_1(t),\dots, z_n(t)) : [0,1] \to \mathbf{C}^n$ where $z_i(t)$ is constant if $i \neq j$. Then

$$\int_{\gamma_j}\left(\int_{\mathscr{K}} K(z,w)\varphi(w)\, dV_w\right) dz_j = \int_{\mathscr{K}}\left(\int_{\gamma_j} K(z,w)\, dz_j\right)\varphi(w)\, dV_w = 0$$

by applying Fubini's theorem and part (a) of (ii). Now we may apply the

Lebesgue dominated convergence theorem to see that the second integral in (3.3) defines a continuous function for $z \in \mathcal{U}_{\mathcal{K}}$; then by applying Morera's theorem we see that $P\varphi$ is a holomorphic function of z_j for each $j = 1, \ldots, n$ and so, by Hartogs theorem on separate analyticity, $P\varphi$ extends to be holomorphic on $\mathcal{U}_{\mathcal{K}}$. In case of condition (b), we have that (b) implies (b′) by shrinking $\mathcal{U}_{\mathcal{K}}$ slightly to get the uniform boundedness hypothesis and so this half of the theorem is proved.

(i) ⇒ (ii). Let $\mathcal{K}$ be a non-empty compact subset of Ω. Choose another compact subset $\mathcal{K}'$ of Ω such that $\mathcal{K} \subset \operatorname{int}(\mathcal{K}')$. To get the open neighborhood $\mathcal{U}_{\mathcal{K}}$ of Ω we will use a Baire category argument.

Following Rudin [16], we let $C^\infty(\mathcal{K}')$ denote the set of $f \in C^\infty(\mathbf{C}^n)$ such that $\operatorname{supp}(f) \subset \mathcal{K}'$. The topology on $C^\infty(\mathcal{K}')$ is given by seminorms

$$\rho_N(f) = \max_{\mathcal{K}'}\{|D^\alpha f| : |\alpha| \le N\},$$

for $N = 1, 2, \ldots,$ and a local basis is given by the sets

$$V_N = \left\{f \in C^\infty(\mathcal{K}') : \rho_N(f) < \frac{1}{N}\right\},$$

for $N = 1, 2, \ldots$. Further, $\{\rho_N\}_{N=1}^\infty$ induces a compatible metric on $C^\infty(\mathcal{K}')$ that makes it a complete metric space.

Now we set up for the application of the Baire category theorem. Let $\{\Omega_j\}_{j=1}^\infty$ be a strictly decreasing sequence of open neighborhoods of $\overline{\Omega}$ such that $\overline{\Omega} = \bigcap_{j=1}^\infty \Omega_j$. Further, we require that for every neighborhood $\mathcal{N}$ of $\overline{\Omega}$, the set Ω_j is contained in $\mathcal{N}$ for j sufficiently large. For each $j, k = 1, 2, \ldots$ define the set

$$E_{jk} = \left\{f \in C^\infty(\mathcal{K}') : Pf \in \mathcal{O}(\Omega_j) \quad \text{and} \quad \sup_{\Omega_j}|Pf| \le k\right\},$$

then by condition Q we have that E_{jk} is nonempty for sufficiently large j, k, say for $j, k \ge M = M_{\mathcal{K}'}$, and

$$C^\infty(\mathcal{K}') = \bigcup_{j,k=M}^{\infty} E_{jk}.$$

We claim that E_{jk} is closed for each $j, k \ge M$. To see this, let $\{f_l\}_{l=1}^\infty$ be a Cauchy sequence in E_{jk} with f_l converging to some $f \in C^\infty(\mathcal{K}')$. Now, for

$z \in \mathscr{K}'$ we have

$$\begin{aligned}|Pf_l(z) - Pf(z)| &= |P(f_l - f)(z)| \\ &= \left| \int_{w \in \mathscr{K}'} K(z,w)(f_l(w) - f(w))\, dV_w \right| \\ &\le \int_{w \in \mathscr{K}'} |K(z,w)|\,|f_l(w) - f(w)|\, dV_w .\end{aligned}$$

Since $|K(z,w)| \le C$ for $(z,w) \in \mathscr{K}' \times \mathscr{K}'$ we have that

$$|Pf_l(z) - Pf(z)| \le C \int_{\mathscr{K}'} |f_l(w) - f(w)|\, dV_w .$$

Therefore, $f_l \to f$ in $C^\infty(\mathscr{K}')$ implies that $Pf_l(z) \to Pf(z)$ for $z \in \mathscr{K}'$. Now we may apply Vitali's theorem to see that $Pf_l(z)$ converges uniformly on compact subsets of Ω_j to some function $g(z) \in \mathscr{O}(\Omega_j)$ with $\sup_{\Omega_j} |g| \le k$. Hence $g(z) = Pf(z)$ for all $z \in \Omega_j$, that is $f \in E_{jk}$, which proves the claim.

Now the Baire category theorem implies that there exist $j, k \in \mathbf{Z}_+$ such that E_{jk} contains an open subset V of $C^\infty(\mathscr{K}')$. Then since E_{jk} is balanced and convex we may choose $n \in \mathbf{Z}_+$ such that $V_N \subset E_{jk}$. So, for any $f \in V_N$ we have

$$Pf \in \mathscr{O}(\Omega_j) \quad \text{and} \quad \sup_{\Omega_j} |Pf| \le k .$$

We set $\delta = \frac{1}{2}\operatorname{dist}(\mathscr{K}, \mathbf{C}^n \setminus \operatorname{int}(\mathscr{K}')) > 0$ and choose an even function $\xi(x) \in C_0^\infty[(-\delta, \delta)]$ with $\xi(x) \ge 0$ for all $x \in \mathbf{R}$. Let

$$\eta_w(w') = \xi(|w - w'|^2)$$

$$\zeta_w(w') = \frac{\eta_w(w')}{2N\rho_N(\eta_w)}$$

for $w \in \mathscr{K}$, $w' \in \mathbf{C}^n$. Notice that for any $w \in \mathscr{K}$, η_w and ζ_w vanish for $w' \in \mathbf{C}^n \setminus \operatorname{int}(\mathscr{K}')$, so that η_w and ζ_w are in $C^\infty(\mathscr{K}')$. Further,

$$\rho_N(\zeta_w) = \rho_N\left(\frac{\eta_w}{2N\rho_N(\eta_w)} \right) = \frac{1}{2N} < \frac{1}{N}$$

so that $\zeta_w \in V_N$ for any $w \in \mathscr{K}$. Also, it is easy to see that

$$a = \int_{\mathbf{C}^n} \zeta_w(w')\, dV_{w'} = \frac{1}{2N\rho_N(\eta_w)} \int_{\mathscr{K}'} \eta_w(w')\, dV_{w'}$$

is independent of w for $w \in \mathscr{K}$. Now we may recover $K(z,w)$ by $K(z,w) = P((1/a)\zeta_w)(z)$. Hence, $aK(z,w) = (P\zeta_w)(z)$ which implies that for any $w \in \mathscr{K}$ we have

$$(3.4) \qquad K(z,w) \in \mathscr{O}(\Omega_j) \quad \text{and} \quad \sup_{\Omega_j} |K(z,w)| \leq \frac{k}{a}$$

where we stress that k/a is independent of w.

Finally we show that $K(z,w) \in C(\Omega_j \times \mathscr{K})$. Let $(z_\nu, w_\nu) \to (z,w)$ in $\mathscr{K}_j \times \mathscr{K}$ where $\mathscr{K}_j$ is a compact subset of Ω_j containing $\mathscr{K}'$. Set $f_\nu(z) = K(z, w_\nu)$. Then by Vitali's theorem, f_ν converges uniformly to $f = K(\cdot, w)$ on $\mathscr{K}_j$, where we take $\mathscr{K}'$ to be our set of uniqueness. So, given an $\varepsilon > 0$ there exists a positive integer N so that $\nu > N$ implies that $|f_\nu(z) - f(z)| < \varepsilon/2$ for all $z \in \mathscr{K}_j$. Now we see that

$$\begin{aligned} |K(z,w) - K(z_\nu, w_\nu)| &\leq |K(z,w) - K(z,w_\nu)| + |K(z,w_\nu) - K(z_\nu,w)| \\ &\quad + |K(z_\nu,w) - K(z_\nu,w_\nu)| \\ &\leq |f(z) - f_\nu(z)| + |f_\nu(z) - f(z_\nu)| \\ &\quad + |f(z_\nu) - f_\nu(z_\nu)| \\ &< \frac{\varepsilon}{2} + |f_\nu(z) - f(z_\nu)| + \frac{\varepsilon}{2} \end{aligned}$$

for $\nu < N$. Now, $f_\nu(z)$ converges to $f(z)$ for all $z \in \mathscr{K}_j$ by Vitali's lemma and $f(z_\nu)$ converges to $f(z)$ by continuity, given by (3.4). So,

$$\lim_{\nu\to\infty} |K(z,w) - K(z_\nu,w_\nu)| < \varepsilon + \lim_{\nu\to\infty} |f_\nu(z) - f(z_\nu)| = \varepsilon + 0 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $K(z,w) \in C(\mathscr{K}_j \times \mathscr{K})$ and now $\mathscr{K}_j$ arbitrary implies that $K(z,w) \in C(\Omega_j \times \mathscr{K})$. Hence, taking $\mathscr{U}_{\mathscr{K}} = \Omega_j$, we are done in the case of (b). As for the case of (b′), measurability is clear and we get uniform boundedness from (3.4). This finishes the proof of Theorem 2.1. □

Proof of Theorem 2.2. (ii) ⇒ (i). This is provided by Theorem 2.1.

(i) ⇒ (ii). Let $\mathscr{K}$ be a compact subset of Ω and let $\mathscr{K}'$ be another compact subset of Ω such that $\mathscr{K} \subset \operatorname{int}(\mathscr{K}')$. Then we let $\mathscr{U}_{\mathscr{K}'}$ be the open neighborhood of $\overline{\Omega}$ associated to $\mathscr{K}'$ which is provided by Theorem 2.1. Next we choose a closed curve γ_j, $j \in \{1, \ldots, n\}$, lying in $\mathscr{K}'$ which is parametrized by a function of the form

$$w(t) = (w_1(t), \ldots, w_n(t)) : [0,1] \to \mathbf{C}^n$$

where $w_i(t)$ is constant if $i \neq j$. Specifically, we are thinking of γ_j as a curve around a point in $\mathscr{K}$. Now define the function

$$f(z) = \int_{\gamma_j} K(z,w)\, d\overline{w}_j, \quad z \in \mathscr{U}_{\mathscr{K}'}.$$

We choose another closed curve Γ_k, $k \in \{1, \dots, n\}$, lying in $\mathscr{U}_{\mathscr{K}'}$ which is parametrized by a function of the form $z(t) = (z_1(t), \dots, z_n(t)) : [0,1] \to \mathbf{C}^n$ where $z_i(t)$ is constant if $i \neq k$. Then we have

$$\int_{\Gamma_k} f(z)\, dz_k = \int_{\gamma_j} \left(\int_{\Gamma_k} K(z,w)\, dz_k \right) d\overline{w}_j = 0$$

where we applied Fubini's theorem and used the fact that $\int_{\Gamma_k} K(z,w)\, dz_k = 0$ since for each fixed $w \in \mathscr{K}'$, $K(z,w)$ is a holomorphic function of $z \in \mathscr{U}_{\mathscr{K}'}$. So, by Morera's theorem, f is a holomorphic function on $\mathscr{U}_{\mathscr{K}'}$. Also, $f(z) = 0$ for each $z \in \Omega$ since $K(z,w)$ is an anti-holomorphic function of w for $w \in \Omega$. We conclude that $f(z) = 0$ for all $z \in \mathscr{U}_{\mathscr{K}'}$ which says that $K(z,w)$ is anti-holomorphic in w for $w \in \mathscr{K}$ by Morera's theorem. Finally, by Hartogs theorem on separate analyticity we have that $K(z,w)$ is holomorphic in z and anti-holomorphic in w on $\mathscr{U}_{\mathscr{K}'} \times \mathscr{K}$. □

Proof of Theorem 2.3. Take a neighborhood $B_\varepsilon(w_0)$, $\varepsilon > 0$, around w_0 such that $\overline{B_\varepsilon(w_0)} \subset \Omega$. By Theorem 2.2, there exists an open neighborhood $\mathscr{U}$ of $\overline{\Omega}$ such that $K(z,w)$ is holomorphic in z and anti-holomorphic in w on $\mathscr{U} \times B_\varepsilon(w_0)$. So, $(\partial^{|\beta|}/\partial \overline{w}^\beta) K(z, w_0)$ is holomorphic for $z \in \mathscr{U}$ and thus has a power series representation about z_0. Therefore there exists a multi-index α such that the conclusion of the theorem holds. □

Proof of Theorem 2.5. Suppose to the contrary that f is a holomorphic function on a neighborhood of $\mathscr{K}$ which cannot be approximated uniformly on $\mathscr{K}$ by functions in S. Then there exists a complex finite Borel measure $d\mu$ on $\mathscr{K}$ such that

$$\int_{\mathscr{K}} \overline{h}\, d\mu = 0 \quad \text{for all } h \in S, \tag{3.5}$$

but $\int_{\mathscr{K}} \bar{f}\, d\mu \neq 0$. Now, by Theorem 2.2, there is an open neighborhood $\mathscr{U}$ of $\overline{\Omega}$ such that $K(w,z)$ extends to be holomorphic in w and anti-holomorphic in z on $\mathscr{U} \times \mathscr{K}$. Hence $(P(d\mu))(w) = \int_{\mathscr{K}} K(w,z)\, d\mu(z)$ extends to be holomorphic on $\mathscr{U} \supset \overline{\Omega}$. Now, (3.5) implies that

$$\frac{\partial^{|\beta|}}{\partial w^\beta} (P(d\mu))(w_0) = \int_{\mathscr{K}} \frac{\partial^{|\beta|}}{\partial w^\beta} K(w_0, z)\, d\mu(z) = \int_{\mathscr{K}} \overline{\frac{\partial^{|\beta|}}{\partial \overline{w}^\beta} K(z, w_0)}\, d\mu(z) = 0,$$

for all multi-indices β with $|\beta| \geq 0$. Therefore we have that $P(d\mu)$ vanishes to infinite order at w_0. But this implies that $P(d\mu) \equiv 0$. So, $d\mu$ is orthogonal to the complex linear span of $\{K(z,w) : w \in \Omega\}$. However, this linear span is easily seen to be dense in $H^2(\Omega)$, and so $d\mu$ is orthogonal to $H^2(\Omega)$. Since $\mathscr{K}$ is H^2-Runge there exists a sequence $\{f_j\} \subset H^2(\Omega)$ such that $f_j \to f$ uniformly on $\mathscr{K}$. So,

$$0 = \int_{\mathscr{K}} \bar{f}_j \, d\mu \to \int_{\mathscr{K}} \bar{f} d\mu \neq 0$$

which gives a contradiction. □

Note added in proof. I wish to thank Harold Boas for pointing out that in the proof of Theorem 2.1 it is possible to work in the normed space $L^2(\mathscr{K}')$ rather than the Frechet space $C^\infty(\mathscr{K}')$ by making the observation that the Bergman projection of a compactly supported L^2 function equals the projection of a compactly supported smooth function, as follows by convolving with a mollifier.

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