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LOWER BOUNDS FOR NORMS ON CERTAIN ALGEBRAS

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Introduction

A Banach algebra $\mathscr{A}$ often carries natural algebra norms other than the complete norm $\|\ \ \|_{\mathscr{A}}$ with which it is equipped. It is then of interest to study the relation of an arbitrary algebra norm on $\mathscr{A}$ to the complete norm of $\mathscr{A}$ (an algebra norm on $\mathscr{A}$ is a norm which satisfies $\|xy\| \leq \|x\|\,\|y\|$, for all $x, y \in \mathscr{A}$). Let us say that an algebra norm $\|\ \ \|$ *dominates* the complete norm $\|\ \ \|_{\mathscr{A}}$ on $\mathscr{A}$ if $\|x\|_{\mathscr{A}} \leq C\|x\|$ for all $x \in \mathscr{A}$ and some constant C. We are now interested in the following property:

(1) *Every algebra norm on $\mathscr{A}$ dominates the complete norm.*

The purpose of this paper is to give a simple argument which suffices to establish property (1) for all noncommutative Banach algebras $\mathscr{A}$ for which it is known to hold, and which also allows us to obtain some new examples.

Let $P = P(\mathscr{A}) = \{q \in \mathscr{A} : qx = x$ for some nonzero $x \in \mathscr{A}\}$ and note that P contains every nonzero idempotent of $\mathscr{A}$. An arbitrary algebra norm $\|\ \ \|$ on $\mathscr{A}$ satisfies $\|q\| \geq 1$, for all $q \in P$: In fact, if $qx = x$, for some nonzero $x \in \mathscr{A}$, then

$$\|x\| = \|qx\| \leq \|q\|\,\|x\|;$$

thus

$$\|q\| \geq 1.$$

This can be exploited as follows: Define

$$\beta(x) = \inf\{\|a\|_{\mathscr{A}}\|b\|_{\mathscr{A}} : a, b \in \mathscr{A} \text{ and } axb \in P\} \quad \text{for all } x \in \mathscr{A}.$$

We set $\beta(x) = \infty$ if there do not exist elements $a, b \in \mathscr{A}$ such that $axb \in P$.

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PROPOSITION 1. *Let $\mathscr{A}$ be a normed algebra. Suppose that C is constant and $\| \ \|$ an algebra norm on $\mathscr{A}$ which satisfies $\|x\| \leq C\|x\|_{\mathscr{A}}$ for all $x \in \mathscr{A}$. Then we have $C^2\beta(x)\|x\| \geq 1$ for all $x \in \mathscr{A}$, with $\beta(x) < \infty$.*

Proof. Suppose that $\beta(x) < \infty$ and let $a, b \in \mathscr{A}$ be such that $axb \in P$. Then

$$1 \leq \|axb\| \leq \|a\| \, \|x\| \, \|b\| \leq C^2\|a\|_{\mathscr{A}} \|b\|_{\mathscr{A}} \|x\|.$$

Taking the infimum over all such $a, b \in \mathscr{A}$ yields the desired inequality. ■

Thus upper bounds for the functional β yield lower bounds for continuous algebra norms on $\mathscr{A}$. Let $S(\mathscr{A}) = \{x \in \mathscr{A} : \|x\|_{\mathscr{A}} = 1\}$ denote the unit sphere of the Banach algebra $\mathscr{A}$. The following theorem allows the reduction from arbitrary norms to continuous norms:

THEOREM 1. *Suppose that the Banach algebra $\mathscr{A}$ satisfies $\beta(x) < \infty$ for all $x \in S(\mathscr{A})$. Then for every algebra norm $\| \ \|$ on $\mathscr{A}$ there exists a continuous algebra norm $\| \ \|_0$ on $\mathscr{A}$ which satisfies $\|x\|_0 \leq \|x\|$ for all $x \in \mathscr{A}$.*

Proof. If $\beta(x) < \infty$, for all $x \in S(\mathscr{A})$, then the Banach algebra $\mathscr{A}$ has property (P) in the sense of [6]. Now use [6, Theorem 2 (C), (E)]. ■

COROLLARY 1. *If the functional β is bounded on the unit sphere $S(\mathscr{A})$, then each algebra norm on $\mathscr{A}$ dominates the complete norm of $\mathscr{A}$.*

Proof. Suppose that $\beta(x) \leq M$, for all $x \in \mathscr{A}$ with $\|x\|_{\mathscr{A}} = 1$. Let $\| \ \|$ be an algebra norm on $\mathscr{A}$ and choose a continuous algebra norm $\| \ \|_0$ on $\mathscr{A}$ such that $\| \ \|_0 \leq \| \ \|$. Choose the constant C such that $\|x\|_0 \leq C\|x\|_{\mathscr{A}}$, for all $x \in \mathscr{A}$. Then, for each $x \in \mathscr{A}$ with $\|x\|_{\mathscr{A}} = 1$, we have

$$1 \leq C^2\beta(x)\|x\|_0 \leq C^2M\|x\|, \text{ that is } \|x\| \geq \frac{1}{C^2M}. \ ■$$

We shall now derive estimates for the functional β for various Banach algebras $\mathscr{A}$. Let $\| \ \| = \| \ \|_{\mathscr{A}}$ in Proposition 1 to note that $\beta(x) \geq 1$ for all $x \in S(\mathscr{A})$.

THEOREM 2. *If $\mathscr{A}$ is a C^*-algebra or $\mathscr{A}$ is the algebra of all bounded linear operators on a Banach space X, then $\beta(x) = 1$, for all $x \in S(\mathscr{A})$.*

According to Corollary 1 these algebras have property (1). This has been known for a long time [2], [14]. However the classical proofs for each case are quite dissimilar.

Let now X be a Banach space and $\mathscr{B} = \mathscr{B}(X)$ the algebra of all bounded linear operators on X. Recall that an operator $t \in \mathscr{B}$ is called *strictly singular* if

$$\inf\{\|tx\| : x \in N \text{ and } \|x\| = 1\} = 0,$$

for each infinite dimensional subspace $N \subseteq X$. The family of strictly singular operators on X is a closed two sided ideal in $\mathscr{B}$ which we denote by S. Let us now estimate the functional β in the quotient algebra $\mathscr{A} = \mathscr{B}/S$. For $t \in \mathscr{B}$ we let $\bar{t}$ denote the coset $t + S \in \mathscr{A}$.

THEOREM 3. *Let $\mathscr{A} = \mathscr{B}/S$ be as above. If $X = l_p$, $1 \le p < \infty$, $X = c_0$ or $X = L^1([0,1])$, then $\beta(x) = 1$, for all $x \in S(\mathscr{A})$.*

If $X = l_p$, $1 \le p < \infty$ or $X = c_0$, then S coincides with the ideal of compact operators on X (the only nontrivial closed two sided ideal in $\mathscr{B}$ in this case). Thus $\mathscr{A}$ is the Calkin algebra on X. This case is also treated in [7].

If $X = L^1([0,1])$, then [8] S coincides with the ideal W of weakly compact operators on X. In this case $\mathscr{A}$ is the weak Calkin algebra $\mathscr{B}/W$ on X.

Finally suppose that $X = C(\Omega)$, where Ω is a compact metric space and $\mathscr{A} = \mathscr{B}/S$ as above. In this case we can only prove a weaker estimate for the functional β on $\mathscr{A}$.

Again [8] the ideal S coincides with the ideal W of weakly compact operators on X so that $\mathscr{A}$ is the weak Calkin algebra on X. Moreover an operator t on X is weakly compact if and only if $tf_n \to 0$, for each bounded sequence $(f_n) \subseteq X = C(\Omega)$, such that $f_n f_m = 0$ for all $n \ne m$ [3, VI.17]. Consequently

$$\Delta(t) = \sup \overline{\lim_{n \uparrow \infty}} \|tf_n\|,$$

where the supremum is taken over all sequences $(f_n) \subseteq X$ with $\|f_n\| = 1$ and $f_n f_m = 0$, for all $n \ne m$, defines a (linear) seminorm on the algebra $\mathscr{B}$, which vanishes exactly on the ideal S. Thus Δ induces a (linear) norm on the quotient $\mathscr{A}$, which we also denote by Δ, by means of $\Delta(\bar{t}) = \Delta(t)$, $t \in \mathscr{B}$.

THEOREM 4. *Let $X = C(\Omega)$, where Ω is a compact metric space and $\mathscr{A} = \mathscr{B}/S$ as above. Then*

$$\beta(x) \le \frac{2}{\Delta(x)} \quad \textit{for all nonzero } x \in \mathscr{A}.$$

Proof of Theorem 2. Assume first that $\mathscr{A}$ is a C^*-algebra and let $x \in S(\mathscr{A})$. Set $u = x^*x$. Then $\rho(u) = \|u\| = \|x\|^2 = 1$. Let now $0 < \alpha < \beta < \gamma < 1$ be arbitrary and choose continuous functions f, g, defined on the complex plane and satisfying $\|f\|_\infty, \|g\|_\infty \le 1$ and

$$f(\lambda) = 0 \text{ for } |\lambda| \le \alpha \quad \text{and} \quad f(\lambda) = 1 \text{ for } |\lambda| \ge \beta,$$
$$g(\lambda) = 0 \text{ for } |\lambda| \le \beta \quad \text{and} \quad g(\lambda) = 1 \text{ for } |\lambda| \ge \gamma.$$

Set $h(\lambda) = f(\lambda)/\lambda$, for all complex numbers λ and note that h is a continuous function satisfying $|h(\lambda)| \le 1/\alpha$ for all λ. The continuous functional calculus now yields elements $b = h[u]$ and $q = g[u]$ in $\mathscr{A}$ which satisfy $\|b\| \le \|h\|_\infty \le 1/\alpha$ and $q \ne 0$ (since $1 \in Sp(q)$, according to the Spectral Mapping Theorem). Since also $\lambda h(\lambda)g(\lambda) = f(\lambda)g(\lambda) = g(\lambda)$ for all λ, we have $ubq = q$; that is, $(x^*xb)q = q$ and consequently $x^*xb \in P(\mathscr{A})$. This shows that

$$\beta(x) \le \|x^*\|_{\mathscr{A}} \|b\|_{\mathscr{A}} \le \frac{1}{\alpha}.$$

The result follows if we let $\alpha \uparrow 1$.

Suppose now that $\mathscr{A} = \mathscr{B}(X)$ is the algebra of all bounded linear operators on some Banach space X and let $t \in S(\mathscr{A})$. Suppose that $0 < \alpha < 1$ and choose a unit vector $u \in X$ with $\|tu\| > \alpha$. Now let $x^* \in X^*$ be a continuous linear function with $\|x^*\| < 1/\alpha$ such that $x^*(tu) = 1$. Let q be the one dimensional operator $b = x^* \otimes u = x^*(\cdot)u \in \mathscr{A}$. Then $\|b\|_{\mathscr{A}} = \|x^*\| < 1/\alpha$ and the operator $tb = x^* \otimes tu$ is a nonzero idempotent. Consequently $tb \in P(\mathscr{A})$ and so

$$\beta(t) \le \|b\|_{\mathscr{A}} < \frac{1}{\alpha}. \quad \blacksquare$$

Let now $\mathscr{A} = \mathscr{B}/S$, where $\mathscr{B} = \mathscr{B}(X)$ is the algebra of all bounded linear operators on X and $S \subseteq \mathscr{B}$ the ideal of strictly singular operators. Recall also that $t \in \mathscr{B} \to \bar{t} \in \mathscr{A}$ denotes the quotient map.

LEMMA 1. *Let $t \in \mathscr{B}$ and suppose that there exist a constant $\rho > 0$, an infinite dimensional subspace $N \subseteq X$ and an idempotent $p \in \mathscr{B}$ such that $p(X) = t(N)$ and $\|tx\| \ge \rho\|x\|$ for all $x \in N$. Then $\beta(\bar{t}) \le \rho^{-1}\|p\|$ in the quotient algebra $\mathscr{A} = \mathscr{B}/S$.*

Proof. In fact the restriction $t|_N : N \to t(N)$ is invertible and satisfies $\|(t|_N)^{-1}\| \le \rho^{-1}$. Moreover $b = (t|_N)^{-1}p$ is a well-defined operator on X which satisfies $\|b\| \le \rho^{-1}\|p\|$ and $tb = p$. The idempotent p has infinite dimensional range and is therefore not strictly singular. Passing to the quotient algebra $\mathscr{A}$ we note that $\overline{tb} = \bar{p}$ is a nonzero idempotent in $\mathscr{A}$. It follows that

$$\beta(\bar{t}) \le \|\bar{b}\|_{\mathscr{A}} \le \|b\| \le \rho^{-1}\|p\|. \quad \blacksquare$$

Let us call a sequence $(f_n) \subseteq L^1([0,1])$ *almost disjointly supported* on $[0,1]$, ([13]), if there exists a sequence $(g_n) \subseteq L^1([0,1])$ with pairwise disjointly supported elements g_n such that $\|f_n - g_n\|_1 \to 0$ as $n \uparrow \infty$. We need the following result from [13].

LEMMA 2. *Let $X = L^1([0,1])$, W the ideal of weakly compact operators on X, and $t \in \mathscr{B}(X)$ such that* $\operatorname{dist}(t, W) = 1$. *Then there exists a normalized sequence $(f_n) \subseteq X$ such that $\|tf_n\|_1 \to 1$ and both the sequences (f_n) and (tf_n) are almost disjointly supported on* $[0,1]$. ■

We also need the following lemma [10, 2.2].

LEMMA 3. *Let $X = L^1([0,1])$ and set*

$$m(\varepsilon) = \frac{1+\varepsilon}{1-\alpha(\varepsilon)} \quad \textit{where } \alpha(\varepsilon) = \frac{\varepsilon(1+\varepsilon)}{1-\varepsilon}$$

for all $0 < \varepsilon < 1/3$. Let $(g_n) \subseteq X$ be a normalized disjointly supported sequence and $(f_n) \subseteq X$ any sequence. Then $\sup_n \|g_n - f_n\|_1 < \varepsilon < 1/3$ *implies that the closed linear* span $N = \overline{\operatorname{span}}(f_n)$ *is $m(\varepsilon)$-complemented in X; that is, there exists an idempotent $p \in \mathscr{B}(X)$ with $\|p\| \leq m(\varepsilon)$ and $p(X) = N$.* ■

Proof of Theorem 3. First, assume that $X = l_p$, $1 \leq p < \infty$ or $X = c_0$, let $x \in S(\mathscr{A})$, choose $t \in \mathscr{B}$ with $x = \bar{t}$ and let $0 < r < 1$. For our Banach space X the ideal S coincides with the ideal of compact operators on X. It is shown in [6] that there exists an infinite dimensional subspace $N \subseteq X$ such that $\|tx\| \geq r\|x\|$, for all $x \in N$. Let $\varepsilon > 0$. Replacing N with a suitable subspace, if necessary, we may assume that N is $(1+\varepsilon)$-complemented in X. Now Lemma 1 shows that

$$\beta(x) = \beta(\bar{t}) \leq \frac{1+\varepsilon}{r}.$$

The result follows if we let $r \uparrow 1$ and $\varepsilon \downarrow 0^+$.

Now, assume that $X = L^1([0,1])$, let $x \in S(\mathscr{A})$ and choose $t \in \mathscr{B}$ with $x = \bar{t}$. The ideal S coincides with the ideal W of weakly compact operators on X. Consequently $\operatorname{dist}(t, W) = \|x\|_{\mathscr{A}} = 1$. By Lemma 2 there exists a normalized sequence $(f_n) \subseteq X$ such that $\lim_n \|tf_n\|_1 = 1$ and such that both the sequences (f_n) and (tf_n) are almost disjointly supported. Choose disjointly supported sequences $(g_n), (h_n) \subset X$ such that $\|f_n - g_n\|_1, \|tf_n - h_n\|_1 \to 0$, as $n \uparrow \infty$. Clearly then $\|g_n\|_1, \|h_n\|_1 \to 1$ and we may assume that the sequences $(g_n), (h_n)$ are normalized.

Let $0 < \varepsilon < 1/3$. Replacing (f_n) by a suitable subsequence (and $(g_n), (h_n)$ by the corresponding subsequences), if necessary, we may assume that

$$\|f_n - g_n\|_1 < \varepsilon \text{ and } \|tf_n - h_n\|_1 < \varepsilon \quad \text{for all } n \geq 1.$$

Let $N = \overline{\text{span}}(f_n)$. We wish to show that

$$\|tf\|_1 \geq \frac{1-\varepsilon}{1+\varepsilon}\|f\|_1 \quad \text{for all } f \in N. \tag{2}$$

It will suffice to show (2) for an arbitrary finite linear combination $f = \Sigma\lambda_n f_n$. Note first that $\|\Sigma\lambda_n g_n\|_1 = \Sigma|\lambda_n|$, since the sequence (g_n) is normalized and disjointly supported. Now the equality $f = \Sigma\lambda_n g_n + \Sigma\lambda_n(f_n - g_n)$ implies that

$$(1-\varepsilon)\sum|\lambda_n| \leq \|f\|_1 \leq (1+\varepsilon)\sum|\lambda_n|. \tag{3}$$

Since $tf = \Sigma\lambda_n tf_n = \Sigma\lambda_n h_n + \Sigma\lambda_n(tf_n - h_n)$ and the sequence (h_n) is normalized and disjointly supported, we obtain similarly

$$(1-\varepsilon)\sum|\lambda_n| \leq \|tf\|_1 \leq (1+\varepsilon)\sum|\lambda_n|. \tag{4}$$

The inequalities (3), (4) now imply that

$$\|tf\|_1 \geq (1-\varepsilon)\sum|\lambda_n| \geq \frac{1-\varepsilon}{1+\varepsilon}\|f\|_1.$$

The subspace $N \subseteq X$ is infinite dimensional and from Lemma 3 we know that there exists an idempotent $p \in \mathscr{B}(X)$ with $\|p\| \leq m(\varepsilon)$ and $p(X) = N$. Here $m(\varepsilon)$ is as in Lemma 3. Note $m(\varepsilon) \to 1$ as $\varepsilon \to 0^+$. According to Lemma 1 we have

$$\beta(x) = \beta(\bar{t}) \leq \frac{1+\varepsilon}{1-\varepsilon}m(\varepsilon).$$

The result follows if we let $\varepsilon \downarrow 0^+$. ■

LEMMA 4. *Let $X = C(\Omega)$, where Ω is a compact metric space, $t \in \mathscr{B}$ and $r < \Delta(t)$. Then there exists a closed subspace $N \subseteq X$ which is isomorphic to c_0 and such that $\|tx\| \geq r\|x\|$ for all $x \in N$.*

Proof. This is a quantitative version of [3, VI.15, p. 159] with similar proof (included for the convenience of the reader). Choose ρ such that $r < \rho < \Delta(t)$ and a sequence $(f_n) \subseteq X$ such that $\|f_n\| = 1$, $f_n f_m = 0$ and $\|tf_n\| > \rho$ for all $n \neq m$.

Now choose continuous linear functionals $x_n^* \in X^*$ with $\|x_n^*\| = 1$ such that $|x_n^*(tf_n)| = |t^*x_n^*(f_n)| > \rho$. Finally let, for each $n \geq 1$, μ_n be the unique regular Borel measure on Ω satisfying

$$t^* x_n^*(f) = \int_\Omega f \, d\mu_n, \quad \text{for all } f \in X.$$

Then $\|\mu_n\| = |\mu_n|(\Omega) = \|t^* x_n^*\| \leq \|t^*\|$, for all $n \geq 1$. Consequently $(|\mu_n|)$ is a uniformly bounded sequence of positive Borel measures on Ω. Here $|\mu_n|$ denotes the total variation of the measure μ_n as usual.

Let $\varepsilon = \rho - r > 0$ and set $G_n = \{|f_n| > 0\}$, for all $n \geq 1$. Since $f_n f_m = 0$ for $n \neq m$, (G_n) is a sequence of disjoint open subsets of Ω. Replacing (G_n) and (f_n) by suitable subsequences if necessary, we may, according to Rosenthal's lemma [3, I.4.1, p. 18], assume that

$$|\mu_n|\Big(\bigcup_{m \neq n} G_m\Big) < \varepsilon \quad \text{for all } n \geq 1.$$

The map $J : (\alpha_n)_{n=1}^\infty \in c_0 \to f = \sum_{n \geq 1} \alpha_n f_n \in X$ defines an isometric embedding of the space c_0 into X. Let $N = J(c_0) \subseteq X$. Recall that $\|f_m\| = 1$, for all $m \geq 1$. Thus, if $f = \sum \alpha_n f_n \in N$, then $\|f\| = \sup_n |\alpha_n|$ and for each $n \geq 1$ we have

$$\begin{aligned} \|tf\| &\geq |x_n^*(tf)| = \left|\int_\Omega f \, d\mu_n\right| = \left|\sum_{m \geq 1} \alpha_m \int_{G_m} f_m \, d\mu_n\right| \\ &\geq \left|\alpha_n \int_{G_n} f_n \, d\mu_n\right| - \sup_m |\alpha_m| \, |\mu_n|\Big(\bigcup_{m \neq n} G_m\Big) \geq |\alpha_n| \, |x_n^*(tf_n)| - \varepsilon\|f\| \\ &\geq \rho|\alpha_n| - \varepsilon\|f\|. \end{aligned}$$

Taking the supremum over all $n \geq 1$ yields $\|tf\| \geq \rho\|f\| - \varepsilon\|f\| = r\|f\|$. ■

Proof of Theorem 4. Let $x \in \mathscr{A}$, $x \neq 0$, and choose an operator $t \in \mathscr{B}$ such that $x = \bar{t}$. Suppose that $r < \Delta(x) = \Delta(t)$. According to Lemma 4 there exists a closed subspace $N \subseteq X$ which is isomorphic to the space c_0 and such that $\|tf\| \geq r\|f\|$, for all $f \in N$. The space $X = C(\Omega)$ is separable and the space c_0 is known to be 2-complemented in every separable space wherein it is contained as a closed subspace [9, 2.f.5]. According to Lemma 1 we have $\beta(x) = \beta(\bar{t}) \leq 2/r$. Now let $r \uparrow \Delta(x)$. ■

Remarks. (A) Theorem 4 would establish property (1) for the weak Calkin algebra $\mathscr{A} = \mathscr{B}(X)/W$ on $X = C(\Omega)$ if one could show that $\Delta(x) \geq \varepsilon$ for some $\varepsilon > 0$ and all $x \in S(\mathscr{A})$. Since $\Delta(x) \leq \|x\|_{\mathscr{A}}$, for all $x \in \mathscr{A}$ this is equivalent with the completeness of $\mathscr{A}$ in the (linear) norm Δ.

(B) A Banach algebra $\mathscr{A}$ which satisfies $\beta(x) < \infty$ for all nonzero $x \in \mathscr{A}$ is semisimple (the Jacobson radical of $\mathscr{A}$ cannot intersect the set $P(\mathscr{A})$). Consequently our arguments cannot be applied to nonsemisimple Banach algebras such as, for example, the Calkin algebra on the Banach space $L^1([0,1])$.

On the other hand we have established the semisimplicity of the weak Calkin algebra $\mathscr{A} = \mathscr{B}/W$, for the Banach spaces $X = L^1([0,1])$ and $X = C(\Omega)$. Let K denote the ideal of compact operators on X, $\mathscr{B}/K$ the Calkin algebra on $X, Q_K : \mathscr{B} \to \mathscr{B}/K$ the quotient map, $R \subseteq \mathscr{B}/K$ the Jacobson radical and $I = Q_K^{-1}(R) \subseteq \mathscr{B}$ the ideal of inessential operators on X. For X as above, $W = S \subseteq I$ [1, 5.6.2]. Now the semisimplicity of the quotient

$$\mathscr{B}/W \cong \mathscr{B}/K\big/W/K$$

implies that $R \subseteq W/K$, that is, $I \subseteq W$. Thus, for the Banach spaces $X = L^1([0,1])$ and $X = C(\Omega)$, Ω a compact metric space, the ideal of weakly compact operators coincides with the ideal of inessential operators.

(C) If the Banach space X is isomorphic to its Cartesian square, then it is known that every homomorphism from $\mathscr{B}(X)$ into any Banach algebra is automatically continuous [4]. This property is inherited by all quotients of the algebra $\mathscr{B}(X)$ and implies that every algebra norm on any quotient $\mathscr{A}$ of $\mathscr{B}(X)$ is continuous (with respect to the quotient norm). In conjunction with property (1) this yields the following strong uniqueness of norm property for $\mathscr{A}$:

Any two algebra norms on $\mathscr{A}$ are equivalent to the complete norm of $\mathscr{A}$ and hence mutually equivalent.

This should be compared with the classical Uniqueness of Norm Theorem: Any two *complete* algebra norms on a semisimple complex algebra are equivalent.

Our results and [4] establish the strong uniqueness of norm property for the following algebras $\mathscr{A}$: $\mathscr{A} = \mathscr{B}(X)$, X a Banach space isomorphic to its Cartesian square (follows also from [14, 4]), $\mathscr{A}$ the Calkin algebra on $X = l_p$, $1 \le p < \infty$, or $X = c_0$ (see also [7]) and $\mathscr{A}$ the weak Calkin algebra on $X = L^1$. It is also known to hold for all simple C^*-algebras $\mathscr{A}$ [5].

An example of a Banach space X such that the Calkin algebra on X carries a continuous algebra norm, which is not equivalent to the quotient norm, is given in [11]. Further interesting constructions can be found in [12].

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References

1. S.R. Caradus, W.E. Pfaffenberger and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Marcel Dekker, New York, 1974.
2. S.B. Cleveland, *Homomorphisms of noncommutative *-algebras*, Pacific J. Math. **13** (1963), 1097–1109.
3. J. Diestel and J. Uhl, *Vector measures*, Mathematical Surveys, no. 15, Amer. Math. Soc., Providence, Rhode Island, 1977.
4. B.E. Johnson, *Continuity of homomorphisms of algebras of operators*, J. London Math. Soc. **42** (1967), 537–541.
5. ______, *Continuity of homomorphisms of algebras of operators II*, J. London Math. Soc. (2) **1** (1969), 81–84.
6. M.J. Meyer, *Minimal incomplete norms on Banach algebras*, Studia Math. **102** (1992), 77–85.
7. ______, *On a topological property of certain Calkin algebras*, Bull. London Math. Soc. **24** (1992), 591–598.
8. A. Pelczynski, *On strictly singular and strictly cosingular operators I, II*, Bull. Acad. Polonaise Sci. **13** (1965), 31–41.
9. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, New York, 1977.
10. H.O. Tylli, *Lifting of non-topological divisors of zero modulo the compact operators*, preprint.
11. K. Astala and H.O. Tylli, *On the bounded compact approximation property and measures of noncompactness*, J. Funct. Anal. **70** (1987), 388–401.
12. H.O. Tylli, *The essential norm of an operator is not self dual*, preprint.
13. L. Weis and M. Wolff, *On the essential spectrum of operators on L^1*, Semesterberichte Funktionalanalysis (Tubingen, Sommersemester 1984), pp. 103–112.
14. B. Yood, *Homomorphisms on normed algebras*, Pacific J. Math. **8** (1954), 373–81.

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