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L_p-REGULARITY OF THE CAUCHY PROBLEM AND THE GEOMETRY OF BANACH SPACES

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Introduction

Let X be a Banach space, L a Banach lattice of functions on $[0, T]$ and A a closed linear operator, defined on a dense domain $D(A) \subset X$, with values in X. We suppose that $-A$ is the generator of an analytic semi-group $S(t)$, $t > 0$.

We consider the vector-valued Cauchy problem

$$\mathrm{CP}_A \begin{cases} u' + Au = f \\ u(0) = 0 \end{cases}$$

where f belongs to $L([0, T]; X)$ and for fixed $t \in [0, T]$, $Au(t) = A(u(t))$. It is known, cf. [P], that CP_A has a mild solution given by:

$$u(t) = \int_0^t S(t-s) f(s)\, ds = \left(\frac{d}{dt} + A\right)^{-1} f$$

We are interested in the *regularity* of the solution: If $L[0, T]$ is one of lattices $L_p[0, T]$, $1 \leq p \leq +\infty$ or $C[0, T]$, then we define CP_A to be L-regular if there exists a constant C such that, for all $f \in L([0, T]; X)$,

$$\|Au\|_{L(X)} = \left\| A\left(\frac{d}{dt} + A\right)^{-1} f \right\|_{L(X)} \leq C\|f\|_{L(X)}$$

It is clear that if A bounded, then CP_A is L-regular for all L.

We are going to consider first the L_2-regularity and prove that Cauchy problems, associated with recent examples of operators, [BC], [G], [V], are L_2-regular. Then we recall a characterization of the L_∞-regularity due to J.B. Baillon [B]. We give next a characterization of the L_1-regularity and we finish

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by a complete study of the regularity of Cauchy problems, associated with diagonal operators on l_p or c_0.

I. L_2-regularity

It is shown in [CL] and in [CV], that for all $p \in]1, +\infty[$, the notions of L_p-regularity are the same. We will call it the L_2-regularity.

If X is a Hilbert space, then CP_A is always L_2-regular; this is a result of [DS].

If X is a UMD-space, (cf. [Bu] and [Bo]), and if A has *well bounded* imaginary powers, namely if $\|A^{is}\| \leq Ce^{\alpha|s|}$ with $\alpha < \pi/2$ and $C > 0$, then CP_A is L_2-regular. This is a result of [DV].

Recently, there were examples of operators on L_p, $1 < p < +\infty$, which are generators of analytic semi-groups and which have not well bounded imaginary powers, see [BC], [G], [V].

A natural question was to know if for these operators on L_p, $p \neq 2$, CP_A is L_2-regular.

Let us describe a version of these examples and show that the answer is yes.

Let $\mathbf{T}$ be the torus and $L_p = L_p(\mathbf{T})$. We know that the trigonometric basis $(e^{in\theta})_{n\in\mathbf{Z}}$ is not unconditional if $p \neq 2$. Thus there exists a choice of signs $(\varepsilon_n)_{n\in\mathbf{N}}$ and a vector $x = \sum_{n=0}^{+\infty} x_n e^{in\theta} \in L_p$ such that

$$\|x\| = \left\| \sum_{n=0}^{+\infty} x_n e^{in\theta} \right\| = 1$$

$$\lim_{N\to+\infty} \left\| \sum_{n=0}^{N} \varepsilon_n x_n e^{in\theta} \right\| = +\infty$$

Let $(k_n)_{n\in\mathbf{N}}$ be an increasing sequence of integers such that, for all $n \in \mathbf{N}$, $\varepsilon_n = e^{ik_n\pi}$.

We define the linear operator A by

$$\forall n \in \mathbf{N},\ A e^{in\theta} = e^{k_n\pi} e^{in\theta}$$

It is easy to verify that $-A$ is generator of the analytic semi-group $S(t)$, $t \in]0, +\infty[$ defined by

$$\forall n \in \mathbf{N},\ S(t)\, e^{in\theta} = e^{t\, e^{k_n\pi}} e^{in\theta}$$

and that A^{is} is defined for all $s \in \mathbf{R}$ by:

$$\forall n \in \mathbf{N},\ A^{is} e^{in\theta} = e^{isk_n\pi} e^{in\theta}$$

By definition of the sequence $(k_n)_{n\in \mathbf{N}}$, A^i is unbounded while A^{2i} is the identity.

This operator A is a multiplier associated with the convolution operator $\mathbf{A}$ on $L_p(\mathbf{T})$ such that $\hat{\mathbf{A}}(n) = e^{k_n\pi}$.

But, for all non decreasing function f, the multiplier

$$\left(\frac{ix}{ix + f(y)}\right)_{x,y\in\mathbf{R}^2}$$

is bounded on $L_p(\mathbf{R}^2)$, because of the Stein's conditions, [S, p. 109, Theorem 6].

Thus, by the transference theorem, cf. [CoV], the multiplier

$$\left(\frac{in}{in + e^{k_m\pi}}\right)_{n,m\in\mathbf{Z}^2}$$

is bounded on $L_p(\mathbf{T}^2)$.

This last multiplier is associated with the convolution operator $d/dt(d/dt + \mathbf{A})^{-1}$ on $L_p(\mathbf{T}^2) = L_p(\mathbf{T}; L_p(\mathbf{T}))$ which is thus bounded on this space.

This is equivalent to the L_p-regularity of CP_A which is equivalent to the L_2-regularity as it was already mentioned.

However the general question is still open, namely:

QUESTION. *Does there exist an unbounded operator A on L_p, $p \neq 2$, such that $-A$ is a generator of an analytic semi-group and* CP_A *is not L_2-regular?*

II. L_∞-regularity

In [B], Baillon shows that if X does not contain c_0, then the L_∞-regularity of CP_A implies that A is bounded. It is also proved that this result is false in c_0. This result is also written in [EG] and with other results in [T]. See also [DG], Proposition 3.11 for the case $X = L_2$.

We know by [LT] that c_0 is always complemented in separable Banach spaces. So, Baillon's result implies the following corollary.

COROLLARY. *Let X be a separable Banach space and A a closed linear operator with dense domain $D(A) \subset X$ and such that $-A$ is a generator of an analytic semi-group. Then the following conditions are equivalent*:

(i) *X does not contain a complemented copy of c_0.*

(ii) *If* CP_A *is L_∞-regular, then A is bounded.*

If X is not separable, the problem is open, namely:

QUESTION. *If X is not separable and contains c_0, does there exist an operator A such that $-A$ is a generator of an analytic semi-group which is unbounded and such that* CP_A *is L_∞-regular?*

Let us mention that if X is the space l_∞, then, since in l_∞, all continuous semi-groups have bounded generators (this is an unpublished result of M. Talagrand), the answer to the previous question is no in this case.

To finish with this notion, let us mention that the same results are true with the C-regularity instead of the L_∞-regularity. In fact, in the original paper [B], it was written with the C-regularity which introduces simply an easy argument of approximation.

III. L_1-regularity

In this part, using Baillon's construction, mentioned in Part II, we are going to prove that it is equivalent to say that X does not contain a complemented copy of l_1 and that the L_1-regularity of CP_A implies that A is bounded.

PROPOSITION 1. *Let A be a closed linear operator with dense domain $D(A) \subset X$ and such that $-A$ is a generator of an analytic semi-group. Suppose that X does not contain a complemented copy of l_1. Then, if there exists a constant $C > 0$ such that, for all $f \in L_1([0,T]; X) = L_1(X)$,*

$$\left\| A\int_0^t S(t-s)f(s)\,ds \right\|_{L_1(X)} \le C\|f\|_{L_1(X)}$$

then A is bounded.

Proof. We recall, as in [B], that if $-A$ is a generator of an analytic semi-group, then there exists a constant C' such that

$$(*)\qquad \|tAS(t)\| \le C' \quad \text{for all } t > 0;\ \text{cf. [P]}.$$

Moreover, if A is not bounded, then

$$(**)\qquad \overline{\lim_{t\to 0}}\, \|tAS(t)\| \ge \frac{1}{e};\ \text{cf. [H] and [Y]}.$$

We are going to proceed by duality and use Baillon's result [B] on the C-regularity:

Let A be an unbounded operator such that $-A$ is a generator of an analytic semi-group, satisfying the inequality of L_1-regularity above and let

$A^\star$ be the adjoint of A, defined on

$$D(A^\star) = \{y^\star \in X^\star,\ x \to \langle Ax, y^\star\rangle \text{ is bounded on } D(A)\}.$$

In general, $A^\star$ is not a generator of a continuous semi-group. However, as it is shown in [C], there is a closed subspace X^0 of $X^\star$, defined by

$$X^0 = \left\{x^\star \in X^\star / \lim_{t\to 0} \|S^\star(t)x^\star - x^\star\| = 0\right\}$$

which is stable by $S^\star(t)$ and $A^\star$ and such that the restriction of $S^\star(t)$ to X^0, noted $S^0(t)$ and called the *dual semi-group* of $S(t)$, is a continuous semi-group on X^0 with generator A^0 which is the restriction of $A^\star$ to $D(A^\star \cap X^0)$.

Moreover, it is easy to see that $S^0(t)$ is an analytic semi-group on X^0 and that A^0 is not bounded on X^0.

LEMMA 2. *If A is as above, let $0 < t_n < t_{n-1} < \cdots < t_1 < t_0 = T$, $\varepsilon_i = \pm 1$ for all $i = 0, 1, \ldots, n$ and $y_i^0 \in X^0$ for $i = 0, 1, \ldots, n$ be given. Then, for all $g \in L_\infty([0, T]; X^0) = L_\infty(X^0)$, such that $\forall t \in [0, T]$, $g(t) = \sum_{i=0}^n \varepsilon_i S(t_i - t)^0 y_i^0 \mathbf{I}_{[t_{i+1}, t_i[}(t)$, we have*

$$\sup_{s\in[0,T]} \left\| \int_s^T A^0 S(t-s)^0 g(t)\, dt \right\|_{X^\star} \leq C\|g\|_{L_\infty(X^\star)}$$

Proof. Let g be as above. Then, for all $t \in [0, T]$, $g(t)$ is in $D(A^0)$. Thus, if f belongs to $L_1(X)$, we can write

$$\begin{aligned}
&\left|\int_0^T \left\langle f(s), \int_s^T A^0 S(t-s)^0 g(t)\, dt\right\rangle ds\right| \\
&= \left|\int_0^T \int_s^T \langle f(s), A^0 S(t-s)^0 g(t)\rangle\, dt\, ds\right| \\
&= \left|\int_0^T \int_0^t \langle f(s), A^0 S(t-s)^0 g(t)\rangle\, ds\, dt\right| \\
&= \left|\int_0^T \int_0^t \langle S(t-s) f(s), A^0 g(t)\rangle\, ds\, dt\right| \\
&= \left|\int_0^T \left\langle \int_0^t S(t-s) f(s)\, ds, A^0 g(t)\right\rangle dt\right| \\
&= \left|\int_0^T \left\langle A\int_0^t S(t-s) f(s)\, ds, g(t)\right\rangle dt\right| \\
&\leq C\|g\|_{L_\infty(X^\star)} \|f\|_{L_1(X)}
\end{aligned}$$

by Fubini's theorem and the hypothesis of Proposition 1.

Thus, for all $s \in [0, T]$ we have

$$\left\| \int_s^T A^0 S(t-s)^0 g(t)\, dt \right\|_{X^\star} \le C \|g\|_{L_\infty(X^\star)}$$

LEMMA 3. *With the same hypothesis on A, X^0 contains c_0.*

Proof. We follow Baillon's proof in [B]: Take $t_0 = T > t_1 > \cdots > t_n > \cdots > 0$ such that for all $i \in \mathbf{N}$, $t_{i+1} \le t_i/2^{i+1}$, and $y_i^0 \in X^0$, $i \in \mathbf{N}$ such that $\|y_i^0\| \le 1$ and $\|t_i A^0 S(t_i)^0 y_i^0\| \ge 1/2e$. This is possible by $(*)$ and $(**)$.
Then, taking g as above in Lemma 2, we get

$$\begin{aligned}\left\| \int_0^T A^0 S(t)^0 g(t)\, dt \right\|_{X^\star} &= \left\| \int_0^T \sum_{i=0}^n \varepsilon_i A^0 S(t_i)^0 y_i^0 \mathbf{I}_{[t_{i+1}, t_i[}(t)\, dt \right\|_{X^\star} \\ &= \left\| \sum_{i=0}^n \varepsilon_i (t_i - t_{i+1}) A^0 S(t_i)^0 y_i^0 \right\| \\ &\ge \left\| \sum_{i=0}^n \varepsilon_i t_i A^0 S(t_i)^0 y_i^0 \right\| - \left\| \sum_{i=0}^n \varepsilon_i t_{i+1} A^0 S(t_i)^0 y_i^0 \right\|\end{aligned}$$

Note that $\|g\|_{L_\infty(X^\star)} \le K$ and

$$\left\| \sum_{i=0}^n \varepsilon_i t_{i+1} A^0 S(t_i)^0 y_i^0 \right\| \le \frac{C'}{2} + \frac{C'}{4} + \cdots \frac{C'}{2^{n+1}} \le C'.$$

Thus, by Lemma 2, we get

$$\left\| \sum_{i=0}^n \varepsilon_i t_i A^0 S(t_i)^0 y_i^0 \right\| \le C' + CK$$

Since the sequence $u_i^0 = t_i A^0 S(t_i)^0 y_i^0$, $i \in \mathbf{N}$ is bounded in X^0 and does not converge to 0 in norm, this inequality proves by Bessaga-Pelcynski's theorem, [BP], that it contains a subsequence which is equivalent to the unit vector basis of c_0. This proves Lemma 3.

End of the proof of Proposition 1. If A is unbounded and CP_A is L_1-regular, then, by Lemmas 2 and 3, X^0 (and also $X^\star$) contains c_0 and thus X contains a complemented copy of l_1, by [LT, p. 103]. This proves Propos ition 1.

PROPOSITION 4. *If X contains a complemented copy of l_1, then there exists an unbounded operator* $\mathbf{A}$ *on X such that* $\mathrm{CP}_\mathbf{A}$ *is L_1-regular.*

Proof. Let $(e_n)_{n\in\mathbf{N}}$ be the unit vector basis of l_1 and define A by: $A(e_n) = \lambda_n e_n$, where $(\lambda_n)_{n\in\mathbf{N}}$ is a sequence of positive numbers, tending to $+\infty$. Then A is unbounded on l_1 and CP_A is L_1-regular: indeed, if $(f_n)_{n\in\mathbf{N}}$ is in $L_1(l_1)$ and $u(t) = ((u_n(t))_{n\in\mathbf{N}}$ is the mild solution of CP_A, by Fubini's theorem we can write

$$\begin{aligned}
\int_0^T \|Au(t)\|_{l_1}\,dt &= \int_0^T \sum_{n=0}^{+\infty} \left| \lambda_n e^{-\lambda_n t} \int_0^t e^{\lambda_n s} f_n(s)\,ds \right| dt \\
&\le \sum_{n=0}^{+\infty} \int_0^T \left(\int_s^T \lambda_n e^{-\lambda_n t}\,dt \right) e^{\lambda_n s} |f_n(s)|\,ds \\
&= \sum_{n=0}^{+\infty} \int_0^T (e^{-\lambda_n s} - e^{-\lambda_n T})\, e^{\lambda_n s} |f_n(s)|\,ds \\
&\le \sum_{n=0}^{+\infty} \int_0^T |f_n(s)|\,ds \\
&= \int_0^T \sum_{n=0}^{+\infty} |f_n(s)|\,ds \\
&= \|(f_n)_{n\in\mathbf{N}}\|_{L_1(l_1)}
\end{aligned}$$

We can consider the operator $\mathbf{A} = A \oplus I_Y$ where Y is a supplementary of l_1 in X. Then it is clear that $\mathbf{A}$ is suitable.

Propositions 1 and 4 together give the announced result.

THEOREM 5. *Let A be a closed linear operator with dense domain $D(A) \subset X$ and such that $-A$ is a generator of an analytic semi-group. Then the following conditions are equivalent*:

(i) *X does not contain a complemented copy of l_1*

(ii) *If CP_A is L_1-regular, then A is bounded.*

IV. A remark on diagonal operators on l_p, $1 \le p < +\infty$ or c_0

In this part, we examinate diagonal operators A on l_p, $1 \le p < +\infty$ or c_0, in terms of L_p-regularity, for $1 \le p \le +\infty$.

Let us consider the linear operator A on l_p, $1 \le p < +\infty$ or c_0 which is defined on the unit vector basis $(e_n)_{n\in\mathbf{N}}$ by

$$A(e_n) = z_n e_n$$

where $(z_n)_{n\in\mathbf{N}}$ is a sequence of complex numbers which lies in a sector D_ϕ, $0 \le \phi < \pi/2$ defined by $D_\phi = \{z \in \mathbf{C}, \arg(z) \le \phi\}$.

Then we have:

THEOREM 6. *On c_0, CP_A is L_∞- and L_2-regular and if A is unbounded, it is not L_1-regular.*

On l_1, CP_A is L_1- and L_2-regular and if A is unbounded, it is not L_∞-regular.

On l_p, $1 < p < +\infty$, CP_A is L_2-regular and if A is unbounded, it is not L_1- and L_∞-regular.

Proof. On c_0, the L_∞-regularity of CP_A is obvious and was mentioned in [B] in the case where $z_n \geq 0$ for all $n \in \mathbf{N}$. Theorem 5 proves that if A is not bounded, then CP_A is not L_1-regular.

Let us prove the L_2-regularity; we have to show that there is a constant C such that, for all $(f_n)_{n \in \mathbf{N}} \in L_2(c_0)$,

$$\int_0^T \left| \operatorname*{Sup}_{n \in \mathbf{N}} z_n\, e^{-z_n t} \int_0^t e^{z_n s} f_x(s)\, ds \right|^2 dt \leq C^2 \int_0^T \left| \operatorname*{Sup}_{n \in \mathbf{N}} f_n(t) \right|^2 dt$$

Let $f(t) = \operatorname{Sup}_{n \in \mathbf{N}} |f_n(t)| = \|f_n(t)\|_{c_0}$ and $\lambda_n = \operatorname{Re} z_n$. Then, f belongs to L_2 and, changing C into $C/\cos\phi$, it is sufficient to prove

$$\int_0^T \left(\operatorname*{Sup}_{n \in \mathbf{N}} \lambda_n\, e^{-\lambda_n t} \int_0^t e^{\lambda_n s} f(s)\, ds \right)^2 dt \leq C^2 \int_0^T f(t)^2\, dt$$

Let us call f_λ the function defined by

$$f_\lambda(t) = \lambda\, e^{-\lambda t} \int_0^t e^{\lambda s} f(s)\, ds.$$

Then, setting $\mu = \lambda t$, for $t > 0$ and $\lambda > 0$ we can write

$$\begin{aligned}
f_\lambda(t) &= \frac{\mu}{t} \int_0^t e^{-\frac{\mu}{t}(t-s)} f(s)\, ds \\
&= \frac{\mu}{t} \int_0^t e^{\frac{\mu}{t} u} f(t-u)\, du \\
&\leq \frac{\mu}{t} \sum_{k=0}^{+\infty} \int_{kt/\mu}^{(k+1)t/\mu} e^{-\frac{\mu}{t} u} f(t-u)\, du \\
&\leq \frac{\mu}{t} \sum_{k=0}^{+\infty} e^{-k} \int_{kt/\mu}^{(k+1)t/\mu} f(t-u)\, du \\
&\leq e \frac{\mu}{t} \sum_{k=0}^{+\infty} e^{-k} \int_0^{kt/\mu} f(t-u)\, du \\
&= e \frac{\mu}{t} \sum_{k=0}^{+\infty} e^{-k} \int_{t-(k/\mu)t}^t f(s)\, ds \\
&= e \sum_{k=0}^{+\infty} k e^{-k} \left(\frac{\mu}{kt} \int_{t-(k/\mu)t}^t f(s)\, ds \right) \\
&\leq e \sum_{k=0}^{+\infty} k e^{-k} (M_- f(t))
\end{aligned}$$

where

$$M_{-}f(t) = \operatorname*{Sup}_{x>0} \frac{1}{x} \int_{t-x}^{t} f(s)\, ds.$$

It is well known that the *maximal unilateral function* M_{-} of Littlewood-Paley is bounded on L_p for $1 < p \leq +\infty$; cf. [S]. So there exists a constant C such that

$$\int_0^T \left(\operatorname*{Sup}_{n\in\mathbf{N}} \lambda_n\, e^{-\lambda_n t} \int_0^t e^{\lambda_n s} f(s)\, ds \right)^2 dt \leq \left(e \sum_{k=0}^{+\infty} k e^{-k} \right) \|M_{-}(f)\|_{L_2}^2$$
$$\leq C^2 \left(e \sum_{k=0}^{+\infty} k e^{-k} \right)^2 \|f\|_{L_2}^2$$

This proves the L_2-regularity of CP_A on c_0.

This computation is also a consequence of Theorem 2 of [S, p. 63].

On l_1, the L_1-regularity is proved in Proposition 4 if $z_n \geq 0$. The proof of the general case is similar. Theorem 8 proves that if A is not bounded, the CP_A is not L_∞-regular.

To prove the L_2-regularity, we proceed by duality, as in Proposition 1. We have to prove that there exists a constant C such that, for all $(f_n)_{n\in\mathbf{N}} \in L_2(l_1)$,

$$\int_0^T \left(\sum_{n=0}^{+\infty} \left| z_n\, e^{-z_n t} \int_0^t e^{z_n s} f_n(s)\, ds \right| \right)^2 dt \leq C^2 \int_0^T \left(\sum_{n=0}^{+\infty} |f_n(t)| \right)^2 dt$$

Define f by

$$f(t) = \sum_{n=0}^{+\infty} |f_n(t)| = \|(f_n)_{n\in\mathbf{N}}\|_{l_1}$$

and, as before, set $\lambda_n = \operatorname{Re} z_n$. Then, f belongs to L_2 and, again changing C into $C/\cos\phi$, it is sufficient to show that

$$\int_0^T \left(\sum_{n=0}^{+\infty} \lambda_n\, e^{-\lambda_n t} \int_0^t e^{\lambda_n s} f(s)\, ds \right)^2 dt \leq C^2 \int_0^T f(t)^2\, dt$$

Let $g(t)$ belong to L_2. Then we can write, for all $\lambda > 0$,

$$\int_0^T \left\langle \lambda\, e^{-\lambda t} \int_0^t e^{\lambda s} f(s)\, ds, g(t) \right\rangle dt = \int_0^T \left\langle f(s), \lambda \int_s^T e^{\lambda(s-t)} g(t)\, dt \right\rangle ds$$

As in the case of c_0, if we replace t by $u = s - t$ and λ by $\mu = \lambda(T - s)$, a similar computation easily gives that for $s < T$,

$$g_\lambda(s) = \lambda \int_s^T e^{\lambda(s-t)} g(t)\, dt \leq \sum_{k=0}^{+\infty} (k+1)\, e^{-(k+1)} \times \left(\frac{\mu}{(k+1)(T-s)} \int_s^{s+((k+1)(T-s)/\mu)} g(t)\, dt \right)$$

$$\leq \sum_{k=0}^{+\infty} (k+1)\, e^{-(k+1)} M_+ g(s)$$

where

$$M_+ g(s) + \operatorname*{Sup}_{x>0} \frac{1}{x} \int_s^{s+(1/x)} g(t)\, dt.$$

As before, the *maximal unilaterial function* M_+ of Littlewood-Paley is bounded on L_p for $1 < p \leq +\infty$ [S] and this prove that there is a constant C such that

$$\|g_\lambda\|_{L_2} \leq C \sum_{k=0}^{+\infty} (k+1)\, e^{-(k+1)} \|g\|_{L_2}$$

Thus, the inequalities of duality, proved above imply that

$$\int_0^T \left(\sum_{n=0}^{+\infty} \lambda_n\, e^{-\lambda_n t} \int_0^t e^{\lambda_n s} f(s)\, ds \right)^2 dt \leq C^2 \left(\sum_{k=0}^{+\infty} (k+1)\, e^{-(k+1)} \right)^2 \int_0^T f(t)^2\, dt$$

which proves the L_2-regularity of CP_A.

On l_p, the L_2-regularity of CP_A is a result of interpolation: the operator $f \to Au$, where u is the mild solution of CP_A is bounded on $L_1(l_1)$ and on $L_\infty(c_0)$ since CP_A is L_1-regular on l_1 and L_∞-regular on c_0. Thus, by interpolation, it is bounded on $L_p(l_p)$, which gives the L_p-regularity on l_p and thus the L_2-regularity as mentioned before. Since, l_p is U.M.D. for $1 < p < +\infty$ and the unit vector basis is 1-unconditional, this result appears also as a consequence of Dore and Venni's theorem [DV].

If A is not bounded, then theorem 5 implies that CP_A is not L_1- or L_∞-regular on l_p.

QUESTION. *It would be interesting to know if the methods of Theorem* 6 *give an answer to the L_2-regularity of* CP_A *on* l_p, $1 \leq p < +\infty$ *or* c_0 *for non-diagonal and unbounded operators* A.

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