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NORM STRUCTURE FUNCTIONS AND EXTREMENESS CRITERIA FOR OPERATORS ON $L_p (p \leq 1)$ OR ONTO $C(K)$

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Introduction

For a Banach or L_p space $\mathbf{E}$ and a Banach space $\mathbf{F}$, let $\mathscr{U} \equiv \mathscr{U}(\mathbf{E}, \mathbf{F})$ be the unit ball of the Banach space $\mathscr{L} \equiv \mathscr{L}(\mathbf{E}, \mathbf{F})$ of bounded linear operators T from $\mathbf{E}$ to $\mathbf{F}$. We study the extreme points of $\mathscr{U}$ when either (i) $\mathbf{E}$ is an L_p space ($p \leq 1$) or (ii) $\mathbf{F}$ is a C-space, i.e., a Banach space $C(K)$ of continuous functions on a compact Hausdorff space K. Extremeness criteria are obtained partly in terms of *norm structure functions* $\delta_1(T)$ and $\delta_\infty(T)$ for the cases (i) and (ii) respectively. The first function $\delta_1(T)$ generalizes the function $|T|^*1$ for the case where $\mathbf{E}$ and $\mathbf{F}$ are L_1 spaces, and the second, $\delta_\infty(T)$, generalizes $|T|1$ for the case where both spaces are C-spaces. Some of their basic properties are studied that are used in tackling the extremeness problems. The scalar field may be the reals or the complexes. The proofs are given for the complex case; the real case follows by minor adjustments.

In the case (i) we obtain, among other things, complete description of extreme contractions in $\mathscr{U}(L_p(\mu), L_q(\nu))$ when $0 < p \leq 1 \leq q < \infty$ in a rather unified manner (Theorem 2.8). Some of our extremeness results for the case $\mathbf{E} = L_1(\mu)$ have points of contact with some results of [Sh2, §2] but the approach and formulation are different. When the scalars are the reals, special cases for $\mathbf{E} = L_1(\mu)$ and $\mathbf{F} = L_1(\nu)$ have been considered in [I, Theorem 2] and, implicitly, in [Ki, Theorem 2]. Concerning case (ii) the problem of characterizing an extreme contraction T between C-spaces $\mathbf{E} = C(H)$ and $\mathbf{F} = C(K)$ have been studied by several researchers. The most desirable criterion for T to be extreme seems to be that T be a composition operator modulated by a unimodular function, which is just criterion $(\infty\infty)'$ in Theorem 3.7. This is equivalent, as is not difficult to show, to T^* mapping all extreme points of the unit ball of $\mathbf{F}'$ to those of that of $\mathbf{E}'$; such a T is said to be *nice*. (The extreme points mentioned are unimodular scalar multiples of evaluation maps.) The criterion, clearly sufficient, is not always necessary

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[Sh3], [Sh4] but it indeed is for close to ten independent known cases, each case with its own conditions imposed on the underlying topological spaces or the operator or both—the scene is a mosaic. See [Sh1], [Ge], the lists in them, and [B]; see also [AL], [BLP] and [CL] for those listed results. See [Gr3] for a related development. Most cases are for both real and complex scalars, but a few are known only for real scalars, the corresponding complex cases being undecidable yet. Our contribution to this intriguing problem, Theorem 3.7 (with $\mathbf{B} = \mathbf{F}$), treats a new case in which T has a linear modulus, a consequent condition in fact of T being a modulated composition operator. This case generalizes three established cases, namely (1) when K is Stonean or equivalently $\mathbf{F}$ is order complete, in [SH1, Theorems 2 and 4] (see also [Ge, Theorem 1.3(ii)] with a different proof), (2) when K is quasi-Stonean and H is metric, in [Ge, Theorem 1.4] and (3) when T is compact, in [BLP, pp. 751–752]. This can be seen from Theorem 3.10 and Remark 3.8(b).

We also consider the related problem of characterizing the extreme points of the set of *positive* contractions. The criteria turn out to be analogous to those for general contractions in the corresponding cases, with similar proofs that require only very little extra effort.

One may note that virtually all our extremeness results are still valid if the sets of contractions being considered are restricted to those of compact ones; this is evident from the proofs. By the same token we may restrict to weakly compact contractions. (If $\mathbf{E}$ or $\mathbf{F}$ is reflexive these are just ordinary contractions [C, Proposition 5.2].)

For related results on extreme contractions between L_p spaces see [H], [K], [Gr1], [Gr2], [Gr4], [Kan3], [Kan4].

1. Norm structure function $\delta_1(T)$

Let $\mathbf{F}$ be an s-normed space ($0 < s \leq 1$). That is, $\mathbf{F}$ is a vector space equipped with a translation-invariant metric $|||f - g|||(f, g \in \mathbf{F})$, for which the associated functional $|||\cdot|||$, called an *s-norm*, satisfies $|||cg||| = |c|^s |||g|||$ ($g \in \mathbf{F}$, c a scalar) [Kö, §15.10]. On $\mathbf{F}$ define the *formal norm* $\|\cdot\| = |||\cdot|||^{1/s}$, which is a true norm if $s = 1$. Similarly let $\mathbf{E}$ be an r-normed space ($0 < r \leq 1$) with an r-norm $\|\cdot\|^r$. The space $\mathscr{L} \equiv \mathscr{L}(\mathbf{E}, \mathbf{F})$ of all linear operators T: $\mathbf{E} \to \mathbf{F}$ with $\|T\| = \sup\{\|Tf\|: f \in \mathbf{E}, \|f\| = 1\} < \infty$ is s-normed by $T \mapsto \|T\|^s$. $\mathscr{L}$ is complete or in particular a Banach space if accordingly so is $\mathbf{F}$. Let $\mathscr{U} \equiv \mathscr{U}(\mathbf{E}, \mathbf{F}) = \{T \in \mathscr{L}: \|T\| \leq 1\}$, the unit ball of $\mathscr{L}$ consisting of all *contractions* from $\mathbf{E}$ to $\mathbf{F}$, and ext $\mathscr{U}$, the set of extreme points of $\mathscr{U}$. This will be the standing setting in the paper. All true or formal norms will be denoted by $\|\cdot\|$ when the reference is clear from the context.

$\mathbf{E} = L_p(\mu) = L_p(X, \mathscr{F}, \mu)$ $(0 < p \leq \infty)$, a usual Lebesgue space, is $\min\{1, p\}$-normed, with $\|f\| = (\int |f|^p \, d\mu)^{1/p}$ $(p <)$ or $\operatorname{ess\,sup}|f|$ $(p = \infty)$. For $p < \infty$ we always assume that $(X, \mathscr{F}, \mu)$ for $L_p(\mu)$ is a direct union of finite measure spaces, without loss of generality; see [Kan3, p. 615] or [L, Corollary to Theorem 15.3]. Then every sub-family $\mathscr{H}$ of $(\mathscr{F}, \mu)$ has a lattice supremum $\sup\mathscr{H}$ (modulo μ-null sets). For $p = \infty$ this assumption would imply that $L_\infty(\mu)$ is the dual space $(L_1(\mu))'$ of $L_1(\mu)$ and an order complete lattice (a property needed to define $\delta_1(T)$). For each $A \in \mathscr{F}$ and each measurable function f on $(X, \mathscr{F}, \mu)$ let f_A be f on A and 0 on A^c, the complement of A. (We can allow for f not defined on A^c.) Let $\mathbf{E}_A = \{f \in \mathbf{E}: \operatorname{supp} f \subset A\}$ where $\operatorname{supp} f = \{f \neq 0\}$, the *support* of f.

$\mathbf{E}_A$ is identified with $L_p(A, \mu)$, and f_A or $1_A f$ with $f|_A$. When $p < \infty$ for each $T \in \mathscr{L}$ define $\mathrm{o}(T) = \sup\{A \in \mathscr{F}: T\mathbf{E}_A = \{0\}\}$ and $s(T) = (\mathrm{o}(T))^c$.

(1.1) PROPOSITION. *Let $0 < p < \infty$, $\mathbf{E} = L_p(\mu)$, $\mathbf{F}$ be an s-normed space $(s \leq 1)$ and $T \in \mathscr{L}$. Then $T\mathbf{E}_{\mathrm{o}(T)} = \{0\}$ and $T\mathbf{E}_A \neq \{0\}$ if $\varnothing \neq A \in \mathscr{F} \cap s(T)$.*

Proof. The result on $s(T)$ follows from definition. For the result on $\mathrm{o}(T)$ let $0 \neq f \in \mathbf{E}_{\alpha(T)}$. There exist $A^1, A^2, \ldots \in \mathscr{F}$ with $T\mathbf{E}_{A^n} = \{0\}$ $(n = 1, 2, \ldots)$, such that $\operatorname{supp} f \subset \bigcup A^n$. Hence

$$\begin{aligned} Tf &= Tf_{A^1} + Tf_{A^2\setminus A^1} + \cdots + Tf_{A^n \setminus (A^1 \cup \cdots \cup A^{n-1})} + \cdots \\ &= 0 + 0 + \cdots = 0. \ \square \end{aligned}$$

Let $p \leq s \leq 1$, $\mathbf{E} = L_p(\mu)$, $\mathbf{F}$ = an s-normed space and $T \in \mathscr{L}$. Define

$$\mathscr{M}(T) = \{\theta \in L_\infty^+(\mu): \|Tf\| \leq \|\theta f\| \ \forall f \in \mathbf{E}\}.$$

Then $\delta_1(T) \equiv \inf \mathscr{M}(T) \in L_\infty^+(\mu)$ exists and is majorized by $1_{s(T)}\|T\| \in \mathscr{M}(T)$, by Proposition 1.1. The $L_\infty(\mu)$ norm, for $\delta_1(T)$ and related functions, will be denoted by $\|\cdot\|_\infty$. An element of $L_\infty(\mu)$ is considered also a multiplication operator that it induces. Denote by $l_p(n)$ the n-dimensional L_p space over the counting measure on n points.

(1.2) THEOREM. *Let $p \leq s \leq 1$, $\mathbf{E} = L_p(\mu)$, $\mathbf{F}$ = an s-normed space and $T \in \mathscr{L}$. Then*:

(a) $\delta_1(T) \in \mathscr{M}(T)$, *i.e.*, $\|Tf\| \leq \|\delta_1(T) f\| \ \forall f \in \mathbf{E}$.

(b) $\operatorname{supp} \delta_1(T) = s(T)$.

(c) $\forall \eta \in L_\infty(\mu)$, (i) $\delta_1(T \circ \eta) = |\eta| \delta_1(T)$ *and* (ii) $\|\eta \delta_1(T)\|_\infty = \|T \circ \eta\|$.

(d) $\delta_1^p(\cdot)$ *is sub-additive on* $\mathscr{L}$.

For $\mathbf{E} = L_p(\mu)$, $\mathbf{F} = L_q(\nu)$ and $T \in \mathscr{L}$ even in the finite-dimensional case $\|\delta_1(T)\|_\infty < \|T\|$ may occur when $p > \min\{1, q\} \equiv s$ instead.

Proof. Observe that since $p/s \leq 1$ we have

$$(1) \qquad \|g + h\|^p \leq (\|g\|^s + \|h\|^s)^{p/s} \leq \|g\|^p + \|h\|^p, \ (g, h \in \mathbf{F}).$$

We first prove (d). Let $S \in \mathscr{L}$, $\xi \in \mathscr{M}(S)$ and $\zeta \in \mathscr{L}(T)$. By (1),

$$\|(S+T)f\|^p \le \|Sf\|^p + \|Tf\|^p \le \|\xi f\|^p + \|\zeta f\|^p = \|(\xi^p + \zeta^p)^{1/p} f\|^p$$

for all $f \in \mathbf{E}$. So $\delta_1^p(S+T) \le \xi^p + \zeta^p$. From this, property (d) follows.

If $\xi, \zeta \in \mathscr{M}(T)$ and $A \in \mathscr{F}$ then for all $f \in \mathbf{E}$ by inequality (1) again,

$$(2) \quad \|Tf\|^p \le \|Tf_A\|^p + \|Tf_{A^c}\|^p \le \|\xi f_A\|^p + \|\zeta f_{A^c}\|^p = \|\xi_A f + \zeta_{A^c} f\|^p.$$

Taking $A = \{\xi < \zeta\}$ in (2) we obtain that $\xi \wedge \zeta = \xi_A + \zeta_{A^c} \in \mathscr{M}(T)$. Thus

(3) *$\mathscr{M}(T)$ is closed under formation of finite lattice infima.*

Assume first that $(X, \mathscr{F}, \mu)$ is σ-finite. Fix some $f_0 \in \mathbf{E}$ with supp $F_0 = X$. For some $\xi_1, \xi_2, \ldots \in \mathscr{M}(T)$ we have

$$\|\xi_n f_0\| \to m \equiv \inf\{\|\xi f_0\| : \xi \in \mathscr{M}(T)\}.$$

By property (3), $\theta_n \equiv \xi_1 \wedge \cdots \wedge \xi_n \in \mathscr{M}(T)$ $(n \ge 1)$. By virtue of the dominated convergence theorem applied on $\|\theta_n f\|$ for any $f \in \mathbf{E}$ and in particular for $f = f_0$ we get $\theta_\infty \equiv \inf\{\theta_n\} \in \mathscr{M}(T)$ and $\|\theta_n f_0\| = m$. For each $\theta = \mathscr{M}(T)$, $\theta \wedge \theta_\infty \in \mathscr{M}(T)$ by (3) and as $\|\cdot f_0\|$ is strictly monotone on $L_\infty^+(\mu)$, $\theta \wedge \theta_\infty = \theta_\infty$. Thus $\theta_\infty = \delta_1(T)$. This proves (a) in the σ-finite case. To prove (a) in the general case we first show that (c)(i) holds when restricted to $\eta = 1_A$ $(A \in \mathscr{F})$. Indeed if $\xi \in \mathscr{M}(T \circ 1_A)$ then $\xi \ge \xi_A = \theta_A \in \mathscr{M}(T \circ 1_A)$ for a $\theta \in \mathscr{M}(T)$, e.g., $\theta = \xi_A + 1_{A^c}\|T\|$ by argument (2) with $\zeta = \|T\|$. Conversely if $\theta \in \mathscr{M}(T)$ then $\theta_A \in \mathscr{M}(T \circ 1_A)$. From these,

$$(4) \quad \delta_1(T \circ 1_A) = \inf\{\xi_A : \xi \in \mathscr{M}(T \circ 1_A)\} = \inf\{\theta_A : \theta \in \mathscr{M}(T)\} = 1_A \delta_1(T).$$

Given $0 \ne f \in \mathbf{E}$ by (4) and the previous case $1_A \delta_1(T) \in \mathscr{M}(T \circ 1_A)$ for $A = \operatorname{supp} f$. This implies (a) (apply the norm inequality for $T \circ 1_A$ on f).

Property (a) and $\delta_1(T) \le 1_{s(T)}\|T\|$ (by Proposition 1.1) imply (b) and equation (c)(ii) for $\eta = 1$. The latter implies (c)(ii) in general if c(i) proved.

Equation (c)(i) is true for $\eta = 1_A (A \in \mathscr{F})$ by (4) and trivially also for η a scalar. Hence by an easy argument (c)(i) holds for η a simple function. The general case of (c)(i) follows by approximations as both sides of the equation are norm-continuous on $\eta \in L_\infty(\mu)$. To see this for the L.H.S. note that for all $\eta, \eta' \in L_\infty(\mu)$ by (d) and (c)(ii) for $\eta = 1$ applied to $T \circ (\eta - \eta')$,

$$\|\delta_1^p(T \circ \eta) - \delta_1^p(T \circ \eta')\|_\infty \le \|\delta_1^p(T \circ (\eta - \eta'))\|_\infty = \|T \circ (\eta - \eta')\|^p \le \|T\|^p \|\eta - \eta'\|_\infty^p.$$

For a non-example when $p > 1$ let $T \in \mathscr{L}(l_p(2), l_q(1))$ map (x, y) to $x + y$. Given $\varepsilon > 0$, by simple calculus $(1+\varepsilon, t_\varepsilon), (t_\varepsilon, 1+\varepsilon) \in \mathscr{M}(T)$ for some t_ε, $1 < t_\varepsilon < \infty$. As $\delta_1(T) \geq (1,1)$ so equality holds. Since $\rho \equiv \|T(1,1)\| / \|(1,1)\| = 2^{1-1/p} > 1$ so $\|T\| > \|\delta_1(T)\|_\infty$. When $p > q$ the same arguments apply to $T = \operatorname{diag}(1,1) \in \mathscr{L}(l_p(2), l_q(2))$, for which $\rho = 2^{1/q-1/p} > 1$. □

(1.3) COROLLARY. *With* $A^{nk} = \{\|T\|(k-1)/2^n < \delta_1(T) \leq \|T\|k/2^n\}$,

$$\begin{aligned}\delta_1(T) &= L_\infty\text{-}\lim_{n\to\infty} \sum_{k=1}^{2^n} 1_{A^{nk}} \|T \circ 1_{a^{nk}}\| \\ &= \inf\{1_A \|T \circ 1_A\| + 1_{A^c}\|T \circ 1_{A^c}\| : A \in \mathscr{F}\}.\end{aligned}$$

Proof. The first equality follows readily from Theorem 1.2(c)(ii), by which the sum of the R.H.S. differs from $\delta_1(T)$ by not more than $\|T\|/2^n$. For the second equality the defining set of functions for $d \equiv$ the stated infimum is a subset of $\mathscr{M}(T)$ by the argument in (2) with $\xi = \|T \circ 1_A\|$ and $\zeta = \|T \circ 1_{A^c}\|$. So $\delta_1(T) \leq d$. But $d \leq$ the displayed limit; note that $T \circ 1_A = O$ for $A = \{\delta_1(T) = 0\}$. The result follows. □

Explicit expressions for $\delta_1(T)$ in some cases will be given in (1.4), (1.6) and (2.2). Define the *atomic segment* $@(\mu)$ of a measure space $(X, \mathscr{F}, \mu)$ to be the supremum of all its atoms (identified as singletons). If $\mathbf{E} = L_p(X, \mathscr{F}, \mu)$ and $A \in \mathscr{F}$ we identity the dual space $(\mathbf{E}_A)'$ with $\{\Lambda \in \mathbf{E}' \colon \Lambda \mathbf{E}_{A^c} = \{0\}\}$.

The main part of Theorem 1.4 below has a close analog [Kan4, Theorem 2.2] for, instead of $\mathscr{F}$, a σ-subalgebra

$$\mathscr{F}(T) = \{A \in \mathscr{F} \colon |Tf| \wedge |Tg| = 0 \ \forall f \in \mathbf{E}_A,\ g \in \mathbf{E}_{A^c}\}$$

in the case $\mathbf{E} = L_p(\mu)$, $\mathbf{F} = L_q(\nu)$, $1 \leq p < q < \infty$ and $O \neq T \in \mathscr{L}$.

(1.4) THEOREM. *Let* $p < s \leq 1$, $\mathbf{E} = L_p(\mu)$, $\mathbf{F}$ *be an s-normed space and* $O \neq T \in \mathscr{L}$. *Then* $(\mathscr{F} \cap s(T), \mu)$ *is purely atomic, i.e.,* $T = T \circ 1_{@(\mu)}$, *and*

$$\delta_1(T)(x) = \|T \circ 1_x\| = \|T1_x\| / \|1_x\| \ \forall x \in \operatorname{supp} \delta_1(T) = s(T) \subset @(\mu).$$

In particular (*cf.* [D1]),

$$\begin{aligned}\mathbf{E}' &= (\mathbf{E}_{@(\mu)})' \\ &= \left\{\textit{function } \xi \textit{ on } @(\mu) \colon \|\xi\|' \equiv \sup_{x \in @(\mu)} |\xi(x)| / (\mu\{x\})^{1/p-1} < \infty\right\}.\end{aligned}$$

Here a member ξ of the last set acts on $f \in \mathbf{E}_{@(\mu)}$ according to the equation

$$\langle f, \xi \rangle = \sum_{x \in @(\mu)} f(x)\xi(x)\mu\{x\},$$

with operator norm $\|\langle \cdot, \xi \rangle\| = \|\xi\|'$. The sum converges absolutely.

Proof. Assume to the contrary $D \equiv s(T) \setminus @(\mu) \neq \varnothing$. There is an $f \in \mathbf{E}_D$ with $\|f\| = 1$ and $Tf \neq 0$. Let $A^0 = \operatorname{supp} f$. Now $\|Tf_{(\cdot)}\|^s$ is subadditive and $\|f_{(\cdot)}\|^p$ is additive, in fact a finite diffuse measure, on $\mathscr{F} \cap A^0$. So if $\varnothing \neq A \in \mathscr{F} \cap A^0$ is partitioned into non-null subsets $B, C \in \mathscr{F}$ then

$$\rho(A) \equiv \|Tf_A\|^s / \|f_A\|^p \leq \max\{\rho(B), \rho(C)\}.$$

Also any such A can be split into two parts of equal diffuse measure (see e.g., [W, p. 100]). Hence there exist inductively $A^1, A^2, \ldots \in \mathscr{F}$ with $A^{n+1} \subset A^n$, $\|f_{A^{n-1}}\|^p = \|f_{A^n}\|^p/2$ and $\rho(A^{n+1}) \geq \rho(A^n)$, $(n \geq 0)$. We have

$$\|T\|^s \geq \|Tf_{A^n}\|^w / \|f_{A^n}\|^s = \rho(A^n)/\|f_{A^n}\|^{s-p} \geq \rho(A^0)2^{n(s-p)/p} \nearrow \infty,$$

a contradiction. Thus $\mathscr{F} \cap s(T)$ is purely atomic. The description of $\delta_1(T)$ then follows easily from Theorem 1.2(b) and (c)(ii).

For the second part the main result with $\mathbf{F} = l_1(1)$ gives $\mathbf{E}' = (\mathbf{E}_{@(\mu)})'$. Further given $\Lambda \in (\mathbf{E}_{@(\mu)})'$ let $\xi(x) = \Lambda 1_x/\mu\{x\}$ $(x \in @(\mu))$. Then

$$\|\xi\|' = \sup_x |\Lambda 1_x| / \|1_x\| = \|\Lambda\| < \infty$$

and

$$\langle f, \xi \rangle = \sum f(x)\Lambda 1_x = \Lambda f (f \in \mathbf{E}_{@(\mu)})$$

by continuity of Λ and by f having σ-finite support. The inverse correspondence is $\xi \mapsto \Lambda = \langle \cdot, \xi \rangle$. To show $\langle \cdot, \xi \rangle \in (\mathbf{E}_{@(\mu)})'$ note that terms of the displayed sum form an element of $l_p(@(\mu))$ (L_p space over counting measure on $@(\mu)$) with formal norm $\leq \|\xi\|'\|f\|$. As $(\cdot)^p$ is countably subadditive on non-negative numbers this implies absolute convergence of that sum. Moreover $\|\langle \cdot, \xi \rangle\| \leq \|\xi\|' = \sup_x |\langle 1_x, \xi \rangle| \, \|1_x\|$ and so equality holds here. □

An *s-normed* (*vector*) *lattice* is a vector lattice equipped with an s-norm that is monotone on the positive cone and invariant under modulus taking. When $\mathbf{E}$ and $\mathbf{F}$ are r- and s-normed lattices with positive cones $\mathbf{E}^+$ and $\mathbf{F}^+$, the (*linear*) *modulus* $|T|$, if it exists, of a $T \in \mathscr{L}$ is a *Positive* operator $S \in \mathscr{L}$ (i.e., one mapping $\mathbf{E}^+$ to $\mathbf{F}^+$) for which $\sup\{|Tg| : |g| \leq f\} = Sf$ exists for each $f \in \mathbf{E}^+$.

Note that Banach lattices $\mathbf{F}$ satisfying the requirement in Theorem 1.5 on norm bounded increasing sequences include reflexive ones due to a theorem of Ogasawara [S, Theorem II.5.11]. Also the requirement on J is satisfied if $\mathbf{F}$ has order continuous norm [S, Theorem II.5.10], or in particular is an L_q space $(1 \le q < \infty)$.

For an element f in a function lattice (normed lattice whose elements are measurable functions) define its *signum* function $\operatorname{sgn} f$ to be $f/|f|$ in $\operatorname{supp} f$ and 0 elsewhere. Denote $\overline{\operatorname{sgn} f}$ by $\overline{\operatorname{sgn}}\, \mathrm{f}$.

(1.5) THEOREM. *Let $p \le s \le 1$, $\mathbf{E} = L_p(\mu)$ and $\mathbf{F}$ be an order-complete vector lattice with a complete, strictly monotone s-norm in which every s-norm bounded increasing sequence is s-norm convergent. Then*:

(a) *$\mathscr{L}$ is a complete s-normed vector lattice under the operator s-norm and the linear modulus. Moreover, $\delta_1(|T|) = \delta_1(T)$ for each $T \in \mathscr{L}$.*

(b) *In the case $p = 1 = s$ for each $T \in \mathscr{L}$, $|T^*|$ also exists. Further, when either the canonical imbedding J: $\mathbf{F} \to \mathbf{F}''$ preserves arbitrary suprema or $\mathbf{F}$ is a function lattice $|T^*| = |T|^*$.*

For $\mathbf{E} = L_p(\mu)$, $\mathbf{F} = L_q(\nu)$ and $T \in \mathscr{L}$ even in the finite-dimensional case $\||T|\| > \|T\|$ may occur when $q < \infty$ and $p > \min\{1, q\} \equiv s$ instead.

Proof. (a) For the lattice part it suffices to show that any given $T \in \mathscr{L}$ has a linear modulus of norm $\|T\|$. Let $f \in \mathbf{E}^+$. For a (finite, measurable) partition $\mathscr{D}$ of $(X, \mathscr{F}, \mu)$ by inequality (1) in the proof of Theorem 1.2,

$$\left\|\sum\{|Tf_A|\colon A \in \mathscr{D}\}\right\| \le \left(\sum\{\|Tf_A\|^p\colon A \in \mathscr{D}\}\right)^{1/p}$$
$$\le \|T\|\left(\sum\{\|f_A\|^p\colon A \in \mathscr{D}\}\right)^{1/p} = \|T\| \cdot \|f\|.$$

The L.H.S. has a finite supremum over all such $\mathscr{D}$ and the sum there increases as $\mathscr{D}$ gets finer. So any maximizing sequence of partitions can be refined to a progressively finer (p.f.) one $\{\mathscr{D}^n\}$, for which $\Sigma\{|Tf_A|\colon A \in \mathscr{D}^n\}$ converges by the given condition to a limit, designated $|T|f$, in $\mathbf{F}^+$ with $\||T|f\|$ = the said supremum $\le \|T\| \cdot \|f\|$. As the formal norm of $\mathbf{F}$ is strictly monotone on $\mathbf{F}^+$, {joint refinement of $\mathscr{D}^n$ and $\mathscr{D}$} for any given $\mathscr{D}$ is a like sequence inducing the same limit $|T|f$. Thus $|T|f = \sup\{\Sigma\{|Tf_A|\colon A \in \mathscr{D}\}\colon \mathscr{D}$ is a partition}. Positive linearity of $|T|$ on $\mathbf{E}^+$ is easy to prove when applied on simple functions; for two such functions use a common p.f. maximizing sequence $\{\mathscr{D}^n\}$ for which each $\mathscr{D}^n$ refines the partitions due to their sets of constancy. The general case follows by approximating $g \in \mathbf{E}^+$ from below by simple $h_n \in \mathbf{E}^+$. Observe that

$$\pm\left(|T|_g - |T|h_n\right) \le |T|(g - h_n),$$

since $\pm(|Tg_A| - |T(h_n)_A|) \le |T(g-h_N)_A|$ $(A \in \mathscr{F})$. We can prove $|Tg| \le |T||g|$ $(g \in \mathbf{E})$ likewise by approximation $(\operatorname{Re} g)^{\pm}$, $(\operatorname{Im} g)^{\pm}$ and hence $|g|$ from below by simple functions. Now $\{|Tg|: |g| \le f\}$ is majorized by $|T|f$ and so has a supremum Sf as $\mathbf{F}$ is order complete. On the other hand given a partition $\{A^1, A^2, \dots, A^N\}$ we have by [S, p. 134] (or [L, p. 9]),

$$\sum |Tf_{A^n}| \equiv \sum \sup_{\theta_n} \operatorname{Re}\left(e^{i\theta_n} Tf_{A^n}\right) = \sup_{\theta_1 \dots \theta_N} \operatorname{Re}\left(T \sum e^{i\theta_n} f_{A^n}\right) \le Sf.$$

It follows that $|T|f = Sf$. Hence $|T|$ extended linearly to an element of $\mathscr{L}$ is indeed the modulus of T and has norm $\|T\|$.

From the equality $|T|f = Sf$ we deduce that for any $A \in \mathscr{F}$, $|T| \circ 1_A = |T \circ 1_A|$. Hence $\delta_1(|T|) = \delta_1(T)$ by Corollary 1.3.

(b) $|T^*|$ exists by [S, Theorem IV.1.5(i)]. For each $f \in \mathbf{E}^+$ and $g \in \mathbf{F}'$,

$$\langle f, |T^*g| \rangle = \langle f \eta T^* g \rangle = \langle T(\eta f), g \rangle \le \langle |T|f, |g| \rangle = \langle f, |T|^* |g| \rangle,$$

where $\eta = \overline{\operatorname{sgn}}\, T^*g$. It follows that $|T|^*|g| \ge |T^*g|$. So $|T|^* \ge |T^*|$.

Conversely for each $h \in \mathbf{E}^+$ and $g \in (\mathbf{F}')^+$, letting $j_1: \mathbf{E} \to \mathbf{E}''$ be the canonical imbedding of $\mathbf{E}$ into its bidual $\mathbf{E}''$ we get for each angle θ,

$$\langle \operatorname{Re}(e^{i\theta} Th), g \rangle = \operatorname{Re}\langle h, e^{i\theta} T^* g \rangle \le \langle h, |T^*| g \rangle = \langle g, |T^*|^* J_1 h \rangle.$$

If J preserves arbitrary suprema this implies

$$|T^*|^* J_1 h \ge \sup_{\theta} J \operatorname{Re}(e^{i\theta} Th) = J \sup_{\theta} \operatorname{Re}(e^{i\theta} Th) = J|Th|.$$

Hence

$$\langle |Th|, g \rangle = \langle g, J|Th| \rangle \le \langle g, |T^*|^* J_1 h \rangle = \langle h, |T^*| g \rangle.$$

If $\mathbf{F}$ is a function lattice the net result $\langle |Th|, g \le \langle h, |T^*| g \rangle$ can be directly proved via using a signum function as in the last paragraph. Consider further each $f \in \mathbf{E}^+$ and any partition $\{A^1, \dots, A^N\}$ of $(X, \mathscr{F}, \mu)$. Applying the result just obtained we get

$$\left\langle \sum |Tf_{A^n}|, g \right\rangle = \sum \langle |Tf_{A^n}|, g \rangle \le \sum \langle f_{A^n}, |T^*| g \rangle = \langle f, |T^*| g \rangle.$$

Since $|Tf|$ is a sequential limit as described in the proof of (a) this implies $\langle |T|f, g \rangle \le \langle f, |T^*| g \rangle$. So $|T|^* g \le |T^*| g$. It now follows that $|T|^* = |T^*|$.

For the non-example part let $T \in \mathscr{L}(l_p(2), l_p(2))$ map $f = (x, y)$ to $(x + ty, y - tx)$ for some $t > 0$. Then $\|Tf\| < \| |T| |f| \|$ if $xy \ne 0$. As T attains its norm it remains to get.

$$\max\{\|T(0,1)\|, \|T(1,0)\|\} < \| |T|(1,1) \| / \|(1,1)\|$$

say, i.e., $(1+t^q)2^{q/p} < 2(1+t)^q$. When $p > 1$ this is true for $t = 1$ and when $p > q$, for t small enough. □

(1.6) COROLLARY. *For each* $T \in \mathscr{L}(L_1(\mu), L_1(\nu))$, $\delta_1(T) = |T|^*1 = |T^*|1$.

Proof. For all $f \in L_1(\mu)$, $\| |T|f| \| \le \langle |T| |f|, 1\rangle = \langle |f|, |T|^*1\rangle = \| f| T|^*1 \|$. Equality holds if $f \ge 0$. So $|T^*|1 = |T|^*1 = \delta_1(|T|) = \delta_1(T)$ (Theorem 1.5(a)(b)). □

(1.7) LEMMA. *Let* $0 \ne f \in \mathbf{E} \equiv L_p(\mu)$ $(0 < p < \infty)$ *and* $A = \operatorname{supp} f$. *Then*

$$\overline{\operatorname{span}}\{f_C : C \in \mathscr{F} \cap A\} = \mathbf{E}_A.$$

Proof. Let $0 \ne g \in \mathbf{E}_A$.
$h \equiv (g/f)_A$ as an element of $L_p(|f|^p\, d\mu)$ is the limit of a sequence $\{h_n \equiv \Sigma_k c_{nk} 1_{A^{nk}}\}$ of simple functions supported in A. (Approximate $(\operatorname{Re} h)^{\pm}$ and $(\operatorname{Im} h)^{\pm}$ from below by non-negative simple functions.) So $\Sigma_k c_{nk} f_{A^{nk}} = h_n f \to hf = g$ in the metric of $\mathbf{E}_A$. □

(1.8) PROPOSITION. *Let* $p \le s \le 1$, $\mathbf{E} = L_p(\mu)$, $\mathbf{F}$ *be an s-normed space,* $T \in \mathscr{L}$ *and* $0 \ne f \in \mathbf{E}$. *Then with* $A = \operatorname{supp} f$,

$$\|T \circ 1_A\| = \sup\{\|Tf_C\| / \|f_C\| : \varnothing \ne C \in \mathscr{F} \cap A\}.$$

Proof. Let $g = \Sigma c_n f_{A^n}$ be a finite sum with disjoint $A^n \in \mathscr{F} \cap A \setminus \{\varnothing\}$ and scalars $c_n \ne 0$. By inequality (1) in the proof of Theorem 1.2, $\|Tg\|^p \le \Sigma |c_n|^p \|Tf_{A^n}\|^p$ while $\|g\|^p = \Sigma |c_n|^p \|f_{A^n}\|^p$. Hence

$$\|Tg\|^p / \|g\|^p \le \|Tf_{A^n}\|^p / \|f_{A^n}\|^p \text{ for some } n.$$

So $\|Tg\| / \|g\| \le$ stated supremum $\le \|T \circ 1_A\|$. These g are dense in $\mathbf{E}_A$ by Lemma 1.7. The result follows. □

(1.9) COROLLARY. *There exists a sequence of countable, measurable partitions* $\mathscr{C}^n$ *of* A *that are progressively more refined such that*

$$1_A \delta_1(T) = L_\infty\text{-}\lim_{n\to\infty} \sum \{1_C \|Tf_C\| / \|f_C\| : C \in \mathscr{C}^n\}.$$

Proof. By Proposition 1.8 and transfinite induction with notation in Corollary 1.3 applied to $T \circ 1_A$ (in place of T), each non-empty A^{nk} admits a countable, well-ordered measurable partition $\mathscr{C}^{nk}$ with

$$\|Tf_C\| / \|f_C\| + 1/n \ge \|T \circ 1_{C'}\|$$

($C \in \mathscr{C}^{nk}$, $C' = A^{nk} \setminus \sup\{$members of $\mathscr{C}^{nk}$ preceding $C\}$), so that $\|Tf_C\|/\|f_C\|$ and $\delta_1(T)$ differ by not more than $\|T \circ 1_A\|/2^n + 1/n$ on C. Now let $\mathscr{C}^n = \bigcup_k \mathscr{C}^{nk}$ ($\mathscr{C}^{nk} = \{\emptyset\}$ if $A^{nk} = \emptyset$). For $n = 2, 3, \ldots$ the construction can be modified so that $A^{nk} \cap \mathscr{C}^{n-1} \subset \mathscr{C}^{nk}$ also. The result follows. □

Let $\mathbf{E} = L_p(\mu)$, $\mathbf{F} = L_q(\nu)$ and $T \in \mathscr{L}$. For $A \in \mathscr{F}$ and $b \in \mathscr{G}$ define $T_{BA} \in \mathscr{L}$ by $T_{BA}f = (Tf_A)_B (f \in \mathbf{E})$.

(1.10) THEOREM. *Let $0 < q < \infty$, $p \leq \min\{1, q\}$, $\mathbf{E} = L_p(\mu)$, $\mathbf{F} = L_q(\nu)$ and $T \in \mathscr{L}$. Then $\delta_1^q(1_{(\cdot)}T)$ is additive on $\mathscr{G}$.*

Proof. Let $\emptyset \neq B \in \mathscr{G}$ be given. We need only prove that

$$(1) \qquad \delta_1^q(T) = \delta_1^q(1_B T) + \delta_1^q(1_{B^c} T).$$

(Replace T by $1_D T$ ($D \in \mathscr{G}$) to get the theorem.) Let $0 \neq f \in \mathbf{E}$ and $A = \operatorname{supp} f$. Take any $\emptyset \neq Z \in \mathscr{F} \cap A$ and $\varepsilon > 0$. Repeated relativized applications of Proposition 1.8 on $1_{B^c}T$ in a transfinite induction process yield a well-ordered countable decomposition $\mathscr{C}$ of Z, such that for all $C \in \mathscr{C}$,

$$\frac{\|T_{B^cC}f\|^q}{\|f_C\|^q} \geq \|1_{B^c}T \circ 1_{C'}\|^q - \varepsilon \geq \inf_A \delta_1^q(1_{B^c}T) - \varepsilon,$$

(where the infimum is taken over A) by Theorem 1.2(c)(ii). Here $C' = Z \setminus \sup($members of $\mathscr{C}$ preceding $C\}$. Also

$$\|T_{BC}f\|/\|f_C\| \geq \|T_{BZ}f\|/\|f_Z\| \text{ for some } C \in \mathscr{C}$$

(extend the argument in the proof of Proposition 1.8 to $f_Z = \Sigma\{f_C : C \in \mathscr{C}\}$).

$$\begin{aligned}\|T \circ 1_A\|^q &\geq \left(\frac{\|Tf_C\|}{\|f_C\|}\right)^q = \left(\frac{\|T_{BC}f\|}{\|f_C\|}\right)^q + \left(\frac{\|T_{B^cC}f\|}{\|f_C\|}\right)^q \\ &\geq \left(\frac{\|T_{BZ}f\|}{\|f_Z\|}\right)^q + \inf_A \delta_1^q(1_{B^c}T) - \varepsilon.\end{aligned}$$

Hence by Theorem 1.2(c)(ii) and Proposition 1.8 again this gives

$$\begin{aligned}(2) \qquad \|1_A \delta_1^q(T)\|_\infty = \|T \circ 1_A\|^q &\geq \|T_{BA}\|^q + \inf_A \delta_1^q(1_{B^c}T) \\ &\geq 1_A \delta_1^q(1_B T) + \inf\nolimits_A \delta_1^q(1_{B^c}T).\end{aligned}$$

Conversely q-additivity of the formal norm on $\mathbf{F}$ and the same theorem imply

$$(3)\qquad \begin{aligned} 1_A\delta_1^q(T) &\le \|T\circ 1_A\|^q \le \|T_{BA}\|^q + \|T_{B^cA}\|^q \\ &= \|1_A\delta_1^q(1_B T)\|_\infty + \|1_A\delta_1^q(1_{B^c}T)\|_\infty. \end{aligned}$$

As A is arbitrary albeit of σ-finite measure inequalities (2) and (3) imply equation (1) by approximations. Indeed given integer $n \ge 1$ partition X into finitely many subsets wherein each of all three terms in (1) has μ-essential oscillations $\le 1/n$. Partition each subset into further subsets A of finite measures. By (2) and (3) considered at points of such A, the two sides of equation (1) differ by not more than $2/n$ a.e. on X. So they are equal a.e. (Note that the special case $p < \min\{q, 1\}$ also follows from Theorem 1.4, and the case $p = 1 = q$ follows from Corollary 1.6.) □

2. Extreme contractions on L_p, $p \le 1$

(2.1) THEOREM. *Let $p \le s \le 1$, $\mathbf{E} = L_p(\mu)$, $\mathbf{F}$ be an s-normed space and $T \in \operatorname{ext} \mathscr{U}$. Then*:

(a) $T\circ 1_A \in \operatorname{ext}\mathscr{U}(\mathbf{A}_A, \mathbf{F})$ *for each* $\varnothing \ne A \in \mathscr{F}$.

(b) $\delta_1(T) = 1$ *when* $p = 1$.

Proof. (a) Let $R \in \mathscr{L}(\mathbf{E}_A, \mathbf{F})$ with $T\circ 1_A \pm R \in \mathscr{U}(\mathbf{E}_A, \mathbf{F})$. Extend R to $R_1 \in \mathscr{L}$ with $R_1\circ 1_{A^c} = O$. We have by Theorem 1.2(c)(i),

$$\delta_1(T \pm R_1) = 1_A\delta_1(T\circ 1_A \pm R) + 1_{A^c}\delta_1(T\circ 1_{A^c}).$$

So $T \pm R_1 \in \mathscr{U}$ by Theorem 1.2(c)(ii). Thus $R_1 = O = R$.

(b) We use Theorem 1.2 freely. We have $\delta \equiv \delta_1(T) \le 1$. Let $g \in \mathbf{F}$ with $\|g\| = 1$ and $S = \langle\,\cdot\,, 1\rangle g$. Then $\delta_1(S) = 1$ and

$$\delta_1(T \pm S\circ(1-\delta)) \le \delta + (1-\delta) = 1.$$

So $T \pm S\circ(1-\delta) \in \mathscr{U}$, whence $S\circ(1-\delta) = O$ and $1 - \delta = \delta_1(S\circ(1-\delta)) = 0$. □

Call an extreme point of the unit ball of an s-normed space $\mathbf{F}$ an *extreme unit vector* of $\mathbf{F}$.

(2.2) THEOREM. *Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space, $\mathbf{E} = L_1(\mu)$, $\mathbf{G}$ a Banach space and $T \in \mathscr{L}(\mathbf{E}, \mathbf{G}')$, where either $\mathbf{G}$ or $T\mathbf{E}$ is separable. Then there exists a μ-essentially unique weak* measurable function $\varphi(T)$ on $(X, \mathscr{F}, \mu)$*

to $\mathbf{G}'$ *such that* $\langle g, \varphi(T)(\cdot)\rangle \in L_\infty(\mu)$ $(\forall g \in \mathbf{G})$ *and*

$$\langle g, Tf\rangle = \int \langle g, \varphi(T)(x)\rangle f(x)\, d\mu(x) \ (\forall f \in \mathbf{E},\, g \in \mathbf{G}).$$

Moreover, $\|T\| = \operatorname{ess\,sup}\|\varphi(T)(\cdot)\|$ *and* $\delta_1(T) = \|\varphi(T)(\cdot)\|$.

Further $T \in \operatorname{ext}\mathscr{U}$ *only if* $\|\varphi(T)(\cdot)\| = 1$ μ*-a.e. Conversely either* $\mathbf{G}$ *or* $\mathbf{E}$ *is separable* $T \in \operatorname{ext}\mathscr{U}$ *if* $\varphi(T)(x)$ *is for* μ*-a.e.* x *an extreme unit vector of* $\mathbf{G}'$.

Proof. The unique existence of the specified $\varphi(T)$ and the formula for $\|T\|$ are given in [DS, VI.8.6-7] (see also [S, Theorem IV.7.6]). The displayed representation equation shows that for each $A \in \mathscr{F}$, $T \circ 1_A$ is similarly induced by $1_A\varphi(T)$ as T is by $\varphi(T)$. It follows that $\delta_1(T) = \|\varphi(T)(\cdot)\|$ since by Theorem 1.2(c)(ii), for all $A \in \mathscr{F}$,

$$\|1_A\delta_1(T)\|_\infty = \|T \circ 1_A\| = \operatorname*{ess\,sup}_{x\in A}\|\varphi(T)(x)\|.$$

The necessary condition for extremeness then follows by Theorem 2.1(b). For the sufficiency part let $R \in \mathscr{L}$ with $T \pm R \in \mathscr{U}$. When $\mathbf{E}$ is separable T and $T \pm R$ all have separable ranges. So under the given conditions all three operators have their respective representations as described in the theorem. By the expression for $\delta_1(\cdot)$ again and Theorem 1.2(c)(ii) we get for μ-a.e. x,

$$\|\varphi(T)(x)\| = 1 \geq \|\varphi(T \pm R)(x)\| = \|\varphi(T)(x) \pm \varphi(R)(x)\|.$$

As $\varphi(T)(x)$ is for μ-a.e. x an extreme unit vector, $\varphi(R)(x) = 0 \in \mathbf{G}'$ for μ-a.e. x. So $R = O$ and T is extreme. □

(2.3) THEOREM. *Let* $\mathbf{E} = L_1(\mu)$ *and* $\mathbf{F}$ *be a normed space with strictly convex bidual* $\mathbf{F}''$. *Then a* $T \in \mathscr{U}$ *is extreme if and only if* $\delta_1(T) = 1$.

Proof. By Theorem 2.1(b) we need only prove the sufficiency. So assume $\delta_1(T) = 1$. We use results in §3. By Corollary 3.2, $\delta_\infty(T^*) = \delta_1(T) = 1$. By Theorem 3.3 applied to T^*, T^* is extreme and so is T. □

In view of Theorems 2.2 and 2.3 we posit a conjecture below, which is true when μ is purely atomic (Corollary 2.6) or when T is a dual operator (Theorem 3.4 and Corollary 3.2). Note that a strictly convex (s.c.) normed space $\mathbf{F}$ may not have a s.c. bidual $\mathbf{F}''$. Indeed l_1, being separable, can be equivalently renormed (e.r.) to be a s.c. space $\mathbf{F}$ [Kö, p. 362, item (5)] but its dual l_∞ cannot be e.r. to be smooth [D2, Theorem 9]. So $\mathbf{F}''$ is not s.c., or else $\mathbf{F}'$ would be smooth by [AB, Footnote 13] (or [Kö, p. 346, item (2)]). Another example is $\mathbf{F} =$ e.r. $c_0(A)$ (A any uncountable index set) [D2, p. 517, comment (ii)].

(2.4) CONJECTURE. *For a strictly convex normed space* $\mathbf{F}$, *a* $T \in \mathscr{U}(L_1(\mu), \mathbf{F})$ *is extreme if and only if* $\delta_1(T) = 1$.

In the general setting of r- and s-normed spaces $\mathbf{E}$ and $\mathbf{F}$ respectively, for each $t \in \mathscr{U}$ define $\mathscr{N}(T) = \{f \in \mathbf{E}\colon \|Tf\| = \|f\|\}$ and

$$\hat{\mathscr{N}}(T) = \{f \in \mathbf{E}\colon f \neq 0 \Leftrightarrow Tf/\|f\| = \textit{an extreme unit vector of } \mathbf{F}\}.$$

One can see that $T \in \operatorname{ext} \mathscr{U}$ if $\mathbf{E} = \overline{\operatorname{span} \hat{\mathscr{N}}(T)}$. The following two results clarify this situation when $\mathbf{E} = L_1(\mu)$ and $\mathbf{F}$ is a normed space. Note also that $T \in \operatorname{ext} \mathscr{U}$ only if $1_x \in \hat{\mathscr{N}}(T)$ $(x \in @(\mu))$ by Theorem 2.1(a).

(2.5) THEOREM. *Let* $\mathbf{E} = L_1(\mu)$, $\mathbf{F}$ *be a normed space and* $T \in \mathscr{U}$ *with* $\hat{\mathscr{N}}(T) \neq \{0\}$. *Then* $\hat{\mathscr{N}}(T) = \bigcup\{\hat{\mathscr{N}}(T) \cap \mathbf{E}_{A^\alpha}\}$ *for a unique disjoint family* $\{A^\alpha\} \subset \mathscr{F} \setminus \{\varnothing\}$ *with* $\sup\{\operatorname{supp} f\colon f \in \hat{\mathscr{N}}(T) \cap \mathbf{E}_{A^\alpha}\} = A^\alpha$ *for each index* α.

Moreover for each index α *there exists* $(\xi_{\overleftarrow{}}\ g_\alpha) \in \mathbf{E}' \times \mathbf{F}$ *with* $|\zeta_\alpha| = 1_{A^\alpha}$ *and* g_α *an extreme unit vector of* $\mathbf{F}$ *such that*

$$T \circ 1_{A^\alpha} = \langle \cdot, \bar{\zeta}_\alpha \rangle g_\alpha$$

and consequently

$$\mathscr{N}(T) \cap \mathbf{E}_{A^\alpha} = \hat{\mathscr{N}}(T) \cap \mathbf{E}_{A^\alpha} = \{ch\,\zeta_\alpha\colon h \in \mathbf{E}^+_{A^\alpha}, c \textit{ a scalar}\}.$$

Proof. Let $f \in \hat{\mathscr{N}}(T)$, $\|f\| = 1$. Let $B \in \mathscr{F} \cap \operatorname{supp} f$. Then

$$\|Tf\| \leq \|Tf_B\| + \|Tf_{B^c}\| \leq \|f_B\| + \|f_{B^c}\| = 1.$$

Thus equalities hold. Hence $f_B \in \mathscr{N}(T)$ and as Tf is an extreme unit vector,

$$Tf_B = \|Tf_B\| Tf = \|f_B\| Tf = \langle f_B, \overline{\operatorname{sgn}} f \rangle Tf.$$

So $T \circ 1_{\operatorname{supp} f} = \langle \cdot, \overline{\operatorname{sgn}} f \rangle Tf$ by Lemma 1.7.

Define an equivalence relation $\sim$ on unit vectors in $\hat{\mathscr{N}}(T)$ by: $f_1 \sim f_2$ if and only if Tf_1 and Tf_2 differ by a scalar factor. It induces a partition $\{\mathscr{N}_\alpha\}$ of these vectors into equivalence classes mod $\sim$. Let $A^\alpha = \sup\{\operatorname{supp} f\colon f \in \mathscr{N}_\alpha\}$. Well order $\{\operatorname{supp} f\colon f \in \mathscr{N}_\alpha\}$ as $\{\operatorname{supp} h_{\alpha\gamma}\}$ for some $h_{\alpha\gamma} \in \mathscr{N}_\alpha$. Inductively for each ordinal β let $B^{\alpha>} = A^\alpha \setminus \sup\{\operatorname{supp} f_{\alpha\beta'}\colon \beta' < \beta\}$ $(B^{\alpha 0} = A^\alpha)$ and let $f_{\alpha\beta}$ be the normalized first (with reference to γ) non-zero $(h_{\alpha\gamma})_{B^{\alpha\beta}}$, until $B^{\alpha\beta}$ becomes $\varnothing$. Each $f_{\alpha\beta} \in \mathscr{N}_A$ and by the result of the last paragraph, $\{\operatorname{supp} f_{\alpha\beta}\}$ partitions A^α. Let $g_\alpha = Tf_{\alpha 0}$. Then $Tf_{\alpha\beta} = c_{\alpha\beta} g_\alpha$ for a unimodular scalar $c_{\alpha\beta}$. Let $\zeta_\alpha = \bar{c}_{\alpha\beta} \operatorname{sgn} f_{\alpha\beta}$ on $\operatorname{supp} f_{\alpha\beta}$ for each β; let $\zeta_\alpha = 0$ on $(A^\alpha)^c$. Now each pair (ζ_α, g_α) is defined. The assertions can then be routinely verified. □

(2.6) COROLLARY. *$Z \equiv \sup\{\operatorname{supp} f\colon\ f \in \hat{\mathscr{N}}(T)\} = \sup\{A^\alpha\}$, $\overline{\operatorname{span}}\hat{\mathscr{N}}(T) = \mathbf{E}_Z$ and $1_Z\delta_1(T) = 1_Z$. Furthermore $T \circ 1_Z \in \operatorname{ext} \mathscr{U}(\mathbf{E}_Z, \mathbf{F})$.*

Proof. The first statement is an easy consequence of last theorem. For the extremeness part, if $R \in \mathscr{L}(\mathbf{E}_Z, \mathbf{F})$ is such that $T \pm R \in \mathscr{U}(\mathbf{E}_Z, \mathbf{F})$ then $R = O$ on $\hat{\mathscr{N}}(T)$, whence on $\mathbf{E}_Z$. □

(2.7) THEOREM. *Let $p < s \le 1$, $\mathbf{E} = L_p(\mu)$, $\mathbf{F}$ be an s-normed space and $T \in \mathscr{U}$. Then $\hat{\mathscr{N}}(T) \subset \mathscr{N}(T) \subset \{c1_x\colon c \text{ a scalar}, x \in @(\mu)\}$.*

Further $\delta_1(T) = 1_{@(\mu)}$ if and only if the last two sets are equal, while all three sets are equal if and only if $T \in \exp \mathscr{U}$. In particular $T \in \operatorname{ext} \mathscr{U}$

(i) *for $\mathbf{F}$ a strictly convex normed space if and only if $\delta_1(T) = 1_{@(\mu)}$, or*

(ii) *for $\mathbf{F}$ M-normed with unit $\mathbf{1}$ if and only if $|T1_x|/\|1_x\| = \mathbf{1}\ \forall x \in @(\mu)$.*

Proof. Let $f \in \mathscr{N}(T)$, $\|f\| = 1$. For each $B \in \mathscr{F} \cap \operatorname{supp} f$,

$$\begin{aligned}\|Tf\|^p \le \left(\|Tf_B\|^s + \|Tf_{B^c}\|^s\right)^{p/s} &\le \left(\|f_B\|^s + \|f_{B^c}\|^s\right)^{p/s}\\ &\le \|f_B\|^p + \|f_{B^c}\|^p = 1.\end{aligned}$$

So equalities hold. As $p/s < 1$, $\|f_B\|$ or $\|f_{B^c}\|$ is 0. Thus $\operatorname{supp} f$ is an atom of $\mathscr{F}$. This proves the second inclusion; the first is trivial. The last two sets are equal if and only if $\delta_1(T) = 1_{@(\mu)}$ by Theorem 1.4. All three sets are equal if and only if $1_x \in \hat{\mathscr{N}}(T)(x \in @(\mu))$, which implies $T \in \operatorname{ext} \mathscr{U}$ as $R = R \circ 1_{@(\mu)}$ $\forall R \in \mathscr{L}$ (Theorem 1.4). The converse holds by Theorem 2.1(a) applied to each atom $A = \{x\}$.

The result (ii) follows as $g \equiv T1_x/\|1_x\| \in \mathbf{F}$ is an extreme unit vector if and only if $|g| = \mathbf{1}$. (Necessity: $|g \pm (\mathbf{1} - |g|)| \le \mathbf{1}$. For sufficiency, imbed $\mathbf{F}$ into $\mathbf{F}^n$, a space $C(K)$ [S, Theorem II.7.4].) (i) is easy. □

Let $\mathbf{E} = L_p(\mu)$ and $\mathbf{F} = L_q(\nu)$. $T \in \mathscr{L}$ is said to be *codisjunctive* if for each $B \in \mathscr{G}$ there is an $A \in \mathscr{F}$ such that $T_{B^cA} = O = T_{BA^c}$. (This was introduced in [Kan2].) Dually T is *disjunctive* (also called *Lamperti* in [Kan1] due to a theorem of J. Lamperti [La]) if $|Tf| \wedge |Tg| = 0$ for all $f, g \in \mathbf{E}$ with $|f| \wedge |g| = 0$. When ν is a direct sum of finite measures (recall our convention in §1) this is equivalent to requiring that given $A \in \mathscr{F}$ there exists $B \in \mathscr{G}$ such that again $T_{B^cA} = O = T_{BA^c}$. It is routine to prove that for $1 \le p, q < \infty$, $T \in \mathscr{L}$ is codisjunctive if and only if T^* is disjunctive. (See also [Kan4, Theorem 2.1].)

When $\mathbf{E} = L_\infty(\mu)$, $\mathbf{F} = L_\infty(\nu)$ and $T \in \mathscr{L}$ is disjunctive we have

(Property D) $T = (T1)\Phi^\#$ and $|Tf| = |T||f|$ $(f \in \mathbf{E})$.

(See Remark 4.1 and Theorem 3.1 in [Kan1].) Here $\Phi^\#$ is the unique linear operator $\in \mathscr{L}$ induced by a Boolean ring homomorphism $\Phi\colon (\mathscr{F}, \mu) \to (\mathscr{G}, \nu)$

satisfying $\Phi^{\#}1_A = 1_{\Phi A}$ $(A \in \mathscr{F})$. We may in face define $\Phi A = \operatorname{supp} T1_A$ $(A \in \mathscr{F})$ if A has finite measure.

$\Phi^{\#}$ is positive and preserves L_∞ convergence. Note that $|\Phi^{\#} f| = \Phi^{\#}|f|$ $(f \in \mathbf{E})$ and so in Property D, $|T| = |T1|\Phi^{\#}$. Also, $\Phi^{\#}$ preserves sequential a.e., not necessarily L_∞, convergence if Φ is a Boolean σ-ring homomorphism. (See also [M, p. 159] or [Do, pp. 453–454].) Such a $\Phi^{\#}$ has the formal properties of a composition operator induced by a measurable transformation and in some cases simplifies to one.

(2.8) THEOREM. *Let $p \le 1 \le q < \infty$, $\mathbf{E} = L_p(\mu)$ and $\mathbf{F} = L_q(\nu)$. Then a $T \in \mathscr{U}$ is extreme if and only if*:

(1^-1^+) *in the case $p < 1 < q$, $\delta_1(T) = 1_{@(\mu)}$;*

(11^+) *in the case $p = 1 < q$, $\delta_1(T) = 1$;*

(1^-1) *in the case $p < 1 = q$, $\delta_1(T) = 1_{@(\mu)}$ and T is codisjunctive;*

(11) *in the case $p = 1 = q$, $\delta_1(T) = 1$ and T is codisjunctive.*

Moreover, criterion (1^-1) is equivalent to $(1^-1)'$, and (11) to $(11)'$ below:

$(1^-1)'$ $T = T \circ 1_{@(\mu)}$ *and* $T1_x = \xi(x)1_{\psi(x)}$, $|\xi(x)| = \|1_x\|/\|1_{\psi(x)}\|$ $(x \in @(\mu))$ *for a function ξ on $@(\mu)$ and a mapping ψ: $@(\mu) \to @(\nu)$.*

$(11)'$ $T^* = h\Psi^{\#}$ *for an $h \in L_x(\mu)$ with $|h| = 1$ and a σ-algebra homomorphism Ψ: $(\mathscr{G}, \nu) \to (\mathscr{F}, \mu)$.*

Proof. The first two parts follow from Theorems 2.7(i) and 2.3. Consider the other two, where $q = 1$.

Necessity. The part on $\delta_1(T)$ is by Theorems 2.1(b) and 2.7. Given $B \in \mathscr{G}$ let $\xi = \delta_1(1_B T)$, $\zeta = \delta_1(1_{B^c}T)$ and $R = 1_B T \circ \zeta - 1_{B^c} T \circ \xi$. Then $0 \le \xi, \zeta \le 1$ and $T \pm R = 1_B T \circ (1 \pm \zeta) + 1_{B^c} T \circ (1 \mp \xi)$. By Theorems 1.10 (or Theorem 1.4 and Corollary 1.6) and 1.2(c)(i),

$$\begin{aligned}\delta_1(T \pm R) &= \delta_1(1_B T \circ (1 \pm \zeta)) + \delta_1(1_{B^c} T \circ (1 \mp \xi)) \\ &= (1 \pm \zeta)\xi + (1 \mp \xi)\zeta = \xi + \zeta = \delta_1(T) \le 1.\end{aligned}$$

So $T \pm R \in \mathscr{U}$ (Theorem 1.2(c)(ii)) and $R = O$, i.e., $1_B T \circ \zeta = O = 1_{B^c} T \circ \xi$. By Theorem 1.2(c)(i), $\xi\zeta = 0$. By Theorem 1.2(b), T is codisjunctive.

Sufficiency. Assume (1^-1). As T is codisjunctive, for each $x \in @(\mu)$, $\operatorname{supp} T1_x$ is an atom of $(Y, \mathscr{G}, \nu)$. So via Theorem 1.4, $1_x \in \hat{\mathscr{N}}(T)$. By Theorem 2.7, T is extreme.

Assume (11). Let $R \in \mathscr{L}$ with $T \pm R \in \mathscr{U}$. Then $|T^* \pm R^*|1 = \delta_1(T \pm R) \le 1$ by Corollary 1.6 and Theorem 1.2(c)(ii), while $|T^*|1 = \delta_1(T) = 1$. But $2|T^*| \le |T^* + R^*| + |T^* - R^*|$. It follows that equalities hold for all three inequalities. So by Property D for T^*,

$$\begin{aligned} 2|T^*g| = 2|T^*||g| &= |T^* - R^*||g| + |T^* - R^*||g| \\ &\geq |T^*g + R^*g| + |T^*g - R^*g| \\ &\geq 2|R^*g|, (g \in \mathbf{F}'). \end{aligned}$$

This implies that $T^* \pm R^*$ and R^* are disjunctive, since T^* is. From $|T^*|1 = 1 = |T^* \pm R^*|1$, we get by Property D again $|T^*1| = 1 = |T^*1 \pm R^*1|$, implying $0 = |R^*1| = |R^*|1$. Hence $R^* = O$, or $R = O$. Thus T is extreme.

Clearly $(1^-1)'$ implies (1^-1) and $(11)'$ implies (11). Condition (1^-1) implies $(1^-1)'$ by Theorem 1.4; see also the pertinent sufficiency part. Condition (11) implies $(11)'$ by Property D and Corollary 1.6; Ψ preserves countable unions as $|T^*| = |T|^*$ is sequentially order-continuous (see e.g. [G, p. 34]). □

(2.9) *Remark.* One can also consider $p \leq q < 1$. The case $p < q < 1$ has the same extremeness criteria as for $p < 1 = q$. When $p = q < 1, T \in \operatorname{ext} \mathscr{U}$ if and only if both $\delta_1(T) = 1_{s(\mathbf{E},\mathbf{F})}$ and T is codisjunctive, where

$$s(\mathbf{E}, \mathbf{F}) = \left(\sup\{A \in \mathscr{F}: \mathscr{L}(\mathbf{E}_A, \mathbf{F}) = \{O\}\}\right)^c.$$

The proofs of these cases especially the latter are more elaborate but partly resemble those for the cases $p \leq 1 = q$. (To show a $T \in \operatorname{ext} \mathscr{U}$ to be codisjunctive change R in the proof of Theorem 2.8 to $R = 1_B T \circ \xi^{1-q}\zeta - 1_{B^c}T \circ \xi\zeta^{1-q}$ and get $\delta_1^q(T \pm R) \leq \delta_1^q(T) \leq 1$ via Theorem 1.10 and Jensen's inequality for $(\cdot)^q$.) These cases $p \leq q < 1$ are less intriguing as $\mathscr{U}$ here is generally not convex.

3. $\delta_\infty(T)$ and extreme contractions into $C(K)$

We follow the terminology of [S] on M-normed and AM-spaces. Let $\mathbf{E}$ be a normed space, $\mathbf{F}$ an M-normed space and $T \in \mathscr{L}$. Define

$$\mathscr{W}(T) = \{|Tf|: f \in \mathbf{E}, \|f\| \leq 1\}$$

and $\delta_\infty(T) = \sup \mathscr{W}(T)$ provided that this exists in $\mathbf{F}^+$; it exists for all $T \in \mathscr{L}$ when $\mathbf{F}$ has a unit and is order complete, or in particular when $\mathbf{F}$ is replaced by $\mathbf{F}''$. Recall that for each subset $\mathbf{H} \subset \mathbf{F}, \mathbf{H}^\perp = \{g \in \mathbf{F}: |g| \wedge |h| = 0\ \forall h \in \mathbf{H}\}$ and so $(T\mathbf{E})^{\perp\perp}$ is the band in $\mathbf{F}$ generated by $T\mathbf{E}$.

An AM-space (complete M-normed space) $\mathbf{F}$ with unit is represented by $C(K)$, the Banach lattice with supremum norm, of continuous functions on a compact Hausdorff space K. In this context we consider multiplication to be defined in $\mathbf{F}$ and elements of $\mathbf{F}$ also as multiplication operators on $\mathbf{F}$.

Some properties of $\delta_1(\cdot)$ in Theorem 1.2 have dualized analogs for $\delta_\infty(\cdot)$.

(3.1) THEOREM. *Let* $\mathbf{E}$ *be a normed space,* $\mathbf{F}$ *an M-normed space with unit* $\mathbf{1}$ *and* $T \in \mathscr{L}$ *for which* $\delta_\infty(T)$ *exists in* $\mathbf{F}^+$.

(a) $\delta_\ddagger(T)$ *is a weak order unit of* $(T\mathbf{E})^{\perp\perp}$ *and* $\|\delta_\infty(T)\| = \|T\|$.

(b) *When* $\mathbf{F}$ *is complete* $\forall \eta \in \mathbf{F}$. (i) *there exists* $\delta_\infty(\eta T) = |\eta|\delta_\infty(T)$ *and* (ii) $\|\eta\delta_\infty(T)\| = \|\eta T\|$.

Proof. Let $d = \delta_\infty(T)$.

(a) By [S, Proposition II.1.5], $d \wedge g = \sup\{h \wedge g: h \in \mathscr{W}(T)\} = 0$ for each $g \in (T\mathbf{E})^\perp$. So $d \in (T\mathbf{E})^{\perp\perp}$. Then the definition of d implies that d is a weak order unit of $(T\mathbf{E})^{\perp\perp}$. We have $d \leq \|T\|\mathbf{1}$ and obviously $\|d\| \geq \|T\|$. Hence $\|d\| = \|T\|$.

(b)(i) Let $\alpha \in \mathbf{F}^+$ majorize $\mathscr{W}(\eta T) = |\eta|\mathscr{W}(T)$. As $\|\eta\|\mathbf{1} \geq |\eta|$ so

$$\alpha + (t\mathbf{1} - |\eta|)d \geq \sup\{|\eta|g + (t\mathbf{1} - |\eta|)g = tg: g \in \mathscr{W}(T)\} = td,$$

where $t = \|\eta\|$. Thus $\alpha \geq |\eta|d$. As $|\eta|d$ majorizes $\mathscr{W}(\eta T)$ so $\delta_\infty(\eta T) = |\eta|d$.

(b)(ii) This follows by (b)(i) and the equation in (a) for ηT. ■

(3.2) COROLLARY. *Let* $\mathbf{E} = L_1(\mu)$, $\mathbf{F}$ *be a normed space and* $T \in \mathscr{L}$. *Then* $\delta_\infty(T^*) = \delta_1(T)$.

Proof. Let $A \in \mathscr{F}$. Then by Theorems 3.1(b)(ii) and 1.2(c)(ii),

$$\|1_A\delta_\infty(T^*)\|_\infty = \|1_A T^*\| = \|T \circ 1_A\| = \|1_A\delta_1(T)\|_\infty.$$

Since A is arbitrary, $\delta_\infty(T^*) = \delta_1(T)$. □

We have an extremeness criterion dual to Theorem 2.5.

(3.3) THEOREM. *Let* $\mathbf{E}$ *be a normed space and* $\mathbf{F}$ *an M-normed space with unit* $\mathbf{1}$. *Then a given* $T \in \mathscr{U}$ *is an* ext $\mathscr{U}$ *only if, and with* $\mathbf{F}$ *complete and* $\mathbf{E}'$ *strictly convex if, there exists* $\delta_\infty(T) = \mathbf{1}$.

Proof. Necessity. Assume $T \in \operatorname{ext}\mathscr{U}$. $\mathbf{1}$ majorizes $\mathscr{W}(T)$. So does $d \wedge \mathbf{1}$ for any majorant $d \in \mathbf{F}^+$. Let $R = \langle\cdot, h\rangle(\mathbf{1} - d \wedge \mathbf{1})$ with $h \in \mathbf{E}'$ of norm 1. If $f \in \mathbf{E}$ and $\|f\| = 1$ then

$$|Tf \pm Rf| \leq |Tf| + (\mathbf{1} - d \wedge \mathbf{1}) \leq d \wedge \mathbf{1} + (\mathbf{1} - d \wedge \mathbf{1}) = \mathbf{1}.$$

Thus $T \pm R \in \mathscr{U}$. Hence $R = O$. So $\mathbf{1} = d \wedge \mathbf{1} \leq d$ and $\delta_\infty(T) = \mathbf{1}$ exists.

Sufficiency. Assume the last three conditions given. Represent $\mathbf{F}$ as $C(K)$ for a compact Hausdorff space K. The pointwise supremum ξ of $\mathscr{W}(T)$ is a lower semicontinuous function on K. $\delta_\infty(T) = \mathbf{1}$ means that $0 \leq \xi(y) \leq 1\ \forall y$ and $\mathbf{1}$ is the least majorant in $\mathbf{F}^+$ of ξ. Let $U^n = \{\xi > 1 - 1/n\}$ $(n = 1, 2, \ldots)$.

Given $y_n \notin \overline{U^n}$ there is $\mathbf{1} \geq \chi_n \in \mathbf{F}^+$ with $\chi_n(y_n) = 1$ and $\chi_n(U^n) = \{0\}$ (Urysohn's Lemma), whence $\xi \leq \mathbf{1} - \chi_n/n \in \mathbf{F}^+$, a contradiction. So each U^n is open and dense in K. Hence $\{\xi = 1\} + \cap U^n$ is dense in K by Baire's category theorem. For each $S \in \mathscr{L}$ there is a weak* continuous function $\theta(S)$: $K \to \mathbf{E}'$ such that $Sf(y) = \langle f, \theta(S)(y)\rangle (f \in \mathbf{E}, y \in K)$ and $\|S\| = \sup_y \|\theta(S)(y)\|$ [DS, Theorem VI.7.1] (replace $\mathbf{E}$ if necessary by its completion and extend S thereto, as this will not change $\mathbf{E}'$ or for $S = T$, $\delta_\infty(T)$); $\theta(S)(y)$ is the S^*-image of the evaluation map at y. So $(\xi(y) = \|\theta(T)(y)\|$. Let $R \in \mathscr{L}$ with $T \pm R \in \mathscr{U}$. Then $\|\theta(T)(y) \pm \theta(R)(y)\| \leq 1$ $\forall y$ as θ is linear on $\mathscr{L}$. Since $\mathbf{E}'$ is strictly convex this implies $\theta(R) = 0$ on $\{\xi = 1\}$. As $\{\xi = 1\}$ is dense so $R = O$ and T is extreme. □

(3.4) THEOREM. *Let* $\mathbf{E}$, $\mathbf{F}$ *be normed spaces with* $\mathbf{F}'$ *an* L_1*-space and* $T \in \mathscr{L}$. *Then* $\delta_\infty(JT) = \delta_\infty(T^{**})$ *for the canonical imbedding* $\mathbf{J}$: $\mathbf{F} \to \mathbf{F}''$. *If either side is the unit* $\mathbf{1}$ *of* $\mathbf{F}''$ *and* $\mathbf{E}'$ *is strictly convex then* T^* *is extreme.*

Proof. For the stated equality let $g \in (\mathbf{F}')^+$ and $h \in \mathbf{E}''$ with $\|h\| \leq 1$, we have by [S, Corollary 3 to Theorem IV.1.8],

$$\langle g, |T^{**}h|\rangle = \sup\{|\langle \gamma, T^{**}h\rangle| : \gamma \in \mathbf{F}', |\gamma| \leq g\} = \lim_{n\to\infty} |\langle T^*\gamma_n, h\rangle|,$$

where each $\gamma_n \in \mathbf{F}'$ and $|\gamma_n| \leq g$. The unit ball of $\mathbf{E}$ is weak* dense in that of $\mathbf{E}''$. So for some $f_n \in \mathbf{E}$ with $\|f_n\| \leq 1$ $(n = 1, 2, \dots)$,

$$\langle g, |T^{**}h|\rangle = \lim_{n\to\infty} |\langle f_n, T^*\gamma_n\rangle| = \lim_{n\to\infty} |\langle \gamma_n, JTf_n\rangle| \leq \langle g, \delta_\infty(JT)\rangle.$$

It follows that $\delta_\infty(T^{**}) \leq \delta_\infty(JT)$. Equality holds as $JT = T^{**}|_{\mathbf{E}}$.

For the last part, JT is extreme by Theorem 3.3 applied on JT. Now $\mathscr{L}(\mathbf{E}, \mathbf{F}'')$ is linearly isometric to $\mathscr{L}(\mathbf{F}', \mathbf{E}')$ with JT corresponding to T^* (forward correspondence: $R \mapsto S = R^*J_1$ and reverse: $S \mapsto R = S^*J$, using the canonical imbedding J_1: $\mathbf{F}' \to \mathbf{F}'''$). So T^* is extreme. □

We have an analog of Corollary 1.6.

(3.5) THEOREM. *Let* $\mathbf{E}$ *be a normed lattice,* $\mathbf{F}$ *an M-normed space and* $T \in \mathscr{L}$ *for which* $|T|$ *exists. Then*:

(a) $\delta_\infty(|T|) = \delta_\infty(T)$ *if either side exists in* $\mathbf{F}^+$.

(b) *For* $\mathbf{E}$ *M-normed with unit* $\mathbf{1}$, (i) *there exists* $\delta_\infty(T) = |T|\mathbf{1}$ *and* (ii) *when* $\mathbf{E}$ *is also complete there exist* $|T \circ \alpha| = |T| \circ |\alpha|$ *and* $\delta_\infty(T \circ \alpha) = |T||\alpha|$ *for all* $\alpha \in \mathbf{E}$.

Proof. (a) Majorants of $\mathscr{W}(T)$ majorize $\mathscr{W}(T; f) \equiv \{|Tg| : g \in \mathbf{E}, |g| \leq f\}$ and so $\sup \mathscr{W}(T; f) = |T|f$, if $f \in \mathbf{E}^+$ and $\|f\| \leq 1$. So they are majorants of $\mathscr{W}(|T|)$, and clearly vice versa. The result follows.

(b)(i) This follows as $\mathscr{W}(T) = \mathscr{W}(T; \mathbf{1})$.

(b)(ii) The second equality follows from the first one and (b)(i) applied to $T \circ \alpha$. The first equality follows since $\mathscr{W}(T \circ \alpha; f)(f \in \mathbf{E}^+)$ is an order dense subset of $\mathscr{W}(T; |\alpha| f)$. To show the denseness represent $\mathbf{E}$ as $C(H)$ for a compact Hausdorff space H. Given $g \in \mathbf{E}$ with $|g| \leq |\alpha| f$, for each $n = 1, 2, \ldots,$ there exists by Ursohn's Lemma $\mathbf{1} \geq \chi_n \in \mathbf{E}^+$ for which χ_n is 1 on $\{|g| \geq 1/n\}$ and 0 on $\{\alpha = 0\}$. Define g_n to be $\chi_n g/\alpha$ on $\{\alpha \neq 0\}$ and 0 on $\{\alpha = 0\}$. Then $g_n \in C(H)$, $|g_n| \leq \chi_n f \leq f$ and $\alpha g_n = \chi_n g$. Hence

$$||T(\alpha g_n)| - |Tg|| \leq |T|\mathbf{1} \cdot \|\chi_n g - g\| \leq |T|\mathbf{1}/n \searrow 0. \qquad \square$$

The concept of a *disjunctive* $T \in \mathscr{L}$ in §2 extends to the case where $\mathbf{E}$ and $\mathbf{F}$ are normed vector lattices. Given topological spaces h, K, a subset $Z \subset K$ and a mapping $\varphi \colon Z \to H$ define for all functions f on H,

$$\varphi^{\circ} f = \begin{cases} f \circ \varphi & \text{on } Z \\ 0 & \text{on } Z^c. \end{cases}$$

(Note. $Z = \varnothing \Rightarrow \varphi$ is the empty mapping $\Rightarrow \varphi^{\circ}\mathrm{f} = 0$.)

We have an extension of Property D (in §2):

(3.6) Lemma. (a) *Let H and K be compact Hausdorff spaces. Then $T \in \mathscr{L}(C(H), C(K))$ is disjunctive if and only if $T = (T1)\varphi^{\circ}$ for a continuous mapping $\varphi \colon \{T1 \neq 0\} \to H$.*

(b) *Let $\mathbf{E}, \mathbf{F}$ be Banach lattices and $T \in \mathscr{L}$ order bounded. Then T is disjunctive if and only if $|T|$ exists and satisfies $|Tf| = |T|\,|f|$ for all $f \in \mathbf{E}$.*

Proof. (a) See [A, Example 2.2.1]. (φ has been modified.)

(b) See [A, Theorem 2.4(i)(v)]. The original statement in [A] mentions $|Tf| = ||T|f| = |T|\,|f|$ but actually the middle term is not essential. (See also [Kan1, Theorem 3.1] for the case of L_∞ spaces $\mathbf{E}, \mathbf{F}$.) $\square$

We have an extended dualized analog of Theorem 2.8 part (11).

(3.7) Theorem. *Let $\mathbf{E}, \mathbf{F}$ be AM-spaces with unit, denoted by $\mathbf{1}$ for either space, and $T \in \mathscr{U}$ for which $|T| \in \mathscr{L}$ exists. Suppose $T\mathbf{E}, |T|\mathbf{E} \subset \mathbf{B}$ for a subalgebra $\mathbf{B}$ of $\mathbf{F}$ containing $\mathbf{1}$. Then $T \in \operatorname{ext} \mathscr{U}_1$ where $\mathscr{U}_1 = \{S \in \mathscr{U} \colon S\mathbf{E} \subset \mathbf{B}\}$ if and only if*

$$(\infty\infty\infty) \qquad \delta_\infty(T) = |T|\mathbf{1} = \mathbf{1} \text{ and } T \text{ is disjunctive.}$$

Further, apart from the requirements on **B**:

(a) *If we have the representations* $\mathbf{E} = C(H)$ *and* $\mathbf{F} = C(K)$ *for compact Hausdorff spaces* H *and* K *then* $(\infty\infty)$ *is equivalent to*

$(\infty\infty)'$ $\quad T = h\varphi^\circ$ *for an* $h \in C(K)$ *with* $|h| = 1$ *and a continuous mapping* $\varphi\colon K \to H$;

(b) *If* $\mathbf{E} = L_\infty(X, \mathscr{F}, \mu)$ *and* $\mathbf{F} = L_\infty(Y, \mathscr{G}, \nu)$ *then* $(\infty\infty)$ *is equivalent to*

$(\infty\infty)''$ $\quad T = h\Phi^\#$ *for an* $h \in L_\infty(\nu)$ *with* $|h| = 1$ *and a Boolean algebra homomorphism* $\Phi\colon (\mathscr{F}, \mu) \to (\mathscr{G}, \nu)$.

Proof. For the equivalence in (a) and (b) clearly each of the conditions $(\infty\infty)'$ and $(\infty\infty)''$ implies condition $(\infty\infty)$. The converse implication for (a) follows from Lemma 3.6(a) and that for (b), from [Kan1, Remark 4.1].

Necessity. Let $T \in \operatorname{ext} \mathscr{U}_1$. By Theorems 3.5(b)(i) and 3.1(a), $|T|\mathbf{1} = \delta_\infty(T) \le \mathbf{1}$. $|T|\mathbf{1} = \mathbf{1}$ by the proof of Theorem 3.3—change $d \wedge \mathbf{1}$ to $|T|\mathbf{1}$ and $\mathscr{U}$ to $\mathscr{U}_1$. Let $\alpha, \beta \in \mathbf{E}^+$ with $\|\alpha\| = \|\beta\| = 1$ and $\alpha \wedge \beta = 0$. T is disjunctive if $\xi\zeta = 0$, where $\xi = |T|\alpha$ and $\zeta = |T|\beta$. Let $R = \zeta T \circ \alpha - \xi T \circ \beta$. Then

$$T \pm R = (\mathbf{1} \pm \zeta)T \circ \alpha + (\mathbf{1} \mp \xi)T \circ \beta + T \circ (\mathbf{1} - \alpha - \beta).$$

Now $\alpha + \beta \le \mathbf{1}$ and $\xi, \zeta \le |T|\mathbf{1} = \mathbf{1}$. So if $f \in \mathbf{E}$ $\|f\| \le 1$ then

$$|(T \pm R)f| \le (\mathbf{1} \pm \zeta)\xi + (\mathbf{1} \mp \xi)\zeta + |T|(\mathbf{1} - \alpha - \beta) = |T|\mathbf{1} = \mathbf{1}.$$

So $T \pm R \in \mathscr{U}_1$ and $R = O$. Represent $\mathbf{E}$ as in (a). There exists $\mathbf{1} \ge \chi_n \in \mathbf{E}^+$ $(n = 1, 2, \dots)$ with values 1 on $\{\alpha \ge 1/n\}$ and 0 on $\{\alpha = 0\} \supset \{\beta \ne 0\}$. Then $O = R \circ \chi_n = \zeta T \circ \alpha\chi_n \to \zeta T \circ \alpha$. So by Theorem 3.5(b)(ii) and 3.1(b)(i), $\zeta\xi = \zeta\delta_\infty(T \circ \alpha) = \delta_\infty(\zeta T \circ \alpha) = \delta_\infty(O) = 0$ as wanted.

Sufficiency. Assume $(\infty\infty)$. Represent $\mathbf{E}$ and $\mathbf{F}$ as in part (a). So $(\infty\infty)'$ holds. Let $R \in \mathscr{L}$ with $T \pm R \in \mathscr{U}_1 \subset \mathscr{U}$. Consider any fixed $y \in K$. It follows from $(\infty\infty)'$ that $Rf(y) = 0$ for any $f \in C(H)$ with $|f(\varphi(y))| = \|f\|$. But all such f span $C(H)$: if $g \in (C(H))^+$ then with $m = g(\varphi(y))$,

$$g = g \wedge m + \|g\| - \left(\|g\| - (g - m)^+\right).$$

And as y is arbitrary, $R = O$. So T is extreme. □

(3.8) *Remark.* (a) Automatically, $|T| = T$ exists if T is a positive operator.

(b) $|T|$ exists when T is compact. Indeed represent $\mathbf{F}$ as $C(K)$, K compact Hausdorff. By [DS, Theorem VI.7.1] for each $S \in \mathscr{L}$ there is a weak* continuous function $\theta(S)$: $K \to \mathbf{E}'$ such that $Sf(y) = \langle f, \theta(S)(y)\rangle$ ($f \in \mathbf{E}$, $y \in K$) and $\|S\| = \sup_y \|\theta(S)(y)\|$. Further S is compact if and only if $\theta(S)$ is continuous with the norm topology in $\mathbf{E}'$. All this implies (using [S, Corollary 3 to Theorem IV.1.8]) that $|T|$ exists and is compact with a norm continuous $\theta(|T|) = |\theta(T)|$ and $\| |T| \| = \|T\|$. Note that a compact T in condition $(\infty\infty)'$ has to satisfy further requirements on φ; see [Ka, Theorem A(1)] in the setting $\mathbf{E} = \mathbf{F}$. Also in Theorem 3.7 we may replace $\mathscr{U}_1$ by $\{S \in \mathscr{U}_1$: *S is compact*$\}$, since the operators R used in the necessity part of the proof are then compact also. When $\mathbf{B} = \mathbf{F}$ this modified Theorem 3.7 is equivalent to [MP, Theorem 4.5] and comes with a simpler proof. (In Theorem 3.3, $\mathscr{U}$ can likewise be replaced by $\{S \in \mathscr{U}$: *S is compact*$\}$.)

(3.9) QUESTION. *In connection with Theorem* 3.7, *what conditions on compact Hausdorff spaces H and K will make* $\mathscr{L}(C(H), C(K))$ *a Banach lattice* (*relative to the operator norm and the linear modulus*)?

Recall that $C(A)$, A compact Hausdorff, is order complete, countably order complete or separable according as A is respectively Stonian, quasi-Stonian or metrizable [S, Propositions II.7.7, Corollary 7.5]. In the light of this the following theorem gives some answers to the interesting question above. (There are likely to be more answers.)

(3.10) THEOREM. *Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be AM-spaces with unit. Then* $\mathscr{L}$ *is a Banach lattice if* (i) $\mathbf{F}$ *is order complete, or* (ii) $\mathbf{E}$ *is separable and* $\mathbf{F}$ *is countably order complete, or* (iii) $\mathbf{E}$ *is finite-dimensional.*

Proof. (i) This is given in [S, Theorem IV.1.5(i), 1.8 Corollary 2].

(ii) Let $T \in \mathscr{L}$. It suffices to show that $|T|$ exists; by Theorems 3.5(b)(i) and 3.1(a), $\| |T| \| = \| |T|\mathbf{1}\| = \|\delta_\infty(T)\| = \|T\|$. Here $\mathbf{1}$ denotes the unit of $\mathbf{E}$, as well as that of $\mathbf{F}$. Represent $\mathbf{E}$ as $C(H)$ for a compact Hausdorff space H. For $|T|$ to exist it suffices that $\sup\{|Tg|: |g| \le f\} \equiv |T|f$ ($f \in \mathbf{E}^+$) exists, as this implies that $|T|$ is positive linear on $\mathbf{E}^+$. (Clearly $|T|(cf) = c|T|f$ for any constant $c \ge 0$. Also $f_1, f_2 \in \mathbf{E}^+$ and $|g| \le f_1 + f_2$ imply $g_1 + g_2 = g$ and $|g_n| \le f_n$ where $g_n = (|g| \wedge f_n)\operatorname{sgn} g \in C(H)$, $(n = 1, 2)$, whence $|T|$ is subadditive. It is easy to deduce that $|T|$ is additive.) As $|T|$ is continuous on $\mathbf{E}^+$ by [S, Theorem II.5.3], linearly extended $|T| \in \mathscr{L}$. So now let $\{e_n\}$ be a countable dense subset of $\mathbf{E}$. Given $0 \ne f \in \mathbf{E}^+$, let

$$h_n = (|e_n| \wedge f)\operatorname{sgn} e_n \in C(H).$$

If $|g| \le f$ then $|g - h_n| \le |g - e_n|$ (by elementary trigonometry in the complex plane) while $||Tg| - |Th_n|| \le \|T(g - h_n)\|\mathbf{1} \le \|T\| \cdot \|g - h_n\|\mathbf{1}$. Hence $\{|Th_n|\}$ is an order dense subset of $\{|Tg|: |g| \le f\}$. Thus $\sup\{|Tg|: |g| \le f\} = \sup\{|Th_n|\}$ exists in $\mathbf{F}^+$ as wanted.

(iii) This is easy; cf. Remark 3.8(b). □

4. Extreme positive contractions

When $\mathbf{E}$ and $\mathbf{F}$ are also vector lattices our extremeness results on $\mathscr{U}$ or $\mathscr{U}_1$ have analogs for $\mathscr{U}^+ \equiv \{S \in \mathscr{U}: S\mathbf{e}^+ \subset \mathbf{F}^+\}$, $\mathscr{U}_1 \cap \mathscr{U}^+$ or some further subset. We may also restrict these to compact operators; cf. Remark 3.8(b).

The following is a $\mathscr{U}^+$ analog of Theorem 2.8.

(4.1) THEOREM. *Let $p \le 1 \le q < \infty$, $\mathbf{E} = L_p(\mu)$ and $\mathbf{F} = L_q(\nu)$. Then a $T \in \mathscr{U}^+$ is in* ext $\mathscr{U}^+$ *if and only if*:

$(1^-1^+)^*$ *in the case $p < 1 < q$, $\delta_1(T) = 1_A$ and $A \subset @(\mu)$;*

$(11^+)^*$ *in the case $p = 1 < q$, $\delta_1(T) = 1_A$, $A \in \mathscr{F}$;*

$(1^-1)^*$ *in the case $p < 1 = q$, $\delta_1(T) = 1_A$, $A \subset @(\mu)$ and T is codisjunctive*:

$(11)^*$ *in the case $p = 1 = q$, $\delta_1(T) = 1_A$, $A \in \mathscr{F}$ and T is codisjunctive.*

Moreover, the criterion $(11^+)^*$ is still valid when $L_q(\nu)$ is generalized to a normed lattice with a strictly convex bidual space.

Proof. Necessity. We have

$$\delta \equiv \delta_1(T) \le 1 \text{ and } T \pm T\circ(1-\delta) \in \mathscr{U}^+$$

as $\delta_1(T \pm T\circ(1-\delta)) = \delta \pm \delta(1-\delta) \le 1$. So

$$T\circ(1-\delta) = O \text{ and } \delta(1-\delta) = \delta_1(T\circ(1-\delta)) = \delta_1(O) = 0,$$

whence $\delta = 1_A$ where $A = \operatorname{supp}\delta = s(T)$ (Theorem 1.2(b)). The part on $@(\mu)$ is from Theorem 1.4. The codisjunctiveness part follows from the proof of Theorem 2.8; just change $\mathscr{U}$ to $\mathscr{U}^+$.

Sufficiency. Let $R \in \mathscr{L}$ be such that $T \pm R \in \mathscr{U}^+$. We have $A = s(T)$. By Proposition 1.1, $\pm R\circ 1_{A^c} = (T \pm R)\circ 1_{A^c} \in \mathscr{U}^+$ and so $R\circ 1_{A^c} = O$. By Theorems 2.8 and 2.3 each of the given conditions implies $T\circ 1_A \in \operatorname{ext}\mathscr{U}(\mathbf{E}_A, \mathbf{F})$. So $R\circ 1_A = O$ also. Thus $R = O$ and $T \in \operatorname{ext}\mathscr{U}^+$. □

(4.2) *Remark.* The part of Theorem 2.8 on equivalence to the criteria $(1^-1)'$ and $(11)'$ has an analog here too. In $(1^-1)'$ just change $@(\mu)$ to $A = s(T)$ with $s(T) \subset @(\mu)$ and $|\xi|$ to ξ. Change $(11)'$ to

$$T^* = \Psi^\# \textit{ for a } \sigma\textit{-ring homomorphism } \Psi: (\mathscr{G}, \nu) \to (\mathscr{F}, \mu).$$

(Note that $\Psi Y = s(T)$ since $1_{\Psi Y} = \Psi^\# 1$ equals $T^*1 = \delta_1(T) = 1_{s(T)}$.)

We have an extended $\mathscr{U}^+$ analog of Theorem 3.3. Note that $\delta_\infty(T)$ exists in $\mathbf{F}^+$ if $\mathbf{F}$ is order complete, if $\mathbf{E}$ is separable and $\mathbf{F}$ is countably order complete, or if $\mathbf{E}$ is an AM-space with unit (Theorem 3.5(b)(i)).

(4.3) THEOREM. *Let* $\mathbf{E}$ *be a normed lattice,* $\mathbf{F}$ *an AM-space with unit* $\mathbf{1}$ *and* $\mathbf{B}$ *a subalgebra of* $\mathbf{F}$ *containing* $\mathbf{1}$. *Then a given* $T \in \mathscr{U}_1^+ \equiv \{S \in \mathscr{U}^+ : S\mathbf{E} \subset \mathbf{B}\}$ *for which there exists* $\delta_\ddagger(T) \in \mathbf{B} \cap \mathbf{F}^+$ *is in* ext $\mathscr{U}_1^+$ *only if, and with* $\mathbf{E}'$ *strictly convex if,* $\delta_\infty(T)$ *is an idempotent element of* $\mathbf{F}^+$.

Proof. Represent $\mathbf{F}$ as $C(K)$, K compact Hausdorff. The criterion means $\delta_\infty(T) = 1_Z$ for a *clopen* (closed and open) set $Z \subset K$. The proof is similar to that of Theorem 4.1, with $\mathscr{U}^+$ changed to $\mathscr{U}_1^-$, $\delta_1(\cdot)$ to $\delta_\infty(\cdot)$, and order of composition reversed. Also change 1_A to 1_Z, 1_{A^c} to 1_{Z^c}, $\mathscr{U}(\mathbf{E}_A, \mathbf{F})$ to $\mathscr{U}(\mathbf{E}, C(Z))$, Proposition 1.1 to Theorem 3.1(a) and Theorem 2.8 and 2.3 to Theorem 3.3. □

Likewise we have an extended $\mathscr{U}^+$ analog of Theorem 3.7. (Case (ii) corresponds to [Ph, Theorem 2.1].)

(4.4) THEOREM. *Let* $\mathbf{E}, \mathbf{F}$ *be AM-spaces with unit, denoted by* $\mathbf{1}$ *for both spaces,* $\mathbf{B}$ *a subalgebra of* $\mathbf{F}$ *containing* $\mathbf{1}$, *and* T *an element of either* (i) $\{S \in \mathscr{U}^+ : S\mathbf{E} \subset \mathbf{B}\}$ *or* (ii) $\{S \in \mathscr{U}^+ : S\mathbf{E} \subset \mathbf{B},\ S\mathbf{1} = \mathbf{1}\}$. *Then* T *is an extreme point of that set if and only if*

$(\infty\infty)^*$ *T is disjunctive and, in case* (i), $(T\mathbf{1})^2 = T\mathbf{1}$.

Further, apart from the requirement $T\mathbf{E} \subset \mathbf{B}$:

(a) *With the representations* $\mathbf{E} = C(H)$ *and* $\mathbf{F} = C(K)$ *for compact Hausdorff spaces* H *and* K, $(\infty\infty)^*$ *is equivalent to*

$$T = \varphi^\circ \text{ for a continuous mapping } \varphi : Z \to H$$

were $Z = K$ *in case* (ii) *and* Z *is a clopen subset of* K *with* $1_Z \in \mathbf{B}$ *in case* (i);

(b) *If* $\mathbf{E} = L_\infty(X, \mathscr{F}, \mu)$ *and* $\mathbf{F} = L_\infty(Y, \mathscr{G}, \nu)$ *then* $(\infty\infty)^*$ *is equivalent to*

$$T = \Phi^\# \text{ for a Boolean ring homomorphism } \Phi : (\mathscr{F}, \mu) \to (\mathscr{G}, \nu).$$

Proof. The necessity is by Theorem 4.3 for the part on $T\mathbf{1}$ in case (i) and by the proof of Theorem 3.7 for disjunctiveness. The sufficiency follows from Theorem 3.7 via the same argument as for Theorem 4.3. □

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