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A CONVERSE OF THE JORDAN-BROUWER THEOREM FOR QUASI-REGULAR IMMERSIONS

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1. Introduction

Suppose that $f\colon S^{n-1} \to S^n$ is a topological embedding. Then it is known as the Jordan-Brouwer Theorem that $f(S^{n-1})$ separates S^n into exactly two connected components. In [BR], C^1-immersions with normal crossings were studied and the following converse of the Jordan-Brouwer Theorem was obtained: if $f\colon S^{n-1} \to S^n$ is a C^1-immersion with normal crossings, then f is an embedding if and only if $f(S^{n-1})$ separates S^n into exactly two connected components. After that, this theorem has been generalized in various settings ([BMS1], [BMS2], [S]); however almost all of them have been involved with immersions *with normal crossings*.

The purpose of this paper is to consider a more general class of immersions than that of immersions with normal crossings, namely the class of quasi-regular immersions [H], and to obtain the converse of the Jordan-Brouwer Theorem. Recall that a C^1-immersion $f\colon M \to N$ into an n-dimensional manifold N is *quasi-regular* if the self-intersection locus $B \subset f(M)$ is an immersed submanifold of N with the property that for each $x \in B$ there is a coordinate system for N valid in a neighborhood U of x so that x corresponds to $0 \in \mathbf{R}^n$ and that the branches of f in U correspond to distinct linear subspaces of $\mathbf{R}^n$; i.e., given a numbering $y_1, y_2, \ldots, y_m$ of the points of $f^{-1}(x)$ there are pairwise disjoint neighborhoods $V_i \subset M$ around y_i so that $U \cap f(M) = U \cap (\cup_{i=1}^m f(V_i))$ is a union of m distinct linear subspaces of $\mathbf{R}^n$. It is clear that an immersion with normal crossings is always quasi-regular.

Our main result of this paper is the following.

THEOREM 1.1. *Let $f\colon M \to N$ be a quasi-regular immersion, where M is a closed connected $(n-1)$-dimensional manifold and N is a connected n-dimensional manifold. Assume that $H_1(M;\mathbf{Z}_2) = 0$ and $H_1(N;\mathbf{Z}) = 0$. Then if f is not an embedding, then $\beta_0(N - f(M)) \geq 3$, where β_0 denotes the number of connected components.*

Note that it has already been known that a proper codimension-1 quasi-regular immersion $f\colon M \to N$ separates N if $H_1(N;\mathbf{Z}_2) = 0$ [NR]. In fact, the same is true for proper C^1-immersions (see [HP], [F]).

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As an immediate corollary, we obtain the following converse of the Jordan-Brouwer Theorem for quasi-regular immersions.

COROLLARY 1.2. *Let M and N be as in Theorem 1.1. Then a quasi-regular immersion $f\colon M \to N$ is an embedding if and only if $\beta_0(N - f(M)) = 2$.*

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2. Proof of Theorem 1.1

Proof of Theorem 1.1. It is known that, under our homological hypothesis, we have $\beta_0(N - f(M)) = 2 + \dim \ker((f|A)_*\colon H_{n-2}(A; \mathbf{Z}_2) \to H_{n-2}(B; \mathbf{Z}_2))$, where

$$A = \{x \in M\colon f^{-1}(f(x)) \neq \{x\}\}$$

is the self-intersection set of f and $B = f(A)$ (for example, see [BMS2, §2]). Thus, for the proof of Theorem 1.1, it suffices to show that $\ker(f|A)_* \neq 0$. By an argument similar to that in [H], we see that there exists an immersion $\varphi\colon X \to M$ of a closed $(n-2)$-dimensional manifold X such that $\varphi(X) = A$. Note that φ is not necessarily a quasi-regular immersion. By the construction of φ, we see that, for $x \in A$, $\sharp f^{-1}(f(x)) = m$ if and only if $\sharp\varphi^{-1}(x) = m - 1$, where $\sharp$ denotes the number of elements in the set. Set $A_m = \{x \in M\colon \sharp f^{-1}(f(x)) = m\}$.

LEMMA 2.1. *If A_m has an interior point in A for an even integer m, then A carries a mod 2 fundamental class $[A] \in H_{n-2}(A; \mathbf{Z}_2)$ which does not vanish.*

Proof. Set $[A] = \varphi_*[X]$, where $[X] \in H_{n-2}(X; \mathbf{Z}_2)$ is the fundamental class of X. Note that, for $x \in A_m$, $\sharp\varphi^{-1}(x) = m - 1$, which is odd by our assumption. Since A_m contains a top dimensional cell of A, we see that $[A] \neq 0$. □

LEMMA 2.2. *We always have $(f|A)_*[A] = 0$ in $H_{n-2}(B; \mathbf{Z}_2)$.*

Proof. We have $(f|A)_*[A] = (f \circ \varphi)_*[X]$. Note that, for $x \in A$, $\sharp f^{-1}(f(x)) = m$ if and only if $\sharp(f \circ \varphi)^{-1}(f(x)) = m(m-1)$. Since $m(m-1)$ is always even, we have the conclusion. □

By Lemmas 2.1 and 2.2, if A_m has an interior point in A for an even integer m, then $\beta_0(N - f(M)) \geq 3$. Thus, in the following, we assume that A_m for m even has no interior points in A. In particular, the dimension of A_2 is less than or equal to $n - 3$.

LEMMA 2.3. *Let $H_1, H_2, \ldots, H_m$ be distinct codimension-1 linear subspaces of $\mathbf{R}^n$ ($n \geq 3$). If $\dim(H_1 \cap H_2 \cap \cdots \cap H_m) < n-2$, then there exists a non-zero vector $w \in \mathbf{R}^n$ such that exactly two of $H_1, H_2, \ldots, H_m$ contain w.*

Proof. First we prove the lemma for $n = 3$. Suppose that there is no non-zero vector w as in the lemma. Let S^2 be the unit sphere centered at the origin in $\mathbf{R}^3$. The intersections of S^2 with H_i induce a natural polyhedral decomposition of S^2. By our assumption, for every vertex of this decomposition, at least 6 edges are incident. We also see easily that every 2-dimensional face of the decomposition has 3 or more boundary edges. Let f, e and v be the numbers of 2-dimensional faces, edges and vertices of the decomposition respectively. Then we have

$$6v \leq 2e \quad \text{and} \quad 3f \leq 2e$$

by the above observation. Since the Euler characteristic of S^2 is equal to 2, we have

$$f - e + v = 2.$$

Then we have $12 = 6f - 6e + 6v \leq 4e - 6e + 2e = 0$, which is a contradiction. This completes the proof for the case $n = 3$.

Now suppose $n \geq 4$ and the lemma is true for $n-1$. Let H be a codimension-1 linear subspace of $\mathbf{R}^n$ different from $H_1, H_2, \ldots, H_m$. Suppose that $H \cap H_i = H \cap H_j$ for $i \neq j$. Then we have

$$H \cap H_i = H \cap H_j = H \cap H_i \cap H_j \subset H_i \cap H_j.$$

Since $\dim(H_i \cap H_j) = \dim(H \cap H_i) = \dim(H \cap H_j) = n-2$, we see that $H_i \cap H_j = H \cap H_i \cap H_j$ and hence $H_i \cap H_j \subset H$.

Case 1. $\dim(H_1 \cap H_2 \cap \cdots \cap H_m) \geq 1$.

Take a codimension-1 linear subspace H of $\mathbf{R}^n$ such that $H \not\supset H_1 \cap H_2 \cap \cdots \cap H_m$, $H_i \cap H_j (i, j = 1, 2, \ldots, m)$. Such a subspace H exists, since the dimension of the codimension-1 subspaces of $\mathbf{R}^n$ is equal to $n-1$, while the dimension of the codimension-1 subspaces containing $H_i \cap H_j$ is equal to 1 and the dimension of the codimension-1 subspaces containing $H_1 \cap H_2 \cap \cdots \cap H_m$ is less than or equal to $n-2$. Then by the above observation, we see that $H \cap H_1, H \cap H_2, \ldots, H \cap H_m$ are distinct codimension-1 subspaces of H and $\dim((H \cap H_1) \cap \cdots \cap (H \cap H_m)) < (n-1)-2$. Then by our induction hypothesis, we see that there exists a non-zero vector $w \in H$ such that exactly two of $H \cap H_1, \ldots, H \cap H_m$ contain w. This vector w is a desired non-zero vector.

Case 2. $H_1 \cap H_2 \cap \cdots \cap H_m = 0$.

Take a codimension-1 subspace H of $\mathbf{R}^n$ such that $H \not\supset H_i \cap H_j (i, j = 1, 2, \ldots m)$. Then $H \cap H_1, \ldots, H \cap H_m$ are distinct codimension-1 subspaces of H and $\dim((H \cap H_1) \cap \cdots \cap (H \cap H_m)) = 0 < (n-1)-2$, since $n \geq 4$. Then our induction hypothesis ensures the existence of a desired non-zero vector. This completes the proof. □

Remark 2.4. In the above lemma, the case where $n = 3$ is equivalent to the well-known Sylvester's problem. For details and the history of this problem, see [G, §2.3]. In fact, the above lemma for $n = 3$ is nothing but Theorem 2.12 of [G]. The above proof is motivated by the proof of an improvement of Sylvester's problem due to Melchior [G, Theorem 2.13].

Now recall that we are assuming that A_2 is of dimension less than $n-2$. Then, by Lemma 2.3 and the definition of a quasi-regular immersion, we see that, for every $x \in B$, $\dim(f(V_1) \cap \cdots \cap f(V_m) \cap U) = n-2$, where $f^{-1}(x) = \{y_1, \ldots, y_m\}$, V_i is a small coordinate neighborhood of y_i in M and U is a small coordinate neighborhood of x in N. Therefore, we see that A is the disjoint union of $A_3, A_5, A_7, \ldots, A_l$ for some odd integer l and each A_m ($m = 3, 5, 7, \ldots, l$) is an $(n-2)$-dimensional closed submanifold of M. Furthermore, $f|A_m$: $A_m \to f(A_m)$ is an m-fold cover.

Let Y be a connected component of $f(A_m)$. If $f^{-1}(Y)$ is not connected, we see easily that $\ker((f|A)_*: H_{n-2}(A; \mathbf{Z}_2) \to H_{n-2}(B; \mathbf{Z}_2)) \neq 0$. Hence we may assume that $f^{-1}(Y)$ is connected. Furthermore, note that A_m is orientable, since it is a codimension-1 embedded submanifold of M with $H_1(M; \mathbf{Z}_2) = 0$. Since $f|A_m$: $A_m \to f(A_m)$ is an odd-fold cover and $f^{-1}(Y)$ is connected for every component Y of $f(A_m)$, $f|A_m$ must be orientation preserving after suitable orientations are given to A_m and $f(A_m)$.

Now suppose that $A_m \neq \emptyset$ for an odd integer m. By the 2-color theorem together with our assumption that $H_1(M; \mathbf{Z}_2) = 0$, there exist two disjoint open sets B_m and W_m of M such that $M - A_m = B_m \cup W_m$ and $\overline{B}_m \cap \overline{W}_m = \partial B_m = \partial W_m = A_m$. Note that M is orientable since $H_1(M; \mathbf{Z}_2) = 0$ and that $\overline{B}_m$ and $\overline{W}_m$ are compact orientable manifolds with boundary. Orient M arbitrarily. Recall that $f|A_m = f|\partial B_m = f|\partial W_m$ is orientation preserving. Hence $f(\overline{B}_m)$ and $f(\overline{W}_m)$ are $(n-1)$-dimensional $\mathbf{Z}_m$-cycles in N. Take a point $x \in f(A_m)$ and take $U, y_1, \ldots, y_m, V_1, \ldots, V_m$ as in the paragraph just after Remark 2.4. We identify U with $\mathbf{R}^n$ and $f(V_1) \cap \cdots \cap f(V_m) \cap U$ with the codimension-2 subspace $\{x_1 = x_2 = 0\}$ of $\mathbf{R}^n$. Set $L = \{x_3 = \cdots = x_n = 0\}$, which is a 2-dimensional subspace of U. We orient N and L arbitrarily. We may assume that $L \cap f(V_i)$ and $L \cap f(V_{i+1})$ are adjacent as in Figure 1 for $i = 1, \ldots, m$ ($V_{m+1} = V_1$) in accordance with the given orientation of L. Since each $f(V_i)$ has a canonical orientation induced by that of M, it has a canonical unit normal vector $v_i \in L \subset T_xN$. Take a connected component C of $L - (f(V_1) \cup \cdots \cup f(V_m))$ bounded by $f(V_i) \cup f(V_{i+1})$. We say that C is *good* if $< v_i, v_{i+1} >$ does not coincide with the given orientation of L (we warn the reader that in Figure 2 the component C is not good). Now suppose that $\beta_0(N - f(M)) = 2$. If there exists a component C of $L - (f(V_1) \cup \cdots \cup f(V_m))$ which is *not* good, then it is not difficult to find a closed oriented smooth curve γ in N which intersects with $f(M)$ transversely in two points with the same sign of intersection (see Figure 2). This contradicts the assumption that $H_1(N; \mathbf{Z}) = 0$. Thus every component of $L - (f(V_1) \cup \cdots \cup f(V_m))$ must be good (see Figure 3). Since $f|A_m$ is orientation preserving, we see that $f(\overline{B}_m \cap V_i) \cap L$ and $f(\overline{B}_m \cap V_{i+1}) \cap L$ is not adjacent in L as in Figure 3. Then it is

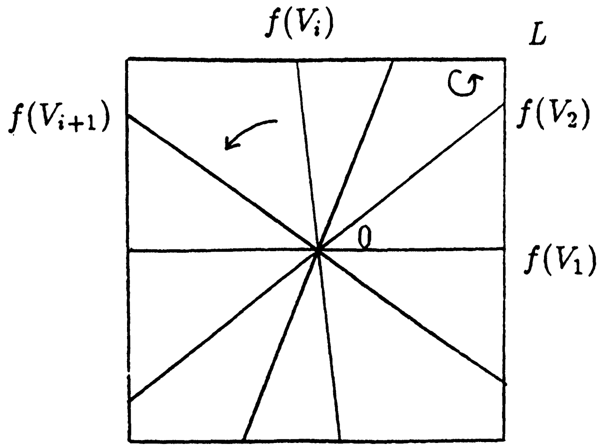

Figure 1

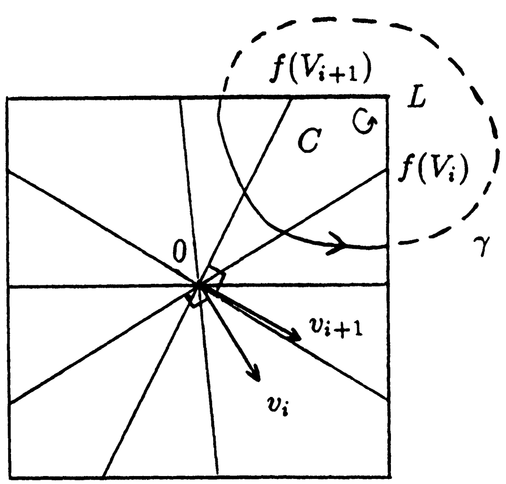

Figure 2

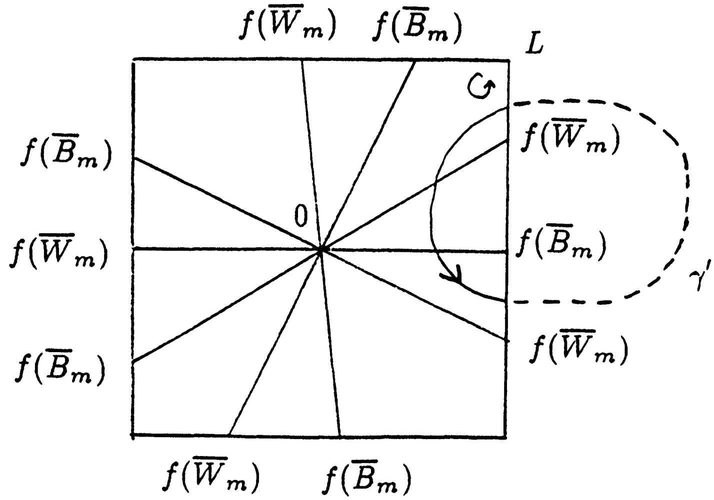

Figure 3

not difficult to find a closed oriented smooth curve γ' in N which intersects $f(\overline{B}_m)$ transversely in one point. This contradicts the assumption that $H_1(N; \mathbf{Z}_m) = 0$. Hence $\beta_0(N - f(M)) \geq 3$. This completes the proof. □

Remark 2.5. When $n = 2$, we have a similar result. In fact, in [N], it is shown that if f: $S^1 \to S^2$ is a continuous map with only finitely many self-intersection points $t_1, \ldots, t_m$ with $\sharp\{f(t_1), \ldots, f(t_m)\} = r$, then $\beta_0(S^2 - f(S^1)) = 2+m-r$. If f is not an embedding, then we have $m - r > 0$ and hence $\beta_0(S^2 - f(S^1)) \geq 3$. Note that, for a quasi-regular immersion f: $S^1 \to S^2$, its self-intersection set is always finite.

Remark 2.6. In Theorem 1.1 and Corollary 1.2, the condition that $H_1(M; \mathbf{Z}_2) = 0$ is essential. In fact, there exists a quasi-regular immersion f: $T^2 \to \mathbf{R}^3$ such that f is not an embedding and $\beta_0(\mathbf{R}^3 - f(T^2)) = 2$, where T^2 is the 2-dimensional torus. See [S, Figure 2].

Remark 2.7. In Theorem 1.1 and Corollary 1.2, the condition that $H_1(N; \mathbf{Z}) = 0$ can be replaced by the conditions that the torsion of $H_{n-2}(N; \mathbf{Z})$ is a 2-group and that $H_{n-1}(N; \mathbf{Z}) = 0$. In this case we have $H_{n-1}(N; \mathbf{Z}_m) = 0$ for every odd integer m by the universal coefficient theorem and the same proof is valid in this case.

Remark 2.8. In [NR], a more general class, namely that of quasi-regular topological immersions, has been studied and it is shown that a proper codimension-

1 quasi-regular topological immersion $f\colon M \to N$ separates N, provided that $H_1(N; \mathbf{Z}_2) = 0$. Note that our results also hold for quasi-regular topological immersions.

The following problem has been given by the referee.

Problem 2.9. Let f be as in Theorem 1.1, but replace the quasi-regular condition with the one that there exist distinct two points p and q in M such that $f(p) = f(q)$ but $f_*(T_p(M)) \neq f_*(T_q(M))$. Is the resulting statement true?

The answer is "yes" if $f^{-1}(f(p)) = \{p, q\}$ (see [S]). The author does not know the answer in general situations.

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