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ON THE η-INVARIANT OF GENERALIZED ATIYAH-PATODI-SINGER BOUNDARY VALUE PROBLEMS

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1. Introduction. η-invariants for Dirac operators on manifolds with boundary

We consider a compact Riemannian manifold M with boundary N, $\dim M = 2k+1$ odd. Moreover let (S, ∇) be a complex Dirac bundle over M (cf. [17, Def. II.5.2]). Then we can form the Dirac operator

$$D\colon\ C_0^\infty(S) \to C_0^\infty(S)$$

associated to this structure. In order to obtain self-adjoint extensions of D we have to impose boundary conditions. We assume that the metric is product near the boundary, i.e., there is a collar $U = [0, 1) \times N$ of the boundary where the metric and the hermitian structure of S are product. Then on U the operator D has the form

$$D = \Gamma \left(\frac{\partial}{\partial x} + A \right), \tag{1.1}$$

where $\Gamma\colon\ S|N \to S|N$ is a unitary bundle automorphism (Clifford multiplication by the inward normal vector) and $A\colon\ C_0^\infty(S|N) \to C_0^\infty(S|N)$ is the corresponding Dirac operator on N. One easily checks the following identities

$$\Gamma^2 = -I,\ \Gamma^* = -\Gamma,\ \Gamma A = -A\Gamma,\ A^* = A. \tag{1.2}$$

In order to define self-adjoint boundary conditions for D we first deal with the case $\ker A = \{0\}$, i.e., A is invertible. This case is most similar to [1] and there is a canonical self-adjoint boundary condition. Let $\Pi_\pm$ be the orthogonal projection onto the positive (negative) spectral subspace of A, i.e. $\Pi_+ = 1_{(0,\infty)}(A)$, $\Pi_- = 1_{(-\infty,0)}(A)$. We use the pseudodifferential operator Π_+ as elliptic boundary condition and put

$$\begin{aligned} D_+ &:= D, \\ \mathcal{D}(D_+) &:= \{s \in H^1(M,S) \mid \Pi_+(s|N) = 0\}. \end{aligned} \tag{1.3}$$

where H^k denotes the k-th Sobolev space and $\mathcal{D}(\cdot)$ denotes the domain of an operator. The elliptic boundary conditions for Dirac operators have been discussed in [3],

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and [3] also shows that D_+ is a self-adjoint operator. In [9] it was shown that the η-function of D_+

$$\eta(D_+, s) = \Gamma((s+1)/2)^{-1} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}(D_+ e^{-tD_+^2})\, dt \tag{1.4}$$

is well-defined for $\operatorname{Re}(s)$ large and has a meromorphic extension to the entire complex plane, regular at $s = 0$. For the last point it is crucial that we have a compatible Dirac operator, since for these operators the local residues of the η-function vanish [4]. Moreover [4] shows that (1.4) converges for $\operatorname{Re} s > -2$ and we thus may also write

$$\eta(D_+, 0) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}(D_+ e^{-tD_+^2})\, dt. \tag{1.5}$$

It is a remarkable fact that η is more or less independent of the length of the boundary cylinder. For $R > 0$ let

$$M_R := ([-R, 0] \times N) \cup M,$$

and

$$M_\infty := ((-\infty, 0] \times N) \cup M$$

be a manifold with cylindrical end. Here the cylinder and M are glued together along the common boundary in the obvious way since M is product near the boundary. By virtue of (1.1) the Clifford structures and D have an obvious extension to M_∞. The manifold M_∞ is complete, thus D is essentially self-adjoint on $C_0^\infty(M_\infty, S)$. This is classical by now. The standard reference is the beautiful paper by Chernoff [8] on hyperbolic equations. Denote by D_∞ this unique self-adjoint extension of D and by D_+^R the operator D on M_R with boundary condition (1.3). It was shown in [14] that $\eta(D_\infty, 0)$, the η-invariant of D_∞, is well defined. Moreover we have:

THEOREM 1.1 ([9, 18, 21]).

$$\lim_{R \to \infty} \eta(D_+^R, 0) = \eta(D_\infty, 0).$$

Modulo integers, $\eta(D_+^R, 0)$ is independent of R.

The situation is different in case of non-trivial kernel of A. (1.3) is not a self-adjoint boundary condition any more and there exist a variety of self-adjoint boundary conditions which we are going to describe now. First we need the Cobordism Theorem for Dirac operators. This is due to Atiyah-Singer and was published in Palais book. But there also exist fairly direct proofs by now.

PROPOSITION 1.2. [20, 13, 15, 3] *We have*

$$\dim(\ker(\Gamma - i) \cap \ker A) = \dim(\ker(\Gamma + i) \cap \ker A).$$

We pick a Lagrangian subspace $\mathcal{L}$ of $\ker A$ with respect to Γ. This means that $\Gamma(\mathcal{L})$ is orthogonal to $\mathcal{L}$ and $\mathcal{L} + \Gamma\mathcal{L} = \ker A$. Let $\pi_{\mathcal{L}}$ be the orthogonal projection onto $\mathcal{L}$ in $\ker A$. $\mathcal{L}$ can equivalently be described by the reflection $\sigma := I - 2\pi_{\mathcal{L}}$. σ is unitary, $\sigma^2 = 1$ and $\mathcal{L}$ is just the -1 eigenspace of σ. Moreover

$$\sigma\Gamma = -\Gamma\sigma. \tag{1.6}$$

Below we identify the Lagrangian subspaces of $\ker A$ with its reflections σ and denote by π_σ the orthogonal projection onto $\ker(\sigma + 1)$. Sometimes we consider π_σ also as projection in $L^2(S|N)$ in the obvious way. Since $\ker A$ consists of smooth sections, this projection has, of course, a smooth kernel. To σ we associate the projection

$$\Pi_\sigma := \Pi_+ + \pi_\sigma \tag{1.7}$$

and define the boundary condition

$$\begin{aligned} D_\sigma &:= D, \\ \mathcal{D}(D_\sigma) &:= \{s \in H^1(M, S) \mid \Pi_\sigma(s|N) = 0\}. \end{aligned} \tag{1.8}$$

Again D_σ is a self-adjoint, unbounded Fredholm operator and the η-function has the same properties as in case of invertible A (see the Appendix A to [9]). A priori there is no canonical choice for σ and the question how η depends on σ naturally arises.

η-invariants for global boundary conditions were first introduced by Cheeger in the context of conical singularities [6], [7], including the emphasis on the role of Lagrangian subspaces in ideal boundary conditions. He studies the η-invariant of the signature operator on manifolds with conic singularities. In order to obtain self-adjoint extensions, Lagrangian subspaces naturally occur. For general 1st order regular singular operators this has been worked out by the first named author [16].

More general any pseudodifferential projection P with the same principal symbol as Π_+ and which satisfies

$$-\Gamma P \Gamma = I - P, \tag{1.9}$$

the equivalent of (1.6) in the terminology of projections, provides us with a self-adjoint elliptic boundary condition. We denote the space of such P by $\mathrm{Ell}^*(D)$. This space was studied in [2] and the Appendix B to [9] (see also [3]), where the homotopy groups of $\mathrm{Ell}^*(D)$ were computed. In particular $\pi_1(\mathrm{Ell}^*(D)) = \mathbb{Z}$.

In the next section we study in detail the case of a cylinder manifold where we can compute $\eta(D_\sigma, 0)$ explicitly. This leads to a formula for the dependence of $\eta(D_\sigma, 0)$ on σ which we then prove in Section 3 in general. Given two reflections σ_1, σ_2 we construct a path connecting σ_1 and σ_2. The main idea then is to transform the resulting family of operators into a family which is constant near the boundary. It seems that many people in the community have the impression that now the result just follows from the standard variation formula for η. Morally, this is correct. But nevertheless, the problem is more subtle since the family of operators (3.7) is not pseudodifferential. Moreover we mention that our formula for the dependence of η on the boundary condition is one of the ingredients of the general glueing formula of the η-invariant, which has been proved in the meantime using our result in the cylinder case [5]; see also [21] for the case of invertible tangential operator.

Finally, we obtain a family of boundary conditions over S^1, which provides us with a generator of $\pi_1(\mathrm{Ell}^*(D))$.

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2. The cylinder case

In this section we discuss in detail the case of the cylinder $M := [0, 1] \times N$. Here we can relax our assumptions on the operators. We just assume that we have a first order symmetric elliptic differential operator of the form (1.1), where $A\colon C_0^\infty(E) \to C_0^\infty(E)$ is a first order symmetric elliptic differential operator over the hermitian vector bundle E and Γ is a unitary 0^{th} order operator, $\Gamma^2 = -I$, $\Gamma A = -A\Gamma$. Furthermore in this situation we have to *assume* that Proposition 1.2 holds. Then we choose two reflections $\sigma_j\colon \ker A \to \ker A$, $j = 0, 1$, as in Section 1 and put

$$\Pi_0 := \Pi_+ + \pi_{\sigma_0},\ \Pi_1 := \Pi_- + \pi_{\sigma_1}$$

and

$$\begin{aligned} D_\sigma &= D \\ \mathcal{D}(D_\sigma) &:= \left\{ f \in H^1(M, E) \,\middle|\, \Pi_0(f|\{0\} \times N) = 0, \Pi_1(f|\{1\} \times N) = 0 \right\}. \end{aligned} \tag{2.1}$$

In this situation we can compute quite explicitly.

THEOREM 2.1. *We put*

$$u := \sigma_0\sigma_1, \quad \textit{and} \quad u_\pm := u|\ker(\Gamma \mp i).$$

Then the η-invariant of D_σ is given by the formula

$$\eta(D_\sigma, 0) = -\frac{1}{\pi} \sum_{\substack{\beta \in (-\pi, \pi) \\ e^{i\beta} \in \operatorname{spec}(-u_+)}} \beta;$$

in particular,

$$\begin{aligned}\eta(D_\sigma, 0) &\equiv -\frac{1}{\pi i}\operatorname{tr}\log(-u_+) + \dim\ker(u_+ - 1) \mod 2\mathbb{Z}\\ &\equiv -\frac{1}{\pi i}\log\det(-u_+) + \dim\ker(u_+ - 1) \mod 2\mathbb{Z}.\end{aligned}$$

Moreover $\dim\ker D_\sigma = \dim\ker(u_+ - 1)$ *and hence the reduced* η*-invariant is given by*

$$\tilde{\eta}(D_\sigma, 0) \equiv \frac{1}{2}(\eta(D_\sigma, 0) + \dim\ker D_\sigma) \mod \mathbb{Z} \equiv -\frac{1}{2\pi i}\log\det(-u_+) \mod \mathbb{Z}.$$

Proof. We choose an orthonormal basis $(\phi_n)_{n=1}^\infty$ of $\operatorname{im}\Pi_+$, consisting of eigensections of A, i.e. $A\phi_n = \lambda_n\phi_n$, $\lambda_n > 0$. Then, since $\Gamma A = -A\Gamma$, $(\Gamma\phi_n)_{n=1}^\infty$ is an orthonormal basis of $\operatorname{im}\Pi_-$, $A\Gamma\phi_n = -\lambda_n\Gamma\phi_n$. Putting

$$V_n = \begin{cases} \operatorname{span}(\phi_n, \Gamma\phi_n), & n \geq 1,\\ \ker A, & n = 0,\end{cases}$$

we have

$$\begin{aligned} L^2(E) &= \bigoplus_{n=0}^\infty V_n\\ L^2([0,1], L^2(E)) &= \bigoplus_{n=0}^\infty L^2([0,1], V_n) \qquad (2.2)\\ D_\sigma &= \bigoplus_{n=0}^\infty D_{\sigma,n}\end{aligned}$$

where for $n \geq 1$ we have

$$\begin{aligned} D_{\sigma,n} &= \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\left(\frac{\partial}{\partial x} + \begin{pmatrix}\lambda_n & 0\\ 0 & -\lambda_n\end{pmatrix}\right)\\ \mathcal{D}(D_{\sigma,n}) &= \{(f, g) \in H^1([0,1], \mathbb{C}^2)\,;\ f(0) = 0,\ g(1) = 0\}\end{aligned}$$

and for $n = 0$,

$$\begin{aligned} D_{\sigma,0} &= \Gamma\frac{\partial}{\partial x}\\ \mathcal{D}(D_{\sigma,0}) &= \{f \in H^1([0,1], \ker A)\,;\ f(0) \in \ker(\sigma_0 - 1),\ f(1) \in \ker(\sigma_1 - 1)\}.\end{aligned}$$

LEMMA 2.2. *For* $n \geq 1$ *the operator* $D_{\sigma,n}$ *is invertible and has symmetric spectrum. In particular,* $\eta(D_{\sigma,n}, s)$ *vanishes.*

Proof. If $D_{\sigma,n}(f,g)=0$ then $f(x)=c_1e^{-\lambda_n x}$, $g(x)=c_2e^{\lambda_n x}$ and the boundary conditions force c_1, c_2 to be 0. The operator

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

obviously leaves the domain of $D_{\sigma,n}$ invariant and anticommutes with $D_{\sigma,n}$, which proves symmetry of the spectrum. □

Now we deal with $D_{\sigma,0}$ and first compile some properties of the unitaries involved.

- $\dim\ker(\sigma_j \pm 1) = 1/2 \dim\ker A$, $\det\sigma_j = (-1)^{1/2\dim\ker A}$.
- $u=\sigma_0\sigma_1$ is unitary, commutes with Γ, $\det u = 1$. Hence $u_\pm := u\,|\ker(\Gamma \mp i)$ is a well-defined unitary.
- $u^* = \sigma_1\sigma_0 = \sigma_1\sigma_0\sigma_1\sigma_1 = \sigma_1 u\sigma_1$, thus $\operatorname{spec} u$ is invariant under complex conjugation.

LEMMA 2.3. *We have*

$$\operatorname{spec} D_{\sigma,0} = \bigcup_{\substack{\alpha\in(-\pi,\pi] \\ e^{i\alpha}\in\operatorname{spec}(u_+)}} \frac{\alpha}{2} + \pi\mathbb{Z},$$

where the multiplicity of the eigenvalue $\alpha/2+\pi k$ is just the multiplicity of the eigenvalue $e^{i\alpha}\in\operatorname{spec}(u_+)$.

Proof. Consider an eigensection $D_{\sigma,0}f=\lambda f$. Then we have obviously

$$f(x) = e^{-\lambda\Gamma x}f_0,\ f_0\in\ker(\sigma_0-1).$$

Decompose $f_0 =: f_{0,+}\oplus f_{0,-}$, $f_{0,\pm}\in\ker(\Gamma\mp i)$. Decomposing $f_{0,+}$ with respect to the spectral decomposition of u_+ w. l. o. g. we may assume, that $f_{0,+}$ is an eigenvector of u_+; i.e., $uf_{0,+}=\mu f_{0,+}$, $\mu\in S^1$. Now one easily checks the relations

$$uf_{0,-} = \bar\mu f_{0,-},\ \sigma_0 f_{0,\pm} = f_{0,\mp},\ \sigma_1 f_{0,+} = \mu f_{0,-},\ \sigma_1 f_{0,-} = \bar\mu f_{0,+}.$$

The boundary condition at 1 shows that λ is an eigenvalue iff

$$e^{2i\lambda} = \mu.$$

Writing $\mu = e^{i\alpha}$, $\alpha\in(-\pi,\pi]$, we obtain the assertion. □

To prove the theorem we have to analyze the analytic continuation of the function

$$f(a,s) := \sum_{n\in\mathbb{Z}} \operatorname{sign}(n+a)|n+a|^{-s},\quad a\in(-1,1)\setminus\{0\}. \tag{2.3}$$

This could be done by differentiating with respect to the parameter a [11, Sec. 1.10]. Here we give a somewhat more general result.

LEMMA 2.4. *Let S be a symmetric elliptic differential operator on the closed manifold M. Denote by $|\lambda_0| > 0$ the first nontrivial eigenvalue of S. Then, for* $0 < |a| < |\lambda_0|$,

$$\begin{aligned}\eta(S+a,0) &= (\dim\ker S)\operatorname{sign}(a) + \eta(S,0) + \sum_{l=1}^{\infty}\frac{a^{2l}}{2l}\operatorname{Res}\,\eta(S)(2l)\\ &\quad -\sum_{l=0}^{\infty}\frac{a^{2l+1}}{l+1/2}\operatorname{Res}\,\zeta(S^2)(l+1/2).\end{aligned}$$

Since η and ζ are holomorphic for Re s *large, the sums are in fact finite.*

Proof. Since $0 < |a| < |\lambda_0|$, for Res large we compute

$$\begin{aligned}\eta(S+a,s) &= (\dim\ker S)\frac{\operatorname{sign}(a)}{|a|^s} + \sum_{\lambda\in\operatorname{spec} S\setminus\{0\}}\operatorname{sign}(\lambda)|\lambda+a|^{-s}\\ &= (\dim\ker S)\frac{\operatorname{sign}(a)}{|a|^s} + \sum_{\lambda\in\operatorname{spec} S\setminus\{0\}}\operatorname{sign}(\lambda)|\lambda|^{-s}\sum_{n=0}^{\infty}\binom{-s}{n}\left(\frac{a}{\lambda}\right)^n\\ &= (\dim\ker S)\frac{\operatorname{sign}(a)}{|a|^s} + \eta(S,s) + \sum_{l=1}^{\infty}\binom{-s}{2l}a^{2l}\eta(S,s+2l)\\ &\quad +\sum_{l=0}^{\infty}\binom{-s}{2l+1}a^{2l+1}\zeta\left(S^2,\frac{s+1}{2}+l\right).\end{aligned}$$

The last series gives the analytic continuation to the entire complex plane and the assertion follows from

$$\binom{-s}{2l}\eta(S,s+2l)\Big|_{s=0} = \frac{1}{2l}\operatorname{Res}\,\eta(S)(2l),$$

$$\binom{-s}{2l+1}\zeta\left(S^2,\frac{s+1}{2}+l\right)\Big|_{s=0} = \frac{-1}{l+1/2}\operatorname{Res}\,\zeta(S^2)(l+1/2). \qquad \square$$

In (2.3) the operator is $S = \frac{1}{i}\frac{\partial}{\partial\varphi}$ on S^1. Its η-function vanishes identically and

$$\zeta(S^2,z) = 2\sum_{n=1}^{\infty}\frac{1}{n^{2z}} = 2\zeta_R(2z),$$

ζ_R the Riemann ζ-function. Since 1 is the only pole of ζ_R and its residue is 1 we find

$$f(a,0) = \operatorname{sign}(a) - 2a. \tag{2.4}$$

With Lemmas 2.3, 2.4 we obtain

$$\begin{aligned}\eta(D_\sigma, s) &= \sum_{\substack{\alpha\in(-\pi,\pi)\\ e^{i\alpha}\in\operatorname{spec}(u_+)}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \operatorname{sign}(n) \left|\frac{\alpha}{2} + \pi n\right|^{-s} \\ &= \sum_{\substack{\alpha\in(-\pi,\pi)\\ e^{i\alpha}\in\operatorname{spec}(u_+)}} \pi^{-s} f\left(\frac{\alpha}{2\pi}, s\right),\end{aligned}$$

hence

$$\eta(D_\sigma, 0) = \sum_{\substack{\alpha\in(-\pi,\pi)\\ e^{i\alpha}\in\operatorname{spec}(u_+)}} \operatorname{sign}(\alpha) - \frac{\alpha}{\pi} = \sum_{\substack{\beta\in(-\pi,\pi)\\ e^{i\beta}\in\operatorname{spec}(-u_+)}} \frac{-\beta}{\pi}$$

and the proof of Theorem 2.1 is complete. □

For the more general Atiyah-Patodi-Singer boundary conditions of D we can at least give a vanishing criterion.

LEMMA 2.5. *Let $M = [0, 1] \times N$ and D be of the form* (1.1). *Moreover let P be a generalized Atiyah-Patodi-Singer condition for D; i.e., P has the same principal symbol as Π_+ and satisfies* (1.9). *Put*

$$\begin{aligned} D_P &= D \\ \mathcal{D}(D_P) &:= \left\{ f \in H^1(M, E) \,\middle|\, P(f|\{0\}\times N) = 0, (I - P)(f|\{1\}\times N) = 0 \right\}.\end{aligned}$$

Then the η-function of D_P vanishes.

Proof. We show that D_P has symmetric spectrum. Put

$$T\colon L^2([0,1], L^2(E|N)) \to L^2([0,1], L^2(E|N)), \quad (Tf)(x) := \Gamma f(1-x).$$

It is clear that T maps $\mathcal{D}(D_P)$ onto itself and anticommutes with D_P. □

An Example. We discuss in some detail an example that explains Theorem 2.1 and which leads to a generator of $\pi_1(\mathrm{Ell}^*(D))$. In the context of the beginning of this section assume $\ker A \neq \{0\}$ and choose an element $\varphi \in \ker A$, $\|\varphi\| = 1$ with $\Gamma\varphi \perp \varphi$. Fix a symmetric boundary condition on the complement of $W_\varphi := C^\infty([0,1] \times N) \otimes \operatorname{span}(\varphi, \Gamma\varphi)$ as in the preceding Lemma. To define self-adjoint boundary conditions we therefore have to fix Lagrangian subspaces in $\operatorname{span}(\varphi, \Gamma\varphi)$. For convenience we work in the base

$$e := \frac{1}{\sqrt{2}}(\varphi - i\Gamma\varphi), \quad f := \frac{1}{\sqrt{2}}(\varphi + i\Gamma\varphi)$$

of $\mathrm{span}(\varphi, \Gamma\varphi)$ consisting of $\pm i$-eigensections of Γ. For simplicity, let

$$\mathcal{L}_0 := \mathrm{span}(e + f),$$

i.e.,

$$\sigma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

and for $0 \le a \le 2\pi$,

$$\mathcal{L}_{1,a} := \mathrm{span}(e + e^{i(\pi-a)} f),$$

i.e.,

$$\sigma_{1,a} = \begin{pmatrix} 0 & -e^{-i(\pi-a)} \\ -e^{i(\pi-a)} & 0 \end{pmatrix}.$$

Let D_a be the operator with symmetric boundary condition on the complement of W_φ and boundary reflection $\sigma_0, \sigma_{1,a}$ on W_φ. We have

$$u_a := \sigma_0\sigma_{1,a} = \begin{pmatrix} e^{i(\pi-a)} & 0 \\ 0 & e^{-i(\pi-a)} \end{pmatrix};$$

thus $e^{i(\pi-a)}$ is the only eigenvalue of $u_{a,+}$ and we obtain from Lemma 2.3 that the spectrum of D_a restricted to W_φ is

$$\frac{\pi - a}{2} + \pi\mathbb{Z}.$$

Hence we have proved:

PROPOSITION 2.6.

$$\eta(D_a, 0) = \begin{cases} \dfrac{a}{\pi}, & 0 \le a < \pi, \\ 0, & a = \pi, \\ \dfrac{a}{\pi} - 2, & \pi < a \le 2\pi, \end{cases}$$

$$\tilde{\eta}(D_a, 0) \equiv \frac{a}{2\pi} \bmod \mathbb{Z}.$$

The reason for the discontinuity of $\eta(D_a, 0)$ is that an eigenvalue crosses the origin as a crosses π. Since exactly one eigenvalue crosses the origin from $+$ to $-$ we obtain that the spectral flow of the family $\{D_a\}_{0\le a\le 2\pi}$ is -1. This makes sense because $D_0 = D_{2\pi}$ and hence we have a family of self-adjoint Fredholm operators over the circle. We state these observations:

COROLLARY 2.7. $\mathrm{sf}\{D_a\} = -1$ *and as a result* $\{D_a\}$ *represents a generator of* $\pi_1(\mathrm{Ell}^*(D))$.

Remark. The spectral flow of the families of boundary value problems over S^1 was also studied by Furutani and Otsuki [10].

More generally, let $\gamma\colon [0,1] \to \mathcal{U}(\ker(\Gamma - i) \cap \ker A)$, $\gamma(0) = \gamma(1) = I$ be a closed path of unitaries. Extend γ to $\ker A$ by I on $\ker(\Gamma + i) \cap \ker A$ and let $\sigma_t := \gamma_t^* \sigma_0 \gamma_t$. This defines a family $\{D_t\}_{0 \le t \le 1}$ of operators with boundary reflections σ_0, σ_t. Since $(\gamma | \ker(\Gamma + i)) = I$ we have

$$u_{+,t} = (\sigma_0 \sigma_t | \ker(\Gamma - i)) = \gamma(t).$$

Thus by Theorem 2.1,

$$\frac{d}{dt}\tilde{\eta}(D_t, 0) = -\frac{1}{2\pi i}\operatorname{tr}(\gamma(t)^* \dot{\gamma}(t)). \tag{2.5}$$

This gives

$$\mathrm{sf}\{D_t\} = -\int_0^1 \frac{d}{dt}\tilde{\eta}(D_t, 0)\, dt = \frac{1}{2\pi i}\int_0^1 \operatorname{tr}(\gamma(t)^* \dot{\gamma}(t))\, dt, \tag{2.6}$$

i.e., the winding number of the path γ, which just gives the isomorphism

$$\pi_1(\mathcal{U}(\ker(\Gamma - i) \cap \ker A)) \to \mathbb{Z}.$$

3. The general case

The aim of this section is to generalize Theorem 2.1 as follows.

THEOREM 3.1. *Let D be a Dirac operator on an odd-dimensional compact Riemannian manifold with boundary as described in Section* 1. *Let σ_0, σ_1 be reflections of* $\ker A$ *satisfying* (1.6). *Then we have*

$$\tilde{\eta}(D_{\sigma_1}, 0) - \tilde{\eta}(D_{\sigma_0}, 0) \equiv \frac{1}{2\pi i}\log\det(\sigma_0\sigma_1 | \ker(\Gamma - i)) \mod \mathbb{Z}.$$

Proof. We choose a self-adjoint endomorphism T of $\ker(\Gamma - i) \cap \ker A$ such that

$$e^{2\pi i T} = \sigma_0\sigma_1 | \ker(\Gamma - i) \text{ and } -1/2 < T \le 1/2,$$

i.e.,

$$T = \frac{1}{2\pi i}\log(\sigma_0\sigma_1 | \ker(\Gamma - i)).$$

We extend T to $L^2(S|N)$ by 0 on the orthogonal complement of $\ker A \cap \ker(\Gamma - i)$. Then we find that

$$V_r := e^{2\pi i r T} \tag{3.1}$$

is a one-parameter family of unitaries commuting with Γ and A and

$$\sigma_r := V_r^* \sigma_0 V_r \tag{3.2}$$

is a one-parameter family of reflections anticommuting with Γ that joins σ_0 and σ_1. Moreover this gives a one-parameter family of operators D_{σ_r}.

One of the main difficulties is that the boundary condition varies with r. Now we introduce a transformation to a family which is constant near the boundary. Choose $f \in C^\infty(\mathbb{R})$ with

$$f(x) = \begin{cases} 1, & 0 \le x \le \varepsilon \\ 0, & x \ge 2\varepsilon \end{cases}, \quad 0 < \varepsilon < \frac{1}{3}. \tag{3.3}$$

Then f extends in an obvious way to a C^∞-function on M. Now define a gauge transformation

$$\begin{aligned} U_r &:= L^2([0,1], L^2(S|N)) \to L^2([0,1], L^2(S|N)) \\ (U_r\varphi)(x) &:= e^{2\pi i r f(x) T} \varphi(x). \end{aligned} \tag{3.4}$$

Since $(U_r\varphi)(x) = \varphi(x)$ for $x \ge 2\varepsilon$ it extends to a unitary one-parameter group on $L^2(M, S)$. Moreover U_r maps $\mathcal{D}(D_{\sigma_r})$ onto $\mathcal{D}(D_{\sigma_0})$ such that

$$D'_{\sigma_r} := U_r D_{\sigma_r} U_r^* \tag{3.5}$$

has fixed domain $\mathcal{D}(D_{\sigma_0})$. On the collar $[0,1) \times N$ we have

$$D'_{\sigma_r} = D - 2\pi i\, r\, f'\, \Gamma\, T, \quad \mathcal{D}(D'_{\sigma_r}) = \mathcal{D}(D_{\sigma_0}) \tag{3.6}$$

hence

$$D'_{\sigma_r} = D_{\sigma_0} - 2\pi i r f' \Gamma T. \tag{3.7}$$

If this were a differential operator, we could apply [9, Prop. 4.4] (see also [9, Appendix]) and would get

$$\frac{d}{dr}\tilde{\eta}(D'_{\sigma_r}, 0) = \frac{d}{dr}\tilde{\eta}(D'_{\sigma_r} \cup (-D_{\sigma_0}), 0), \tag{3.8}$$

where $D'_{\sigma_r} \cup (-D_{\sigma_0})$ is a Dirac operator on the double $\tilde{M}$ of M. By [11, Lemma 1.10.3] this would be

$$-\pi^{-1/2} \int_{\tilde{M}} a_m \left(p, \frac{d}{dr}\left(D'_{\sigma_r} \cup (-D_{\sigma_0})\right), \left(D'_{\sigma_r} \cup (-D_{\sigma_0})\right)^2\right) dp \tag{3.9}$$

where a_m is a local invariant in the jets of the symbol of the operators involved. Since a_m is local in the jets of the symbols, it is supported on supp $f' \subset [\varepsilon, 2\varepsilon] \times N$, hence

$$
\begin{aligned}
(3.10) \qquad \frac{d}{dr}\tilde{\eta}(D_{\sigma_r}, 0) &= \frac{d}{dr}\tilde{\eta}(D'_{\sigma_r}, 0) \\
&= 2\pi i r \pi^{-1/2} \int_{[\varepsilon, 2\varepsilon]\times N} a_m(p, f'\, \Gamma T, (D'_{\sigma_r})^2)\, dp.
\end{aligned}
$$

Now T is not a differential operator and thus we cannot argue in this way. But of course it gives us an idea what to do. In the next section we will prove:

MAIN LEMMA 3.2. *Let D_r: $C^\infty(S^1 \times N, S|N) \to C^\infty(S^1 \times N, S|N)$ denote the operator*

$$
\begin{aligned}
D_r &:= D - 2\pi i\, r\, f'\Gamma T, \\
D &= \Gamma\left(\frac{\partial}{\partial x} + A\right),
\end{aligned}
$$

where S^1 is $\mathbb{R}/\mathbb{Z}$ here. Then we have

$$\frac{d}{dr}\tilde{\eta}(D'_{\sigma_r}, 0) = \frac{d}{dr}\tilde{\eta}(D_r, 0).$$

This formula shows that the variation of η is independent of the rest of the manifold. We can argue now in two ways. We could compute $\frac{d}{dr}\tilde{\eta}(D_r, 0)$ explicitly as in Section 2 or we can point out, that we can make the same considerations as above for the operator on the cylinder. Consider the operator D on the cylinder $[0, 1] \times N$. Let D_r^{cyl} be the operator D with boundary condition as in (2.1) where the boundary reflection on $\{0\} \times N$ is σ_r and the boundary reflection on $\{1\} \times N$ is σ_0. Then the above consideration yields

$$(3.11) \qquad \frac{d}{dr}\tilde{\eta}(D_{\sigma_r}, 0) = \frac{d}{dr}\tilde{\eta}(D_r^{\text{cyl}}, 0).$$

By Theorem 2.1 we have

$$\tilde{\eta}(D_r^{\text{cyl}}, 0) \equiv -\frac{1}{2\pi i} \operatorname{tr} \log(-\sigma_r \sigma_0 | \ker(\Gamma - i)) \bmod \mathbb{Z}.$$

An easy calculation shows

$$\sigma_r \sigma_0 | \ker(\Gamma - i) = e^{-2\pi i r T}$$

thus

$$\tilde{\eta}(D_r^{\text{cyl}}, 0) \equiv r \operatorname{tr} T - \frac{1}{2} \dim(\ker(\Gamma - i) \cap \ker A) \bmod \mathbb{Z}$$

and

$$\frac{d}{dr}\tilde{\eta}(D_r^{\mathrm{cyl}}, 0) = \operatorname{tr} T. \tag{3.12}$$

Together with (3.11) we obtain

$$\begin{aligned}\tilde{\eta}(D_{\sigma_1}, 0) - \tilde{\eta}(D_{\sigma_0}, 0) &\equiv \int_0^1 \frac{d}{dr}\tilde{\eta}(D_{\sigma_r}, 0)\, dr \bmod \mathbb{Z}\\ &\equiv \operatorname{tr} T \equiv \frac{1}{2\pi i}\log\det(\sigma_0\sigma_1 \,|\, \ker(\Gamma - i)) \bmod \mathbb{Z}\end{aligned}$$

and Theorem 3.1 is proved. We note an immediate corollary which is the generalisation of (2.6) to arbitrary manifolds with boundary.

THEOREM 3.3. *Under the assumptions of Theorem* 3.1 *let* σ_0 *be a reflection and*

$$\gamma\colon [0, 1] \to \mathcal{U}(\ker(\Gamma - i) \cap \ker A), \gamma(0) = \gamma(1) = I$$

be a closed path of unitaries. Put $\sigma_r := \gamma(r)^*\sigma_0\gamma(r)$. *Then the spectral flow of* (D_{σ_r}) *is the negative of the homotopy class of the path* γ *which is given by the winding number; i.e.,*

$$\operatorname{sf}(D_{\sigma_r}) = -\frac{1}{2\pi i}\int_0^1 \operatorname{tr}(\gamma(r)^*\dot{\gamma}(r))\, dr.$$

In particular, if $\ker A \neq \{0\}$ *there exists a family* (D_{σ_r}) *of spectral flow* 1, *i.e., such a family represents a generator of the fundamental group of the space of generalized self-adjoint boundary conditions of Atiyah-Patodi-Singer type introduced in* [2].

Proof. The proof is immediate from Theorem 3.1 analogously to the computations after Corollary 2.7. □

Remarks. 1. The formula for $\eta(D_\sigma, 0)$ in Theorem 2.1 is somehow related to the Maslov index; cf. [5].

2. In his recent work, L. Nicolaescu deals with the generalizations of the Maslov index to the infinite dimensional context. Let us observe that in this case still the spectral flow is equal to the Maslov index as it was described earlier in the work of Furutani and Otsuki [10]. We refer to the forthcoming paper of Nicolaescu [19] for details.

4. Proof of the Main Lemma

For the proof of the Main Lemma we proceed along the lines of [9] with suitable modifications due to the fact that T is not a differential operator.

PROPOSITION 4.1. *There exist positive constants c_1, c_2, c_3 and a natural number l, such that for any $(u, x), (v, y) \in S^1 \times N$,*

$$\|D_r e^{-tD_r^2}((u, x), (v, y))\| \leq c_1 t^{-l} e^{-c_2 t} e^{-c_3(u-v)^2/t}. \tag{4.1}$$

Proof. We decompose D_r as in (2.2) into

$$D_r = \bigoplus_{n=0}^{\infty} D_{r,n},$$

where for $n \geq 1$ the operator $D_{r,n}$ is as in (2.2) and

$$D_{r,0} = \Gamma \frac{\partial}{\partial u} - 2\pi i r f' \Gamma T.$$

By the Sobolev embedding theorem and Gårding's inequality we have an estimate

$$\|\phi_n(y)\| \leq c(1 + \lambda_n^{2k}) \tag{4.2}$$

with c independent of n and y and where $2k = \dim N$.

Now we find for $n \geq 1$

$$\begin{aligned}
&\|D_{r,n} e^{-tD_{r,n}^2}((u,x),(v,y))\| \\
&= \|\Gamma(\partial_u + A)e^{-t\partial_u^2}(u,v)e^{-t\lambda_n^2}\{\phi_n(x) \otimes \phi_n(y) + \Gamma\phi_n(x) \otimes \Gamma\phi_n(y)\}\| \\
&\leq \|\partial_u e^{-t\partial_u^2}(u,v)e^{-t\lambda_n^2}\{\Gamma\phi_n(x) \otimes \phi_n(y) - \phi_n(x) \otimes \Gamma\phi_n(y)\}\| \\
&\quad + \|e^{-t\partial_u^2}(u,v)\lambda_n e^{-t\lambda_n^2}\{-\Gamma\phi_n(x) \otimes \phi_n(y) + \phi_n(x) \otimes \Gamma\phi_n(y)\}\| \\
&\leq c|\partial_u e^{-t\partial_u^2}(u,v)|(1+\lambda_n)e^{-t\lambda_n^2}(1+\lambda_n^{2k})^2 \\
&\leq \frac{c_1}{t} e^{-c_2(u-v)^2/t}(1+\lambda_n)(1+\lambda_n^{2k})^2 e^{-t\lambda_n^2}.
\end{aligned}$$

Summing from 1 to ∞ standard estimates of the heat kernel of A^2 at 0 and ∞ yield

$$\sum_{n=1}^{\infty} \|D_{r,n} e^{-tD_{r,n}^2}((u,x),(v,y))\| \leq c_1 t^{-l} e^{-c_2 t} e^{-c_3(u-v)^2/t}$$

and it remains to investigate the operator $D_{r,0}$. But $D_{r,0}$ is just an elliptic operator on $C_0^\infty(S^1, \ker A)$. Since $\ker A$ is finite-dimensional, standard elliptic theory gives us the estimate (4.1) (cf. [9, Section 1]). □

Now we use Duhamel's principle to investigate the heat kernel $D'_{\sigma_r} e^{-t(D'_{\sigma_r})^2}$. Let $E_1(t; z, w)$ denote the kernel of the operator $e^{-t\tilde{D}^2}$, where $\tilde{D} := D_{\sigma_0} \dot{\cup} (-D_{\sigma_0})$ is the double of the operator D_{σ_0} (cf. (3.8)). $E_2^r(t; z, w)$ denotes the kernel of the operator $e^{-tD_r^2}$ on $S^1 \times N$ and $E_3(t; z, w)$ is the kernel of the operator $e^{-tD_{\sigma_0}^2}$ on the infinite cylinder $[0, \infty) \times N$. Finally let $\mathcal{E}_r(t; z, w)$ be the kernel of the operator $e^{-t(D'_{\sigma_r})^2}$, which we are really interested in. First of all, by (3.6), each $\mathcal{E}_r(t; z, w)$ is unitarily equivalent to the kernel of the operator $D_{\sigma_r} e^{-t D^2_{\sigma_r}}$. Hence we have from [9, Theorem 4.1]

$$\| D_r \mathcal{E}_r(t; z, w) \| \le c_1 t^{-(k+1)} e^{-c_2 t} e^{-c_3 d(z,w)^2/t}. \tag{4.3}$$

Now we describe the parametrix Q_r for $\mathcal{E}_r$. Analogous to [9, Section 2] we consider partitions of unity $\{\phi_1, \phi_2, \phi_3\}$, $\{\psi_1, \psi_2, \psi_3\}$ on M as follows:

- $0 \le \phi_j \le 1$,
- $\operatorname{supp}(\phi_1) \subset M \setminus [0, 1 - \varepsilon/2] \times N$,
- $\operatorname{supp}(\phi_2) \subset [\varepsilon/4, 1 - \varepsilon/4] \times N$,
- $\operatorname{supp}(\phi_3) \subset [0, \varepsilon/2] \times N$,
- On the cylinder all functions depend on the normal variable only.

The ψ_j have the same properties as the ϕ_j and

- $\psi_j \equiv 1$ in a neighborhood of $\operatorname{supp}(\phi_j)$, i.e.,

$$\operatorname{dist}\left(\operatorname{supp}(\phi_j), \operatorname{supp}\left(\frac{\partial}{\partial u}\psi_j\right)\right) \ge \delta > 0.$$

Now we define

$$Q_r(t; z, w) := \sum_{j=1}^{3} \psi_j(z) E_j(t; z, w) \phi_j(w) \tag{4.4}$$

and as in [9, Section 4] we have

$$\begin{aligned} \mathcal{E}_r(t; z, w) &= Q_r(t; z, w) + (\mathcal{E}_r \# C)(t; z, w), \\ D_r \mathcal{E}_r(t; z, w) &= D_r Q_r(t; z, w) + (\mathcal{E}_r \# DC)(t; z, w), \end{aligned}$$

where C is the "error-term"

$$C(t; z, w) = -\sum_{j=1}^{3} \left\{ \frac{\partial \psi_j}{\partial u}(z) \frac{\partial E_j}{\partial u}(t; z, w) \phi_j(w) + \frac{\partial^2 \psi_j}{\partial u}(z) E_j(t; z, w) \phi_j(w) \right\}.$$

Now the choice of the ϕ_j, ψ_j and the estimates we have proved give:

PROPOSITION 4.2. *There exist positive constants c_4, c_5, c_6 such that*

$$\| (\mathcal{E}_r \# DC)(t; z, w) \| \le c_4 e^{-c_5 t} e^{-c_6 \delta / t}.$$

Now we can prove the Main Lemma assuming that D_{σ_r} is invertible, namely

$$\begin{aligned}
\frac{d}{dr}\eta(D_{\sigma_r},0) &= \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\,\mathrm{Tr}(\dot D'_{\sigma_r}e^{-t(D'_{\sigma_r})^2} - 2t\dot D'_{\sigma_r}(D'_{\sigma_r})^2e^{-t(D'_{\sigma_r})^2})\,dt \\
&= \frac{2}{\sqrt{\pi}}t^{1/2}\,\mathrm{Tr}(\dot D'_{\sigma_r}e^{-t(D'_{\sigma_r})^2})\Big|_0^\infty \\
&= -\frac{2}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\,\mathrm{Tr}(\dot D'_{\sigma_r}e^{-\varepsilon(D'_{\sigma_r})^2}) \\
&= -\frac{2}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\mathrm{Tr}\{\sqrt{\varepsilon}\dot D'_{\sigma_r}Q_r(\varepsilon) + \sqrt{\varepsilon}\dot D'_{\sigma_r}(\mathcal{E}_r\#C)(\varepsilon)\} \\
&= -\frac{2}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\,\mathrm{Tr}(\dot D'_{\sigma_r}Q_r(\varepsilon)) \\
&= \frac{2}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}2\pi i\,\mathrm{Tr}(f'\Gamma T\psi_2E_2(t;\cdot,\cdot)\phi_2) \\
&= \frac{d}{dr}\eta(D_r,0)
\end{aligned}$$

as asserted. If D_{σ_r} is not invertible we choose $\lambda > 0$ which is not in the spectrum of D'_{σ_r} for r in a small interval. Let

$$E_r^\lambda := 1_{(-\infty,-\lambda)\cup(\lambda,\infty)}(D'_{\sigma_r})$$

be the spectral projection of D'_{σ_r} onto the eigenspaces to eigenvalues of modulus $> \lambda$. Then

$$\eta^\lambda(D'_{\sigma_r},0) := \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\,\mathrm{Tr}(E_r^\lambda D'_{\sigma_r}e^{-t(D'_{\sigma_r})^2})\,dt$$

differs from $\eta(D'_{\sigma_r},0)$ by an integer. A simple computation with the resolvent (cf. [11, Sec. 1.10]) shows

$$\frac{d}{dr}\,\mathrm{Tr}(E_r^\lambda D'_{\sigma_r}e^{-t(D'_{\sigma_r})^2}) = \mathrm{Tr}(E_r^\lambda(\dot D'_{\sigma_r}e^{-t(D'_{\sigma_r})^2} - 2t\,\dot D'_{\sigma_r}(D'_{\sigma_r})^2e^{-t(D'_{\sigma_r})^2}))$$

hence as before

$$\begin{aligned}
\frac{d}{dr}\eta(D_{\sigma_r},0) = \frac{d}{dr}\eta^\lambda(D_{\sigma_r},0) &= -\frac{2}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\,\mathrm{Tr}(E_r^\lambda\dot D'_{\sigma_r}e^{-\varepsilon(D'_{\sigma_r})^2}) \\
&= -\frac{2}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\,\mathrm{Tr}(\dot D'_{\sigma_r}e^{-\varepsilon(D'_{\sigma_r})^2})
\end{aligned}$$

since $I - E_r^\lambda$ is of finite rank and hence

$$\mathrm{Tr}((I - E_r^\lambda)D'_{\sigma_r}e^{-\varepsilon(D'_{\sigma_r})^2})$$

is bounded as $\varepsilon \to 0$. Now we can proceed as before and the Main Lemma is proved.

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