WEIERSTRASS LOCI FOR VECTOR BUNDLES ON CURVES

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The thesis in [3] contains a very satisfactory theory of Weierstrass loci for spanned vector bundles on a smooth curve (at least in characteristic 0). This thesis will appear soon [4]. Fix a smooth projective curve C, a vector bundle E on C and a vector space $V \subseteq H^0(C, E)$ such that V spans E; set $r := \mathrm{rank}(E)$ and $v := \dim(V)$. Hence the pair (E, V) induces a morphism $h := h_{E,V}$ from C to the Grassmannian $G = G(r, v)$ of the r-dimensional quotient vector spaces of V. Let

$$(1) \qquad\qquad 0 \to S_G \to V \otimes O_G \to Q_G \to 0$$

be the tautological exact sequence on G with Q_G the universal quotient bundle and S_G the universal subbundle (hence $E \cong h^*(Q_G)$). The main point of the theory in [3] is to associate to the pair (E, V) not only a family of ramification loci, but also a family of vector bundles $\{E_i\}$, $i \geq 0$, with $E_0 = E$ and E_{i+1} quotient of E_i. Here is the construction. Since the tangent bundle TG is isomorphic to $\mathrm{Hom}(S_G, Q_G)$, the differential map $dh: TC \to h^*(TG)$ defines a morphism $\partial := \partial_{E,V}: TC \otimes h^*(S_G) \to E$. Let ϕ_1 be the torsion sheaf of $E/\mathrm{Im}(\partial)$ and set $E_1 := (E/|\mathrm{Im}(\partial))/\phi_1$; E_1 is a vector bundle on C. Since E_1 is a quotient of E, it is a quotient of $V \otimes O_C$ and (if $E_1 \neq 0$) we may apply the same construction to the pair (E_1, V) to find another torsion sheaf ϕ_2 and another bundle E_2, and so on. The torsion sheaf ϕ_i is called the *i-th torsion sheaf* (or the *i-th torsion*) of the pair (E, V); E_i is called the *i-th derived bundle* of (E, V); set $r_i := \mathrm{rank}(E_i)$; $r_{i-1} - r_i$ is called the *i-th differential rank* of (E, V). It was proved in [3] (see [4], Theorem 5.1) that these ranks give a local normal form for the morphism h at a general point of C.

The aim of this paper is to investigate the generic behavior of differential ranks and torsion. Theorem 0.1 shows that, very often, if $v \geq 2r$ the first differential rank is r, i.e., $E_1 = 0$ and if $v > 2r$ there is no first order torsion. A result like this has strong geometric implications. For instance if $v = 2r$, E has no trivial factor and $E_1 = 0$, then $\deg(\phi_1) = r(2g - 2) + 2\deg(E)$. Hence if $v = 2r$, E has no trivial factor and $E_1 = 0$ then we have $\phi_1 = 0$ if and only if $C \cong \mathbf{P}^1$, E is the direct sum of r line bundles of degree 1 and $V = H^0(\mathbf{P}^1, E)$; this result was proved in [4], Example 8.2.

Theorem 0.2 concerns the case where $v \le 2r$. In general, if $r < v < 2r$ then $E_1 \ne 0$, and we cannot expect that for $i \ge 2$ the i-th differential rank is the maximal possible compatible with the ranks of the previous subbundle $\ker(V \otimes O_C \to E_{i-1})$ because the sequence of differential ranks $\{r_{j-1} - r_j\}_{j>0}$ is non-increasing ([3], Cor. II.3.3, or [4], Cor. 4.5); for instance if $v = r + 1$ each differential rank is one. However, Theorem 0.2 shows that very often if $r < v < 2r$, then for every $i \ge 0$ the $(i+1)$-th differential rank $r_i - r_{i+1}$ is $\min\{r_i, r_{i-1} - r_i, v - r_i\}$ (with $r_0 := r$), i.e. it is as large as possible with the only obstruction that the sequence $\{r_{j-1} - r_j\}_{j>0}$ is non-increasing. We will use in an essential way the following duality theorem which is one of the key results of [3] (see [4], Theorem 7.1). Let $u: V \to E$, $V \subseteq H^0(C, E)$ be a surjection; set $S := \ker(u)$; let $v: V^* \to S^*$ be the dual surjection of u; let $P^1(E)$ be the bundle of principal parts of E, $\alpha: V \to P^1(E)$ the first Taylor expansion map induced by u and $G^1(E) := \operatorname{Im}(\alpha)$ the first osculating bundle of u. Then ([4], Theorem 7.1) $\ker(V \to G^1(E)) := \ker(\alpha) = ((S^*)_1)^*$ and $\ker(V \to E_1) = (G^1(S^*))^*$.

Now we describe the range of (C, E, V) with which we will prove these statements.

Fix integers r and v with $v > r > 0$; set $s = v - r$ and let m be the maximal common divisor of v and r; take non-negative integers d and g; set $x := [gm/(r+s)]$, $x' := x$ if $x(r+s) = gm$, $x' := x + 1$ otherwise; and assume $d \ge rs + x'(sr/m)$. In [1], §2 (mimicking the case of a projective space done in [2]), we defined an irreducible component, $W(d, g; r, v)$, of the Hilbert scheme of curves of degree d and genus g in the Grassmannian $G(r, v)$ of $r - 1$ dimensional linear subspaces of $\mathbf{P}^{v-1}$. This component has very nice properties (see [1], §2): it is generically smooth of the expected dimension and parametrizes curves with general moduli. If C is a curve contained in a Grassmannian $G(r, v)$ we take as bundle and as spanning vector space the restriction to C of the corresponding objects on $G(r, v)$ appearing in the tautological exact sequence (1).

THEOREM 0.1. *Fix integers v, r, d, g such that $W(d - r, g; r, v)$ is defined. Then for a general $C \in W(d, g; r, v)$ and the corresponding pair (E, V) associated to the embedding $C \subset G(r, v)$ we have:*

   (i) *If $v \ge 2r$ the first derived bundle of (E, V) is 0.*
   (ii) *If $v > 2r$ the first torsion sheaf of (E, V) is 0.*
   (iii) *If $v = 2r$ the first torsion sheaf of (E, V) has degree $2 \deg(E) - 2r + 2rg$.*
   (iv) *If $v < 2r$ the first derived bundle of (E, V) has rank $2r - v$.*

THEOREM 0.2. *Fix integers r and v with $1 < r < v < 2r$. Set $a := 2r - v$, $b := [(r + a - 1)/a]$ and $r' := ab$. Fix integers d, g such that $W(d - r', g; r, v)$ is defined. Then for a general $C \in W(d, g; r, v)$ the i-th differential rank $r_{i-1} - r_i$ of the associated (E, V) is a if $i \le b$, $r' - r$ if $i = b + 1$ and 0 if $i > b + 1$.*

We hope that this paper will be the first part of a joint project.

1. Notations and key lemmas

For simplicity we always assume that the algebraically closed base field has characteristic zero. This assumption will be used in an essential way in the proof of Theorem 0.2. In this paper C will always denote a smooth projective connected curve of genus g.

Let Q be a vector bundle on C and $V \subseteq H^0(C, Q)$ a vector space spanning Q; $S_{Q,V}$ will denote the kernel of the evaluation map $V \otimes O_C \to Q$; $S_{Q,V}$ is a vector bundle of rank $\dim(V) - \mathrm{rank}(Q)$.

(1.1) If U is a quasi-projective smooth curve contained in a Grassmannian G, we may construct the map ∂ as above, restricting the tautological exact sequence of G to U; we get a first derived bundle and for each connected component, U', of U an integer: the first differential rank for U'. In particular this construction may be applied to $(A)_{\mathrm{reg}} \subset G$ for every reduced curve $A \subset G$; we will call the maximum of the first differential ranks of the irreducible components of A the *maximal first differential rank*. This obvious generalization has the following two motivations. Sometimes, a variety $Y \subset \mathbf{P}^N$ with $\dim(Y) > 1$ has a good behavior outside a reducible curve $Y \cap \Pi$, with Π projective space with $\dim(\Pi)$ much smaller than N. The second motivation comes from the use of deformation theory and reducible curves to obtain results on smooth curves (see the proofs of 0.1 and 0.2).

PROPOSITION 1.2.   *Fix a spanned vector bundle E on C, and vector spaces $W \subseteq V \subseteq H^0(C, E)$ with V spanning E and W spannig a locally free subsheaf f of E with $\mathrm{rank}(F) = \mathrm{rank}(E)$. Then the first derived bundle of (E, V) has rank at most the rank of the first derived bundle of (F, W); if these ranks are equal, then the first torsion sheaf of (E, V) is a subsheaf of the first torsion sheaf of (F, W).*

Proof.   By the snake lemma applied to the triples $(S_{F,W}, W, F)$ and $(S_{E,V}, V, E)$ the inclusion of W into V induces an inclusion as sheaves of $S_{F,W}$ into $S_{E,V}$. This inclusion is compatible with $\partial_{F,W}$ and $\partial_{E,V}$ by the functoriality properties of the differential of a morphism. Hence for every $P \in C$ we have $\mathrm{rank}(\partial_{E,W})(P) \leq \mathrm{rank}(\partial_{E,W})(P)$, as wanted. $\square$

We will use the following particular case of 1.2.

COROLLARY 1.3.   *If (with the notations of 1.2) the first derived bundle of (F, W) is 0, then the first derived bundle of (E, V) is 0.*

Note that a general subspace W of V with $\dim(W) > \mathrm{rank}(E)$ spans E because $\dim(C) = 1$.

The proof of 1.2 (changing only th notations) also gives the following result.

PROPOSITION 1.4.   *Let E be a rank r vector bundle on C and $V \subseteq M \subseteq H^0(E)$ vector spaces with E spanned by V. Then the difference between the first differential rank of (E, M) and the first differential rank of (E, V) is at most $\dim(M) - \dim(V)$.*

The following result is very useful because by [1], §2, $W(d, g; r, v)$ contains many reducible smoothable nodal curves.

LEMMA 1.5.   *Let A be a reduced connected curve in a Grassmannian G. Assume that A is smoothable. Let $f: Y \to T$ be a flat family of curves on G parametrized by an integral affine variety and with a point $o \in T$ with $A = f^{-1}(o)$ and $f^{-1}(y)$ smooth for a general $y \in T$. Then for every irreducible component B of A, the first derived bundle of B_{reg} has rank at least the rank of the first derived bundle of $f^{-1}(x)$ for a general $x \in T$.*

Proof.   Fix a general $x \in T$. Since there is an integral curve contained in T and containing $\{o, x\}$, we reduce to the case $\dim(T) = 1$. Taking the normalization we reduce to the case T smooth at o. We fix an irreducible component B of A and a general $P \in B$. Note that every smooth morphism has local sections for the étale topology. Hence with a base change with respect to a dominant étale map with image o, we may assume that f has a section s with $s(o) = P$. Now it is sufficient to use the definition of the differential ∂ and an obvious semicontinuity result for the matrix of first order derivatives for $f^{-1}(t)$ at the points of $s(t)$ with $t \in T$, t near o.   □

2. Proof of Theorems 0.1 and 0.2

In the first part of this section we consider briefly the case of rational curves. This case is important for its own sake. Here we could say much more than in the case of curves of higher genera. However, for this paper this case is also a key technical tool for the higher genus case (and its more elementary properties were used in [1] to define $W(d, g; r, v)$). Let $O(j)$ be the degree j line bundle on $\mathbf{P}^1$.

LEMMA 2.1.   *Let Q be a rank r spanned vector bundle on $\mathbf{P}^1$. Set $V := H^0(\mathbf{P}^1, Q)$. Then $S_{Q,V} \cong tO(-1)$ with $t = \dim(V) - r$.*

Remark 2.2.   For every integer $t > 0$ the bundle of t-th order principal parts $P^t(O(t))$ of the degree t line bundle $O(t)$ on $\mathbf{P}^1$ is the trivial rank $t + 1$ bundle and the Taylor expansion morphism $H^0(\mathbf{P}^1, O(t)) \otimes O \to P^t(O(t))$ is an isomorphism.

LEMMA 2.3.   *Fix integers a, b, v with $a > 0, b > 0, ab < v \le ab + a$. Set $F := aO(b)$ and let H be a general subspace of $H^0(\mathbf{P}^1, F)$ with $\dim(H) = v$. Then the Taylor expansion map $e_{Hb}: H \otimes O \to P^b(O(b))$ is an embedding and for every integer $j < b$ the Taylor expansion map $e_{Hj}: H \otimes P^j(O(b))$ is surjective outside finitely many points of $\mathbf{P}^1$.*

Proof.   The first assertion is a particular case of Remark 2.3. To prove the second one, we fix a point $P \in \mathbf{P}^1$. Look at the fibers over P of the surjections $P^{t+1}(O(b)) \to$

$P^t(O(b))$ for $t = b - 1, b - 2, \ldots, 0$, and use the generality of H to obtain the surjectivity of $e_{H,j}$ for $j < b$ at the point P.   □

PROPOSITION 2.4.   *Fix an integer* $r > 0$ *and a rank* r *ample vector bundle* E *on* $\mathbf{P}^1$. *Set* $V := H^0(\mathbf{P}^1, E)$. *Then the first derived bundle of* $(E, V))$ *is* 0 *and* (E, V) *has no first order torsion.*

Proof.   The second assertion is obvious since $\mathbf{P}^1$ and E are homogeneous. Since E is ample, $rO(1)$ is a subsheaf of E. Hence by 1.2 we reduce the first assertion to the case $E = rO(1)$. Hence we have $\dim(V) = 2r$, $S_{E,V} \cong E^*$ and $S \otimes T\mathbf{P}^1 \cong E$. Now we may use 2.2 and the duality theorem ([4], Theorem 7.1) stated in the introduction.
    □

Proof of Theorem 0.1.   (a) First we will prove the assertions (i), (ii), (iii) and (iv) of the statement of 0.1 for every ample vector bundle E on $\mathbf{P}^1$. Note that if $v \geq 2r$ we may take E ample. Hence $rO(1)$ is a subsheaf of E. Thus part (i) follows from 1.2 and 2.4. As in the proof of 2.4 for parts (iii) and (iv) use 1.2 and the fact that if $r < v \leq 2r$ then $(2r - v)O \oplus (v - r)O(1)$ is a subsheaf of E. By the ampleness of E, the generality of V and Corollary 1.3 to prove (i) and (ii) we reduce to the case $v = 2r$, $E = rO(1)$, which was proved in 2.2. If $v = 2r$, then (iii) follows from (i) because $\text{rank}(S_{E,V}) = r$ and $\deg(S_{E,V} \otimes T\mathbf{P}^1) = 2r - \deg(E)$; the same is true for part (iii) of 0.1 for any curve C. Again, for (iv) we reduce to the case $E \cong rO(1)$ and then apply 1.4 taking $M := H^0(\mathbf{P}^1, E)$.
    (b) By a very particular case of [1], Lemma 2.8, and the assumption on the integers $d - r$, g, r and v, the component $W(d, g; r, v)$ contains a reducible nodal curve $A \cup B$ with A smooth, $A \in W(d - r, g; r, v)$, $\text{card}(A \cap B) = 1$, $B \cong \mathbf{P}^1$, $Q_{G(r,v)}|B \cong rO(1)$ and B "non degenerate or linearly normal" (i.e., such that the evaluation map $V \otimes O_B \to Q_{G(r,v)}|B$ is injective if $v \leq 2r$ and surjective if $v > 2r$). Hence parts (i) and (iv) of Theorem 0.1 follow from 1.5 applied to $A \cup B$ and 2.4 applied to B. For part (iii) use part (i) and Riemann-Roch. Now we wil check part (ii). Take $W \subset H^0(C, E)$ with $\dim(W) = 2r$ and (E, W) satisfying part (iii); let J be the support of the first torsion sheaf of (E, W). Take a general V with $\dim(V) = v$ and $W \subset V$. Then one can use a local calculation with the first order Taylor map around each point of J and [4], Cor. 2.3 (or the more general duality theorem for all the torsion sheaves given in [4], Prop. 8.4.1).   □

Proof of Theorem 0.2.   As in part (b) of the proof of 0.1, by a very particular case of [1], Lemma 2.8, and the assumption on the integers $d - r'$, g, r and v, the component $W(d, g; r, v)$ contains a reducible nodal curve $A \cup B$ with A smooth, $A \in W(d - r, g; r, v)$, $\text{card}(A \cap B) = 1$, $B \cong \mathbf{P}^1$, $Q_{G(r,v)}|B \cong rO(1)$ and such that the evaluation map $V \otimes O_B \to Q_{G(r,v)}|B$ is injective. By 1.5 we are reduced to proving the corresponding assertions on $\mathbf{P}^1$ for a rank r ample bundle of degree r'. We will take as V the dual of a general v-dimensional subspace H of $H^0(\mathbf{P}^1, aO(b))$;

we will take as E the dual of the kernel of the evaluation map $H \otimes O \to aO(b)$. By the theory of duality ([3], Cor. II.6.2, or [4], Theorem 7.1.2) the i-th differential rank $r_{i-1} - r_i$ of the associated (E, V) is a if $i \leq b$, $r' - r$ if $i = b + 1$ and 0 if $i > b + 1$, as wanted. $\square$

References

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