THE PRIME ELEMENT THEOREM IN ADDITIVE ARITHMETIC SEMIGROUPS. I

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1. Introduction

As is well known, the abstract prime number theorem for an algebraic function field is proved in the context of an additive arithmetic semigroup since the concept of the latter was introduced by Knopfmacher [7], [8]. Thus it is essentially a theorem about prime elements in additive arithmetic semigroups.

We recall that an additive arithmetic semigroup G is, by definition, a free commutative semigroup with identity element 1 such that G has a countable free generating set P of "primes" p and such that G admits an integer-valued degree mapping $\partial\colon G \to \mathbf{N} \cup \{0\}$ satisfying

(1) $\partial(1) = 0$ and $\partial(p) > 0$ for all $p \in P$,
(2) $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$, and
(3) the total number $\bar{G}(n)$ of elements of degree n in G is finite for each $n \geq 0$.

In [6], [11], abstract prime number theorems are proved under a variety of conditions. The theorems assume mainly that

$$\bar{G}(n) = Aq^n + O(q^{\nu n}) \quad \text{as } n \to \infty \tag{1.1}$$

with constants $A > 0$, $q > 1$, and $0 \leq \nu < 1$, and that the generating function $Z^{\#}(y)$ of $\bar{G}(n)$ has no zeros on the circle $|y| = q^{-1}$, or assume that $0 \leq \nu < \frac{1}{2}$ in (1.1). Let $\bar{P}(n)$ be the total number of primes of degree n in G. The theorems state that

$$\bar{P}(n) = q^n/n + O(q^{\theta n}) \quad \text{as } n \to \infty,$$

where $\nu < \theta < 1$, or, equivalently, that

$$\bar{\Lambda}(n) := \sum_{\substack{p \in P, r \geq 1 \\ \partial(p^r) = n}} \partial(p) = \sum_{r \mid n} r \bar{P}(r) = q^n + O(q^{\theta n}).$$

In [12], elementary proofs of the abstract prime number theorem are given.

Also, in [11], subject to the weak condition

$$\bar{G}(n) = Aq^n + O(q^n n^{-\gamma})$$

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with the constant $\gamma > 1$, an upper estimate $\bar{P}(n) \ll q^n/n$, or, equivalently, $\bar{\Lambda}(n) \ll q^n$ is proved. From the upper estimate, Chebyshev type estimates for $\psi(n) := \sum_{s=1}^{n} \bar{\Lambda}(s)$, i.e., $q^n \ll \psi(n) \ll q^n$, can be deduced.

It is now interesting, by analogy with the theory of Beurling generalized prime numbers [2], to investigate the prime element theorem and the upper estimate in additive arithmetic semigroups in which the more general condition

$$\bar{G}(n) = q^n \sum_{\nu=1}^{r} A_\nu n^{\rho_\nu - 1} + O(q^{\upsilon n}) \tag{1.2}$$

or

$$\bar{G}(n) = q^n \sum_{\nu=1}^{r} A_\nu n^{\rho_\nu - 1} + O(q^n n^{-\gamma}) \tag{1.3}$$

is given, where $\rho_1 < \cdots < \rho_r$ and $A_1, \ldots, A_r$ are arbitrary real numbers, $q > 1, \rho_r > 0, A_r > 0, 0 \le \upsilon < 1$, and $\gamma > 1$. In this case, a generalization of the abstract prime number theorem (henceforth, P.E.T.) states that $\bar{P}(n) \sim \rho_r q^n n^{-1}$, or equivalently, $\bar{\Lambda}(n) \sim \rho_r q^n$.

We recall that if $f(n)$ and $g(n)$ are two arithmetic functions defined for all non-negative integers, the functions $h(n)$ defined by setting

$$h(n) = \sum_{k=0}^{n} f(k)g(n-k), \qquad n = 0, 1, 2, \ldots$$

is called the additive convolution of f and g and denoted by $f * g$. As in [10], an operator L on all arithmetic function f is defined by setting

$$(Lf)(n) = nf(n), \qquad n = 0, 1, 2, \ldots.$$

The arithmetic function $\bar{G}(n)$ is an elementary combinatorial function of $\bar{P}(n)$. This can be expressed explicitly by

$$Z^{\#}(y) := \sum_{n=0}^{\infty} \bar{G}(n)y^n = \prod_{m=1}^{\infty} (1 - y^m)^{-\bar{P}(m)}. \tag{1.4}$$

From (1.4), we can deduce

$$\sum_{n=1}^{\infty} \bar{\Lambda}(n)y^n \sum_{n=0}^{\infty} \bar{G}(n)y^n = \sum_{n=1}^{\infty} n\bar{G}(n)y^n.$$

This can be rewritten in an additive convolution version as

$$\bar{\Lambda} * \bar{G} = L\bar{G}, \tag{1.5}$$

an analog of Chebyshev's identity in the classical prime number theory.

Thus we are essentially dealing with two nonnegative arithmetic functions $\bar{G}(n)$ and $\bar{\Lambda}(n)$, not necessarily integer-valued, which are related by (1.5). This consideration leads to an investigation of a general formulation of the prime element theorem in additive arithmetic semigroups. In this investigation, we shall release the constraint $q > 1$ and assume that $q > 0$ although results for $q > 1$ are still of main interest. As in [11], we shall only prove results for $\bar{\Lambda}(n)$. When $\bar{P}(n)$ is the main object of interest, the results on $\bar{\Lambda}(n)$ can be easily converted to ones on $\bar{P}(n)$ (subject the condition $q > 1$, say) by using

$$\bar{P}(n) = \frac{1}{n} \sum_{r|n} \bar{\Lambda}(r)\mu(n/r).$$

This is particularly true when we apply the results to prime elements in additive arithmetic semigroups.

The present paper is the first part of the results of this investigation. In this paper, we shall establish the P.E.T. in Theorems 6.1, 6.2, and 6.4. The key to the P.E.T. is to determine the number of zeros of the generating function $Z^\#(y)$ on the circle $|y| = q^{-1}$. We shall show in Theorem 4.1 that the "total number" of zeros of $Z^\#(y)$ on the circle is at most ρ_r in some sense. Thus, Theorems 6.2 and 4.1 are analogs of Beurling's Théorèmes VI and II' respectively [2]. Theorem 4.1 has a general formulation in Theorem 4.2. In this theorem we shall depart from the framework of additive arithmetic semigroups since the subject matter rather belongs to the theory of holomorphic functions. The periodicity in θ of the generating function $Z^\#(re^{i\theta})$, or, in a different way, the periodicity in t of the associated "zeta function" $\zeta(\sigma + it) :=$ $Z^\#(q^{-\sigma-it})$, shows a major divergence from the classical zeta functions and represents new difficulties. Thus, our proof of Theorem 4.2 depends on a representation of the solution set of a linear diophantine equation with real coefficients. As in the theory of Beurling generalized prime numbers, we shall show in Example 6.5 that in case $\rho_r > 1$ even the hypothesis (1.2) with zero remainder term does not generally entail the P.E.T. We remark that this example presents new arithmetic features in some sense too.

Since this paper may be regarded as a continuation of [11], we shall continue the same notations. In particular, the generating function $Z^\#(y)$ is defined in (1.4).

The constant γ in the remainder term of (1.3) determines the "degree" of smoothness of $Z^\#(y)$ on the circle $|y| = q^{-1}$. It is subjected to different conditions through this paper.

2. An equivalent form to the asymptotic expansion

The essential part of the asymptotic expansion (1.3) (or (1.2)) is the partial sum of the terms with ρ_v positive. In this section we shall introduce an equivalent form to

the essential part in Lemma 2.1. This form is easy to deal with by using generating function techniques. We shall work with it in most parts of this paper.

As usual, we define

$$\binom{x}{n} = \begin{cases} \dfrac{x(x-1)\cdots(x-n+1)}{n!}, & \text{if } n > 0; \\ 1, & \text{otherwise} \end{cases} \tag{2.1}$$

for real numbers x and nonnegative integers n.

LEMMA 2.1.   *Let $\rho_1, \ldots, \rho_r$ be positive numbers with $\rho_1 < \cdots < \rho_r$ and $A_1, \ldots, A_r$ be arbitrary real numbers.*

(1) If $\rho_1, \ldots, \rho_r$ are all integers then there exist positive integers $\tau_1 < \cdots < \tau_s$ with $\tau_s = \rho_r$ and $\tau_\mu = \rho_\nu - k$ for some nonnegative integer k such that

$$\sum_{\nu=1}^{r} A_\nu n^{\rho_\nu - 1} = \sum_{\mu=1}^{s} B_\mu \binom{n + \tau_\mu - 1}{n}, \tag{2.2$_1$}$$

where $B_s = A_r(\rho_r - 1)!$.

(2) If $\rho_1, \ldots, \rho_r$ are not all integers, then for any positive integer m there exist a positive integer $s = s(m)$, real numbers $\tau_1 < \cdots < \tau_s$ with $\tau_s = \rho_r$ and $\tau_\mu = \rho_\nu - k$ for some nonnegative integer $k (\le m + [\rho_\nu] - 2)$ such that, as $n \to \infty$,

$$\sum_{\nu=1}^{r} A_\nu n^{\rho_\nu - 1} = \sum_{\mu=1}^{s} B_\mu \binom{n + \tau_\mu - 1}{n} + O(n^{-m+\alpha}), \tag{2.2$_2$}$$

where $\alpha = \max\{\rho_\nu - [\rho_\nu], \nu = 1, \ldots, r\}$, $B_s = A_r\Gamma(\rho_r)$, and Γ is the Euler gamma function.

Equality (2.2)$_2$ must be known. However we are unable to locate where this occurs. As substitution, here we give a proof of it with the help of the next lemma.

LEMMA 2.2.   *Let ρ be a real number. There exist coefficients $a_k = a_k(\rho)$, $k = 0, 1, \ldots$ such that:*

(1) If $\rho = l$ is a nonnegative integer then

$$n^l = \sum_{k=0}^{l} a_k \binom{n+l-k}{n} = \sum_{k=0}^{l} a_k \binom{n+l-k}{l-k}, \quad n \ge 1, \tag{2.3}$$

where $a_0 = l!$ $(0! = 1)$ and a_k are all integers.

(2) If ρ is not an integer then, for any nonnegative integer m,

$$n^\rho = \sum_{k=0}^{m} a_k \binom{n+\rho-k}{n} + O(n^{\rho-m-1}) \quad \text{as} \quad n \to \infty, \tag{2.4}$$

where $a_0 = \Gamma(\rho + 1)$. The O-constant in (2.4) depends on ρ and m only.

Proof. The identity (2.3) is a specific form of a well-known finite-difference formula [3]. It is a representation of the integer-valued polynomial x^l in terms of integer-valued polynomials $\binom{x+k}{k}$ [5]. Dividing both sides of (2.3) by n^l and taking limits as $n \to \infty$, we obtain

$$1 = a_0 \frac{1}{l!}; \quad \text{i.e., } a_0 = l!.$$

To prove (2.4), it is sufficient to show that there exist coefficients $c_k = c_k(\rho)$ such that, for any nonnegative integer m,

$$\binom{n+\rho}{n} = \sum_{k=0}^{m} c_k n^{\rho-k} + O(n^{\rho-m-1}) \quad \text{as } n \to \infty \tag{2.5}$$

with $c_0 = 1/\Gamma(\rho+1)$. Then,

$$n^\rho = \Gamma(\rho+1) \binom{n+\rho}{n} - \sum_{k=0}^{m-1} c_{k+1} \Gamma(\rho+1) n^{\rho-1-k} + O(n^{\rho-m-1})$$

and (2.4) follows by induction on m.

We begin with the well-known formula

$$\Gamma(\rho) = \frac{1}{\rho} e^{-\gamma\rho} \prod_{k=1}^{\infty} \left(1 + \frac{\rho}{k}\right)^{-1} e^{\rho/k}.$$

It follows that

$$\binom{n+\rho}{n} = \frac{(1+\rho)(2+\rho)\cdots(n+\rho)}{n!}$$

$$= \frac{1}{\Gamma(\rho+1)} \exp\left\{\rho\left(\sum_{k=1}^{n} 1/k - \gamma\right)\right\} \prod_{k=n+1}^{\infty} \left(1 + \frac{\rho}{k}\right)^{-1} e^{\rho/k}. \tag{2.6}$$

We have [4]

$$\sum_{k=1}^{n} \frac{1}{k} - \gamma = \log n + \sum_{k=1}^{m} d_k n^{-k} + O(n^{-m-1})$$

and hence

$$\exp\left\{\rho\left(\sum_{k=1}^{n} 1/k - \gamma\right)\right\} = n^\rho \exp\left\{\sum_{k=1}^{m} d_k' n^{-k} + O_\rho(n^{-m-1})\right\}. \tag{2.7}$$

Also, we have

$$
\prod_{k=n+1}^{\infty} \left(1 + \frac{\rho}{k}\right)^{-1} e^{\rho/k} = \exp\left\{ \sum_{k=n+1}^{\infty} \left(-\log(1 + \frac{\rho}{k}) + \frac{\rho}{k}\right) \right\}
$$

$$
= \exp\left\{ \sum_{k=n+1}^{\infty} \sum_{t=2}^{\infty} \frac{(-1)^t}{t} \left(\frac{\rho}{k}\right)^t \right\}
$$

$$
= \exp\left\{ \sum_{t=2}^{m+1} \frac{(-1)^t}{t} \rho^t \sum_{k=n+1}^{\infty} \frac{1}{k^t} + O_{\rho,m}(n^{-m-1}) \right\}
$$

$$
= \exp\left\{ \sum_{k=1}^{m} d_k'' n^{-k} + O_{\rho,m}(n^{-m-1}) \right\} \tag{2.8}
$$

since, by the Euler-MacLaurin summation formula [10],

$$
\sum_{k=n+1}^{\infty} \frac{1}{k^t} = \sum_{s=0}^{m-t+1} c_{ts} n^{-t-s+1} + O(n^{-m-1}).
$$

From (2.6), (2.7), and (2.8), we arrive at

$$
\binom{n+\rho}{n} = \frac{1}{\Gamma(\rho+1)} n^\rho \left\{ 1 + \sum_{k=1}^{m} c_k' n^{-k} + O_{\rho,m}(n^{-m-1}) \right\}
$$

$$
= \sum_{k=0}^{m} c_k n^{\rho-k} + O(n^{\rho-m-1})
$$

with $c_0 = 1/\Gamma(\rho+1)$. $\square$

Proof of Lemma 2.1. If $\rho_1, \ldots, \rho_r$ are all positive integers, we simply set $\rho_\nu - 1$ in place of ρ in (2.3) and (2.2)$_1$ follows directly.

If ρ_ν is not an integer, we set $\rho_\nu - 1$ in place of ρ and replace m by $m + [\rho_\nu] - 2$ in (2.4). We note that (2.4) is obviously true when m is a negative integer. Thus we obtain

$$
n^{\rho_\nu - 1} = \sum_{k=0}^{m+[\rho_\nu]-2} a_k \binom{n + \rho_\nu - k - 1}{n} + O(n^{-[\rho_\nu]+\rho_\nu - m}). \tag{2.9}
$$

Then (2.2)$_2$ follows from (2.3) and (2.10). $\square$

3. The prime element theorem and zeros of the generating function

If $\bar{G}(n) \ll q^n n^\rho$ for some constant ρ, the power series $\sum_{n=0}^{\infty} \bar{G}(n) y^n$ converges and hence $Z^\#(y)$ is holomorphic in the disk $|y| < q^{-1}$.

THEOREM 3.1.   *Suppose that $\bar{G}(n) \ll q^n n^\rho$ for some constant ρ. Then $Z^\#(y)$ has no zeros in the disk $|y| < q^{-1}$.*

Proof.   Let $\Lambda^\#(y)$ be the generating function of $\bar{\Lambda}(n)$. From (1.5), we have $\bar{\Lambda}(n) \le \bar{\Lambda} * \bar{G}(n) = L\bar{G}(n) \ll q^n n^{\rho+1}$ since $\bar{\Lambda}(n)$ and $\bar{G}(n)$ are nonnegative. Hence $\Lambda^\#(y)$ is holomorphic in the disk $|y| < q^{-1}$. Also, from (1.5),

$$\frac{y \dfrac{d}{dy} Z^\#(y)}{Z^\#(y)} = \Lambda^\#(y). \tag{3.1}$$

Thus, $Z^\#(y)$ has no zeros in the disk.   □

Suppose that (1.3) holds with $\gamma > 1$. Then without loss of generality, we may assume $\rho_1 = 0$ in (1.3). From (1.3) and (2.2) with $m = 2$, we have

$$Z^\#(y) = \sum_{\mu=1}^{s} B_\mu \frac{1}{(1 - qy)^{\tau_\mu}} + A_1 \log(1 - qy) + R(y), \tag{3.2}$$

where $R(y) = \sum_{n=0}^{\infty} r_n q^n y^n$ with $r_n = O(n^{-\beta})$ and

$$\beta = \min\{\gamma, 2 - \tau_\mu + [\tau_\mu], \ \mu = 1, \ldots, s\} > 1.$$

Thus $Z^\#(y)$ has a continuous continuation on $\{y \in \mathbf{C}: |y| \le q^{-1}$ and $y \ne q^{-1}\}$. Although $Z^\#(y)$ has no zeros in the disk $|y| < q^{-1}$, it may have zeros on the circle $|y| = q^{-1}$. The connection of the P.E.T. with the zeros of $Z^\#(y)$ on the circle $|y| = q^{-1}$ is of interest and is given in the following Theorems 3.2 and 3.4, which are generalizations of Theorems 2.1 and 2.3 in [11] respectively.

THEOREM 3.2.   *Suppose that (1.3) holds with $\gamma > 1$. If $\bar{\Lambda}(n) \sim \rho_r q^n$ (P.E.T.) then the generating function $Z^\#(y)$ has no zeros on the circle $|y| = q^{-1}$.*

To prove Theorem 3.2, we need the following lemma.

LEMMA 3.3.   *Let $\tau > 0$ and*

$$(1 - qy)^\tau \log(1 - qy) = \sum_{n=1}^{\infty} a_n q^n y^n, \quad |y| < q^{-1}.$$

Then

$$a_n = O_\epsilon(n^{-\tau+\epsilon-1}) \tag{3.3}$$

for any $\epsilon > 0$.

Proof.   It is easy to see that

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{(1-z)^\tau \log(1-z)}{z^{n+1}} \, dz, \tag{3.4}$$

where $0 < r < 1$. For $n > 1 + \tau$, we can shift the integration contour in (3.4) to the one consisting of the upper and lower edges of the cut of the complex plane along the real axis from 1 to ∞. Then we have

$$|a_n| \ll \int_1^\infty \frac{(x-1)^\tau (|\log(x-1)|+1)}{x^{n+1}} \, dx \ll_\epsilon \int_1^\infty \frac{(x-1)^{\tau-\epsilon} + (x-1)^{\tau+\epsilon}}{x^{n+1}} \, dx,$$

since $|\log(x-1)| \ll_\epsilon (x-1)^{-\epsilon} + (x-1)^\epsilon$ for $1 < x < \infty$. Finally we have

$$\int_1^\infty \frac{(x-1)^{\tau-\epsilon}}{x^{n+1}} \, dx = B(\tau - \epsilon + 1, n - \tau + \epsilon)$$

$$= \frac{\Gamma(\tau - \epsilon + 1)\Gamma(n - \tau + \epsilon)}{\Gamma(n+1)}$$

$$= O_\epsilon(n^{-\tau+\epsilon-1}),$$

where B and Γ are Euler's beta and gamma functions.   □

Proof of Theorem 3.2.   Let D be the domain formed by cutting the complex plane along the real axis from q^{-1} to ∞. If τ_μ is not an integer, the function

$$\frac{1}{(1-qy)^{\tau_\mu}} = \exp\{\tau_\mu(\log|1-qy| + i \arg(1-qy))\}$$

assumes 1 at $y = 0$ and is the single-valued branch in D of the associated multiple-valued function. We consider the function

$$Z(y) := (1-qy)^\tau Z^\#(y)$$

$$= B_s + \sum_{\mu=1}^{s-1} B_\mu (1-qy)^{\tau-\tau_\mu} + (1-qy)^\tau (A_1 \log(1-qy) + R(y)), \tag{3.5}$$

where $\tau = \rho_r = \tau_s$, which has a continuous continuation to the circle $|y| = q^{-1}$. It is sufficient to show that $Z(y)$ has no zeros on $|y| = q^{-1}$.

On the one hand, the generating function $\Lambda^\#(y)$ of the arithmetic function $\bar{\Lambda}(n)$ satisfies

$$\Lambda^\#(y) = y \frac{\frac{d}{dy} Z^\#(y)}{Z^\#(y)} = \tau \frac{qy}{1-qy} + y \frac{Z'(y)}{Z(y)}, \qquad |y| < q^{-1}.$$

Hence,

$$y\frac{Z'(y)}{Z(y)} = \sum_{n=1}^{\infty}(\bar{\Lambda}(n) - \tau q^n)y^n.$$

As in the proof of Theorem 2.1 in [11], we then obtain

$$|Z(re^{i\theta})| \geq e^{-c}(1 - rq)^{\epsilon}, \tag{3.6}$$

where $c = c(\epsilon)$ is a constant.

On the other hand, we have

$$(1 - qy)^{\tau-\tau_\mu} = \sum_{n=0}^{\infty}\binom{\tau - \tau_\mu}{n}(-1)^n q^n y^n \tag{3.7}$$

and

$$(-1)^n\binom{\tau - \tau_\mu}{n} = \binom{n - \tau + \tau_\mu - 1}{n} \sim \frac{n^{-\tau+\tau_\mu-1}}{\Gamma(-\tau + \tau_\mu - 1)}$$

$$= O(n^{-\tau+\tau_{s-1}-1}) \tag{3.8}$$

as $n \to \infty$ if $\tau - \tau_\mu$ is not an integer. Also, we have

$$(1 - qy)^{\tau}R(y) = \left(\sum_{n=0}^{\infty}\binom{\tau}{n}(-1)^n q^n y^n\right)\left(\sum_{n=0}^{\infty}r_n q^n y^n\right) = \sum_{n=0}^{\infty}a_n q^n y^n, \tag{3.9}$$

where

$$a_n = \sum_{k=0}^{n}(-1)^k\binom{\tau}{k}r_{n-k}$$

$$\ll n^{-\beta}\sum_{k=1}^{\infty}\frac{k^{-\tau-1}}{|\Gamma(-\tau - 1)|} + n^{-\tau-1}\sum_{k=[n/2]+1}^{n}|r_{n-k}| \ll n^{-\beta} + n^{-\tau-1}, \tag{3.10}$$

since

$$(-1)^n\binom{\tau}{n} \sim \frac{n^{-\tau-1}}{\Gamma(-\tau - 1)}$$

as $n \to \infty$ if τ is not an integer. It follows, from (3.5), (3.7), (3.8), (3.3), (3.9), and (3.10), that

$$Z(y) = \sum_{n=0}^{\infty}b_n q^n y^n$$

with $b_n = O(n^{-\delta})$ and $\delta = \min\{\beta, 1 + \tau - \tau_{s-1}\} > 1$. Hence,

$$|Z(r_1 e^{i\theta}) - Z(re^{i\theta})| \ll \sum_{n=1}^{\infty} \frac{1}{n^{\delta}}((qr_1)^n - (qr)^n) \ll (qr_1 - qr)^{\delta-1} \qquad (3.11)$$

for $0 \leq r < r_1 \leq q^{-1}$, by Lemma 2.2 in [11].

Now suppose Theorem 3.2 is false and $Z(q^{-1}e^{i\theta}) = 0$. Then, letting $r_1 = q^{-1}$ in (3.11), we would obtain

$$|Z(re^{i\theta})| \ll (1 - qr)^{\delta-1}.$$

Taking $\epsilon = (\delta - 1)/2$ in (3.6), we would have

$$e^{-c}(1 - qr)^{(\delta-1)/2} \leq K(1 - qr)^{\delta-1},$$

or

$$e^{-c}/K \leq (1 - qr)^{(\delta-1)/2};$$

this is certainly absurd for r sufficiently close to q^{-1}.   □

Conversely, we have the following result which is a "conditional" prime element theorem and an inverse of Theorem 3.2 in some sense.

THEOREM 3.4. *Suppose that*

$$\sum_{n=1}^{\infty} n^2 \left| \bar{G}(n)q^{-n} - \sum_{\nu=1}^{r} A_{\nu} n^{\rho_{\nu}-1} \right|^2 < \infty, \qquad (3.12)$$

where $\rho_1 < \cdots < \rho_r$ and $A_1, \ldots, A_r$ are arbitrary real numbers, $\rho_r = \tau > 0$, $A_r > 0$. If $(1 - qy)^{\tau} Z^{\#}(y)$ is continuous on the closed disk $|y| \leq q^{-1}$ and has no zero on the circle $|y| = q^{-1}$ then

$$\bar{\Lambda}(n) \sim \rho_r q^n \qquad (P.E.T.).$$

Proof. Without loss of generality, we may assume $\rho_1 \geq -\frac{1}{2}$ in (3.12). From (3.12) and (2.2) with $m = 3$, we have

$$Z(y) := (1 - qy)^{\tau} Z^{\#}(y)$$

$$= B_s + \sum_{\mu=1}^{s-1} B_{\mu}(1 - qy)^{\tau - \tau_{\mu}} + (1 - qy)^{\tau}$$

$$\times \left[A \log(1 - qy) + \sum_{-\frac{1}{2} \leq \rho_{\nu} < 0} A_{\nu} \sum_{n=1}^{\infty} n^{\rho_{\nu}-1} q^n y^n + R(y) \right],$$

where $A = A_\nu$ in case $\rho_\nu = 0$ and $A = 0$ otherwise, and where

$$R(y) = \sum_{n=1}^{\infty} \left(\bar{G}(n) - q^n \sum_{\nu=1}^{r} A_\nu n^{\rho_\nu - 1} \right) y^n + \sum_{n=0}^{\infty} r_n q^n y^n$$

with $r_n = O(n^{-\beta})$ and $\beta = \min\{3 - \tau_\mu + [\tau_\mu], \ \mu = 1, \dots, s\} > 2$ if ρ_ν are not all integers and $r_n = 0$ otherwise. Then, as in the proof of Theorem 3.2,

$$\Lambda^{\#}(y) = \tau \frac{qy}{1 - qy} + y \frac{Z'(y)}{Z(y)} = \sum_{n=1}^{\infty} \bar{\Lambda}(n) y^n, \qquad |y| < q^{-1}.$$

It suffices to show that

$$\int_{|y|=r} \frac{Z'(y)}{Z(y)} y^{-n} \, dy = o(q^n) \tag{3.13}$$

as $n \to \infty$, where $0 < r < q^{-1}$.

The function $Z(y)$ is continuous on the disk $|y| \leq q^{-1}$ and has no zeros there. For $|y| < q^{-1}$, we have

$$Z'(y) = -\sum_{\mu=1}^{s-1} B_\mu (\tau - \tau_\mu) q (1 - qy)^{\tau - \tau_\mu - 1}$$

$$-q(1 - qy)^{\tau - 1}(A\tau \log(1 - qy) + H(y)) + (1 - qy)^\tau R'(y), \tag{3.14}$$

where

$$H(y) := \tau \left[\sum_{-\frac{1}{2} \leq \rho_\nu < 0} A_\nu \sum_{n=1}^{\infty} n^{\rho_\nu - 1} q^n y^n + R(y) \right] + A$$

$$- \sum_{-\frac{1}{2} \leq \rho_\nu < 0} A_\nu \left[1 + \sum_{n=1}^{\infty} ((n+1)^{\rho_\nu} - n^{\rho_\nu}) q^n y^n \right].$$

To show (3.13), we first note that for $0 < \tau - \tau_\mu < 1$, in a small neighborhood of $\theta = 0$,

$$|(1 - e^{i\theta})^{\tau - \tau_\mu - 1}| \sim |\theta|^{\tau - \tau_\mu - 1}$$

as $\theta \to 0$. Hence, $\int_{-\pi}^{\pi} (1 - e^{i\theta})^{\tau - \tau_\mu - 1} \, d\theta$ converges absolutely. We consider a contour $C_{\delta, \epsilon}$ consisting of the part of the circle C_δ: $|y| = \delta(< q^{-1})$ which is outside the small circle c_ϵ: $|y - q^{-1}| = \epsilon$, and the part of the circle c_ϵ which is inside the circle C_δ. Then

$$\int_{|y|=r} \frac{(1 - qy)^{\tau - \tau_\mu - 1}}{Z(y)} y^{-n} \, dy = \int_{C_{\delta, \epsilon}} \frac{(1 - qy)^{\tau - \tau_\mu - 1}}{Z(y)} y^{-n} \, dy.$$

Upon letting $\delta \to q^{-1}-$, we obtain

$$\int_{|y|=r} \frac{(1-qy)^{\tau-\tau_\mu-1}}{Z(y)} y^{-n}\, dy$$

$$= \left(\int_{-\pi}^{-\alpha(\epsilon)} + \int_{\alpha(\epsilon)}^{\pi}\right) \frac{(1-e^{i\theta})^{\tau-\tau_\mu-1}}{Z(q^{-1}e^{i\theta})} q^{n-1} e^{-i(n-1)\theta} i\, d\theta$$

$$+ \int_{c_\epsilon'} \frac{(1-qy)^{\tau-\tau_\mu-1}}{Z(y)} y^{-n}\, dy,$$

where $\alpha(\epsilon)$ and $-\alpha(\epsilon)$ are the arguments of intersection points above and below the x-axis of c_ϵ with the circle $|y| = q^{-1}$ respectively, and where c_ϵ' is the part of c_ϵ inside the circle $|y| = q^{-1}$. The modulus of the integrand of the last integral is

$$\ll_n |q^{-1} - y|^{\tau-\tau_\mu-1} = \epsilon^{\tau-\tau_\mu-1}.$$

Since $\tau - \tau_\mu > 0$, the last integral tends to zero as $\epsilon \to 0$. Letting $\epsilon \to 0$, we conclude that

$$\int_{|y|=r} \frac{(1-qy)^{\tau-\tau_\mu-1}}{Z(y)} y^{-n}\, dy = iq^{n-1} \int_{-\pi}^{\pi} \frac{(1-e^{i\theta})^{\tau-\tau_\mu-1}}{Z(q^{-1}e^{i\theta})} e^{-i(n-1)\theta}\, d\theta. \quad (3.15)$$

Plainly, (3.15) holds for $1 \le \tau - \tau_\mu$.

We then note that $R(y)$ has a continuous continuation on the closed disk $|y| \le q^{-1}$. Actually,

$$|R(re^{i\theta}) - R(\rho e^{i\theta})| \le \sum_{n=1}^{\infty} \left| \bar{G}(n) - q^n \sum_{\nu=1}^{r} A_\nu n^{\rho_\nu-1} \right| |r^n - \rho^n|$$

$$+ \sum_{n=0}^{\infty} |r_n| q^n |r^n - \rho^n|$$

$$\le \left\{ \sum_{n=1}^{\infty} n^2 \left| \bar{G}(n) q^{-n} - \sum_{\nu=1}^{r} A_\nu n^{\rho_\nu-1} \right|^2 ((qr)^n - (q\rho)^n)^2 \right\}$$

$$\times \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=0}^{\infty} |r_n| |(qr)^n - (q\rho)^n|$$

by the Cauchy–Schwarz inequality and hence $|R(re^{i\theta}) - R(\rho e^{i\theta})| \to 0$ uniformly as $r, \rho \to q^{-1}-$. This implies the existence of the continuous continuation on the circle $|y| = q^{-1}$. Thus $H(y)$ has a continuous continuation on the same disk too. An

argument similar to the one deducing (3.15) shows that

$$\int_{|y|=r} \frac{(1-qy)^{\tau-1}(A\tau \log(1-qy) + H(y))}{Z(y)} y^{-n} \, dy$$

$$= iq^{n-1} \int_{-\pi}^{\pi} \frac{(1-e^{i\theta})^{\tau-1}(A\tau \log(1-e^{i\theta}) + H(q^{-1}e^{i\theta}))}{Z(q^{-1}e^{i\theta})} e^{-i(n-1)\theta} \, d\theta. \quad (3.16)$$

We finally note that

$$\int_{-\pi}^{\pi} |R'(re^{i\theta}) - R'(\rho e^{i\theta})|^2 \, d\theta = \sum_{n=1}^{\infty} n^2 r_n^2 q^{2n}(r^{n-1} - \rho^{n-1})^2 \to 0$$

as $r, \rho \to q^{-1}-$. There exists a function $F(\theta) \in L_2[-\pi, \pi]$ such that $R'(re^{i\theta}) \to F(\theta)$ in $L_2[-\pi, \pi]$ as $r \to q^{-1}$. Thus,

$$\frac{(1-qre^{i\theta})^{\tau} R'(re^{i\theta})}{Z(re^{i\theta})} \to \frac{(1-e^{i\theta})^{\tau} F(\theta)}{Z(q^{-1}e^{i\theta})} \quad (3.17)$$

in $L_2[-\pi, \pi]$ as $r \to q^{-1}-$.

Now, from (3.14) to (3.17), we obtain

$$\int_{|y|=r} \frac{Z'(y)}{Z(y)} y^{-n} \, dy$$

$$= iq^{n-1} \int_{-\pi}^{\pi} \frac{1}{Z(q^{-1}e^{i\theta})} \left\{ -\sum_{\mu=1}^{s-1} B_\mu(\tau - \tau_\mu)q(1 - e^{i\theta})^{\tau-\tau_\mu-1} - q(1-e^{i\theta})^{\tau-1} \right.$$

$$\left. \times (A\tau \log(1-e^{i\theta}) + H(q^{-1}e^{i\theta})) + (1-e^{i\theta})^{\tau} F(\theta) \right\} e^{-i(n-1)\theta} \, d\theta$$

$$= o(q^n),$$

for the last integral tends to zero as $n \to \infty$ by the Riemann-Lebesgue lemma. $\quad \square$

COROLLARY 3.5. *Suppose that (1.3) holds with $\gamma > \frac{3}{2}$. If $Z^{\#}(y)$ has no zeros on the circle $|y| = q^{-1}$ then*

$$\bar{\Lambda}(n) \sim \rho_r q^n \quad (P.E.T.).$$

4. The total number of zeros of the generating function

The generating function $Z^{\#}(y)$ has no zeros in the disk $|y| < q^{-1}$ but may have zeros on the circle $|y| = q^{-1}$. If $q^{-1}e^{2\pi i\theta}$ is a zero of $Z^{\#}(y)$, the number

$$\alpha(\theta) := \sup \left\{ \alpha : \limsup_{r \to q^{-1}-} (q^{-1} - r)^{-\alpha} |Z^{\#}(re^{2\pi i\theta})| < \infty \right\}, \quad (4.1)_1$$

or equivalently, following Beurling [2],

$$\alpha(\theta) := \liminf_{r \to q^{-1}-} \frac{\log |Z^{\#}(re^{2\pi i\theta})|}{\log(q^{-1} - r)}, \tag{4.2)$_2$}$$

is called, by definition, the order of $q^{-1}e^{2\pi i\theta}$.

In this section, we shall prove the following theorem which gives the "total number" of zeros of $Z^{\#}(y)$. This theorem is an analog of Beurling's Théorème II' [2]; the latter is a generalization of a theorem of Hadamard.

THEOREM 4.1. *Suppose that (1.3) holds with* $\gamma > 1$. *The "total number" of zeros of* $Z^{\#}(y)$ *on the circle* $|y| = q^{-1}$ *is at most* $\tau = \rho_r$ *in the sense that*

$$\alpha\left(\frac{1}{2}\right) + 2 \sum_{0 < \theta < \frac{1}{2}} \alpha(\theta) \leq \tau \tag{4.2)$_1$}$$

or

$$2 \sum_{0 < \theta < \frac{1}{2}} \alpha(\theta) \leq \tau \tag{4.2)$_2$}$$

according as $-q^{-1}$ *is or is not a zero of* $Z^{\#}(y)$, *where the summation is taken over all zeros of* $Z^{\#}(y)$ *on the upper half of the circle* $|y| = q^{-1}$.

Remark. The upper bound of the total number of zeros given in (4.2) is best possible as Example 4.1 in [11] and Examples 6.5 and 6.7 of this paper show. We note that $\alpha(\frac{1}{2})$ is the order of zero $-q^{-1}$.

We shall first prove the following general formulation of Theorem 4.1 and then Theorem 4.1 follows directly.

THEOREM 4.2. *Let* $f(z)$ *be a function continuous on* $\{z \in \mathbf{C}: |z| \leq 1 \ and \ z \neq 1\}$ *and holomorphic in the disk* $|z| < 1$. *Suppose that* $f(z)$ *has no zeros in the disk and*

$$\log f(z) = \sum_{k=1}^{\infty} c_k z^k, \qquad |z| < 1 \tag{4.3}$$

with coefficients $c_k \geq 0$ *and that, for some constant* $\tau > 0$,

$$\lim_{r \to 1-} f(r)(1 - r)^{\tau} \tag{4.4}$$

exists and is positive. Let $0 < \theta_1 < \cdots < \theta_k < 1$ *be arbitrary. Then*

$$\sum_{j=1}^{k} \liminf_{r \to 1-} \frac{\log |f(re^{2\pi i\theta_j})|}{\log(1 - r)} \leq \tau. \tag{4.5}$$

The following proof of Theorem 4.2 is a simplification of the author's original proof of Theorem 4.1 due to suggestions of Warlimont.

We begin with the representation of the solution set of the diophantine equation $\alpha_1\theta_1 + \cdots + \alpha_k\theta_k = m$.

LEMMA 4.3.   *Given $\theta = (\theta_1, \ldots, \theta_k) \in \mathbf{R}^k$ with $0 < \theta_1 < \cdots < \theta_k < 1$ arbitrary, let*

$$S = S(\theta) := \{\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbf{Z}^k \colon \langle \alpha, \theta \rangle \in \mathbf{Z}\} \tag{4.6}$$

where $\langle \alpha, \theta \rangle = \alpha_1\theta_1 + \cdots + \alpha_k\theta_k$. If $S \neq \{0\}$ then there exist a positive integer $m \leq k$ (m is the "dimension" of S) and a matrix $C = C(\theta) \in \mathbf{M}(m \times k, \mathbf{Q})$ of rank m such that

$$S = \{\beta C \colon \beta \in \mathbf{Z}^m\}.$$

Proof.   We consider two possible cases separately.

Case I. The equation $\alpha_1\theta_1 + \cdots + \alpha_k\theta_k = z$ has no solutions in $\mathbf{Z}^k$ for all $z \in \mathbf{Z}$, $z \neq 0$. Then the homogeneous equation $\alpha_1\theta_1 + \cdots + \alpha_k\theta_k = 0$ has non-zero solutions. We consider a maximal subset of elements linearly independent over $\mathbf{Q}$ of the set $\{\theta_1, \ldots, \theta_k\}$. Upon changing the subscripts, we may assume $\{\theta_1, \ldots, \theta_l\}$ is such a subset. Then $l < k$. There exist $a_{s,t} \in \mathbf{Q}$, $t = 1, \ldots, l$, $s = l+1, \ldots, k$ such that

$$\theta_{l+1} = a_{l+1,1}\theta_1 + \cdots + a_{l+1,l}\theta_l,$$

$$\vdots \tag{4.7}$$

$$\theta_k = a_{k,1}\theta_1 + \cdots + a_{k,l}\theta_l.$$

If $\alpha \in S$, then

$$0 = \alpha_1\theta_1 + \cdots + \alpha_k\theta_k$$

$$= (\alpha_1 + \alpha_{l+1}a_{l+1,1} + \cdots + \alpha_k a_{k,1})\theta_1 + \cdots + (\alpha_l + \alpha_{l+1}a_{l+1,l} + \cdots + \alpha_k a_{k,l})\theta_l.$$

It follows that

$$\alpha_1 + \alpha_{l+1}a_{l+1,1} + \cdots + \alpha_k a_{k,1} = 0,$$

$$\vdots$$

$$\alpha_l + \alpha_{l+1}a_{l+1,l} + \cdots + \alpha_k a_{k,l} = 0,$$

and hence

$$\alpha_1 = -\alpha_{l+1}a_{l+1,1} - \cdots - \alpha_k a_{k,1},$$

$$\vdots$$

$$\alpha_l = -\alpha_{l+1}a_{l+1,l} - \cdots - \alpha_k a_{k,l}.$$

Let $m = k - l$,

$$C = \begin{pmatrix} -a_{l+1,1} & \cdots & -a_{l+1,l} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_{k,1} & \cdots & -a_{k,l} & 0 & \cdots & 1 \end{pmatrix}$$

and then $S = \{\beta C \colon \beta \in \mathbf{Z}^m\}$.

Case II. The equation $\alpha_1\theta_1 + \cdots + \alpha_k\theta_k = z$ has solutions in $\mathbf{Z}^k$ for some $z \in \mathbf{Z}$, $z \neq 0$. Then the set $M = \{z \colon 0 < z = \langle \alpha, \theta \rangle$ for some $\alpha \in S\}$ is non-empty. Let $z_0 = \min\{z \colon z \in M\}$ and $\alpha^0 = (\alpha_1^0, \ldots, \alpha_k^0) \in S$ such that $\alpha_1^0\theta_1 + \cdots + \alpha_k^0\theta_k = z_0$. We claim that, for each $\alpha \in \mathbf{Z}^k$, $\alpha \in S$ if and only if there exist $t \in \mathbf{Z}$ and $\eta = (\eta_1, \ldots, \eta_k) \in \mathbf{Z}^k$ such that $\alpha = t\alpha^0 + \eta$ and

$$\eta_1\theta_1 + \cdots + \eta_k\theta_k = 0$$

and that if $\alpha \in S$ the representation $\alpha = t\alpha^0 + \eta$ is unique. Actually, if $\alpha_1\theta_1 + \cdots + \alpha_k\theta_k = 0$ then $\alpha = 0\alpha^0 + \eta$ with $\eta = \alpha$. If $\alpha_1\theta_1 + \cdots + \alpha_k\theta_k = z \in \mathbf{Z}$ and $z \neq 0$, then $z = tz_0$ for some $t \in \mathbf{Z}$. Otherwise, $z = tz_0 + r$ with $t, r \in \mathbf{Z}$ and $0 < r < z_0$. Then $\alpha' = \alpha - t\alpha^0 \in \mathbf{Z}^k$ and

$$\langle \alpha', \theta \rangle = z - tz_0 = r.$$

This contradicts the definition of z_0. Thus $z = tz_0$ and $\eta = \alpha - t\alpha^0$ satisfies $\langle \eta, \theta \rangle = 0$. Plainly, t and η are unique and the claims hold.

Now, if the homogeneous equation $\alpha_1\theta_1 + \cdots + \alpha_k\theta_k = 0$ has only the solution $(0, \ldots, 0) \in \mathbf{Z}^k$ then $\eta = 0$ and $\alpha = t\alpha^0$. Let $m = 1, C = \alpha^0$ and then $S = \{\beta C \colon \beta \in \mathbf{Z}\}$.

Thus we may assume that the homogeneous equation $\alpha_1\theta_1 + \cdots + \alpha_k\theta_k = 0$ has non-zero solutions in $\mathbf{Z}^k$. Then, as in *Case I*, we may assume that $\{\theta_1, \ldots, \theta_l\}$ be a maximal subset of elements linearly independent over $\mathbf{Q}$ of the set $\{\theta_1, \ldots, \theta_k\}$ and that there exist $a_{s,t}, t = 1, \ldots, l, s = l + 1, \ldots, k$ such that (4.7) hold. Let $m = k - l + 1$ and

$$C = \begin{pmatrix} \alpha_1^0 & \cdots & \alpha_{k-l}^0 & \alpha_{k-l+1}^0 & \cdots & \alpha_k^0 \\ -a_{l+1,1} & \cdots & -a_{l+1,l} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -a_{k,1} & \cdots & -a_{k,l} & 0 & \cdots & 1 \end{pmatrix}$$

and then $S = \{\beta C : \beta \in \mathbf{Z}^m\}$. Finally, it is easy to show that the rank of C is $m = k - l + 1$. $\square$

We now turn to the proof of Theorem 4.2.

For $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$, $y = (y_1, \ldots, y_k) \in \mathbf{R}^k$, we set $\|x\| = \max_{1 \le j \le k} |x_j|$ and $\langle x, y \rangle = x_1 y_1 + \cdots + x_k y_k$. Let K be a positive integer. We consider the inequality

$$0 \le \left| \sum_{\substack{l \in \mathbf{N}^k \\ \|l\| \le K}} e^{i \langle l, x \rangle} \right|^2 = \sum_{\alpha \in \mathbf{Z}^k} n(\alpha, K) \cos \langle \alpha, x \rangle \tag{4.8}$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$ and

$$n(\alpha, K) := \sum_{\substack{l, l' \in \mathbf{Z}^k \\ \|l\|, \|l'\| \le K \\ l - l' = \alpha}} 1 = \begin{cases} 0, & \text{if } \|\alpha\| \ge K, \\ \prod_{j=1}^{k} (K - |\alpha_j|), & \text{if } \|\alpha\| < K. \end{cases} \tag{4.9}$$

When $k = 1$, this inequality reduces to the one in [1],

$$0 \le K + \sum_{l=1}^{K-1} 2(K - l) \cos(lx).$$

Using the inequality (4.8), we can give a more direct proof of the theorem of Beurling mentioned before. Here we shall utilize (4.8) to prove Theorem 4.2.

Proof of Theorem 4.2.   Let $x = 2\pi n (\theta_1, \ldots, \theta_k)$ with $n \in \mathbf{N}$ in (4.8). We multiply both sides of (4.8) by $c_n r^n$ and sum up over n. Then we obtain, from (4.3), for $0 < r < 1$,

$$0 \le \sum_{\alpha \in \mathbf{Z}^k} n(\alpha, K) \log |f(re^{2\pi i \langle \alpha, \theta \rangle})|$$

$$= \left(\sum_{\substack{\alpha \in \mathbf{Z}^k \\ \langle \alpha, \theta \rangle \in \mathbf{Z}}} n(\alpha, K) \right) \log f(r) + \sum_{\substack{\alpha \in \mathbf{Z}^k \\ \langle \alpha, \theta \rangle \notin \mathbf{Z}}} n(\alpha, K) \log |f(re^{2\pi i \langle \alpha, \theta \rangle})|. \tag{4.10}$$

It follows that

$$\sum_{\substack{\alpha \in \mathbf{Z}^k \\ \langle \alpha, \theta \rangle \notin \mathbf{Z}}} n(\alpha, K) \frac{\log |f(re^{2\pi i \langle \alpha, \theta \rangle})|}{\log(1 - r)} \le \left(\sum_{\substack{\alpha \in \mathbf{Z}^k \\ \langle \alpha, \theta \rangle \in \mathbf{Z}}} n(\alpha, K) \right) \frac{\log f(r)}{\log \frac{1}{1-r}}. \tag{4.11}$$

Let

$$N(\theta, K) := \sum_{\substack{\alpha \in \mathbf{Z}^k \\ \langle \alpha, \theta \rangle \in \mathbf{Z}}} n(\alpha, K) = \sum_{\alpha \in S} n(\alpha, K),$$

where S is defined in (4.6), and

$$N_j(\theta, K) := \sum_{\substack{\alpha \in \mathbf{Z}^k \\ \langle \alpha, \theta \rangle - \theta_j \in \mathbf{Z}}} n(\alpha, K) = \sum_{\alpha \in S_j} n(\alpha, K), \quad j = 1, \ldots, k \qquad (4.12)$$

where

$$S_j = S_j(\theta) := \{\alpha \in \mathbf{Z}^k \colon \langle \alpha, \theta \rangle - \theta_j \in \mathbf{Z}\}, \quad j = 1, \ldots, k.$$

We note that $S, S_j, j = 1, \ldots, k$ are mutually disjoint and that

$$S_j = \{\alpha + e_j \colon \alpha \in S\}, \quad j = 1, \ldots, k,$$

where e_j is the *jth* vector of the standard basis of $\mathbf{R}^k$. From (4.11), we have

$$\sum_{j=1}^{k} \frac{N_j(\theta, K)}{N(\theta, K)} \frac{\log |f(re^{2\pi i\theta_j})|}{\log(1-r)} + \sum_{\substack{\alpha \in \mathbf{Z}^k \\ \alpha \notin S \cup (\cup_{j=1}^k S_j)}} \frac{n(\alpha, K)}{N(\theta, K)} \frac{\log |f(re^{2\pi i\langle \alpha, \theta \rangle})|}{\log(1-r)} \leq \frac{\log f(r)}{\log \frac{1}{1-r}}.$$

$$(4.13)$$

We note that, from (4.4),

$$\lim_{r \to 1-} \frac{\log f(r)}{\log \frac{1}{1-r}} = \tau$$

and that

$$\liminf_{r \to 1-} \frac{\log |f(re^{2\pi i\langle \alpha, \theta \rangle})|}{\log(1-r)} \geq 0$$

for $\langle \alpha, \theta \rangle \notin \mathbf{Z}$ since $f(z)$ is continuous on $\{z \in \mathbf{C} \colon |z| \leq 1, z \neq 1\}$. It follows, from (4.13), that

$$\sum_{j=1}^{k} \frac{N_j(\theta, K)}{N(\theta, K)} \liminf_{r \to 1-} \frac{\log |f(re^{2\pi i\theta_j})|}{\log(1-r)} \leq \tau. \qquad (4.14)$$

We claim that

$$\frac{N_j(\theta, K)}{N(\theta, K)} = 1 + O(K^{-1}), \quad j = 1, \ldots, k. \qquad (4.15)$$

Actually, if $S = \{0\}$, from (4.9) we have

$$N(\theta, K) = n(0, K) = K^k$$

and

$$N_j(\theta, K) = n(e_j, K) = K^{k-1}(K - 1).$$

If $S \neq \{0\}$, by Lemma 4.3, there exists a matrix $C = C(\theta) \in M(m \times k, \mathbf{Q})$ of rank $m > 0$ such that $S = \{\beta C: \beta \in \mathbf{Z}^m\}$. We note that

$$\sum_{\substack{\beta \in \mathbf{Z}^m \\ \|\beta C\| \leq K}} 1 \gg_\theta K^m$$

since $\|\beta C\| \leq \|\beta\| \sum_{i=1}^m \sum_{j=1}^k |c_{ij}|$. Hence

$$N(\theta, K) = \sum_{\substack{\beta \in \mathbf{Z}^m \\ \|\beta C\| \leq K}} \prod_{j=1}^k (K - |\beta C_j|) \gg_\theta K^{m+k}$$

where C_j is the *jth* column of C. Also,

$$N_j(\theta, K) = \sum_{\substack{\beta \in \mathbf{Z}^m \\ \|\beta C + e_j\| \leq K}} \left(\prod_{\substack{i=1 \\ i \neq j}}^k (K - |\beta C_j|) \right) (k - |\beta C_j + 1|)$$

$$= N(\theta, K) + O(K^{m+k-1}), \qquad j = 1, \ldots, k.$$

Thus the claim holds.

Now, (4.5) follows from (4.14) and (4.15) by letting $K \to \infty$ on the left-hand side of (4.14).   □

Proof of Theorem 4.1.   We consider $f(z) := Z^\#(q^{-1}z)$. Then, from (3.5),

$$f(z) = \frac{Z(q^{-1}z)}{(1-z)^\tau},$$

where $Z(q^{-1}z)$ is continuous on the disk $|z| \leq 1$ and holomorphic in the disk $|z| < 1$ and $Z(q^{-1}) = B_s = A_r\Gamma(\tau) > 0$. It follows that

$$\lim_{r \to 1-} f(r)(1-r)^\tau = Z(q^{-1}) > 0.$$

Also, it is easy to see that, from (3.1),

$$\log f(z) = \log Z^\#(q^{-1}z) = \sum_{k=1}^\infty \frac{\bar{\Lambda}(k)}{k} q^{-k} z^k, \qquad |z| < 1$$

and (4.3) holds with $c_k = q^{-k}\bar{\Lambda}(k)k^{-1} \geq 0$. Thus (4.2) follows from Theorem 4.2.
   □

5. The orders of zeros of the generating function

If $\gamma > 1 + \tau$ then the orders of zeros of the generating function $Z^{\#}(y)$ are positive integers as the following theorem shows. This theorem is an analog of a result of Beurling [2] and its proof follows the general idea of Beurling too. It is essentially best possible as Example 5.4 will show.

THEOREM 5.1. *Let $q^{-1}e^{2\pi i\theta}$ be a zero of $Z^{\#}(y)$ with order $\alpha = \alpha(\theta)$ where $0 < \theta < 1$.*
(1) *If (1.3) holds with $\gamma > 1$ then $\alpha \geq \min\{1, \gamma - 1\}$.*
(2) *If (1.3) holds with $\gamma > 1 + \alpha$, in particular, if $\theta \neq \frac{1}{2}$ and (1.3) holds with $\gamma > 1 + \frac{\tau}{2}$ or if $\theta = \frac{1}{2}$ and (1.3) holds with $\gamma > 1 + \tau$ then α is a positive integer. Moreover,*

$$\lim_{r \to q^{-1}-} \frac{Z^{\#}(re^{2\pi i\theta})}{(q^{-1} - r)^{\alpha}e^{2\pi i\alpha\theta}} = \frac{(-1)^{\alpha}}{\alpha!} Z^{\#(\alpha)}(q^{-1}e^{2\pi i\theta}) \neq 0$$

and $Z^{\#}(y)/(q^{-1}e^{2\pi i\theta} - y)^{\alpha}$ is continuous on $\{y: |y| \leq q^{-1}, |y - q^{-1}e^{2\pi i\theta}| < \epsilon\}$ for some $\epsilon > 0$.

As in Section 4, we shall prove a general formulation of Theorem 5.1 in the following theorem and then Theorem 5.1 follows directly.

THEOREM 5.2. *Let $f(z) = S(z) + R(z)$ where $S(z)$ is holomorphic in the disk $|z| < 1$ and*

$$R(z) = \sum_{n=0}^{\infty} r_n z^n, \qquad |z| < 1$$

with $r_n = O(n^{-\gamma})$.
(1) *Suppose that $S(z)$ is also holomorphic in a neighborhood of $e^{2\pi i\theta}$ and $e^{2\pi i\theta}$ is a zero of $f(z)$. If $\gamma > 1$ then*

$$\alpha = \alpha(\theta) := \sup\{\beta: (1 - r)^{-\beta}|f(re^{2\pi i\theta})| \ll 1\}$$

satisfies $\alpha \geq \min\{1, \gamma - 1\}$.
(2) *Furthermore, suppose that $S(z)$ is continuous on $\{z: |z| \leq 1 \quad and\ z \neq 1\}$ and for some constant $\tau > 0$,*

$$\lim_{r \to 1-} S(r)(1 - r)^{\tau}$$

exists and is positive, and that $f(z) \neq 0$ in the disk $|z| < 1$ and

$$\log f(z) = \sum_{k=1}^{\infty} c_k z^k, \qquad |z| < 1$$

with coefficients $c_k \geq 0$. If $\gamma > 1 + \alpha$, in particular, if $0 < \theta < 1, \theta \neq \frac{1}{2}$ and $\gamma > 1 + \frac{\tau}{2}$ or if $\theta = \frac{1}{2}$ and $\gamma > 1 + \tau$, then α is a positive integer. Moreover,

$$\lim_{r \to 1-} \frac{f(re^{2\pi i\theta})}{(1-r)^\alpha e^{2\pi i\alpha\theta}} = \frac{(-1)^\alpha}{\alpha!} f^{(\alpha)}(e^{2\pi i\theta}) \neq 0 \tag{5.1}$$

and $f(z)/(e^{2\pi i\theta} - z)^\alpha$ is continuous on $\{z : |z| \leq 1, |z - e^{2\pi i\theta}| < \epsilon\}$ with some $\epsilon > 0$.

   Proof of Theorem 5.2.   To prove (1), we note that the function $R(z)$ is continuous on the disk $|z| \leq 1$. Hence

$$\begin{aligned}
|f(re^{2\pi i\theta})| &= |f(re^{2\pi i\theta}) - f(e^{2\pi i\theta})| \\
&= |(S(re^{2\pi i\theta}) - S(e^{2\pi i\theta})) + (R(re^{2\pi i\theta}) - R(e^{2\pi i\theta}))| \\
&\ll_\theta (1-r) + (1-r) \sum_{1 \leq n < M} n^{-\gamma+1} + \sum_{n \geq M} n^{-\gamma} \\
&\ll (1-r) + (1-r)M^{-\gamma+2} + M^{-\gamma+1}.
\end{aligned}$$

Let $M = (1-r)^{-1}$. Then we obtain

$$|f(re^{2\pi i\theta})| \ll (1-r) + (1-r)^{\gamma-1} \ll (1-r)^{\min\{1, \gamma-1\}}$$

and $\alpha \geq \min\{1, \gamma - 1\}$ follows.

   To prove (2), without loss of generality, we may assume that γ is not an integer. We have to show that α is an integer. Suppose on the contrary that α is not an integer. Then, from the definition of α,

$$\lim_{r \to 1-} \frac{|f(re^{2\pi i\theta})|}{(1-r)^k} = 0 \tag{5.2}$$

for all integers $k < \alpha$.

   We note that all derivatives $f^{(k)}(z), k < \gamma - 1$, are continuous on $\{z : |z| \leq 1, |z - e^{2\pi i\theta}| < \epsilon\}$ for some $\epsilon > 0$. Hence, for $1 \leq k < \gamma - 1$, we have the Taylor formula

$$f(re^{2\pi i\theta}) = \sum_{n=1}^{k-1} \frac{1}{n!} f^{(n)}(e^{2\pi i\theta}) e^{2\pi i n\theta} (r-1)^n$$

$$+ \frac{1}{(k-1)!} \int_1^r f^{(k)}(te^{2\pi i\theta}) e^{2\pi i k\theta} (r-t)^{k-1} dt. \tag{5.3}$$

Moreover, if $k = [\gamma] - 1$, then

$$\begin{aligned}
|f^{(k)}(te^{2\pi i\theta}) - f^{(k)}(e^{2\pi i\theta})| &= |(S^{(k)}(te^{2\pi i\theta}) - S^{(k)}(e^{2\pi i\theta})) + (R^{(k)}(te^{2\pi i\theta}) \\
&\quad - R^{(k)}(e^{2\pi i\theta}))| \\
&\ll_\theta (1-t) + (1-t)^{\gamma-[\gamma]} \ll (1-t)^{\gamma-[\gamma]},
\end{aligned}$$

since

$$|R^{(k)}(te^{2\pi i\theta}) - R^{(k)}(e^{2\pi i\theta})| = \left| \sum_{n=k}^{\infty} r_n n(n-1) \cdots (n-k+1) e^{2\pi i(n-k)\theta} (1-t^{n-k}) \right|$$

$$\ll (1-t) \sum_{k \leq n < M} n^{-\gamma+k+1} + \sum_{n \geq M} n^{-\gamma+k}$$

$$\ll (1-t) M^{-\gamma+k+2} + M^{-\gamma+k+1}$$

$$\ll (1-t)^{\gamma-k-1} = (1-t)^{\gamma-[\gamma]}$$

with $M = (1-t)^{-1}$. Hence, if $k = [\gamma] - 1$, the last term of (5.3) equals

$$\frac{1}{k!} f^{(k)}(e^{2\pi i\theta}) e^{2\pi ik\theta} (r-1)^k + O((1-r)^{\gamma-1})$$

and we obtain

$$f(re^{2\pi i\theta}) = \sum_{n=1}^{[\gamma]-1} \frac{1}{n!} f^{(n)}(e^{2\pi i\theta}) e^{2\pi in\theta} (r-1)^n + O((1-r)^{\gamma-1}). \tag{5.4}$$

Since $\alpha < \gamma - 1$, from (5.2) and (5.4), we obtain, by induction,

$$f^{(n)}(e^{2\pi i\theta}) = 0, \qquad \text{for} \qquad 1 \leq n \leq [\alpha]. \tag{5.5}$$

Now, from (5.4) and (5.5), if $[\alpha] < [\gamma] - 1$ then

$$\frac{|f(re^{2\pi i\theta})|}{(1-r)^{[\alpha]+1}} \ll 1$$

and if $[\alpha] = [\gamma] - 1$ then

$$\frac{|f(re^{2\pi i\theta})|}{(1-r)^{\gamma-1}} \ll 1.$$

This implies that $\alpha \geq \min\{[\alpha] + 1, \gamma - 1\}$ which contradicts $\alpha < \gamma - 1$. Therefore, α must be a positive integer.

Finally, since α is a positive integer and $\alpha < \gamma - 1$, (5.1) follows from (5.4). Moreover, for z in $\{z: |z| \leq 1, |z - e^{2\pi i\theta}| < \epsilon\}$ with some $\epsilon > 0$, we have the Taylor formula

$$f(z) = \sum_{n=\alpha}^{k-1} \frac{1}{n!} f^{(n)}(e^{2\pi i\theta})(z - e^{2\pi i\theta})^n$$

$$+ \frac{1}{(k-1)!} \int_0^1 f^{(k)}((1-t)e^{2\pi i\theta} + tz)(z - e^{2\pi i\theta})^k (1-t)^{k-1} dt$$

with $k = [\gamma] - 1$ and the continuity of $f(z)/(e^{2\pi i\theta} - z)^\alpha$ follows. $\quad\square$

Proof of Theorem 5.1.   Without loss of generality, we may assume that $\rho_1 > -\gamma + 1$ in (1.3). Let $f(z) := Z^{\#}(q^{-1}z)$. Then, from (1.3) and (2.2) with $m \geq \gamma + \max\{\rho_\nu - [\rho_\nu], \nu = 1, \ldots, r\}$, we can write

$$f(z) = S(z) + R(z),$$

where the function

$$S(z) = \sum_{\mu=1}^{s} B_\mu \frac{1}{(1-z)^{\tau_\mu}} + A\log(1-z) + \sum_{k=2}^{t} C_k \sum_{n=1}^{\infty} \frac{z^n}{n^k}, \quad |z| < 1$$

with $\tau_1 < \cdots < \tau_s, \tau = \tau_s > 0, A = A_\nu$ in case $\rho_\nu = 0, C_k = A_\nu$ in case $\rho_\nu = -k + 1$ and where the function

$$R(z) = \sum_{n=0}^{\infty} r_n z^n$$

with $r_n = O(n^{-\gamma})$. It is easy to show, by induction, that $\sum_{n=1}^{\infty} z^n n^{-k}$ is holomorphic in the domain D formed by cutting the complex plane along the real axis from 1 to ∞. Hence so is $S(z)$. Moreover, plainly,

$$\lim_{r \to 1-} S(r)(1-r)^{\tau} = B_s = A_r \Gamma(\tau) > 0.$$

Thus $f(z)$ satisfies the hypotheses of Theorem 5.2 and Theorem 5.1 follows directly. $\qquad\square$

Theorem 5.1 or Theorem 5.2 has the following immediate consequence.

COROLLARY 5.3.   *If* $0 < \tau < 1$ *and* $\gamma > 1 + \frac{\tau}{2}$ *then, for* $0 < \theta < 1, \theta \neq \frac{1}{2}$,

$$Z^{\#}(q^{-1}e^{2\pi i\theta}) \neq 0 \qquad (or\ f(e^{2\pi i\theta}) \neq 0).$$

If $\gamma > 1 + \tau$ *then* $Z^{\#}(y)$ *(or* $f(z)$*) has no zeros on the circle* $|y| = q^{-1}$ *(*$|z| = 1$*).*

The following example shows that $\alpha(\frac{1}{2})$ may not be integral if (1.3) holds with $\gamma < 1 + \tau$ and hence the result in Theorem 5.1 is essentially best possible in some sense.

EXAMPLE 5.4.   Let k and m be arbitrary positive integers such that $m > 4k$. We set $q = m^2, \alpha = (\frac{m-1}{m})^2$, and

$$\bar\Lambda(n) = q^n(k + (-1)^{n+1}(k + \alpha - 1)), \qquad n = 1, 2, \ldots.$$

Then $\bar{\Lambda}(n)$ are all positive integers. We have $\bar{P}(1) = \bar{\Lambda}(1) > 0$ and, for $n \geq 2$,

$$\bar{P}(n) = \sum_{r|n} \bar{\Lambda}(r)\mu(n/r) \geq \bar{\Lambda}(n) - \sum_{1 \leq r \leq n/2} \bar{\Lambda}(r)$$

$$\geq q^n(1 - \alpha) - 2k \sum_{1 \leq r \leq n/2} q^r \geq q^{n/2}\left\{q(1 - \alpha) - \frac{2kq}{q - 1}\right\}$$

$$\geq q^{n/2}(2m - 1 - 4k) > 0.$$

Thus $\bar{P}(n)$ are all positive integers too.

It is easy to see that

$$y\frac{\frac{d}{dy}Z^{\#}(y)}{Z^{\#}(y)} = \Lambda^{\#}(y) = \frac{kqy}{1 - qy} + \frac{(k + \alpha - 1)qy}{1 + qy}$$

and hence

$$Z^{\#}(y) = \frac{(1 + qy)^{k-1+\alpha}}{(1 - qy)^k}, \qquad |y| < q^{-1}$$

which has a zero $y = -q^{-1}$ with non-integral order $k - 1 + \alpha$.

Then we have

$$\bar{G}(n) = \frac{1}{2\pi i}\int_{|y|=r_1} \frac{(1 + qy)^{k-1+\alpha}}{y^{n+1}(1 - qy)^k}\,dy = \frac{q^n}{2\pi i}\int_{|z|=r} \frac{(1 + z)^{k-1+\alpha}}{z^{n+1}(1 - z)^k}\,dz$$

where $0 < r_1 < q^{-1}, 0 < r < 1$, and hence

$$\bar{G}(n) = q^n\left\{P_k(n) + \frac{\sin(k - 1 + \alpha)\pi}{\pi}\int_{-\infty}^{-1} \frac{|1 + x|^{k-1+\alpha}}{x^{n+1}(1 - x)^k}\,dx\right\}, \qquad (5.6)$$

where

$$P_k(n) = \text{Res}_{z=1} \frac{(1 + z)^{k-1+\alpha}}{z^{n+1}(1 - z)^k}$$

$$= \sum_{l+m=k-1} 2^{k+\alpha-1}\binom{k + \alpha - 1}{m}(-1)^m 2^{-m}\binom{n + l}{l} = \sum_{\nu=1}^{k} A_\nu n^{\nu-1}$$

with $A_k = 1/(k - 1)! > 0$. The integral in (5.6), denoted by I, equals

$$(-1)^{n+1}\int_0^\infty \frac{v^{k-1+\alpha}}{(1 + v)^{n+1}(2 + v)^k}\,dv.$$

Let

$$I_1 = \int_0^\infty \frac{v^{k-1+\alpha}}{(1 + v)^{n+1+k}}\,dv.$$

Then

$$2^{-k} I_1 \le |I| \le I_1$$

since $2 + v < 2(1 + v)$ for $v \ge 0$. We have

$$
\begin{aligned}
I_1 &= B(k + \alpha, n + 1 - \alpha) = \frac{\Gamma(k + \alpha)\Gamma(n + 1 - \alpha)}{\Gamma(n + k + 1)} \\
&= \frac{\pi}{\sin \alpha \pi} \frac{(\alpha + 1) \cdots (\alpha + k - 1)(1 - \alpha)(2 - \alpha) \cdots (n - \alpha)}{(n + k)!} \\
&\sim \frac{\pi}{\sin \alpha \pi} \frac{(\alpha + 1) \cdots (\alpha + k - 1)}{\Gamma(1 - \alpha)} n^{-k-\alpha},
\end{aligned}
$$

where B and Γ are Euler's beta and gamma functions. Therefore

$$\bar{G}(n) = \sum_{\nu=1}^{k} A_\nu n^{\nu-1} + O(n^{-k-\alpha}).$$

In this example, $\tau = k$. For any $\gamma < 1 + \tau = 1 + k$, we can choose m sufficiently large so that $k + \alpha > \gamma$. Then (1.3) holds with $\gamma < 1 + \tau$ and $Z^\#(y)$ has a zero at $y = -q^{-1}$ with non-integral order.

6. A generalization of the abstract prime number theorem

The key to establishing the abstract prime number theorem is to show that the generating function $Z^\#(y)$ has no zeros on the circle $|y| = q^{-1}$. For $0 < \tau < 1$, from Corollary 5.3, $Z^\#(y)$ has no zeros on the circle $|y| = q^{-1}$ provided that $\gamma > 1 + \tau$. Thus, the following theorem follows directly from Corollaries 5.3 and 3.5.

THEOREM 6.1.   *If* (1.3) *holds with* $\tau = \rho_r < 1$ *and* $\gamma > \frac{3}{2}$, *then* $\bar{\Lambda}(n) \sim \tau q^n$ (*P.E.T.*).

However, for $\tau > 1$, the Examples 6.5 and 6.7 show that, in the general case, even a zero remainder of $\bar{G}(n)$ does not guarantee that $Z^\#(y)$ is nonvanishing. Thus, the following theorem is our best knowledge about $\bar{\Lambda}(n)$ in the general case. This theorem together with Theorem 6.1 is an analog of Beurling's generalization of the classical prime number theorem.

THEOREM 6.2.   *For* $\tau = \rho_r > 1$, *the hypothesis*

$$\bar{G}(n) = q^n \sum_{\nu=1}^{r} A_\nu n^{\rho_\nu - 1}$$

does not generally entail $\bar{\Lambda}(n) \sim \tau q^n$. *However, for* $\tau \ge 1$:

(1) *If the condition (1.3) holds with $\gamma > 1 + \tau$ then there exist a nonnegative integer k with $k \leq [(\tau + 1)/2]$, k real numbers $\theta_1, \ldots, \theta_k$ with $0 < \theta_1 < \cdots < \theta_k \leq 1/2$, and k positive integers $\alpha_1, \ldots, \alpha_k$ such that*

$$\bar{\Lambda}(n) = q^n \left(\tau - 2 \sum_{\nu=1}^{k-1} \alpha_\nu \cos 2n\pi\theta_\nu - (-1)^n \alpha_k + o(1) \right) \tag{6.1}$$

and

$$\alpha_k + 2 \sum_{\nu=1}^{k-1} \alpha_\nu \leq [\tau] \tag{6.2}$$

if $\theta_k = 1/2$ or such that

$$\bar{\Lambda}(n) = q^n \left(\tau - 2 \sum_{\nu=1}^{k} \alpha_\nu \cos 2n\pi\theta_\nu + o(1) \right) \tag{6.3}$$

and

$$2 \sum_{\nu=1}^{k} \alpha_\nu \leq [\tau] \tag{6.4}$$

if $\theta_k < 1/2$.

(2) *If the generating function $Z^{\#}(y)$ of $\bar{G}(n)$ has no zero at $y = -q^{-1}$ and (1.3) holds with $\gamma > 1 + \frac{\tau}{2}$ then (6.3) and (6.4) hold with $k \leq [\tau/2]$.*

Remark. We note that $q^{-1} e^{\pm 2\pi i\theta_\nu}$, $\nu = 1, \ldots, k$ are zeros of $Z^{\#}(y)$. In particular, if $Z^{\#}(y)$ has no zeros on the circle $|y| = q^{-1}$ then $k = 0$ and $\bar{\Lambda}(n) \sim \tau q^n$ as $n \to \infty$. However, this result is weaker than Theorem 3.4.

Proof. We shall give only the proof of (1). The proof of (2) is almost the same. Without loss of generality, we may assume that $1 + \tau < \gamma < [\tau] + 2$. By Theorem 5.1, the generating function $Z^{\#}(y)$ has at most $[\tau]$ zeros on the circle $|y| = q^{-1}$.

Assume the case that $-q^{-1}$ is one of its zero. Then $Z^{\#}(y)$ has $2k - 1$ distinct zeros $q^{-1} e^{\pm 2\pi i\theta_\nu}$, $\nu = 1, \ldots, k - 1$, and $-q^{-1}$ with $0 < \theta_1 < \cdots < \theta_k = 1/2$. By Theorem 5.1, the orders α_ν of $q^{-1} e^{2\pi i\theta_\nu}$ are all positive integers. Hence, $2k - 1 \leq \tau$, i.e. $k \leq [(\tau + 1)/2]$, and (6.2) holds.

Let

$$F(y) = \frac{Z^{\#}(y)(1 - qy)^{\tau}}{(1 + qy)^{\alpha_k} \displaystyle\prod_{\nu=1}^{k-1} (1 - qye^{2\pi i\theta_\nu})^{\alpha_\nu} (1 - qye^{-2\pi i\theta_\nu})^{\alpha_\nu}}. \tag{6.5}$$

Then $F(y)$ is holomorphic in the disk $|y| < q^{-1}$ and, by Theorem 5.1, is continuous on the disk $|y| \leq q^{-1}$ and has no zeros there. Hence

$$\Lambda^{\#}(y) = \tau \frac{qy}{1-qy} - \sum_{\nu=1}^{k-1} \alpha_\nu \left(\frac{qye^{2\pi i\theta_\nu}}{1 - qye^{2\pi i\theta_\nu}} + \frac{qye^{-2\pi i\theta_\nu}}{1 - qye^{-2\pi i\theta_\nu}} \right)$$

$$+ \alpha_k \frac{qy}{1+qy} + y\frac{F'(y)}{F(y)}.$$

Therefore,

$$\bar{\Lambda}(n) = q^n \left(\tau - 2\sum_{\nu=1}^{k-1} \alpha_\nu \cos 2n\pi\theta_\nu - (-1)^n \alpha_k \right) + \frac{1}{2\pi i} \int_{|y|=r} \frac{F'(y)}{F(y)} y^{-n} \, dy \quad (6.6)$$

where $0 < r < q^{-1}$. It remains to show that the last integral, denoted by I, is $o(q^n)$.

We note that $F'(y)$ is continuous up to the circle $|y| = q^{-1}$ except at points $\pm q^{-1}$ and $q^{-1}e^{\pm 2\pi i\theta_\nu}$, $\nu = 1, \ldots, k-1$. Consider a contour $C_{\delta,\epsilon}$ comprising the parts of the circle $C_\delta : |y| = \delta \, (< q^{-1})$ which are outside all small circles $c_0 : |y - q^{-1}| = \epsilon_0$, $c_\nu : |y - q^{-1}e^{2\pi i\theta_\nu}| = \epsilon_1$, $c_{-\nu} : |y - q^{-1}e^{-2\pi i\theta_\nu}| = \epsilon_1$, $\nu = 1, \ldots, k-1$, and $c_k : |y + q^{-1}| = \epsilon_1$, and the parts of these small circles which are inside the circle C_δ. Then we can shift the integration contour of I to $C_{\delta,\epsilon}$ and obtain

$$I = \int_{C_{\delta,\epsilon}} \frac{F'(y)}{F(y)} y^{-n} \, dy. \quad (6.7)$$

Upon letting $\delta \to q^{-1}-$, we obtain

$$I = \int_{C_\epsilon} \frac{F'(y)}{F(y)} y^{-n} \, dy, \quad (6.8)$$

where the contour C_ϵ is comprised in much the same way as $C_{\delta,\epsilon}$ but with the circle $|y| = q^{-1}$ in place of C_δ.

We claim that, in a small neighborhood of $y_0 = q^{-1}e^{2\pi i\theta_\nu}$ or $q^{-1}e^{-2\pi i\theta_\nu}$, $\nu = 1, \ldots, k$,

$$|F'(y)| \ll |y - y_0|^{\gamma - [\tau] - 2} \quad (6.9)$$

holds. To be concrete, we consider $y_0 = q^{-1}e^{2\pi i\theta_\nu}$; a similar argument applies to the case $y_0 = q^{-1}e^{-2\pi i\theta_\nu}$. It suffices to show that, in the neighborhood,

$$\left| \frac{d}{dy} \left(\frac{Z^{\#}(y)}{(1 - qye^{-2\pi i\theta_\nu})^{\alpha_\nu}} \right) \right| \ll |y - y_0|^{\gamma - [\tau] - 2}.$$

and then the claim follows plainly. Actually, in the neighborhood,

$$Z^{\#}(y) = Z^{\#}(y_1) + \frac{1}{1!}Z^{\#\prime}(y_1)(y - y_1) + \cdots + \frac{1}{(m-1)!}Z^{\#(m-1)}(y_1)(y - y_1)^{m-1}$$

$$+ \frac{1}{(m-1)!} \int_{y_1}^{y} Z^{\#(m)}(u)(y - u)^{m-1} \, du$$

where $m = [\tau]$ and $|y|, |y_1| < q^{-1}$. Upon letting $y_1 \to y_0$, we obtain

$$Z^{\#}(y) = \frac{1}{\alpha!}Z^{\#(\alpha)}(y_0)(y - y_0)^{\alpha} + \cdots + \frac{1}{(m-1)!}Z^{\#(m-1)}(y_0)(y - y_0)^{m-1}$$

$$+\frac{1}{(m-1)!}\int_{y_0}^{y} Z^{\#(m)}(u)(y - u)^{m-1}\, du,$$

where $\alpha = \alpha_\nu$, since $Z^{\#}(y_0) = \cdots = Z^{\#(\alpha-1)}(y_0) = 0$. Then

$$Z^{\#}(y) = \frac{1}{\alpha!}Z^{\#(\alpha)}(y_0)(y - y_0)^{\alpha} + \cdots + \frac{1}{m!}Z^{\#(m)}(y_0)(y - y_0)^{m}$$

$$+\frac{1}{(m-1)!}\int_{y_0}^{y}(Z^{\#(m)}(u) - Z^{\#(m)}(y_0))(y - u)^{m-1}\, du.$$

Thus

$$\frac{d}{dy}\left(\frac{Z^{\#}(y)}{(1 - qye^{-2\pi i\theta_\nu})^{\alpha}}\right)$$

$$= (-1)^{\alpha}y_0^{\alpha}\left\{P(y) + \frac{1}{(m-1)!}\frac{d}{dy}\left(\frac{1}{(y-y_0)^{\alpha}}\right.\right.$$

$$\left.\left.\times\int_{y_0}^{y}(Z^{\#(m)}(u) - Z^{\#(m)}(y_0))(y - u)^{m-1}\, du\right)\right\}, \qquad (6.10)$$

where $P(y)$ is a polynomial in y. The derivative on the right-hand side of (6.10) equals

$$-\frac{\alpha}{(y-y_0)^{\alpha+1}}\int_{y_0}^{y}(Z^{\#(m)}(u) - Z^{\#(m)}(y_0))(y - u)^{m-1}\, du$$

$$-\frac{m-1}{(y-y_0)^{\alpha}}\int_{y_0}^{y}(Z^{\#(m)}(u) - Z^{\#(m)}(y_0))(y - u)^{m-2}\, du. \qquad (6.11)$$

By integration by substitution, the first integral in (6.11) equals

$$\int_{0}^{1}(Z^{\#(m)}(y_0 + t(y - y_0)) - Z^{\#(m)}(y_0))(1 - t)^{m-1}(y - y_0)^{m}\, dt$$

and hence its modulus is

$$\ll |y - y_0|^{\gamma-1}\int_{0}^{1}t^{\gamma-m-1}(1 - t)^{m-1}\, dt \ll |y - y_0|^{\gamma-1},$$

since

$$|Z^{\#(m)}(y_0 + t(y - y_0)) - Z^{\#(m)}(y_0)| \ll |t(y - y_0)|^{\gamma-m-1},$$

which can be shown as we did in the proof of Theorem 5.2. Similarly, the modulus of the second integral in (6.11) is

$$\ll |y - y_0|^{\gamma - 2}.$$

It follows that, in a small neighborhood of $y = q^{-1} e^{2\pi i \theta_\nu}$,

$$\left| \frac{d}{dy} \left(\frac{Z^\#(y)}{(1 - qye^{-2\pi i \theta_\nu})^\alpha} \right) \right| \ll |y - y_0|^{\gamma - \alpha - 2} \ll |y - y_0|^{\gamma - [\tau] - 2}$$

from (6.10) and (6.11).

Now, since $\gamma > 1 + \tau$, $[\tau] + 2 - \gamma < 1$, from (6.9),

$$\int_{c_\nu'} \frac{F'(y)}{F(y)} y^{-n} \, dy, \qquad \int_{c_{-\nu}'} \frac{F'(y)}{F(y)} y^{-n} \, dy \to 0$$

as $\epsilon_1 \to 0$, where c_ν' and $c_{-\nu}'$ are the parts of c_ν and $c_{-\nu}$ inside the circle $|y| = q^{-1}$ respectively. Upon letting $\epsilon_1 \to 0$ on the right-hand side of (6.8), we reach

$$I = \int_{C_{\epsilon_0}} \frac{F'(y)}{F(y)} y^{-n} \, dy, \tag{6.12}$$

where C_{ϵ_0} comprises the part of the circle $|y| = q^{-1}$ which is outside the small circle $c_0 \colon |y - q^{-1}| = \epsilon_0$, and the part of c_0 which is inside the circle $|y| = q^{-1}$.

Finally, in a small neighborhood of $y = q^{-1}$,

$$\frac{F'(y)}{F(y)} = \frac{Z'(y)}{Z(y)} - \frac{Q'(y)}{Q(y)},$$

where $Z(y) := (1 - qy)^\tau Z^\#(y)$ and

$$Q(y) = (1 + qy)^{\alpha_k} \prod_{\nu=1}^{k-1} (1 - qye^{2\pi i \theta_\nu})^{\alpha_\nu} (1 - qye^{-2\pi i \theta_\nu})^{\alpha_\nu}.$$

Note that $Q(y)$ is holomorphic and nonvanishing in the neighborhood. Also, note that, from (1.3) and (2.2) with $m = 3$,

$$Z(y) = B_s + \sum_{\mu=1}^{s-1} B_\mu (1 - qy)^{\tau - \tau_\mu} + (1 - qy)^\tau$$

$$\times \left(A \log(1 - qy) + \sum_{k=2}^{t} C_k \sum_{n=1}^{\infty} \frac{q^n y^n}{n^k} + R(y) \right),$$

where $R(y) = \sum_{n=0}^{\infty} r_n q^n y^n$ with $r_n = O(n^{-\beta})$ and $\beta > 2$, is continuous and has no zeros on $\{y \colon |y| \le q^{-1}, |y - q^{-1}| < \eta\}$ for some $\eta > 0$. Thus, an argument involving

$Z(y)$ similar to the one setting up (3.15) and (3.16) in the proof of Theorem 3.4 shows that, upon letting $\epsilon_0 \to 0$ in (6.12),

$$I = iq^{n-1} \int_{-\pi}^{\pi} \frac{F'(q^{-1}e^{i\theta})}{F(q^{-1}e^{i\theta})} e^{-i(n-1)\theta} \, d\theta, \tag{6.13}$$

where $F'(q^{-1}e^{i\theta}) \in L[-\pi, \pi]$. Hence $I = o(q^n)$ for the integral in (6.13) tends to zero as $n \to \infty$ by the Riemann-Lebesgue lemma. This proves the theorem in the case that $y = -q^{-1}$ is a zero of $Z^{\#}(y)$.

Similarly, we can prove the theorem in the case that $y = -q^{-1}$ is not a zero of $Z^{\#}(y)$.   □

The following corollary is an immediate consequence of Theorem 6.2.

COROLLARY 6.3.   *If* $1 \le \tau < 2$ *and if the condition* (1.3) *holds with* $\gamma > 1 + \tau$, *then*

$$\bar\Lambda(n) = q^n(\tau - (-1)^n + o(1)) \qquad or \qquad q^n(\tau + o(1))$$

depending upon whether $-q^{-1}$ *is or is not a zero of* $Z^{\#}(y)$.

Proof.   Since $1 \le \tau < 2$, the number k in Theorem 6.2 is 0 or 1. If $k = 0$, $\bar\Lambda(n) = q^n(\tau + o(1))$. If $k = 1$, then $\theta_1 = 1/2$ and $\alpha_1 = 1$, and hence $\bar\Lambda(n) = q^n(\tau - (-1)^n + o(1))$.   □

If $\rho_1, \ldots, \rho_r$ are all positive integers and $q > 1$, the remainder terms in (6.1) and (6.3) can be improved.

THEOREM 6.4.   *If*

$$\bar G(n) = q^n \sum_{\nu=1}^{r} A_\nu n^{k_\nu - 1} + O(q^{\nu n}) \tag{6.14}$$

holds with constant $q > 1$, *positive integers* $k_1 < \cdots < k_r$, *constant* $A_1, \ldots, A_{r-1}$ *and* $A_r > 0$, *and constant* $v < 1$, *then*

$$\bar\Lambda(n) = q^n \left(k - 2 \sum_{\nu=1}^{l-1} \alpha_\nu \cos 2n\pi\theta_\nu - (-1)^n \alpha_\nu \right) + O(q^{un})$$

or

$$\bar\Lambda(n) = q^n \left(k - 2 \sum_{\nu=1}^{l} \alpha_\nu \cos 2n\pi\theta_\nu \right) + O(q^{un})$$

for some u *with* $v < u < 1$, *where* $k = k_r$ *and* l *is a nonnegative integer with* $l \le [(k+1)/2]$. *In particular, if the generating function* $Z^{\#}(y)$ *has no zeros on the circle* $|y| = q^{-1}$ *then* $l = 0$ *and*

$$\bar\Lambda(n) = kq^n + O(q^{un}).$$

Remark.   Theorem 6.4 may give rise to the problem of determining the quantity u more precisely in terms of v. However, in the general case, there is not too much we can say about this, as we know from [11].

Proof.   From (6.14) and (2.2)$_1$, we have

$$Z^{\#}(y) = \sum_{\nu=1}^{s} B_\mu \frac{1}{(1 - qy)^{\tau_\mu}} + R(y),$$

where τ_μ are all positive integers, $\tau_1 < \cdots < \tau_s$ and $\tau_s = k_r = k$, and where $R(y)$ is holomorphic in the disk $|y| < q^{-v}$. Thus, $(1 - qy)^k Z^{\#}(y)$ is holomorphic in the same disk and hence so is the function $F(y)$ defined in (6.5). Since $F(y)$ has no zeros in the disk $|y| \le q^{-1}$, there exists some constant u' with $v < u' < 1$ such that $F(y)$ has no zeros in the disk $|y| < q^{-u'}$. If we shift the integration path in the formula (6.6) to a circle with $r = q^{-u}$, where $u' < u < 1$, then we arrive at the conclusion. $\square$

We conclude our discussion by giving the examples mentioned at the beginning of this section. These examples are of independent interest, though they are artificial constructions.

EXAMPLE 6.5.   We consider an additive arithmetic semigroup with $\bar{G}(n)$ and $\bar{P}(n)$ given as follows. Let $(x, y, z) = (a_\nu, b_\nu, c_\nu)$, $\nu = 1, 2, \ldots$ be solutions of the diophantine equation $x^2 + y^2 = z^2$ such that

$$(a_\nu, b_\nu) = 1, \quad 1 \le a_\nu < b_\nu < c_\nu, \quad c_1 < c_2 < \cdots.$$

Let θ_ν be given by $\cos \theta_\nu = b_\nu/c_\nu, 0 < \theta < \frac{\pi}{2}$. Then the θ_ν are all different. We set $q = \prod_{\nu=1}^{k} c_\nu^4$ and

$$\bar{P}(n) = \frac{1}{n} \sum_{r|n} \left(2 \sum_{\nu=1}^{k} (1 - \cos r\theta_\nu) q^r \right) \mu(n/r), \quad n = 1, 2, \ldots,$$

where μ is the Möbius function. Then

$$\bar{\Lambda}(n) = \sum_{r|n} r \bar{P}(r) = 2 \sum_{\nu=1}^{k} (1 - \cos n\theta_\nu) q^n.$$

Thus, the $P.E.T.$ does not hold since $\sum_{\nu=1}^{k} \cos n\theta_\nu$ has no limit as $n \to \infty$. Actually, on the one hand, an application of Dirichlet's approximation theorem implies that

$$\limsup_{n \to \infty} \sum_{\nu=1}^{k} \cos n\theta_\nu = k.$$

On the other hand, it is easy to see that $\cos n\theta_1$ has no limit as $n \to \infty$ and hence

$$\liminf_{n\to\infty} \sum_{\nu=1}^{k} \cos n\theta_\nu < k.$$

It is easy to see that

$$
\begin{aligned}
y\frac{\frac{d}{dy}Z^\#(y)}{Z^\#(y)} &= \Lambda^\#(y) = \sum_{n=1}^{\infty} 2\sum_{\nu=1}^{k}(1 - \cos n\theta_\nu)q^n y^n \\
&= \sum_{\nu=1}^{k}\left(\frac{2qy}{1-qy} - \frac{e^{i\theta_\nu}qy}{1-e^{i\theta_\nu}qy} - \frac{e^{-i\theta_\nu}qy}{1-e^{-i\theta_\nu}qy}\right) \\
&= \frac{2kqy}{1-qy} - \sum_{\nu=1}^{k}\left(\frac{e^{i\theta_\nu}qy}{1-e^{i\theta_\nu}qy} + \frac{e^{-i\theta_\nu}qy}{1-e^{-i\theta_\nu}qy}\right).
\end{aligned}
$$

Hence

$$Z^\#(y) = \frac{\prod_{\nu=1}^{k}(1-e^{i\theta_\nu}qy)(1-e^{-i\theta_\nu}qy)}{(1-qy)^{2k}}$$

which has $2k$ zeros $y = q^{-1}e^{\pm i\theta_\nu}$, $\nu = 1, \ldots, k$.

Let

$$Z^\#(y) = \sum_{n=0}^{\infty} \bar{G}(n)y^n$$

and

$$\frac{(1-e^{i\theta_\nu}qy)(1-e^{-i\theta_\nu}qy)}{(1-qy)^2} = \sum_{n=0}^{\infty} \bar{G}_\nu(n)y^n, \quad \nu = 1, \ldots, k.$$

Then $\bar{G}_\nu(0) = 1$ and

$$\bar{G}_\nu(n) = 2(1-\cos\theta_\nu)nq^n = 2\frac{c_\nu - b_\nu}{c_\nu}nq^n, \quad n = 1, 2, \ldots \tag{6.15}$$

are positive integers and so is

$$\bar{G}(n) = \bar{G}_1 * \cdots * \bar{G}_k(n).$$

We can show, by induction on k, that

$$\bar{G}(n) = q^n \sum_{\nu=1}^{2k} A_\nu n^{\nu-1}, \quad n = 1, 2, \ldots \tag{6.16}$$

and hence the "remainder term" of $\bar{G}(n)$ is zero. Actually, for $k = 1$, from (6.15), (6.16) is certainly true. Now, suppose

$$\bar{G}_1 * \cdots * \bar{G}_h(n) = q^n \sum_{\nu=1}^{2h} C_\nu n^{\nu-1}, \quad n = 1, 2, \ldots.$$

Then, from (6.15),

$$\bar{G}_1 * \cdots * \bar{G}_h * \bar{G}_{h+1}(n) = (\bar{G}_1 * \cdots * \bar{G}_h) * \bar{G}_{h+1}(n)$$

$$= q^n \left\{ \sum_{\nu=1}^{2h} C_\nu n^{\nu-1} + 2\frac{c_{h+1} - b_{h+1}}{c_{h+1}} \left(\sum_{\nu=1}^{2h} C_\nu \sum_{m=1}^{n-1} m^{\nu-1}(n-m) + n \right) \right\}. \quad (6.17)$$

We note that

$$\sum_{m=1}^{n-1} m^{\nu-1}(n-m) = n\sum_{m=1}^{n-1} m^{\nu-1} - \sum_{m=1}^{n-1} m^\nu$$

$$= \frac{n}{\nu}(B_\nu(n) - B_\nu(1)) - \frac{1}{\nu+1}(B_{\nu+1}(n) - B_{\nu+1}(1)),$$

where $B_k(x)$ is the kth Bernoulli polynomial of degree k [9], [10]. Hence

$$\sum_{m=1}^{n-1} m^{\nu-1}(n-m) = Q(n),$$

where $Q(x)$ is a polynomial of degree $\nu+1$ with the leading coefficient $(\nu(\nu+1))^{-1}$. It follows that

$$\bar{G}_1 * \cdots * \bar{G}_h * \bar{G}_{h+1}(n) = q^n \sum_{\nu=1}^{2(h+1)} A_\nu n^{\nu-1}$$

with

$$A_{2(h+1)} = \frac{2^{h+1}}{(2h+1)!} \prod_{\nu=1}^{h+1} \frac{c_\nu - b_\nu}{c_\nu}.$$

We shall show that $\bar{P}(n)$ is actually a positive integer. To this end, we first show that $\bar{P}(n)$ is positive. It suffices to show that

$$\sum_{r|n}(1 - \cos r\theta_\nu)q^r \mu(n/r) > 0, \quad \nu = 1, \ldots, k.$$

We note that $\sin n\theta_\nu \neq 0$ since

$$\sin n\theta_\nu = \operatorname{Im}\frac{(b_\nu + ia_\nu)^n}{c_\nu^n} = \operatorname{Im}\frac{i^n}{c_\nu^n}(a_\nu - ib_\nu)^n$$

$$= \begin{cases} \pm\frac{1}{c_\nu^n}(a_\nu^n - \binom{n}{2}a_\nu^{n-2}b_\nu^2 + \cdots), & \text{if } n \text{ is odd,} \\ \frac{1}{c_\nu^n}(\binom{n}{1}b_\nu^{n-1}a_\nu - \binom{n}{3}b_\nu^{n-3}a_\nu^3 + \cdots), & \text{if } n \text{ is even,} \end{cases}$$

and $(a_\nu, b_\nu) = 1$. Hence $1 - \cos n\theta_\nu > 0$. We have

$$1 - \cos n\theta_\nu = \frac{1}{c_\nu^n}(c_\nu^n - \mathrm{Re}(b_\nu + ia_\nu)^n) \geq \frac{1}{c_\nu^n}, \quad n = 1, 2, \ldots$$

since $c_\nu^n - \mathrm{Re}(b_\nu + ia_\nu)^n \in \mathbf{Z}$. We conclude that

$$\sum_{r|n}(1 - \cos r\theta_\nu)q^r \mu(n/r) \geq (1 - \cos n\theta_\nu)q^n - 2\sum_{1 \leq r \leq n/2} q^r$$

$$\geq q^{3n/4} - \frac{2q^{1+n/2}}{q-1} \geq q^{3n/4} - 4q^{n/2} > 0$$

since $q \geq 5^4$.

It remains to show that $\bar{P}(n) \in \mathbf{Z}$. This can be done directly through an elementary argument. Let $n = p^\alpha m$ where p is an ordinary prime number, $\alpha \geq 1$, and $(p, m) = 1$. Then

$$\bar{P}(n) = \frac{1}{n}\sum_{r|n}\bar{\Lambda}(r)\mu\left(\frac{n}{r}\right) = \frac{1}{p^\alpha m}\sum_{r|m}(\bar{\Lambda}(p^\alpha r) - \bar{\Lambda}(p^{\alpha-1}r))\mu\left(\frac{m}{r}\right).$$

It suffices to show that $\bar{\Lambda}(p^\alpha r) - \bar{\Lambda}(p^{\alpha-1}r) \equiv 0 \pmod{p^\alpha}$. If $p|q$, this is trivial. Otherwise, $(p, q) = 1$, and this can be shown by computation based on a well-known theorem of Euler [5]. The computation of this proof is lengthy by comparison with the following argument suggested by Warlimont. Thus we shall prove the following proposition and then the assertion $\bar{P}(n) \in \mathbf{Z}$ follows.

PROPOSITION 6.6.   *Let the sequences $a(n)$ and $b(m)$ be formally related by*

$$1 + \sum_{n=1}^{\infty}a(n)t^n = \prod_{m=1}^{\infty}(1 - t^m)^{-b(m)}.$$

Then $a(n) \in \mathbf{Z}$ for all n if and only if $b(m) \in \mathbf{Z}$ for all m.

Proof.   Assume that $a(n) \in \mathbf{Z}$ for all n. We shall prove, by induction, that $b(m) \in \mathbf{Z}$ for all m. The implication of the opposite direction is trivial.
Actually, we first have

$$1 + \sum_{n=1}^{\infty}a(n)t^n = (1 - t)^{-b(1)}(1 + ct^2 + \cdots)$$

$$= (1 + b(1)t + \cdots)(1 + ct^2 + \cdots)$$

$$= 1 + b(1)t + \cdots.$$

Therefore $b(1) = a(1) \in \mathbf{Z}$. Then, assume $b(m) \in \mathbf{Z}$ for $1 \le m < k$. We have

$$\left(1 + \sum_{n=1}^{\infty} a(n)t^n\right) \prod_{m=1}^{k-1} (1 - t^m)^{b(m)} = (1 - t^k)^{-b(k)}(1 + dt^{k+1} + \cdots)$$

$$= (1 + b(k)t^k + \cdots)(1 + dt^{k+1} + \cdots)$$

$$= 1 + b(k)t^k + \cdots.$$

The left-hand side is the product of two power series with integer coefficients and hence is itself a power series with integer coefficients. Therefore, $b(k) \in \mathbf{Z}$. $\square$

Now, to show that $\bar{P}(n) \in \mathbf{Z}$, we simply take $a(n) = \bar{G}(n)$ and $b(m) = \bar{P}(m)$ and note that $\bar{G}(n) \in \mathbf{Z}, n \ge 1$.

EXAMPLE 6.7.   Consider

$$\bar{P}(n) = \frac{1}{n} \sum_{r|n} \left(2 \sum_{v=1}^{k}(1 - \cos r\theta_v) + 1\right) q^r \mu(n/r)$$

where $\theta_v, v = 1, \ldots, k$ and q are defined in Example 6.5. Then,

$$Z^{\#}(y) = \frac{\prod_{r=1}^{k}(1 - e^{i\theta_v}qy)(1 - e^{-i\theta_v}qy)}{(1 - qy)^{2k+1}}.$$

Similarly, consider

$$\bar{P}(n) = \frac{1}{n} \sum_{r|n} \left(2 \sum_{v=1}^{k}(1 - \cos r\theta_v) + 1 - (-1)^r\right) q^r \mu(n/r).$$

Then

$$Z^{\#}(y) = \frac{(1 + qy) \prod_{r=1}^{k}(1 - e^{i\theta_v}qy)(1 - e^{-i\theta_v}qy)}{(1 - qy)^{2k+1}}.$$

Using the facts given in Example 6.5, we can show that in both cases $\bar{P}(n)$ and $\bar{G}(n)$ are both positive integers and

$$\bar{G}(n) = q^n \sum_{v=1}^{2k+1} A_v n^{v-1}.$$

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