ILLINOIS JOURNAL OF MATHEMATICS
Volume 40, Number 2, Summer 1996

MEAGER-NOWHERE DENSE GAMES (VI): MARKOV k-TACTICS

MARION SCHEEPERS

Let J be the ideal of nowhere dense subsets of a T_1-space. Then $\langle J \rangle$, the σ-completion of J, denotes the collection of meager subsets. Two players, ONE and TWO, play the following game of length ω: In the n-th inning, ONE first chooses a meager subset O_n; TWO responds by choosing a nowhere dense subset T_n of X. TWO wins the play $(O_1, T_1, \ldots, O_n, T_n, \ldots)$ if $\cup_{n=1}^{\infty} O_n \subseteq \cup_{n=1}^{\infty} T_n$. TWO has a winning perfect information strategy. Does TWO really need so much information to win?

This question has been considered for games of this sort in the papers [1], [2] and [4] through [9]. We now continue these studies by considering strategies for TWO that use as information the number of the inning in progress, as well as a bounded number of earlier moves of ONE. Telgársky calls a strategy of the form $T_k = F(O_k, k)$ for the second player a *Markov* strategy. Fix a positive integer k. By analogy we define a *Markov k-tactic* for TWO to be a function F such that $T_j = F(O_1, \ldots, O_j, j)$ for $j \leq k$, and $T_{m+k} = F(O_{m+1}, \ldots, O_{m+k}, m + k)$ for each m. A strategy for TWO which depends on only the $\leq k$ most recent moves of ONE (and not also the number of the inning in progress) is said to be a *k-tactic*. For both of these notions we omit mention of k when $k = 1$; thus, "1-tactic" is replaced by "tactic".

Various special versions of the game described above result from imposing additional constraints on the players. One such game is denoted $MG(J)$: For each n player ONE is required to choose O_{n+1} in such a way that $O_n \subset O_{n+1}$. Here, as everywhere else in the paper, $\subset$ means *proper* subset of. Another such game is denoted $WMEG(J)$: For each n ONE is required to choose O_{n+1} such that $O_n \subseteq O_{n+1}$ *and* TWO wins if $\cup_{n=1}^{\infty} O_n = \cup_{n=1}^{\infty} T_n$.

The paper is organized as follows. In Section 1 we introduce the coherent *assignment* problem for partially ordered sets. In Section 2 we use coherent assignments in conjunction with coherent *decompositions*. In Section 3 we recall a few relevant facts from [3] about a Ramseyan type partition relation. These facts are used in the fifth section. In Section 4 we study the existence of winning Markov k-tactics for TWO in the game $MG(J)$. In Section 5 we prove theorems concerning the existence of winning k-tactics in the game $MG(J)$.

Received April 4, 1994.
1991 Mathematics Subject Classification. Primary 90D44; Secondary 04A99.

Notation, terminology and conventions follow those of [1]. All mentioned consistency results presuppose the consistency of **ZF**. For a cardinal number λ we use the following notation: $\lambda^{+0} = \lambda$, and for each $n < \omega$, $\lambda^{+(n+1)}$ is the least cardinal number larger than λ^{+n}; $\lambda^{+\omega}$ is the supremum of the set $\{\lambda^{+n}: n < \omega\}$. We used topological terminology in the description of our game. This description is equivalent to the combinatorial one where J is taken to be a (proper) free ideal on a set and $\langle J \rangle$ is its σ–completion. Throughout this paper we reserve the symbol J to denote a proper free ideal. The symbol $add(J)$ denotes the least cardinality of a subcollection of J whose union is not a member of J. The symbols R and N denote the sets of real numbers and of natural numbers, respectively.

1. The coherent assignment problem

Let $(P, <)$ be a partially ordered set and let $Q \subseteq P$ be a cofinal set. A function $K: P \to Q$ with the properties that $p \leq K(p)$ for each $p \in P$ and $K(p) \leq K(q)$ whenever $p \leq q$ is said to be a *coherent assignment*. A set $Q \subseteq P$ is said to be *representative* if Q is cofinal in P and there is a coherent assignment $K: P \to Q$. The *coherent assignment problem for* $(P, <)$ is the question of whether every cofinal subset of $(P, <)$ is representative.

LEMMA 1. *If $(P, <)$ has a cofinal chain, then every cofinal subset of $(P, <)$ is representative.*

Proof. Let $Q \subseteq P$ be a cofinal set of minimal cardinality, κ. Let $(c_\alpha: \alpha < \lambda)$ enumerate a cofinal chain such that $c_\alpha < c_\beta$ whenever $\alpha < \beta < \lambda$, and λ is the minimal cardinal for which this is possible. Then $\kappa = \lambda$ and κ is a regular cardinal number. Choose a sequence $(q_\alpha: \alpha < \kappa)$ from Q such that $\alpha < \beta$ implies that $q_\alpha < q_\beta$ and $c_\beta < q_\beta$. Then $\{q_\alpha: \alpha < \kappa\}$ is cofinal in P. For $p \in P$, define $K(p)$ to be q_α, where α is minimal with $p \leq q_\alpha$. □

Let $\lambda \leq \kappa$ be infinite cardinal numbers. The symbol $CA_\lambda(\kappa)$ denotes the assertion:

> *For every set X such that $|X| \leq \kappa$, every cofinal subset of $[X]^\lambda$ is representative.*

LEMMA 2. *If $CA_\lambda(\kappa)$ is true, then so is $CA_\lambda(\kappa^+)$.*

Proof. By Lemma 1 we may assume that $\lambda < \kappa$. Let $\mathcal{A}$ be a cofinal subset of $[\kappa^+]^\lambda$. Define a function $f: \kappa^+ \to \kappa^+$ so that for each $X \in [\alpha]^\lambda$ there is an $A \in \mathcal{A}$ such that $X \subseteq A \subseteq f(\alpha)$.

There is a closed, unbounded set $C \subseteq \kappa^+$ such that $f(\gamma) \leq \delta$ whenever $\gamma, \delta \in C$ and $\gamma < \delta$. The set $B = \{\beta \in C: cof(\beta) = \lambda^+\}$ is unbounded in κ^+. For each

$\beta \in B$ let $\mathcal{A}_\beta = \{A \in \mathcal{A}\colon A \subseteq \beta\}$; then $\mathcal{A}_\beta$ is cofinal in $[\beta]^\lambda$. For each $\beta \in B$ choose a coherent assignment $K_\beta\colon [\beta]^\lambda \to \mathcal{A}_\beta$.

For each $\beta \in B$, choose another coherent assignment $\pi_\beta\colon [\beta]^\lambda \to \mathcal{A}_\beta$ as follows: When $\beta = \min(B)$, let $\pi_\beta = K_\beta$. Assume that $\beta \in B$ is larger than $\min(B)$, and that for each $\delta \in \beta \cap B$ we have already chosen a coherent assignment $\pi_\delta\colon [\delta]^\lambda \to \mathcal{A}_\delta$ such that if $\gamma, \delta \in \beta \cap B$ and $\gamma \leq \delta$, then $\pi_\gamma \subseteq \pi_\delta$. Then π_β is selected thus: Consider $X \in [\beta]^\lambda$. Let $\xi(X)$ be the least element of B such that $|X \cap \xi(X)| = \lambda$, and define

$$\pi_\beta(X) = \begin{cases} K_\beta(X) & \text{if } \xi(X) = \beta, \\ \pi_\delta(X) & \text{if } \delta \in [\xi(X), \beta) \cap B \text{ and } X \in [\delta]^\lambda, \\ K_\beta(X \cup (\bigcup\{\pi_\gamma(X \cap \gamma)\colon & \\ \quad \gamma \in [\xi(X), \rho) \cap B\})) & \text{otherwise.} \end{cases}$$

Then π_β is a coherent assignment which extends π_γ for each $\gamma \in \beta \cap B$, and so the inductive selection procedure continues. Then $K = \bigcup_{\rho \in B} \pi_\rho$ is a coherent assignment. □

THEOREM 3. *$CA_\lambda(\kappa)$ is true whenever $\kappa < \lambda^{+\omega}$.*

Proof. Lemma 1 and Lemma 2. □

PROBLEM 1. *Is every cofinal family $\mathcal{A} \subseteq \langle \mathcal{NWD}_{\mathsf{R}} \rangle$ a representative family?*

2. Coherent decompositions

Let $\mathcal{B}$ be a subset of J. A family $\mathcal{A} \subseteq \langle J \rangle$ has a *coherent decomposition in terms of $\mathcal{B}$* if for each $A \in \mathcal{A}$ there is a sequence $(A_n\colon n \in \mathsf{N})$ such that:

(1) $A = \bigcup_{n=1}^{\infty} A_n$.
(2) If $m < n$, then $A_m \subseteq A_n$.
(3) Each A_m is in $\mathcal{B}$.
(4) If $A \subseteq B$ are in $\mathcal{A}$, then there is an m such that $A_n \subseteq B_n$ for all $n \geq m$.

If $\langle J \rangle$ has a coherent decomposition in terms of J, then J is said to *have the coherent decomposition property*, and $\langle J \rangle$ is said to *have a coherent decomposition.* These two notions were introduced in [1] because of their relevance to the construction of certain sorts of winning strategies in the game $MG(J)$.

THEOREM 4. *The following statements are equivalent.*

(1) *There is a cofinal subset of $\langle J \rangle$ which is representative and which also has a coherent decomposition in terms of J.*

(2) $\langle J \rangle$ *has a coherent decomposition in terms of* J.

(3) *There is for each* $A \in \langle J \rangle$ *a function* f_A: $A \to \omega$ *such that*:

(a) *for each* n, $\{x \in A\colon f_A(x) \leq n\} \in J$, *and*

(b) *if* $A \subset B$, *then there is an* $m < \omega$ *such that* $\{x \in A\colon f_A(x) < f_B(x)\} \subseteq \{x \in B\colon f_B(x) \leq m\}$.

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{C}$ be a cofinal subset of $\langle J \rangle$ satisfying the two hypotheses. Let $K\colon \langle J \rangle \to \mathcal{C}$ be a coherent assignment. For each $C \in \mathcal{C}$, choose a sequence $(C_n\colon 0 < n < \omega)$ such that the chosen sequences witness the existence of a coherent decomposition in terms of J. For $X \in \langle J \rangle$ and for $0 < n < \omega$, define $X_n = X \cap K(X)_n$. This defines a coherent decomposition for $\langle J \rangle$ in terms of J.

(2) $\Rightarrow$ (3) For each $A \in \langle J \rangle$, select a sequence $(A_n\colon n < \omega)$ of sets from J such that the selected sequences witness the existence of a coherent decomposition for $\langle J \rangle$. Then define $f_A(x) = \min\{n\colon x \in A_n\}$.

(3) $\Rightarrow$ (1) Let $(f_A\colon A \in \langle J \rangle)$ be as in (3). For each $n < \omega$ and for each $A \in \langle J \rangle$ define $A_n = \{x \in A\colon f_A(x) \leq n\}$. □

Let λ be an uncountable cardinal number of countable cofinality. We use Theorem 4 to show that $[\lambda^+]^\lambda$ has a coherent decomposition in terms of $[\lambda^+]^{<\lambda}$. To begin, fix an increasing sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ of infinite cardinal numbers which converges to λ. Let $\mathcal{C} = \{\alpha\colon \lambda \leq \alpha < \lambda^+\}$. Then $\mathcal{C}$ is a cofinal chain in $[\lambda^+]^\lambda$, and so is representative (Lemma 1). We check that $\mathcal{C}$ has a coherent decomposition in terms of $[\lambda^+]^{<\lambda}$: For each $\alpha \in \mathcal{C}$, choose a preliminary sequence $(S_{\alpha,n}\colon n < \omega)$ such that $\alpha = \bigcup_{n<\omega} S_{\alpha,n}$, if $m < n$ then $S_{\alpha,m} \subset S_{\alpha,n}$, and for each n, $|S_{\alpha,n}| = \lambda_n$. Then modify each as follows:

$$T_{\alpha,n} = S_{\alpha,n} \cup (\bigcup_{\beta \in \mathcal{C} \cap S_{\alpha,n}} S_{\beta,n}).$$

The set $\{(T_{\alpha,n}\colon n \in \mathbb{N})\colon \alpha \in \mathcal{C}\}$ defines a coherent decomposition for $\mathcal{C}$ in terms of $[\lambda^+]^{<\lambda}$. (Though this argument was given for $\lambda > \aleph_0$, a fairly similar argument shows that $[\omega_1]^{\aleph_0}$ has a coherent decomposition in terms of $[\omega_1]^{<\aleph_0}$.)

In Theorem 4, $3(b)$ cannot in general be replaced by the following condition:

> $3(b')$: *if* $A \subset B$, *then there is an* $m < \omega$ *such that* $\{x \in A\colon f_A(x) \leq f_B(x)\} \subseteq \{x \in B\colon f_B(x) \leq m\}$.

To see this, take a cardinal $\lambda > 2^{\aleph_0}$ of countable cofinality. Then $[\lambda^+]^\lambda$ has a coherent decomposition in terms of $[\lambda^+]^{<\lambda}$. Suppose that there were functions f_A: $A \to \omega$ which witness this, and satisfy $3(a)$ and $3(b')$. Choose an ascending sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ of cardinals which converges to λ. Define a coloring $\Phi\colon [\lambda^+ \setminus \lambda]^2 \to \omega$ so that for $\alpha < \beta$, $\Phi(\{\alpha, \beta\}) = \min\{n\colon |\{x < \alpha\colon f_\alpha(x) \leq f_\beta(x)\}| \leq \lambda_n\}$. As a weak consequence of the Erdös-Rado theorem we find an ascending sequence $\lambda < \alpha_1 < \cdots < \alpha_m < \cdots < \lambda^+$, and an $n < \omega$, such that

$\Phi(\{\alpha_m, \alpha_{m+1}\}) = n$ for all m. But then we find $x \in \alpha_1$ such that $f_{\alpha_m}(x) > f_{\alpha_{m+1}}(x)$ for each m, a contradiction.

PROPOSITION 5. *For an infinite cardinal number κ the following are equivalent*:

(1) $[\kappa]^{\aleph_0}$ *has a coherent decomposition in terms of* $[\kappa]^{<\aleph_0}$.
(2) *For each* $A \in [\kappa]^{\aleph_0}$ *there is a finite-to-one* f_A: $A \to \omega$ *such that* $\{x \in A\colon f_A(x) < f_B(x)\}$ *is finite whenever* $A \subset B$.
(3) *For each* $A \in [\kappa]^{\aleph_0}$ *there is a finite-to-one* f_A: $A \to \omega$ *such that* $\{x \in A\colon f_A(x) \leq f_B(x)\}$ *is finite whenever* $A \subset B$.
(4) *Let* $(L, <)$ *be a linear order of countable cofinality. For each* $A \in [\kappa]^{\aleph_0}$ *there is a finite-to-one function* f_A: $A \to L$ *which has only finitely many values below each* $q \in L$, *such that* $\{x \in A\colon f_A(x) \leq f_B(x)\}$ *is finite whenever* $A \subset B$.

Proof. (1) $\Rightarrow$ (2) Let $(A_n\colon n < \omega)$ be a decomposition of A into finite sets such that these decompositions witness the existence of coherent decomposition. Then define $f_A(x) = \min\{m\colon x \in A_m\}$. Consider $A \subset B$, and $x \in A$. If $f_A(x) < f_B(x)$, then $A_{f_A(x)} \not\subseteq B_{f_A(x)}$; by hypothesis there are only finitely many such events.

(2) $\Rightarrow$ (3) For each A, let h_A: $A \to \omega$ be as in (2). By Theorem 12 of [9] there is for each countable subset A of κ a function g_A: $\omega \to \omega$ such that if $A \subset B$, then $g_B(n) < g_A(n)$ for all but finitely many n. By replacing each g_A by g'_A which is defined so that $g'_A(n) = n + \sum_{j \leq n} g_A(j)$, we see that we may assume that each g_A is increasing. Now put $f_A = g_A \circ h_A$ for each A. Then the family $(f_A\colon A \in [\kappa]^{\aleph_0})$ is as required.

(3) $\Rightarrow$ (4) Let $(\ell_n\colon n < \omega)$ enumerate in increasing order a cofinal subset of L. Let $(f_A\colon A \in [\kappa]^{\aleph_0})$ be as in (3). Setting $f'_A(x) = \ell_{f_A(x)}$ for each $A \in [\kappa]^{\aleph_0}$ and for each $x \in A$ works.

(4) $\Rightarrow$ (1) Let f_A: $A \to L$ be as in (4). Let $\{\ell_n\colon n < \omega\}$ enumerate in increasing order a cofinal subset of L. For each A and each $n < \omega$, let $A_n = \{x \in A\colon f_A(x) \leq \ell_n\}$. Then this defines a coherent decomposition for the countable subsets of κ. □

In [2], Koszmider introduces the notion of a *coherent family of finite-to-one functions*: Let κ be an infinite cardinal number. A family $(f_A\colon A \in [\kappa]^{\aleph_0})$ is a coherent family of finite-to-one functions if for each A, f_A: $A \to \omega$ is a finite-to-one function, and for all A and B, $\{x \in A \cap B\colon f_A(x) \neq f_B(x)\}$ is finite. He then proves there is a coherent family of finite-to-one functions on $[\kappa]^{\aleph_0}$ for each infinite $\kappa < \aleph_\omega$ and, granting additional hypotheses, for each κ there is a coherent family of finite-to-one functions on $[\kappa]^{\aleph_0}$. Applying (2) of Proposition 5, we see:

COROLLARY 6. *If there is a coherent family of finite-to-one functions on* $[\kappa]^{\aleph_0}$, *then* $[\kappa]^{\aleph_0}$ *has a coherent decomposition in terms of* $[\kappa]^{<\aleph_0}$.

PROBLEM 2. *Is the existence of a coherent decomposition for* $[\kappa]^{\aleph_0}$ *in terms of* $[\kappa]^{<\aleph_0}$ *equivalent to the existence of a coherent family of finite-to-one functions*?

3. The ω-path partition relation

Let $(P, <)$ be a partially ordered set and let κ be a cardinal number. For positive integer r the symbol $(P, <) \to (\omega - \text{path})^r_{\kappa/<\omega}$ denotes the statement:

For any coloring $\Phi\colon [P]^r \to \kappa$, there is a strictly increasing ω-path $p_1 < p_2 < \cdots < p_k < \cdots$ in P, such that the set $\{\Phi(\{p_{i+1}, \ldots, p_{i+r}\})\colon i < \omega\}$ is a finite subset of κ.

The negation of this assertion is denoted $(P, <) \not\to (\omega - \text{path})^r_{\kappa/<\omega}$.

There exists a least ordinal α such that $\alpha \to (\omega - \text{path})^2_{\kappa/<\omega}$ (by the Erdös-Rado theorem); it is denoted $M(\kappa)$. It was proven in [3] (see Corollary 14 there) that $M(\kappa)$ is at at least κ^{++} and is at most $(2^\kappa)^+$.

We begin by recalling Proposition 15 of [3]:

THEOREM 7. *Let λ be an infinite cardinal. Then for every infinite set X,*

$$([X]^{\leq\lambda}, \subset) \not\to (\omega - path)^2_{\lambda/<\omega}.$$

By making the necessary minor changes in the proof of Proposition 17 of [3], one obtains:

THEOREM 8. *Let $(P, <)$ be a partially ordered set of cardinality κ. Then the following statements are equivalent.*

(1) $(P, <) \not\to (\omega - path)^2_{\omega/<\omega}$.
(2) *There is a function $\Phi\colon P \to {}^{\omega}\kappa^+$ such that:*
   (a) *for each $p \in P$, $\Phi(p)$ is weakly increasing, and*
   (b) *if $p < q$, then there is an $m < \omega$ such that $\Phi(q)(n) < \Phi(p)(n)$ whenever $n \geq m$.*

4. Markov k-tactics in the game $MG(J)$

In the game $MG(J)$ player ONE is required to choose meager sets O_n such that $O_n \subset O_{n+1}$ for each n. If we are interested in Markov k-tactics for player TWO, this requirement on ONE may be somewhat relaxed to requiring only that $O_n \subseteq O_{n+1}$ for each n (the game is then denoted $WMG(J)$), and the requirements on TWO may be made more demanding, by specifying that TWO wins a play exactly when $\bigcup_{n=1}^\infty O_n = \bigcup_{n=1}^\infty T_n$ (in which case the game is denote $WMEG(J)$); from the point of view of Markov k-tactics these are all equivalent games. Since this remark is used below only for the case when $k = 1$, we indicate a proof for only that case.

The following statements are equivalent:

(1) TWO has a winning Markov tactic in $WMEG(J)$.
(2) TWO has a winning Markov tactic in $MG(J)$.

To see that (2) implies (1), let F be a winning Markov tactic for TWO in $MG(J)$. We may assume that $\langle J\rangle$ is a proper ideal. For each $B \in \langle J\rangle$ choose a sequence $(x_n^B\colon n \in \mathbb{N})$ from $S \setminus B$ such that $x_n^B \neq x_m^B$ whenever $m \neq n$. Define for each n, $\sigma(B, n) = B \cap (\cup_{i=1}^n F(B, i) \cup F(B \cup \{x_1^B, \dots, x_i^B\}, i))$. Then σ is a winning Markov tactic for TWO in $WMEG(J)$.

4.1. Markov tactics. Fix a J for which TWO has a winning Markov tactic in the game $WMEG(J)$ and let F be such a winning Markov tactic for TWO.

THEOREM 9. *There are subsets $X_1, X_2, \dots, X_n, \dots$ of X such that: $X = \cup_{n=1}^\infty X_n$ and for each n, $[X_n]^{<add(\langle J\rangle)} \subseteq J$.*

Proof. Observe that for each x, there is a $B_x \in \langle J\rangle$ and an $n_x \in \mathbb{N}$ such that $x \in B_x$, and for each $C \in \langle J\rangle$ such that $B_x \subseteq C$, we have $x \in F(C, n_x)$. (If not, consider a contrary x. Then, for each $n \in \mathbb{N}$, and each $B \in \langle J\rangle$ such that $x \in B$, there is a $C \in \langle J\rangle$ such that $B \subseteq C$, and $x \notin F(C, n)$. Let ONE choose $O_1 \subseteq O_2 \subseteq \cdots \subseteq O_n \subseteq \cdots$ such that $x \in O_1$, but $x \notin F(O_n, n)$ for all n. Then TWO loses the play $(O_1, F(O_1, 1), O_2, F(O_2, 2), \dots, F(O_n, n), \dots)$, a contradiction.)

For each x, choose the least n_x as above, and a corresponding $B_x \in \langle J\rangle$. For each n, let $X_n = \{x \in X\colon n_x = n\}$. Consider a subset Y of X_n, with $|Y| < add(\langle J\rangle)$. Then $C = \cup_{x\in Y} B_x \in \langle J\rangle$, and $Y \subseteq F(C, n)$. □

COROLLARY 10. *TWO does not have a winning Markov tactic in $MG(\mathcal{NWD}_{\mathbb{R}})$.*

Proof. Consider a partition $\mathbb{R} = \cup_{n=1}^\infty X_n$. By the Baire Category Theorem there is an n, and a nonempty open interval J such that $X_n \cap J$ is a dense subset of J. But then X_n contains a countable set which is not nowhere dense. Now apply Theorem 9. □

COROLLARY 11. *Let λ be an infinite cardinal number of countable cofinality. Then TWO does not have a winning Markov tactic in $MG([\kappa]^{<\lambda})$ for any $\kappa > \lambda$.*

Proof. Let $\kappa > \lambda$ be given, and consider a partition $\kappa = \cup_{n=1}^\infty X_n$. Then there is an n such that X_n has cardinality larger than λ. Apply Theorem 9. □

4.2. Markov k-tactics for $k > 1$. Let $\mathcal{A}$ be a subset of $\langle J\rangle$. Then the games $WMG(\mathcal{A}, J)$ and $WMEG(\mathcal{A}, J)$ are the versions of $WMG(J)$ and $WMEG(J)$ respectively in which ONE must choose from $\mathcal{A}$.

THEOREM 12. *If $\mathcal{A} \subseteq \langle J \rangle$ has a coherent decomposition in terms of J, then TWO has a winning Markov 2-tactic in $WMEG(\mathcal{A}, J)$.*

Proof. For each $A \in \mathcal{A}$, choose a decomposition $(A_n\colon 0 < n < \omega)$ such that the family of sequences so chosen witnesses the existence of a coherent decomposition. To define a winning Markov 2-tactic F for TWO, consider sets $A \subseteq B$ from $\mathcal{A}$, and $n \in \mathsf{N}$. Then define

$$F(A, B, n) = \begin{cases} B_n & \text{if } A = B \\ B_{\min\{m \geq n\colon (\forall j \geq m)(A_j \subseteq B_j)\}} & \text{otherwise.} \end{cases} \qquad \square$$

If there is a representative family in $\langle J \rangle$ which also has a coherent decomposition, then $\langle J \rangle$ has a coherent decomposition; in this case we can conclude that TWO has a winning Markov 2-tactic in $WMEG(J)$.

Since it is consistent that the ideal of meager subsets of the real line has a cofinal chain, it is consistent that TWO has a winning Markov 2-tactic in $WMG(\mathcal{NWD}_{\mathsf{R}})$. In [1], Theorem 15, it was shown that there is a cofinal family $\mathcal{A} \subseteq \langle \mathcal{NWD}_{\mathsf{R}} \rangle$ which has a coherent decomposition. However, it is not known if this $\mathcal{A}$ is a representative family. If it were representative, that would solve the following problem positively.

PROBLEM 3. *Is it a theorem of* ZFC *that TWO has a winning Markov 2-tactic in $WMG(\mathcal{NWD}_{\mathsf{R}})$?*

Let $(S_n\colon n \in \mathsf{N})$ be a sequence of pairwise disjoint infinite sets. For each $n \in \mathsf{N}$ let J_n be a free proper ideal on the set S_n. Then define J so that $X \in J$ if for each n, $X \cap S_n \in J_n$. The symbol $\sum_{n=1}^{\infty} J_n$ will be used to denote J, the *sum* of the J_n's.

In the next proposition we use the following fact:

LEMMA 13. *If TWO has a winning Markov k–tactic in $WMEG(J)$, then TWO has a winning Markov k–tactic G in $WMEG(J)$ such that for all $X_1 \subseteq \cdots \subseteq X_k$, and for all $\ell \geq k$,*

$$G(X_1, 1) \cup \cdots \cup G(X_1, \ldots, X_k, k) \cup \cdots \cup G(X_1, \ldots, X_k, \ell - 1) \subseteq G(X_1, \ldots, X_k, \ell).$$

Proof. Let F be a winning Markov k–tactic for TWO in $WMEG(J)$. Define G by recursion on i so that $G(X_1, \ldots, X_i, i) = F(X_1, 1) \cup \ldots \cup F(X_1, \ldots, X_i, i)$ when $i \leq k$, and so that $G(X_1, \ldots, X_k, i) = (\cup_{j \leq k} G(X_1, \ldots, X_j, j)) \cup (\cup_{k < j < i} G(X_1, \ldots, X_k, j)) \cup F(X_1, \ldots, X_k, i)$ for $i > k$. $\square$

PROPOSITION 14. *Let k be a positive integer. If, for each n, TWO has a winning Markov k–tactic in $WMEG(J_n)$, then TWO has a winning Markov k-tactic in $WMEG(\sum_{n=1}^{\infty} J_n)$.*

Proof. For each n, let F_n be a winning Markov k–tactic for TWO in $WMEG(J_n)$. We may assume that each F_n has the property described in Lemma 13.

For $m \leq k$, $X_1 \subseteq \cdots \subseteq X_m$ in $\langle \sum_{n=1}^{\infty} J_n \rangle$ and for $m \leq \ell < \omega$ we define

$$F(X_1, \ldots, X_m, \ell) = \begin{cases} \cup_{j \leq k} F_j(S_j \cap X_1, \ldots, S_j \cap X_m, m) & \text{if } m = \ell \leq k \\ \cup_{j \leq \ell} F_j(S_j \cap X_1, \ldots, S_j \cap X_k, \ell) & \text{if } m = k \leq \ell. \end{cases}$$

Then F is a winning Markov k–tactic for TWO. □

COROLLARY 15. *If λ is a cardinal of countable cofinality then TWO has a winning Markov 2-tactic in $WMEG([\kappa]^{<\lambda})$ for each $\kappa \leq \lambda^{+\omega}$.*

Proof. Consider an infinite cardinal number $\kappa < \lambda^{+\omega}$. By Corollaries 1 and 8 of [1], some cofinal subset of $[\kappa]^{\lambda}$ has a coherent decomposition in terms of $[\kappa]^{<\lambda}$. Then by Theorems 3 and 4, $[\kappa]^{\leq\lambda}$ has a coherent decomposition in terms of $[\kappa]^{<\lambda}$. Apply Theorem 12.

For $\lambda^{+\omega}$, the result follows from Proposition 14 and what had just been proved. □

PROBLEM 4. *Is it true that whenever λ has countable cofinality, then for each κ TWO has a winning Markov 2-tactic in $WMEG([\kappa]^{<\lambda})$?*

5. k-tactics in $MG(J)$

Let $\mathcal{C}$ be a subset of $\langle J \rangle$ such that $\mathcal{C}$ has no maximal element. The game $MG(\mathcal{C}, J)$, introduced in [1], is played just like $MG(J)$ except that now ONE must pick from $\mathcal{C}$. The following theorem distills the essential features from most constructions of winning k-tactics in the game $MG(J)$ carried out in papers in the bibliography.

THEOREM 16. *Let $k \geq 2$ be an integer and let $J \subset \mathcal{P}(S)$ be a free ideal. If $\langle J \rangle$ has a representative cofinal subset $\mathcal{C}$ then* (1) *implies* (2), *where*:

(1) *TWO has a winning 2-tactic in $MG(\mathcal{C}, J)$ and for each $C \in \mathcal{C}$ TWO has a winning k-tactic in $MG(J\lceil_C)$.*
(2) *TWO has a winning k-tactic in $MG(J)$.*

Proof. For each $C \in \mathcal{C}$, let F_C be a winning k-tactic for TWO in $MG(J\lceil_C)$. Also, let G be a winning 2-tactic for TWO in $MG(\mathcal{C}, J)$. Let $K\colon \langle J \rangle \to \mathcal{C}$ be a coherent assignment.

Define a k-tactic F for TWO as follows: Let $j \leq k$, and let $X_1 \subset \cdots \subset X_j$ be given. Then

$$F(X_1, \ldots, X_j) = \begin{cases} \emptyset & \text{if } j < k \\ G(K(X_{k-1})) \cup G(K(X_{k-1}), K(X_k)) & \text{if } K(X_{k-1}) \subset K(X_k) \\ F_{K(X_k)}(X_1) \cup \cdots \cup F_{K(X_k)}(X_1, \ldots, X_k) & \text{otherwise.} \end{cases}$$

To see that F is a winning k-tactic for TWO, consider an F-play $O_1, T_1, \ldots, O_m, T_m, \ldots$ of $MG(J)$. Then compute $K(O_1) \subseteq \cdots \subseteq K(O_m) \subseteq \cdots$; either this sequence is eventually constant, or else it has infinitely many terms.

In the first case, select the first $m \geq k$ such that $K(O_j) = K(O_m)$ ($= C$ say) for all $j \geq m$. Put $O'_n = O_{n+m-k+1}$ for each n. Then $O'_1, O'_2, \ldots$ is a sequence of moves by ONE in the game $MG(J\lceil_C)$, and by the third clause in the definition of F, TWO has played against these moves using the winning k-tactic F_C. Thus, TWO wins such plays of $MG(J)$.

In the second case, the infinitely many terms of the sequence $K(O_1), \ldots, K(O_n), \ldots$ constitute a sequence of moves by ONE in the game $MG(\mathcal{C}, J)$; by the second clause of the definition of F, TWO has played according to the winning 2-tactic G against these moves. Thus, TWO also wins such plays. □

THEOREM 17. *Let λ be a cardinal number of countable cofinality and let $k > 1$ be an integer. The following statements are equivalent*:

(1) *TWO has a winning k-tactic in* $MG([\lambda^+]^{<\lambda})$.
(2) $([\lambda^+]^{\leq\lambda}, \subset) \not\rightarrow (\omega - path)^k_{\omega/<\omega}$.
(3) $\lambda^+ \not\rightarrow (\omega - path)^2_{\omega/<\omega}$ *and* $(\mathsf{P}(\lambda), \subset) \not\rightarrow (\omega - path)^k_{\omega/<\omega}$.
(4) *TWO has a winning k-tactic in* $MG([\lambda^{+n}]^{<\lambda})$ *for all* $n < \omega$.

Proof. When $\lambda = \aleph_0$, each of items (1), (2) and (3) is individually a theorem of **ZFC**: (1) is a special case of Corollary 4 of [4], (2) is a special case of Theorem 7, and (3) is implied by (2) and Corollary 10 of [3]. For uncountable λ the equivalence of these three items was proven in Theorem 23 of [1]. It is also clear that (4) implies (1). We show that (3) implies (4).

Fix $1 < n < \omega$, let $J = [\lambda^{+n}]^{<\lambda}$ and assume (3). It follows from Proposition 3 of [4] that TWO has a winning k-tactic in $MG(J\lceil_A)$ for each $A \in \langle J\rangle$. By cardinality considerations there is a cofinal subset $\{D_\alpha \colon \alpha < \lambda^{+n}\} \subset [\lambda^{+n}]^\lambda$ of $\langle J\rangle$. Inductively choose $C_\alpha \in [\lambda^{+n}]^\lambda$ such that $D_\alpha \subseteq C_\alpha \not\subseteq \cup_{\beta<\alpha} C_\beta$. The family $\mathcal{C} = (C_\alpha \colon \alpha < \lambda^{+n})$ is cofinal in $\langle J\rangle$ and is well-founded under the $\subset$ relation. Observe that for each $B \in \langle J\rangle$, $|\{C \in \mathcal{C} \colon C \subseteq B\}| \leq \lambda$. Since the rank function for $(\mathcal{C}, \subset)$ embeds it in λ^+, we see that $(\mathcal{C}, \subset) \not\rightarrow (\omega - \text{path})^2_{\omega/<\omega}$. By Corollaries 1 and 6 of [1] for $\lambda = \aleph_0$ or Theorem 7 of [1] for λ uncountable, $\mathcal{C}$ also has a coherent decomposition in terms of J. Then Theorem 16 of [1] implies that TWO has a winning 2-tactic in $MG(\mathcal{C}, J)$. By Theorem 3, $\mathcal{C}$ is also representative. Apply Theorem 16. □

Theorem 17 extends Theorem 23 of [1] in that it gives another non–trivial equivalence of the assertions of Theorem 23. In [2], Koszmider proved that 4 of Theorem 17 is true for $\lambda = \aleph_0$.

Let λ be a cardinal number of countable cofinality. We saw in Corollary 15 that TWO always has a winning Markov 2–tactic in $WMEG([\lambda^{+\omega}]^{<\lambda})$. If λ is uncountable then existence of winning k–tactics for TWO in $MG([\lambda^+]^{<\lambda})$ is independent of **ZFC**:

this follows from Theorem 17 and the remarks near the bottom of p. 59 and at the top of p. 60 of [3], and Theorem 23 of [1].

PROBLEM 5. *Let λ be a cardinal number of countable cofinality. If TWO has a winning k-tactic in $MG([\lambda^+]^{<\lambda})$, does TWO then have a winning k-tactic in $MG([\kappa]^{<\lambda})$ for each infinite κ?*

For $\lambda = \aleph_0$ item 4 of Theorem 17 could also be obtained in another way:

THEOREM 18. *Let $J \subset \mathcal{P}(S)$ be a free ideal such that $(\langle J \rangle, \subset) \not\to (\omega - path)^k_{\omega/<\omega}$, and $\langle J \rangle$ has a coherent decomposition in terms of J. Then TWO has a winning k-tactic in $MG(J)$.*

Proof. For each $A \in \langle J \rangle \setminus J$ choose a sequence $(A_n\colon 0 < n < \omega)$ such that the set of selected sequences witnesses the existence of a coherent decomposition of $\langle J \rangle$. Also choose a coloring $\Phi\colon [\langle J \rangle]^k \to \omega$ which witnesses the partition relation for $(\langle J \rangle, \subset)$.

For $j \leq k$ and for $X^1 \subset \cdots \subset X^j \in \langle J \rangle$, define

$$F(X^1, \ldots, X^j) = \begin{cases} X^j & \text{if } X^j \in J \\ \emptyset & \text{if } X^j \notin J \text{ and } |\{i \leq j\colon X_i \notin J\}| < k \\ X^k_{\substack{\min\{n \geq \Phi(\{X^1,\ldots,X^k\})\colon \\ (\forall i \geq n)(X^1_i \subseteq \cdots \subseteq X^k_i)\}}} & \text{otherwise} \end{cases}$$

To see that F is a winning k-tactic for TWO, consider an F-play $(O_1, T_1, \ldots, O_m, T_m, \ldots)$ of $MG(J)$. We may assume that no O_m is in J.

For each n let

$$x_{n+k} = \min\{p \geq \Phi(\{O_{n+1}, \ldots, O_{n+k}\})\colon (\forall j \geq p)(O_{n+1,j} \subseteq \cdots \subseteq O_{n+k,j})\}.$$

Because $\langle J \rangle$ has a coherent decomposition, each x_{n+k} is well defined. Because $\langle J \rangle$ satisfies the negative partition relation, the set $\{x_{n+k}\colon n < \omega\}$ is infinite. Consider O_m. Every time x_{i+k}, $i \geq m$ reaches a new record high, $O_{m,x_{i+k}}$ is a subset of TWO's response. We see that TWO covers O_m. Since m was arbitrary it follows that TWO wins. □

Now for $J = [\kappa]^{<\aleph_0}$ Koszmider's result follows from Theorem 18 in the following way: By Theorem 7 we know that for every infinite cardinal number κ, $([\kappa]^{\leq\aleph_0}, \subset) \not\to (\omega - \text{path})^2_{\omega/<\omega}$. Thus, one of the hypotheses of Theorem 18 is satisfied. If κ is less than $\aleph_\omega$, then by cardinality considerations and Theorem 3 also the second hypothesis of Theorem 18 is satisfied.

The existence of a winning 2-tactic for TWO in the game $MG(J)$ was described in combinatorial terms as follows in Proposition 1 of [9]: Let ω_α be the cardinality of $\langle J \rangle$. Then there are functions $f_A\colon A \to \omega_{\alpha+1}$, $A \in \langle J \rangle$, such that if $A \subset B$ are in $\langle J \rangle$ then $\{x \in A\colon f_A(x) \leq f_B(x)\}$ is in J. Here is how one constructs such a family of functions directly from the hypotheses of Theorem 18 for $k = 2$: Since $\langle J \rangle$ has a coherent decomposition, for each $A \in \langle J \rangle$ fix a function $g_A\colon A \to \omega$ as in Theorem 4.3. From the hypothesis that $(\langle J \rangle, \subset) \not\to (\omega - \text{path})^2_{\omega/<\omega}$, for each $A \in \langle J \rangle$, by Theorem 8, we find a function $h_A\colon \omega \to \omega_{\alpha+1}$ having properties (2) (a) and (b) of that theorem. For each A, put $f_A = h_A \circ g_A$.

Also, note the distinction between Proposition 1(b) of [9], and Proposition 5.3 above: starting from a coherent decomposition of $[\kappa]^{\aleph_0}$ in terms of $[\kappa]^{<\aleph_0}$ we obtain functions which witness the existence of a winning 2-tactic, with ranges ω instead of $\omega_{\alpha+1}$! This raises the following question:

PROBLEM 6. *Does the existence of a winning 2-tactic for TWO in $MG([\kappa]^{<\aleph_0})$ imply the existence of a coherent decomposition for $[\kappa]^{\aleph_0}$ in terms of $[\kappa]^{<\aleph_0}$?*

Acknowledgement. I would like to acknowledge support from Idaho State Board of Education Grant 94–051, as well as the referee's contribution towards improving the quality of this paper.

REFERENCES

[1] T. Bartoszynski, W. Just and M. Scheepers, *Covering games and the Banach-Mazur game: k-tactics*, Canad. J. Math. **45** (1993), 897–929.

[2] P. Koszmider, *On coherent families of finite-to-one functions*, J. Symbolic Logic **58** (1993), 128–138.

[3] M. Scheepers, *A partition relation for partially ordered sets*, Order **7** (1990), 41–64.

[4] ———, *Meager-nowhere dense games (I): n-tactics*, Rocky Mountain J. Math. **22** (1992), 1011–1055.

[5] ———, *Meager-nowhere dense games(II): coding strategies*, Proc. Amer. Math. Soc. **112** (1991), 1107–1115.

[6] ———, *Meager-nowhere dense games (III): remainder strategies*, Topology Proc. **17** (1992), 215–231.

[7] ———, *Meager-nowhere dense games (IV): n-tactics (continued)*, J. Symbolic Logic **59** (1994), 603–605.

[8] ———, *Meager-nowhere dense games (V): coding strategies again*, Quaestiones Math., to appear.

[9] ———, *Concerning n-tactics in the countable-finite game*, J. Symbolic Logic **56** (1991), 786–794.

BOISE STATE UNIVERSITY
BOISE, IDAHO