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COMPOSITION OPERATORS ON SMALL WEIGHTED HARDY SPACES

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1. Introduction

Let φ be an analytic map of the unit disk D into itself and define $C_\varphi(f) = f \circ \varphi$ whenever f is analytic on D. We are interested here in studying basic properties (e.g., boundedness, compactness) of the composition operator C_φ acting on weighted Hardy spaces $H^2(\beta)$, defined from a weight sequence $\{\beta(n)\}_0^\infty$ satisfying $\beta(0) = 1$, $\beta(n) > 0$ and $\lim_{n\to\infty} \beta(n)^{\frac{1}{n}} = 1$. Given such a sequence, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $H^2(\beta)$ if and only if

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta(n)^2 < \infty.$$

Note that $H^2(\beta)$ will be a Hilbert space of analytic functions on D with inner product

$$\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} c_n z^n \rangle_\beta = \sum_{n=0}^{\infty} a_n \bar{c}_n \beta(n)^2,$$

for which the monomials $\{z^n\}_0^\infty$ form a complete set of non-zero orthogonal vectors.

The terminology "weighted Hardy space" comes of course from the observation that if $\beta(n) \equiv 1$, then $H^2(\beta)$ is the usual Hardy Hilbert space $H^2(D)$. For other particular choices of $\{\beta(n)\}_0^\infty$, the corresponding space $H^2(\beta)$ may turn out to be a familiar space; we will note these as the occasion arises. If $\{\beta_1(n)\}$ and $\{\beta_2(n)\}$ are two weight sequences with

$$\frac{1}{c}\, \beta_2(n) \le \beta_1(n) \le c\, \beta_2(n) \ \text{ for some } \ c \in (0, +\infty),$$

then $H^2(\beta_1) = H^2(\beta_2)$, with equivalent norms.

Our principal interest in this paper will be with "small" weighted Hardy spaces. The precise meaning of small will vary somewhat from theorem to theorem. At the very least we will require that

$$\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} < \infty,$$

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so that functions in $H^2(\beta)$ extend continuously to $\overline{D}$. Usually we will in fact want to move beyond the realm of spaces which come from the choices $\beta(n) = (n+1)^a$ for some real a, so a typical hypothesis will be that

$$\frac{n^A}{\beta(n)} \to 0 \text{ as } n \to \infty \quad \text{for all } A > 0.$$

Note that this is equivalent to the requirement that

$$\sum_{n=0}^{\infty} \frac{n^{2k}}{\beta(n)^2} < \infty \quad \text{for all } \ k \geq 0.$$

Much of our work on small spaces is motivated by some known results for certain "large" weighted Hardy spaces. For example, the following theorem due to T. Kriete and B. MacCluer [7] gives a necessary condition for C_φ to be bounded on $H^2(\beta)$ when $\beta(n)$ tends to 0 sufficiently rapidly so that $\lim_{n\to\infty} n^A \beta(n) = 0$ for all $A > 0$. In the statement of the theorem, the notation $|\varphi'(\zeta)|$ refers to the angular derivative of φ at $\zeta \in \partial D$, defined by

$$|\varphi'(\zeta)| = \liminf_{z \to \zeta} \frac{1 - |\varphi(z)|}{1 - |z|}$$

where $z \to \zeta$ unrestrictedly in D. This can be finite only if φ has non-tangential limit of modulus 1 at ζ and in this case $|\varphi'(\zeta)| = |\lim_{r\to 1} \varphi'(r\zeta)|$. The basic facts concerning angular derivative can be found in Section 2.3 of [2].

THEOREM A (KREITE AND MACCLUER). *Suppose $H^2(\beta)$ is a weighted Hardy space such that* $\lim_{n\to\infty} n^A \beta(n) = 0$ *for every $A > 0$. If φ: $D \to D$ is analytic and satisfies $|\varphi'(\zeta)| < 1$ at some ζ in the circle then C_φ does not map $H^2(\beta)$ into itself.*

In Section 2 we will obtain a dual result to this, for certain appropriately defined small spaces, in which the condition "$|\varphi'(\zeta)| < 1$" will be replaced by "$|\varphi'(\zeta)| > 1$".

It is well known that the angular derivative plays an important role in the study of compactness of composition operators on the Hardy space $H^2(D)$ and closely related weighted Hardy spaces (e.g., the Bergman space which corresponds to $\beta(n) = 1/\sqrt{n+1}$). At issue in these standard spaces is finiteness of $|\varphi'(\zeta)|$. In moving from standard spaces to large spaces actual values of the angular derivative play a role in question of both compactness and boundedness. Beyond Theorem A, most of the work which has been done on composition operators on large spaces has been in the more restrictive (and more structured) setting of large weighted Bergman spaces $A^2_G(D)$, defined where $G(r)$ is positive, continuous and non-increasing on $(0, 1)$ with

$$\int_0^1 G(r) r\, dr < \infty \quad \text{and} \quad \lim_{r\to 1} \frac{G(r)}{(1-r)^A} = 0 \quad \text{for all } A > 0\,,$$

with $G(r)/(1-r)^A$ decreasing for r near 1. Such a function $G(r)$ is called a fast regular weight. Then

$$A_G^2(D) = \left\{ f \text{ analytic in } D\colon \ \|f\|_G^2 \equiv \int_D |f(z)|^2 G(|z|) \frac{dA(z)}{\pi} < \infty \right\},$$

where dA denotes Lebesgue area measure on D. Clearly, $A_G^2(D) = H^2(\beta)$ where $\beta(0) = 1$, $\beta(n)^2 = 2p_n/c$ with

$$p_n = \int_0^1 r^{2n+1} G(r)\, dr \quad \text{and} \quad c = 2\int_0^1 G(r) r\, dr.$$

The hypothesis that $G(r)$ is a fast regular weight guarantees that $n^A \beta(n) \to 0$, for all $A > 0$, as $n \to \infty$. Compact composition operators on $A_G^2(D)$ where G is fast and regular are characterized precisely in [7]:

THEOREM B (KRIETE AND MACCLUER). *Let G be a fast regular weight and let $\varphi\colon D \to D$ be analytic. Then C_φ is compact on $A_G^2(D)$ if and only if $|\varphi'(\zeta)| > 1$ for all $\zeta \in \partial D$.*

We will discuss a partial dual result for small spaces in Section 2.

This duality between certain large and small weighted Hardy spaces is most complete in the case that φ is a linear fractional map of D into D, where we are able to use techniques of P. Hurst [5] to make the desired comparisons. In particular we are able to resolve completely the issue of which automorphisms give bounded composition operators on any space for which

$$\frac{n^A}{\beta(n)} \to 0 \text{ as } n \to \infty \text{ for all } A > 0, \text{ and } \frac{\beta(n+1)}{\beta(n)} \to 1.$$

Our results are less complete when we consider symbols which are not linear fractional and in general here we are only able to give necessary conditions, in terms of the values of the angular derivative, for C_φ to be bounded or compact on the appropriately defined small space. The very strong parallels with results for large spaces, however, suggest natural conjectures for sufficient conditions.

We close this section with a brief discussion of various measures of smallness of the spaces $H^2(\beta)$ as expressed by properties of the weight sequence $\{\beta(n)\}$. To avoid certain pathologies we will often assume that $\{\beta(n)\}$ is monotone increasing (or at least eventually increasing, since changing a finite number of the weights produces the same space with an equivalent norm). Additional conditions which have been used to define small weighted Hardy spaces include

(1) $\frac{n^A}{\beta(n)} \to 0$ as $n \to \infty$ for all $A > 0$,

(2) $\sum_{n=0}^{\infty} \frac{n^{2k}}{\beta(n)^2} < \infty$ for all $k \geq 0$,

(3) $\sum_{n=0}^{\infty} \frac{\beta([\alpha n])}{\beta(n)} < \infty$ for some (or all) $0 < \alpha < 1$,
(4) $\beta(n) = e^{h(n)}$, where $\{h(n)\}$ is concave and satisfies $\sum_{n=0}^{\infty} \frac{h(n)}{n^{3/2}} = \infty$.

Our principal interest here will be with conditions (1)–(3). As previously noted, it is easy to see that (1) and (2) are equivalent. Assuming (2), an application of the Cauchy-Schwarz inequality shows that if $f \in H^2(\beta)$ then $f \in C^\infty(\overline{D})$. In general (1) implies neither (3) nor (4); for example, consider $\beta(n) = e^{(\log n)^b}$, $1 < b < 2$. Then (1) holds but the sum in (3) is infinite for all $0 < \alpha < 1$. When $b = 2$, (3) holds only for $0 < \alpha < 1/\sqrt{e}$. If $\{\beta(n)\}$ is increasing and (3) holds for some α, then (1) holds as well (see, for example, the proof of Corollary 7.15 in [2]). The choice $\beta(n) = e^{(\log n)^3}$ shows that (3) does not imply (4). Condition (4) appears in the work of Carleson [1], where it is shown that when (4) holds $H^2(\beta)$ is a quasi-analytic class on $\overline{D}$. An important class of examples to which conditions (1)-(4) all apply are the weight sequences

$$\beta(n) = e^{n^a}, \quad \frac{1}{2} \leq a < 1.$$

The dual condition to (1), namely $n^A \beta(n) \to 0$ as $n \to \infty$ for all $A > 0$, is used in [7] as a defining condition for "large" weighted Hardy spaces. Condition (3) appears in [8] to study maps of the form $rz + (1 - r)$ $(0 < r < 1)$ on small spaces. It is not difficult to show that condition (3) holds for some $\alpha \in (0, 1)$ if and only if there exists an integer $k \geq 2$ so that

$$\sum_{n=0}^{\infty} \frac{\beta(n)}{\beta(kn)} < \infty.$$

The dual condition for large weighted Bergman spaces can be found in Section 5 of [7].

2. Necessary conditions for boundedness and compactness

We give necessary conditions, in terms of the angular derivative $|\varphi'(\zeta)|$, for C_φ to be bounded or compact on a small space $H^2(\beta)$. The first of these requires only the weak hypothesis that

$$\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} < \infty. \tag{2.1}$$

Note that (2.1) implies that every function in $H^2(\beta)$ extends continuously to $\overline{D}$.

Recall that any analytic map φ: $D \to D$ with no fixed point in D has a distinguished fixed point (in the radial limit sense) on ∂D, called the Denjoy-Wolff point, at which the angular derivative is less than or equal to 1. If φ is not the identity nor an elliptic automorphism of D, its iterates converge, uniformly on compact subsets of D, to the interior fixed point of φ (if there is one) or to its Denjoy-Wolff point.

THEOREM 2.1. *Suppose* $\varphi: D \to D$ *is analytic with* $|\varphi'(\zeta)| > 1$ *for some* $\zeta \in \partial D$ *satisfying* $|\varphi(\zeta)| = 1$. *Then* C_φ *is not compact on* $H^2(\beta)$ *whenever* $\{\beta(n)\}$ *satisfies* (2.1).

Proof. Suppose $\varphi(\zeta) = \eta \in \partial D$. Let $\psi(z) = \zeta\overline{\eta}\varphi(z)$ so that $\psi(\zeta) = \zeta$. Notice that $\psi'(\zeta) = |\psi'(\zeta)| > 1$. But ψ must either have an interior fixed point, or Denjoy-Wolff point on ∂D, which cannot be ζ. Call this point a. We have $C_\psi^*(K_\zeta) = K_{\psi(\zeta)} = K_\zeta$ and $C_\psi^*(K_a) = K_{\psi(a)} = K_a$, where K_w denotes the kernel function for evaluation at $w \in \overline{D}$ (see Theorem 2.10 of [2]). Thus if C_φ, and hence also C_ψ, is compact on $H^2(\beta)$, then $\dim\ker(C_\psi - 1) = \dim\ker(C_\psi^* - 1) \geq 2$. But if $f \in \ker(C_\psi - 1)$ then $f \circ \psi_n = f$, where ψ_n is the n^{th} iterate of ψ. If $a \in \partial D$, continuity of f on $\overline{D}$ implies that f is constant. The same conclusion holds if $a \in D$, since ψ is not an elliptic automorphism of D. Thus C_φ cannot be compact on $H^2(\beta)$. □

An alternate proof can be obtained by applying Proposition 1 of [11] to see that under the hypothesis (2.1), if C_φ is compact on $H^2(\beta)$, then φ has a unique fixed point in $\overline{D}$.

The next result applies to small spaces defined slightly more restrictively by requiring that functions in the space have derivative which extends continuously to $\overline{D}$.

THEOREM 2.2. *Suppose* $\varphi: D \to D$ *is analytic with* $|\varphi'(\zeta)| = 1$ *for some* $\zeta \in \partial D$ *with* $|\varphi(\zeta)| = 1$. *If*

$$\sum_{n=0}^{\infty} \frac{n^2}{\beta(n)^2} < \infty$$

then C_φ *is not compact on* $H^2(\beta)$.

Proof. We normalize by choosing $e^{i\theta}$ so that $\psi = e^{i\theta}\varphi$ fixes ζ and consequently $\psi'(\zeta) = 1$. As before, $C_\psi^* K_\zeta = K_\zeta$, where K_ζ is the kernel function for evaluation at ζ.

If $K_\zeta^{(1)}$ is the kernel function for the evaluation of the first derivative at ζ, it is easy to see (p. 266, [2]) that

$$C_\psi^*(K_\zeta^{(1)}) = \overline{\psi'(\zeta)} K_{\psi(\zeta)}^{(1)} = K_\zeta^{(1)}.$$

Thus 1 is an eigenvalue of C_ψ^* with multiplicity at least 2. As in the proof of Theorem 2.1 this shows that C_ψ (and hence C_φ) cannot be compact, since $\dim\ker(C_\psi - 1) = 1$. □

The next result, due to J. Shapiro [8], provides a stepping stone for obtaining a necessary condition, in terms of the angular derivative, for C_φ to be bounded on $H^2(\beta)$.

THEOREM C (SHAPIRO). *Suppose* $\{\beta(n)\}$ *is increasing and*

$$\sum_{n=0}^{\infty} \frac{\beta([\alpha n])}{\beta(n)} < \infty \text{ for some } \alpha = \alpha_0, 0 < \alpha_0 < 1. \tag{2.2}$$

If $\varphi(z) = rz + (1-r)$, *where* $0 < r < \alpha_0$, *then* C_φ *is compact on* $H^2(\beta)$.

THEOREM 2.3. *Assume* $\{\beta(n)\}$ *is increasing with* (2.2). *Suppose* ψ *maps* D *into* D *with* $|\psi'(\zeta)| > 1$ *at some* $\zeta \in \partial D$ *with* $|\psi(\zeta)| = 1$. *Then* C_ψ *is not bounded on* $H^2(\beta)$.

The result should be compared with Theorem A which shows that no map with angular derivative *less than* 1 gives a bounded composition operator on a large space ($n^A\beta(n) \to 0$ for all $A > 0$).

Proof. Consider the map $\overline{\eta}\psi(\zeta z)$ where $\psi(\zeta) = \eta$. This map fixes 1 and has derivative greater than 1 at 1. Since rotations induce unitary composition operators on any weighted Hardy space, this shows that we may assume without loss of generality that $\psi(1) = 1$ and $\psi'(1) > 1$.

Set $r_n = (\psi'(1))^{-n}$. Fix n sufficiently large so that $r_n < \alpha_0$ and, for this n set $\varphi(z) \equiv r_n z + (1 - r_n)$. If ψ_n denotes the n^{th} iterate of ψ and $\tau = \varphi \circ \psi_n$, we have $\tau(1) = 1$ and $\tau'(1) = \varphi'(1)\psi_n'(1) = r_n(\psi'(1))^n = 1$. Thus C_τ is not compact on $H^2(\beta)$, by Theorem 2.2, which applies since (2.2) in fact implies $\sum_{n=0}^{\infty} n^{2k}/(\beta(n))^2 < \infty$ for all k (see, for example, the proof of Corollary 7.15 in [2]).

On the other hand, $C_\tau = C_{\psi_n} \circ C_\varphi$ and C_φ is compact by Theorem C. Therefore C_ψ cannot be bounded, which is the desired conclusion. □

As a corollary to this last result, note that if $\{\beta(n)\}$ is increasing and satisfies (2.2), then $H^2(\beta)$ supports no bounded composition operators with automorphism symbol, other than those induced by rotations. Theorem 3.3 in the next section will generalize this to a wider range of small spaces.

The conclusion of Theorem C, and therefore also of Theorem 2.3, may hold for certain small spaces which are less restrictively defined than by the condition (2.2). Examples will be given in Section 3.

For small spaces, as was the case for large spaces (see [7]), the angular derivative alone does *not*, in general, determine boundedness. We give one example now which shows this; in Section 3 other examples, with linear fractional symbol, will be given.

Example. Let φ: $D \to D$ be defined by

$$\varphi(z) = \frac{(z+1)^2}{4}.$$

Then $\varphi(1) = 1$, $\varphi'(1) = 1$ and $|\varphi(\zeta)| < 1$ for $\zeta \in \partial D\backslash\{1\}$. Consider C_φ acting on $H^2(\beta)$ where $\beta(n) = e^{n^a}$, $\frac{1}{2} < a < 1$. We claim that C_φ is unbounded. To see this let e_n be the unit vector $\frac{z^n}{\beta(n)}$ and compute, using the binomial expansion,

$$\begin{aligned}\|C_\varphi(e_n)\|_\beta &= \frac{1}{\beta(n)}\frac{1}{4^n}\|(z+1)^{2n}\|_\beta \\ &= \frac{1}{\beta(n)}\frac{1}{4^n}\left\|\sum_{k=0}^{2n} C(2n,k)z^k\right\|_\beta \\ &= \frac{1}{\beta(n)}\frac{1}{4^n}\left(\sum_{k=0}^{2n} C(2n,k)^2\beta(k)^2\right)^{1/2} \\ &\geq \frac{1}{\beta(n)}\frac{1}{4^n} C(2n, n+[\sqrt{n}])\,\beta(n+[\sqrt{n}]).\end{aligned}$$

For large n, by using Stirling's formula to estimate $C(2n, n+[\sqrt{n}])$, we see that this is bounded below by

$$\frac{c}{\beta(n)}\frac{1}{4^n}\left(\frac{4^n}{\sqrt{n}}\frac{1}{e^{\sqrt{n}+1}e^{-\sqrt{n}+1}}\right)\beta(n+[\sqrt{n}]) = \frac{c}{e^{n^a}}\frac{e^{(n+[\sqrt{n}])^a}}{\sqrt{n}},$$

where c is a positive constant. When $a > 1/2$ this tends to ∞ as $n \to \infty$. Thus C_φ is not bounded.

By contrast, any rotation (angular derivative 1 everywhere) gives a bounded (in fact unitary) composition operator on any weighted Hardy space.

3. Linear fractional maps

In this section we continue our study of composition operators acting on small spaces by restricting our attention to symbols which are linear fractional maps of D into D. Our main tool will be a modest extension of a recent result due to P. Hurst [5], [6], which gives a method of relating linear fractional composition operators on different weighted Hardy spaces. Hurst's work has its origins in C. Cowen's study of linear fractional composition operators on $H^2(D)$ (see [4]). We begin with a few ideas from that work.

If

$$\varphi(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$$

maps D into D, then the map

$$\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$$

also takes D into D (Lemma 1, [4]). We call σ the dual map to φ. Note that φ is the dual map to σ so there is no ambiguity in referring to $\{\varphi, \sigma\}$ as a dual pair, and that φ is an automorphism of D if and only if σ is. If $z_0 \in \mathbf{C} \cup \infty$ (where $\mathbf{C}$ denotes the complex plane) is a fixed point of φ, then (Theorem 2.2, [5])

$$\frac{1}{\overline{z_0}} \text{ is a fixed point of } \sigma \tag{3.3}$$

and

$$\sigma'\left(\frac{1}{\overline{z_0}}\right) = \frac{1}{\overline{\varphi'(z_0)}} \tag{3.4}$$

where ′ denotes the ordinary derivative.

Hurst's theorem sets up a relationship between C_φ and C_σ on appropriate weighted Hardy spaces. To state his result (Theorem 5, [6]) we use the notation T_g for the operator of multiplication by g, where g is analytic in D, and let

$$\mu(z) = \frac{1}{-\overline{b}z + \overline{d}} \quad \text{and} \quad \nu(z) = \frac{\overline{ad - bc}}{(-\overline{b}z + \overline{d})^2}\,. \tag{3.5}$$

THEOREM D (HURST). *Suppose* $\varphi(z) = \frac{az+b}{cz+d}$ *maps* D *into* D *with dual map* $\sigma(z)$. *Let* $H^2(\gamma)$ *be a weighted Hardy space, defined from a positive sequence* $\{\gamma(n)\}$ *with* $\gamma(0) = 1$ *and* $\lim_{n\to\infty} \gamma(n)^{1/n} = 1$. *Define* $\{\beta(n)\}$ *by* $\beta(0) = 1$ *and* $\beta(n) = 1/\gamma(n-1)$ *for* $n \geq 1$. *If* C_φ *is bounded on* $H^2(\beta)$ *and* C_σ *and* T_μ *are bounded on* $H^2(\gamma)$, *then the restriction of* C_φ^* *to the invariant subspace* $zH^2(\beta)$ *is unitarily equivalent to* $T_\nu C_\sigma$ *acting on* $H^2(\gamma)$.

We need a slight extension of this result which is obtained by showing that T_μ (and hence T_ν) is automatically bounded on the weighted Hardy spaces of interest to us, and that the boundedness of both C_φ and C_σ on their respective weighted Hardy spaces need not be assumed.

LEMMA 3.1. *Suppose* $\{\gamma(n)\}$ *is a positive sequence so that either* $\gamma(n)$ *is eventually monotone decreasing and* $\lim_{n\to\infty} \gamma(n)^{1/n} = 1$ *or* $\lim_{n\to\infty} \frac{\gamma(n+1)}{\gamma(n)} = 1$. *Let* $\varphi(z) = \frac{az+b}{cz+d}$ *map* D *into* D *with* $ad - bc \neq 0$. *Then* T_μ *and hence* T_ν *are bounded on* $H^2(\gamma)$.

Proof. Since φ maps D into D so does $\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$. Thus $-\overline{b}z + \overline{d} = 0$ has no solution in $\overline{D}$ (since $\overline{ad} - \overline{bc} \neq 0$) and $\mu(z) = 1/(-\overline{b}z + \overline{d})$ is analytic in a neighborhood of $\overline{D}$. Either hypothesis on $H^2(\gamma)$ guarantees that multiplication by z, and hence $T_{(-\overline{b}z+\overline{d})}$, is bounded on $H^2(\gamma)$ (Proposition 2.7, [2]). We will show that $T_{(-\overline{b}z+\overline{d})}$ is in fact invertible on $H^2(\gamma)$. Since

$$T_{(-\overline{b}z+\overline{d})} = T_{-\overline{b}(z-\overline{d}/\overline{b})}$$

it is enough to show that $\overline{d}/\overline{b}$ is not in the spectrum of T_z on $H^2(\gamma)$. We use the spectral radius formula to show that T_z has spectral radius $\rho(T_z)$ at most 1 and recall that $|\overline{d}/\overline{b}| > 1$ to get the desired conclusion. Now using the norm estimate by A. Shields [9], we have

$$\|T_z^n\| = \|T_{z^n}\| = \sup_k \frac{\gamma(k+n)}{\gamma(k)}, \quad \text{for } n = 1, 2, \ldots.$$

If $\{\gamma(n)\}$ decreasing then clearly $\|T_{z^n}\| \leq 1$ and $\rho(T_z) = \lim_{n\to\infty} \|T_{z^n}\|^{1/n} \leq 1$. If on the other hand, $\{\gamma(n)\}$ satisfies the hypothesis $r_n = \frac{\gamma(n+1)}{\gamma(n)} \to 1$, we have

$$\begin{aligned} \frac{\gamma(k+n)}{\gamma(k)} &= \frac{\gamma(k+n)}{\gamma(k+n-1)} \frac{\gamma(k+n-1)}{\gamma(k+n-2)} \cdots \frac{\gamma(k+1)}{\gamma(k)} \\ &= r_{k+n-1} r_{k+n-2} \cdots r_{k+1} r_k. \end{aligned}$$

Given $\epsilon > 0$, choose N so that $r_j \leq 1+\epsilon$ for $j \geq N$. Let $r = \max\{r_j \colon 1 \leq j \leq N\}$. Then for all k and n, the estimate

$$\frac{\gamma(k+n)}{\gamma(k)} \leq r^N (1+\epsilon)^n$$

holds. So

$$\begin{aligned} \rho(T_z) &= \lim_{n\to\infty} \|T_z^n\|^{1/n} \\ &= \lim_{n\to\infty} \left(\sup_k \frac{\gamma(k+n)}{\gamma(k)} \right)^{1/n} \\ &\leq \lim_{n\to\infty} (r^N(1+\epsilon)^n)^{1/n} = 1+\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary we conclude $\rho(T_z) \leq 1$. Thus $T_{(-\overline{b}z+\overline{d})} = T_{-\overline{b}(z-\overline{d}/\overline{b})}$ is invertible on $H^2(\gamma)$. Its inverse must be T_μ. □

We use Lemma 3.1 to extend Hurst's Theorem. Let ν be as in (3.5).

THEOREM D′. *Let $\varphi(z) = \frac{az+b}{cz+d}$ map D into D with dual map $\sigma(z)$ and let $\{\gamma(n)\}$ satisfy the hypothesis of Lemma* 3.1. *Define $\beta(n)$ by $\beta(0) = 1$ and $\beta(n) = 1/\gamma(n-1)$ for $n \geq 1$. If C_σ is bounded on $H^2(\gamma)$, then C_φ is bounded on $H^2(\beta)$, with the restriction of C_φ^* to $zH^2(\beta)$ unitarily equivalent to $T_\nu C_\sigma$ on $H^2(\gamma)$. Conversely, if C_φ is bounded on $H^2(\beta)$, then C_σ is bounded on $H^2(\gamma)$ (and the same unitary equivalence holds).*

COROLLARY 3.2. *Let φ, σ, $\{\gamma(n)\}$, $\{\beta(n)\}$ be as above. Then C_φ is compact on $H^2(\beta)$ if and only if C_σ is compact on $H^2(\gamma)$.*

Proof of Theorem D′. With the result of Lemma 3.1 in hand, the proof of Theorem D′ is almost exactly like that of Theorem D. For completeness we sketch the argument here, but refer the reader to [5] and [6] for further details.

Assume C_σ is bounded on $H^2(\gamma)$ and let U be the unitary map of $H^2(\gamma)$ onto $zH^2(\beta)$, which takes z^n to $\frac{\gamma(n)}{\beta(n+1)} z^{n+1} = \frac{1}{\beta(n+1)^2} z^{n+1}$.

For $m,\ n \geq 1$ we have

$$\begin{aligned}
\langle UT_\nu C_\sigma U^* z^n, z^m\rangle_{H^2(\beta)} &= \beta(n)^2\beta(m)^2 \langle T_\nu C_\sigma z^{n-1}, z^{m-1}\rangle_{H^2(\gamma)} \\
&= \beta(n)^2\beta(m)^2\gamma(m-1)^2 \langle T_\nu C_\sigma z^{n-1}, z^{m-1}\rangle_{H^2(D)} \quad (3.6)\\
&= \beta(n)^2 \langle T_z^* C_\varphi^*|_{zH^2(D)} T_z z^{n-1}, z^{m-1}\rangle_{H^2(D)} \quad (3.7)\\
&= \beta(n)^2 \langle z^n, C_\varphi(z^m)\rangle_{H^2(D)} \\
&= \langle z^n, C_\varphi(z^m)\rangle_{H^2(\beta)} ,
\end{aligned}$$

where (3.6) $\Rightarrow$ (3.7) follows from the fact (Lemma 4, [6]) that

$$T_z^* C_\varphi^*|_{zH^2(D)} T_z = T_\nu C_\sigma \quad \text{on} \quad H^2(D).$$

If P is the projection of $H^2(\beta)$ onto $zH^2(\beta)$ this shows that $PC_\varphi|_{zH^2(\beta)}$ agrees with the bounded operator $(UT_\nu C_\sigma U^*)^*$ on the polynomials in $zH^2(\beta)$, a dense set in $zH^2(\beta)$. Hence $PC_\varphi|_{zH^2(\beta)}$ extends to a bounded operator on $zH^2(\beta)$.

Finally we conclude that C_φ is bounded on $zH^2(\beta)$, since $f \circ \varphi = P(f \circ \varphi) + f \circ \varphi(0)$, and thus C_φ is bounded on $H^2(\beta)$. This gives the first part of the theorem.

For the converse statement, if C_φ is bounded on $H^2(\beta)$ then $U^* C_\varphi^*|_{zH^2(\beta)} U$ is bounded on $H^2(\gamma)$, with U defined as above (since $zH^2(\beta)$ is an invariant subspace for C_φ^*). Calculations similar to those above show that $U^* C_\varphi^*|_{zH^2(\beta)} U$ agrees with $T_\nu C_\sigma$ on polynomials, a dense subset of $H^2(\gamma)$. Thus $T_\nu C_\sigma$ is bounded on $H^2(\gamma)$, and since T_ν is invertible in $H^2(\gamma)$, C_σ is bounded as well. □

In the next result we apply Theorem D′ to consider the question of automorphism invariance (boundedness of composition operators with automorphism symbols) on weighted Hardy spaces. The automorphisms of D are the one-to-one analytic maps of D onto D; by the Schwarz Lemma they are all linear fractional maps.

THEOREM 3.3. *Suppose $\{\gamma(n)\}$ is a weighted sequence satisfying $\frac{\gamma(n+1)}{\gamma(n)} \to 1$ as $n \to \infty$. Then:*

(a) *If $\lim_{n\to\infty} n^A\gamma(n) = 0$ for all $A > 0$, no non-rotation automorphism of D gives a bounded composition operator on $H^2(\gamma)$.*

(b) *If* $\lim_{n\to\infty} \frac{n^A}{\gamma(n)} = 0$ *for all* $A > 0$, *no non-rotation automorphism gives a bounded composition operator on* $H^2(\gamma)$.

(c) *If there exists a real number* A *and* $c > 0$ *so that* $\lim_{n\to\infty} n^A \gamma(n) = c$, *then* $H^2(\gamma)$ *is automorphism invariant.*

Proof. Since a non-rotation automorphism has angular derivative less than 1 at some point of ∂D, part (a) follows from Theorem (1.1)′ of [7].

For (b) we first make the observation that it is enough to show that

$$\sigma(z) = \sigma_r(z) = \frac{z+r}{1+rz} \quad (0 < r < 1)$$

cannot induce a bounded composition operator on $H^2(\gamma)$, since any non-rotation automorphism can be written as $U_1\sigma_r U_2$ where U_1 and U_2 are rotations and $r \in (0, 1)$ (see, e.g. [3]).

Set $\varphi(z) = \frac{z-r}{1-rz}$ so that σ, φ are a dual pair. Let $\{\beta(n)\}$ be associated to $\{\gamma(n)\}$ as in Theorem D′. Let us assume, for a contradiction, that C_σ is bounded on $H^2(\gamma)$. Then C_φ is bounded on $H^2(\beta)$. But φ is a non-rotation automorphism and $n^A\beta(n) \to 0$ as $n \to \infty$ for all $A > 0$. This contradicts part (a) and therefore C_σ cannot be bounded on $H^2(\gamma)$.

Part (c) follows from the observation that if $n^A\gamma(n) \to c$ where c is finite positive constant, then $H^2(\gamma) = H^2(\tau)$, up to equivalent norm, where $\tau(n) = (n+1)^{-A}$. Since $H^2(\tau)$ is automorphism invariant [10], so is $H^2(\gamma)$. □

Next we consider linear fractional maps of D which are not automorphisms. If $\lim_{n\to\infty} \beta(n)^{1/n} = 1$, then any linear fractional map φ with $\|\varphi\|_\infty < 1$ induces a compact (indeed trace class) composition operator on $H^2(\beta)$ (Theorem 4.7, [2]). Thus we are interested in the case $\|\varphi\|_\infty = 1$. We distinguish several cases via the fixed point structure of φ. Relative to $\mathbf{C} \cup \{\infty\}$, and including multiplicity, φ has 2 fixed points. For linear fractional maps taking D to D, various possibilities exist for the location of these fixed points.

If φ has one fixed point in D and one in ∂D, then by consideration of properties (3.3) and (3.4), the dual map must have Denjoy-Wolff point in ∂D with angular derivative strictly less than 1 there, since the derivative of φ at the boundary fixed point must be greater than 1 by Wolff's Lemma (Theorem 2.48, [2]). So C_σ cannot be bounded on any space $H^2(\beta)$ where $n^A\beta(n) \to 0$ as $n \to \infty$ for all $A > 0$. By Theorem D′, C_φ is not bounded on any small space $H^2(\gamma)$ where $n^A/\gamma(n) \to 0$ for all $A > 0$ and $\{\gamma(n)\}$ satisfies the hypothesis of Lemma 3.1.

If φ has one fixed point in ∂D and one outside $\overline{D}$, its dual map σ has one fixed point on ∂D and one in D. By Wolff's Lemma σ has angular derivative strictly greater than 1 at its boundary fixed point, and φ has angular derivative strictly less than 1 at its boundary fixed point. Since $|\sigma(\zeta)| = 1$ only for ζ equal to boundary fixed point of σ (we are assuming φ, and therefore σ, is not an automorphism of D)

we know (by Theorem B) that C_σ is compact on any large weighted Bergman space $A^2_G(D)$ defined from a fast regular weight $G(r)$. Setting $\gamma(n) = \|z^n\|_G$ and noting that $\gamma(n)$ is monotone decreasing with $\lim_{n\to\infty} \gamma(n)^{1/n} = 1$ (for the latter fact, see Dynkin's Lemma, p. 182, [7]), we conclude that C_φ is compact on any small space $H^2(\beta)$ where

$$\gamma(n-1)\,\beta(n) \to c \in (0,\infty) \text{ as } n \to \infty.$$

The affine maps $\varphi(z) = rz + 1 - r$ $(0 < r < 1)$ fall into this category (fixed points ∞ and 1) and we are able to conclude that C_φ is compact on certain small weighted Hardy spaces not covered by Theorem C.

For a specific example, consider the fast regular weights

$$G(r) = e^{-(\log \frac{1}{1-r})^\alpha} \quad \text{for } 1 < \alpha < 2.$$

The next result estimates the weight sequence $\{\gamma(n)\}$ for these choices of G.

LEMMA 3.4. *There are positive constants c_1, c_2 so that*

$$\frac{c_2}{2n+2}\, e^{-[\log(2n+2)]^\alpha} \le \gamma(n)^2 \le \frac{c_1}{2n+2}\, e^{-[\log(2n+2)-\alpha\log\log(2n+2)]^\alpha}$$

for large n.

Proof. For the lower estimate we have

$$\int_0^1 r^{2n+1} G(r)\,dr \ge \int_0^\delta r^{2n+1} G(r)\,dr \ge G(\delta)\,\frac{\delta^{2n+2}}{2n+2}.$$

Setting $\delta = 1 - \frac{1}{2n+2}$ $(n$ large$)$ gives the lower bound.

For the upper estimate,

$$\begin{aligned}\int_0^1 r^{2n+1} G(r)\,dr &= \int_0^\lambda r^{2n+1} G(r)\,dr + \int_\lambda^1 r^{2n+1} G(r)\,dr \\ &\le G(0)\,\frac{\lambda^{2n+2}}{2n+2} + G(\lambda)\,\frac{1-\lambda^{2n+2}}{2n+2}\end{aligned}$$

Choosing $\lambda = 1 - \frac{[\log(2n+2)]^\alpha}{2n+2}$ $(n$ large$)$ gives the upper estimate. □

LEMMA 3.5. *If $H^2(\beta)$ is the small weighted Hardy space with*

$$\beta(0) = 1, \quad \beta(n) = \frac{1}{\gamma(n-1)}$$

where $\{\gamma(n)\}$ is the weight sequence for $G(r) = e^{-(\log \frac{1}{(1-r)})^\alpha}$ $(1 < \alpha < 2)$, then for $\varphi(z) = rz + (1-r)$ $(0 < r < 1)$, C_φ is compact on $H^2(\beta)$ yet

$$\sum_{n=0}^\infty \frac{\beta([an])}{\beta(n)} = \infty \quad \textit{for all } 0 < a < 1.$$

Proof. The first part of the conclusion follows from Corollary 3.2: $\{\gamma(n)\}$ is decreasing with $\lim_{n\to\infty}\gamma(n)^{1/n} = 1$. The dual map σ induces a compact composition operator on $H^2(\gamma)$ (by Theorem B), so C_φ is compact on $H^2(\beta)$.

For the second part of the conclusion we use Lemma 3.4 to estimate

$$\sum_{n=0}^{\infty}\frac{\beta([an])}{\beta(n)} = \sum_{n=0}^{\infty}\frac{\gamma(n-1)}{\gamma([an]-1)}.$$

Using the lower estimate for $\gamma(n-1)^2$ and the upper estimate for $\gamma([an]-1)^2$ we see that

$$\frac{\gamma(n-1)^2}{\gamma([an]-1)^2} \geq c\, e^{(\log 2[an] - \alpha\log\log 2[an])^\alpha - (\log(2n))^\alpha}.$$

For large n this is bounded below by $c\,\frac{1}{n}$, since $1 < \alpha < 2$. Therefore

$$\sum_{n=0}^{\infty}\frac{\beta([an])}{\beta(n)} = \infty. \qquad \square$$

The affine maps $\varphi(z) = az + b$ mapping D into D give bounded composition operators on any small space $H^2(\beta)$ for which $\beta(n)$ is (eventually) increasing with $\lim_{n\to 1}\beta(n)^{1/n} = 1$. To see this, note that the dual map

$$\sigma(z) = \frac{\bar{a}z}{-\bar{b}z + 1}$$

takes 0 to 0. A result of C. Cowen (Cor. 3.3, [2]) shows that C_σ is bounded on $H^2(\gamma)$, where

$$\gamma(0) = 1,\ \gamma(n) = \frac{1}{\beta(n+1)},$$

since $\{\gamma(n)\}$ is (eventually) decreasing. By Theorem D′, C_φ is bounded on $H^2(\beta)$.

If φ has one fixed point on ∂D of multiplicity two, then the derivative at the fixed point is 1. Notice that the dual map σ has exactly the same fixed point character. Conjugating by a rotation, we may assume the fixed point is 1. The map then corresponds, under the conformal map $z \to \frac{1+z}{1-z}$ of D onto $RHP = \{z\colon Re(z) > 0\}$ to a translation $w \to w + a$ of RHP, with $Re(a) \geq 0$. When $Re(a) = 0$, φ is an automorphism of D, so we are interested in the case $Re(a) > 0$. We have

$$\varphi(z) = \frac{(2-a)z + a}{-az + (2+a)} = 1 - \frac{2}{a} + \frac{4}{a}\,\frac{1}{(-az + 2 + a)}.$$

If a is real it is not hard to verify that

$$M(r) = \max_\theta |\varphi(re^{i\theta})|$$

is attained for $z = r$ and thus

$$M(r) = \frac{2r - ar + a}{-ar + 2 + a}.$$

We wish to apply results of Kriete and MacCluer (Theorems 5.8 and 5.9, [2]) to study C_φ on large weighted Bergman spaces A^2_G. To this end we compute

$$\frac{1}{1 - M(r)} - \frac{1}{1 - r} = \frac{a}{2}. \tag{3.8}$$

Consider the following families of fast regular weights:

$$G(r) = e^{-c(\frac{1}{1-r})^\alpha} \quad (c > 0, \ \alpha > 0) \tag{3.9}$$

$$G(r) = e^{-c(\log \frac{1}{1-r})^\alpha} \quad (c > 0, \ \alpha > 1) \tag{3.10}$$

Using (3.8) and the Mean Value Theorem to estimate $\frac{G(r)}{G(M(r))}$ we see that for $G(r)$ in (3.9) we have

$$\lim_{r \to 1} \frac{G(r)}{G(M(r))} = \begin{cases} 1, & \text{if } 0 < \alpha < 1 \\ e^{\frac{ac}{2}}, & \text{if } \alpha = 1 \\ \infty, & \text{if } \alpha > 1 \end{cases} .$$

By the results mentioned above (Theorems 5.8 and 5.9, [2]), this says that C_φ is bounded (but not compact) when $0 < \alpha \leq 1$ and unbounded for $\alpha > 1$. Similarly, for $G(r)$ in (3.10) we see that $\frac{G(r)}{G(M(r))} \to 1$, and thus C_φ is bounded on A^2_G.

The computation of $M(r)$ is considerably more complicated when $Re(a) > 0$ but a is not purely real. Nevertheless, *Mathematica* calculations verify that the real part $g(r)$ of the point $z(r)$, where $M(r)$ is attained, satisfies

$$\lim_{r \to 1} \frac{1 - g(r)}{1 - r} = 1 \quad \text{and} \quad \lim_{r \to 1} \frac{r - g(r)}{(1 - r)^2} = \frac{[\mathrm{Im}(a)]^2}{2[\mathrm{Re}(a)]^2} .$$

From this it can be shown that

$$\lim_{r \to 1} \left(\frac{1}{1 - M(r)} - \frac{1}{1 - r} \right) = \frac{\mathrm{Re}(a)}{2} + \frac{[\mathrm{Im}(a)]^2}{2\mathrm{Re}(a)} .$$

Thus, as before, $\lim_{r \to 1} G(r)/G(M(r))$ is finite for G as in (3.9) with $\alpha \leq 1$ or for G as in (3.10), while $\lim_{r \to 1} G(r)/G(M(r)) = \infty$ for G as in (3.9) with $\alpha > 1$. Hence we reached exactly the same conclusions about C_φ acting on A^2_G as we did for the case $\mathrm{Im}(a) = 0$. Once we understand the properties of C_φ on $A^2(G)$, Theorem D′ allows us to draw conclusions about C_σ on the small spaces $H^2(\beta)$, $\beta(n) \cong 1/\gamma(n - 1)$ (where $\{\gamma(n)\}$ is the weight sequence for the fast regular weights $G(r)$ in (3.9) and (3.10)), as we did for C_φ on $A^2_G = H^2(\gamma)$.

Finally, a linear fractional map φ of D into D may have one fixed point in D and one outside $\overline{D}$. If $\|\varphi\|_\infty = 1$, there exists $e^{i\theta}$ so that $\psi = e^{i\theta}\varphi$ has a fixed point on ∂D and ψ will be in one of the three previously discussed cases. The boundedness and compactness properties of C_φ and C_ψ are the same on any weighted Hardy space, and the angular derivative of φ at the point ζ where $|\varphi(\zeta)| = 1$ is the same as the angular derivative of ψ at its boundary fixed point. All three cases can occur. For example, if

$$\varphi(z) = -\frac{1}{2}z + \frac{1}{2},$$

then φ fixes $\frac{1}{3}$ and ∞ while $\psi(z) = -\varphi(z)$ fixes -1 and ∞ (one boundary point and one exterior point). If

$$\varphi(z) = \frac{-z}{2+z},$$

then φ fixes 0 and -3 while $\psi(z) = -\varphi(z)$ fixes 0 and -1 (one boundary point and one interior point). If

$$\varphi(z) = \frac{z+1}{z-3},$$

then φ fixes $2 \pm \sqrt{5}$ while $\psi(z) = -\varphi(z)$ has fixed point one of multiplicity 2.

This concludes the discussion of all possible configurations for the fixed points of a non-automorphism φ which is a linear fractional map of D into D.

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