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ON THE $L^{N/2}$-NORM OF SCALAR CURVATURE

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1. Introduction

Let M be a compact n-manifold without boundary. For a Riemannian metric g on M, the curvature tensor, Ricci curvature tensor and scalar curvature of g are denoted by $R(g)$, $\mathrm{Ric}(g)$ and $S(g)$, respectively. A natural and interesting problem in Riemannian geometry is the relations between the topology of the manifold M and curvatures of g. Often the topology of M would impose certain restrictions on the behavior of curvatures of the metric g. The Gauss-Bonnet theorem provides a beautiful relation in this direction. As complexity of the Gauss-Bonnet integrand increases with dimension, it would be desirable to obtain simpler but not "sharp" relations. Indeed, there have been many interests on $L^{\frac{n}{2}}$-curvature pinching and bounds on topological quantities by integral norms of curvatures. In this article, we study some questions on obtaining lower bounds on $L^{\frac{n}{2}}$-norms of the Ricci curvature and scalar curvature. There are some rather general and well-known problems: given a compact n-manifold M, for a sufficiently large class of Riemannian metrics g on M, are there positive lower bounds on

(1) $\mathrm{Vol}\,(M, g)$, provided $K_g \ge -1$ or $\mathrm{Ric}(g)_{ij} \ge -(n-1)g_{ij}$ or $S(g) \ge -n(n-1)$, where K_g is the sectional curvature of (M, g),
(2) $\int_M |S(g)|^{\frac{n}{2}}\, dv_g$ or
(3) $\int_M |\mathrm{Ric}(g)|^{\frac{n}{2}}\, dv_g$?

We note that (2) and (3) are both scale invariant, while a lower bound on curvature is required in (1) so that $\mathrm{Vol}\,(M, g)$ will not go to zero by scaling. As a flat torus would not have positive lower bounds on (1), (2) and (3), some restrictions are needed on the manifold M. Some suggestions are:

(a) M admits a locally symmetric metric of strictly negative sectional curvature;
(b) M admits an Einstein metric of negative sectional curvature;
(c) or simply M admits a metric of negative sectional curvature.

Recently, Besson, Courtois and Gallot [5], [6] have demonstrated that if (M, h) is a compact hyperbolic n-manifold ($n \ge 3$), then for any Riemannian metric g on M with $\mathrm{Ric}(g) \ge -(n-1)g$, one has $\mathrm{Vol}(M, g) \ge \mathrm{Vol}\,(M, h)$ and equality holds if

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and only if (M, g) is isometric to (M, h). In this note, we mainly consider question (2) and (3), under one of the conditions in (a), (b) or (c) and with restrictions on the choices of the Riemannian metric g by certain curvature assumptions or in certain conformal classes. Our method is to investigate relations between the $L^{\frac{n}{2}}$-norms of scalar curvatures for different metrics with that of a standard metric.

The Gauss-Bonnet theorem for two-manifolds shows that if M is a compact surface and h is a metric on M with constant negative curvature $S(h)$ then

$$\int_M |S(g)|\, dv_g \geq \int_M |S(h)|\, dv_h. \tag{1.1}$$

Let $\chi(M)$ be the Euler characteristic of M. The Gauss-Bonnet theorem for higher dimensions (n even) [16] states that

$$\begin{aligned} c_n \chi(M) = \int_M \sum_{\sigma \in \mathcal{C}_n} \sum_{\tau \in \mathcal{C}_n} \varepsilon(\sigma)\varepsilon(\tau) R(g)_{\sigma(1)\sigma(2)\tau\circ\sigma(1)\tau\circ\sigma(2)} \cdots \\ R(g)_{\sigma(n-1)\sigma(n)\tau\circ\sigma(n-1)\tau\circ\sigma(n)}\, dv_g, \end{aligned} \tag{1.2}$$

where c_n is a dimension constant, $\mathcal{C}_n$ is the set of all permutations on $\{1, 2, \ldots, n\}$ and $\varepsilon(\tau)$ is the sign of $\tau \in \mathcal{C}_n$. A decomposition of the curvature tensor gives

$$R(g)_{ijkl} = W(g)_{ijkl} + Z(g)_{ijkl} + U(g)_{ijkl}, \tag{1.3}$$

where $W(g)$ is the Weyl curvature tensor and

$$U(g)_{ijkl} = \frac{S(g)}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}), \tag{1.4}$$

$$Z(g)_{ijkl} = \frac{1}{n-2}(\mathbf{z}(g)_{ik}g_{jl} + \mathbf{z}(g)_{jl}g_{ik} - \mathbf{z}(g)_{il}g_{jk} - \mathbf{z}(g)_{jk}g_{il}), \tag{1.5}$$

where $\mathbf{z}(g)$ is the trace-free Ricci tensor given by

$$\mathbf{z}(g)_{ij} = \mathrm{Ric}(g)_{ij} - \frac{S(g)}{n} g_{ij}. \tag{1.6}$$

Let $x \in M$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis for the tangent space of M above x. We have

$$U(g)_{ijkl} = \frac{S(g)}{n(n-1)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \quad \text{at } x.$$

If we apply (1.3), then at the point x we have

$$\begin{aligned} \sum_{\sigma \in \mathcal{C}_n} \sum_{\tau \in \mathcal{C}_n} \varepsilon(\sigma)\varepsilon(\tau) R(g)_{\sigma(1)\sigma(2)\tau\circ\sigma(1)\tau\circ\sigma(2)} \cdots R(g)_{\sigma(n-1)\sigma(n)\tau\circ\sigma(n-1)\tau\circ\sigma(n)} \\ = C_o S(g)^{\frac{n}{2}} + P(W(g)_{ijkl}, Z(g)_{ijkl}, U(g)_{ijkl}, g_{ij}), \end{aligned} \tag{1.7}$$

where P is a certain polynomial function and C_o is a constant that depends on n only. Putting (1.7) into the Gauss-Bonnet formula, we have

$$\begin{aligned}\chi(M) &= \int_M C_o S(g)^{\frac{n}{2}}\, dv_g + \int_M P(W(g)_{ijkl}, Z(g)_{ijkl}, U(g)_{ijkl}, g_{ij})\, dv_g \\ &= \int_M C_o S(g')^{\frac{n}{2}}\, dv_{g'} + \int_M P(W(g')_{ijkl}, Z(g')_{ijkl}, U(g')_{ijkl}, g'_{ij})\, dv_{g'},\end{aligned}$$

where g' is another Riemannian metric on M. In general, the above formula is too complicated to given effective bounds on $L^{\frac{n}{2}}$-norms of scalar curvatures.

THEOREM 1. *Let (M, h) be a compact hyperbolic n-manifold with n being even.*

(1) *Let $n = 4$. For any conformally flat metric g on M, we have*

$$\int_M |S(g)|^2\, dv_g \geq \int_M |S(h)|^2\, dv_h,$$

and equality holds if and only if g is, up to a positive constant, isometric to h.

(2) *Let $n \geq 4$. For any conformally flat metric g on M, we have*

$$\int_M |\operatorname{Ric}(g)|^{\frac{n}{2}}\, dv_g \geq c_n \int_M |\operatorname{Ric}(h)|^{\frac{n}{2}}\, dv_h,$$

where c_n is a positive constant that depends on n only.

THEOREM 2. *Let (M, h) be a compact hyperbolic n-manifold with n being even. There exists a positive constant c'_n which depends on n only such that for any metric g on M with non-positive sectional curvature we have*

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \geq c'_n \int_M |S(h)|^{\frac{n}{2}}\, dv_h.$$

Besson, Courtois and Gallot [4] have shown that if (M, g) is a compact Einstein manifold with negative sectional curvature, then for any metric g' in a neighborhood of g, we have

$$\int_M |S(g')|^{\frac{n}{2}}\, dv_{g'} \geq \int_M |S(g)|^{\frac{n}{2}}\, dv_g. \tag{1.8}$$

In the proof of this result, they investigated the following.

(I) (1.8) holds whenever g' is conformal to g; i.e., if $g' = u^{\frac{4}{n-2}} g$ for some smooth function $u > 0$ and if $S(g)$ is a *negative* constant, then we have

$$\int_M |S(g')|^{\frac{n}{2}}\, dv_{g'} \geq \int_M |S(g)|^{\frac{n}{2}}\, dv_g.$$

Then they used the second variation formula to investigate the local behavior of the $L^{n/2}$-norm of $S(g)$. Partially motivated by their results, we consider the change of

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \qquad \text{and} \qquad \int_M |\operatorname{Ric}(g)|^{\frac{n}{2}}\, dv_g$$

under Ricci flow and conformal change of metrics when $S(g)$ is a *positive* constant. The Ricci flow have been considered by Hamilton [11] and many other authors. It has been proven to be very useful in deforming metrics into standard metrics, especially when the original metric is close to a standard metric. For example, it has been shown in [14] and [17] that the Ricci flow starting near a Einstein metric of negative sectional curvature always converges to it. We obtain the following behaviors of $L^{\frac{n}{2}}$-norms on curvatures under the Ricci flow.

THEOREM 3. *Let (M, g) be a compact Riemannian manifold with $S(g) < 0$. Let g_t be the Ricci flow starting at g. If $S(g_t) \leq 0$ then*

$$\frac{d}{dt}\int_M |S(g_t)|^{\frac{n}{2}}\, dv_{g_t} \leq 0.$$

If we assume that the sectional curvature K_g of g is suitably pinched

$$-1-\epsilon \leq K_g \leq -1+\epsilon$$

for some $\epsilon > 0$ then

$$\frac{d}{dt}\int_M |\operatorname{Ric}(g_t)|^{\frac{n}{2}}\, dv_{g_t} \leq 0.$$

Under the above conditions, if the Ricci flow converges to a smooth metric g_o on M then

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \geq \int_M |S(g_o)|^{\frac{n}{2}}\, dv_{g_o}$$

and

$$\int_M |\operatorname{Ric}(g)|^{\frac{n}{2}}\, dv_g \geq \int_M |\operatorname{Ric}(g_o)|^{\frac{n}{2}}\, dv_{g_o}.$$

In particular, we provide an alternative proof to (1.8). In the last section, we consider conformal change of metrics when the scalar curvature is positive. An interesting question is whether Besson-Courtois-Gallot's result holds for positive scalar curvature: namely, if g' is conformal to g and g has constant positive scalar curvature, does the inequality

$$\int_M |S(g')|^{\frac{n}{2}}\, dv_{g'} \geq \int_M |S(g)|^{\frac{n}{2}}\, dv_g$$

hold?

THEOREM 4. *Let (M, g_o) be an n-manifold with $b^2 g \ge \text{Ric}\,(g) \ge a^2 g$ for some positive numbers a and b. Then for any metric $g = u^{\frac{4}{n-2}} g_o$, $u > 0$, we have*

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \ge c_n \int_M |S(g_o)|^{\frac{n}{2}}\, dv_{g_o},$$

where c_n is a positive constant that depends on a, b and n only. In general, $c_n < 1$. For the special cases that (i) g is an Einstein metric with positive scalar curvature and $g = u^{\frac{4}{n-2}} g_o$, $u > 0$, or (ii) (M, g) is a compact conformally flat manifold with positive Ricci curvature and g_o has constant positive sectional curvature then we have

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \ge \int_M |S(g_o)|^{\frac{n}{2}}\, dv_{g_o}.$$

2. Gauss-Bonnet formula

Given a compact n-manifold M with $n \ge 4$ and a Riemannian metric g on M, the Weyl conformal curvature tensor can be defined by

$$W(g)_{ijkl} = R(g)_{ijkl} - Z(g)_{ijkl} - U(g)_{ijkl},$$

where $Z(g)$ and $U(g)$ are defined in (1.4) and (1.5), respectively. Using the fact that $g^{ij}\mathbf{z}(g)_{ij} = 0$ and $g^{ik}g^{jl}R(g)_{ijkl} = S(g)$, it is easy to show that $g^{ik}g^{jl}W(g)_{ijkl} = 0$ and $g^{ik}W(g)_{ijkl} = 0$. And we have

$$|R(g)|^2 = |W(g)|^2 + |Z(g)|^2 + |U(g)|^2. \tag{2.1}$$

A direct calculation shows that

$$|U(g)|^2 = \frac{2S(g)^2}{n(n-1)}, \tag{2.2}$$

$$|Z(g)|^2 = \frac{4}{(n-2)}|\mathbf{z}(g)|^2 \qquad \text{and} \qquad |\,\text{Ric}(g)|^2 = |\mathbf{z}(g)|^2 + \frac{S(g)^2}{n}.$$

In dimension four, the Gauss-Bonnet formula [3] takes the form

$$\chi(M) = \frac{1}{8\pi^2}\int_M (|U(g)|^2 - |Z(g)|^2 + |W(g)|^2)\, dv_g, \tag{2.3}$$

where $\chi(M)$ is the Euler characteristic of M. Let h be a hyperbolic metric on M. Then

$$\chi(M) = \frac{1}{48\pi^2}\int_M S(h)^2\, dv_h, \tag{2.4}$$

where $S(h) = -4\cdot 3 = -12$. In dimension bigger than or equal to four, a Riemannian metric g is conformally flat if and only if $W(g) \equiv 0$. Then (2.2), (2.3) and (2.4) show that if g is any conformally flat metric on M, we have

$$\int_M S(g)^2\, dv_g \geq \int_M S(h)^2\, dv_h.$$

Furthermore, equality holds if and only if $\mathbf{z}(g) \equiv 0$ and $W(g) \equiv 0$, i.e., (M, g) is a hyperbolic metric. By the Mostow rigidity theorem, (M, g) is isometric to (M, h) up to a positive constant.

THEOREM 2.5. *Let (M, h) be a compact hyperbolic n-manifold and $n \geq 4$, n even. For any conformally flat metric g on M, we have*

$$\int_M |\operatorname{Ric}(g)|^{\frac{n}{2}}\, dv_g \geq c_n \int_M |\operatorname{Ric}(h)|^{\frac{n}{2}}\, dv_g,$$

where c_n is a positive constant that depends on n only.

Proof. As $n \geq 4$, the metric g is conformally flat if and only if $W(g) \equiv 0$. Therefore $R(g) = Z(g) + U(g)$. Applying the Gauss-Bonnet theorem we have

$$\chi(M) = C(n) \int_M P(Z(g), U(g))\, dv_g,$$

where $C(n)$ is a constant that depends on n only and P is a homogeneous polynomial of degree $n/2$ in the components of $Z(g)$ and $U(g)$. There exist positive constants $C_1, C_1, C_2, \ldots, C_{\frac{n}{2}}$, which depend on n only, such that

$$\begin{aligned} |\chi(M)| \leq \int_M &(C_0|Z(g)|^{\frac{n}{2}} + C_1|Z(g)|^{\frac{n}{2}-1}|U(g)| \\ &+ C_2|Z(g)|^{\frac{n}{2}-2}|U(g)|^2 + \cdots + C_{\frac{n}{2}}|U(g)|^{\frac{n}{2}})\, dv_g. \end{aligned}$$

Using (2.2) we have

$$|\operatorname{Ric}(g)| \geq (n-2)/\sqrt{4(n-2)}|Z(g)|$$

and

$$|\operatorname{Ric}(g)| \geq \sqrt{(n-1)/2}|U(g)|;$$

we have

$$|\chi(M)| \leq C \int_M |\operatorname{Ric}(g)|^{\frac{n}{2}}\, dv_g,$$

where C is a constant that depends on n only. For the hyperbolic metric h, we have $W(h) \equiv 0$, $Z(h) \equiv 0$ and $|\operatorname{Ric}(h)|^2 = S(h)^2/n$. The Gauss-Bonnet theorem gives

$$|\chi(M)| = C'(n) \int_M |S(h)|^{\frac{n}{2}}\, dv_h = C''(n) \int_M |\operatorname{Ric}(h)|^{\frac{n}{2}}\, dv_h,$$

where $C'(n)$ and $C''(n)$ are positive constants that depends on n only. Combining the two formulas we have the result. □

THEOREM 2.6. *Let (M, h) be a compact hyperbolic n-manifold of even dimension. There exists a positive constant c_n, which depends on n only, such that for any Riemannian metric g on M with nonpositive sectional curvature we have*

$$\int_M S(g)^{\frac{n}{2}}\, dv_g \geq c_n \int_M S(h)^{\frac{n}{2}}\, dv_h.$$

Proof. By (1.1), we may assume that $n \geq 4$. As h is a hyperbolic metric, $W(h) \equiv 0$ and $Z(h) \equiv 0$. The Gauss-Bonnet formula (1.2) gives

$$\chi(M) = \int_M C_o S(h)^{\frac{n}{2}}\, dv_h,$$

where C_o is a non-zero constant that depends on n only (its value can be found by applying the Gauss-Bonnet formula on S^n and the fact that $\chi(S^n) = 2$ if n is even). For the Riemannian metric g, making use of the fact that $R(g) = W(g) + Z(g) + U(g)$, the Gauss-Bonnet formula gives

$$\chi(M) = \int_M (C_o S(g)^{\frac{n}{2}} + P(W(g)_{ijkl}, Z(g)_{ijkl}, U(g)_{ijkl}, g_{ij}))\, dv_g,$$

where P is a certain polynomial such that each term contain exactly $n/2$ terms of $W(g)_{jklk}$, $Z(g)_{ijkl}$ or $U(g)_{ijkl}$. Therefore we have

$$|\chi(M)| \leq \int_M C_o |S(g)|^{\frac{n}{2}}\, dv_g + \int_M |P(W(g)_{ijkl}, Z(g)_{ijkl}, U(g)_{ijkl}, g_{ij}))|\, dv_g.$$

From (2.1), $|R(g)| \geq |W(g)|$, $|R(g)| \geq |Z(g)|$ and $|R(g)| \geq |U(g)|$, there exists a positive constant C_n that depends on n only such that

$$|P(W(g)_{ijkl}, Z(g)_{ijkl}, U(g)_{ijkl}, g_{ij}))| \leq C_n |R(g)|^{\frac{n}{2}}.$$

Given a point $x \in M$, we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for the tangent space of M above x. Let σ_{ij} be the sectional curvature of the plane spanned by e_i and e_j, $i \neq j$, with respect to the Riemannian metric g on M. Assume that $\sigma_{ij} \leq 0$. We may also assume that σ_{12} is the minimum of the sectional curvatures at the point x. We have

$$|S(g)| = \left| \sum_{i,j,i \neq j} \sigma_{ij} \right| \geq |\sigma_{12}|.$$

Let $\sigma(\mathbf{u}, \mathbf{v})$ be the sectional curvature of the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ in the tangent space of M above x. Then [7] we have

$$\begin{aligned} R(g)_{ijkl} = & \frac{1}{6}\{4[\sigma(e_i+e_l, e_j+e_k) - \sigma(e_j+e_l, e_i+e_k)] \\ & -2[\sigma(e_i, e_j+e_k) + \sigma(e_j, e_i+e_l) + \sigma(e_k, e_i+e_l) + \sigma(e_l, e_j+e_k)] \\ & +2[\sigma(e_i, e_j+e_l) + \sigma(e_j, e_k+e_l) + \sigma(e_k, e_j+e_l) + \sigma(e_l, e_i+e_k)] \\ & +\sigma_{ik} + \sigma_{jl} - \sigma_{il} - \sigma_{jk}\}. \end{aligned}$$

There exists a positive constant C' which depends on n only, and with $g_{ij} = \delta_{ij}$, such that we obtain

$$|R(g)|^2 = \sum_{ijkl} R_{ijkl}R_{ijkl} \leq C'(\sigma_{12})^2 \leq C'|S(g)|^2,$$

and so

$$|P(W(g)_{ijkl}, Z(g)_{ijkl}, U(g)_{ijkl}, g_{ij}))| \leq C_n C'|S(g)|^{\frac{n}{2}}.$$

Thus

$$|\chi(M) \leq (|C_o| + C_n C') \int_M |S(g)|^{\frac{n}{2}}\, dv_g,$$

or

$$\int_M |S(h)|^{\frac{n}{2}}\, dv_h \leq C \int_M |S(g)|^{\frac{n}{2}}\, dv_g,$$

where $C = 1 + C_n C'/|C_o|$ is a positive constant that depends on n only. □

Remark. From the proof of the above theorem, one can replace the condition of non-positive sectional curvature by a pinching condition that the absolute value of sectional curvature of any 2-plane above a point $x \in M$ is lesser than or equal to $c_n|S(g)(x)|$, a positive constant times the absolute value of the scalar curvature at that point. Then we have

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \geq c' \int_M |S(h)|^{\frac{n}{2}}\, dv_h,$$

where c' is now a constant that depends both on n and c_n.

Remark. It is easy to see that the same result in theorem 2.6 holds for *conformally flat* metrics of nonpositive Ricci curvature.

The Gauss-Bonnet formula yields the following estimate on the $L^{n/2}$-norm of scalar curvature.

LEMMA 2.7. *For an even integer n bigger than two, let (M, g) be a compact n-manifold with $\chi(M) \neq 0$. Then there exist positive constants δ_n and ϵ_n, depending on n, which can be chosen a priori such that if*

$$\int_M |Z(g)|^{\frac{n}{2}}\, dv_g \leq \delta_n \qquad \text{and} \qquad \int_M |W(g)|^{\frac{n}{2}}\, dv_g \leq \epsilon_n$$

then

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \geq c_n,$$

where c_n is a positive constant that depends on n only.

Proof. As $R(g) = W(g) + Z(g) + U(g)$, applying the Gauss-Bonnet theorem we have

$$\chi(M) = C(n) \int_M P(W(g), Z(g), U(g))\, dv_g,$$

where $C(n)$ is a non-zero constant that depends on n only and P is a homogeneous polynomial of degree $n/2$ in the components of $W(g)$, $Z(g)$ and $U(g)$. There exist positive constants $C'_o, C_o, C_1, C_2, \ldots, C_{\frac{n}{2}}$ and $C(n_1, n_2, n_3)$, which depend on n, n_1, n_2 and n_3 only, such that

$$\begin{aligned}(2.8)\quad |\chi(M)| \leq & \int_M (C'_0 |U(g)|^{\frac{n}{2}} \\ & + \sum_{n_1,n_2,n_3} C(n_1,n_2,n_3) \int_M |U(g)|^{n_1} |Z(g)|^{n_2} |W(g)|^{n_3})\, dv_g \\ & + \int_M (C_o |Z(g)|^{\frac{n}{2}} + C_1 |Z(g)|^{\frac{n}{2}-1} |W(g)| \\ & + C_2 |z(g)|^{\frac{n}{2}-2} |W(g)|^2 + \cdots + C_{\frac{n}{2}} |W(g)|^{\frac{n}{2}})\, dv_g,\end{aligned}$$

where n_1, n_2, and n_3 are positive integers such that $n_1 + n_2 + n_3 = n/2$ and $n_1 < n/2$. For positive numbers s, t, p and q such that

$$s + t = \frac{n}{2} \qquad \text{and} \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

a calculation shows that if $tq = n/2$, then we have $sp = n/2$ as well. Appling the Hölder's inequality to (2.8) (twice to the terms with n_1, n_2, and n_3), we have

$$\begin{aligned}(2.9)\quad |\chi(M)| \leq\ & C'_o \int_M |U(g)|^{\frac{n}{2}}\, dv_g \\ & + \sum_{n_1,n_2,n_3} C(n_1,n_2,n_3) \left(\int_M |U(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{pn_1}} \left(\int_M |Z(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{rn_2}}\end{aligned}$$

$$\times \left(\int_M |W(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{q_{n_3}}} + C_o \int_M |Z(g)|^{\frac{n}{2}}\, dv_g$$
$$+ C_1 \left(\int_M |Z(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{p_1}} \left(\int_M |W(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{q_1}} + \cdots$$
$$+ C_{\frac{n}{2}-1} \left(\int_M |Z(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{p_{\frac{n}{2}-1}}} \left(\int_M |W(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{q_{\frac{n}{2}-1}}}$$
$$+ C_{\frac{n}{2}} \int_M |W(g)|^{\frac{n}{2}}\, dv_g,$$

where

$$p_{n_1}, r_{n_2}, q_{n_3}, \quad p_1, \ldots, p_{\frac{n}{2}-1} \quad \text{and} \quad q_1, \ldots, q_{\frac{n}{2}-1}$$

are positive constants specifed in the Hölder's inequality. If we choose δ_n and ϵ_n (which depend on $C_o, C_1, \ldots, C_{n/2}$, i.e., depend on n only) sufficiently small so that

$$\int_M |Z(g)|^{\frac{n}{2}}\, dv_g \le \delta_n \qquad \text{and} \qquad \int_M |W(g)|^{\frac{n}{2}}\, dv_g \le \epsilon_n$$

then

$$C_o \int_M |Z(g)|^{\frac{n}{2}}\, dv_g + C_1 \left(\int_M |Z(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{p_1}} \left(\int_M |W(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{q_1}} + \cdots$$
$$+ C_{\frac{n}{2}-1} \left(\int_M |Z(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{p_{\frac{n}{2}-1}}} \left(\int_M |W(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{1}{q_{\frac{n}{2}-1}}}$$
$$+ C_{\frac{n}{2}} \int_M |W(g)|^{\frac{n}{2}}\, dv_g \le \frac{1}{2},$$

and the fact that

$$|U(g)|^2 = \frac{2S(g)^2}{n(n-1)}$$

(2.9) gives

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \ge 2c_n \left(|\chi(M)| - \frac{1}{2}\right) \ge c_n,$$

as $\chi(M) \neq 0$ and hence $|\chi(M)| \ge 1$. Here c_n is a positive constant that depends on n only. □

COROLLARY 2.10. *For an even integer n bigger than two, let (M, g) be a compact Einstein n-manifold with* $\mathrm{Ric}(g) = \pm(n-1)g$. *If $\chi(M) \neq 0$ and*

$$\int_M |W(g)|^{\frac{n}{2}}\, dv_g \le \frac{1}{2C_{\frac{n}{2}}}$$

then $\text{Vol}(M, g) \geq c'_n$, *where* $C_{\frac{n}{2}}$ *is the same constant as in* (2.9) *and* c'_n *is a positive constant that depends on n only.*

Proof. As (M, g) is an Einstein manifold, we have $Z(g) = 0$. Therefore in (2.9), the terms involving $Z(g)$ vanish and we just need

$$\int_M |W(g)|^{\frac{n}{2}}\, dv_g \leq \frac{1}{2C_{\frac{n}{2}}}$$

to conclude that

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \geq c_n.$$

Using the fact that

$$|S(g)| = n(n-1)$$

for an Einstein manifold with $\text{Ric}(g) = \pm(n-1)g$, we obtain the result. □

COROLLARY 2.11. *For an even integer n bigger than two, let* (M, g) *be a compact Einstein n-manifold with* $\text{Ric}(g) = (n-1)g$ *and* $\chi(M) \neq 0$. *Then there exists a positive number* ε_n, *which depends on n only, such that if*

$$\int_M |W(g)|^{\frac{n}{2}}\, dv_g \leq \varepsilon_n,$$

then g has constant positive sectional curvature. In the case $n = 4$, *we can drop the assumption that* $\chi(M) \neq 0$.

Proof. If we take $\varepsilon_n < 1/(2C_{\frac{n}{2}})$, then Corollary (2.9) shows that Vol $(M, g) \geq c'_n$ for some positive constant c'_n that depends on n only. A result in [15] shows that there exists a positive constant c''_n which depends on n only, such that if

$$\int_M |W(g)|^{\frac{n}{2}}\, dv_g \leq c''_n \text{Vol}(M, g),$$

then g is a metric of constant sectional curvature one. We can take $\epsilon_n = \min\{c'_n c''_n, 1/(2C_{\frac{n}{2}})\}$. If $n = 4$, then the Gauss-Bonnet formula for an Einstein metric has the form

$$\chi(M) = \frac{1}{8\pi^2} \int_M (|U(g)|^2 + |W(g)|^2)\, dv_g.$$

It follows that $\chi(M) \neq 0$ if $\text{Ric}(g) = (n-1)g$. □

Remark. Similar pinching results are obtained in [12] and [9].

3. Ricci curvature flow

Let (M, g_o) be a compact Riemannian manifold. In this section we consider the Ricci curvature flow

$$\frac{\partial g}{\partial t} = -2\mathbf{z}(g) - \frac{2\delta S(g)}{n} g, \qquad g(0) = g_o, \tag{3.1}$$

where $\mathbf{z}(g) = \mathrm{Ric}(g) - [S(g)/n]g$ is the trace free Ricci tensor as in Section 1 and

$$\delta S(g) = S(g) - \frac{\int_M S(g)\, dv_g}{\int_M dv_g}.$$

The Ricci curvature flow has been studied extensively by Hamilton, Huisken, Margerin, Nishikawa, Shi, Ye, and many others in respect to the questions of long time existence and convergence; we refer to [17] for comprehensive references. It has been shown that if (M, h) is a compact Einstein manifold of strictly negative sectional curvature, then there exists an open neighborhood of h in the space of smooth metrics with C^∞-norm such that each metric g_o in that open neighborhood converges to h under the Ricci curvature flow (3.1) [14], [17]. Furthermore, we can choose an open neighborhood such that the Ricci curvature remains negative during the Ricci curvature flow.

LEMMA 3.2. *For $n \geq 4$, let M be a compact n-manifold. Let g be a solution to the Ricci curvature flow equation* (3.1) *on the time interval* $(0, t')$, *where t' may equal infinity. Assume that* $\lim_{t \to t'} g = g'$ *is a smooth Riemannian metric on M. If* $S(g) < 0$ *for $t \in (0, t')$, then*

$$\frac{d}{dt} \int_M |S(g)|^{\frac{n}{2}}\, dv_g \leq 0 \qquad \textit{for all} \quad t \in (0, t').$$

Hence

$$\int_M |S(g_o)|^{\frac{n}{2}}\, dv_{g_o} \geq \int_M |S(g')|^{\frac{n}{2}}\, dv_{g'}.$$

Proof. From (3.1) (see [11], [16]) we have

$$\frac{dS(g)}{dt} = \Delta S(g) + 2|\mathbf{z}|^2 + \frac{2\delta S(g)}{n} S(g). \tag{3.3}$$

As $\mathbf{z}(g)$ is trace-free, we have

$$(dv_g)' = \frac{1}{2}\mathrm{tr}_g \left(\frac{dg}{dt}\right) dv_g = -\delta S(g). \tag{3.4}$$

Therefore

$$\begin{aligned}
\frac{d}{dt}\int_M |S(g)|^{\frac{n}{2}}\,dv_g &= \int_M \frac{n}{2}|S(g)|^{\frac{n}{2}-1}\frac{d}{dt}|S(g)|\,dv_g + \int_M |S(g)|^{\frac{n}{2}}(dv_g)' \\
&= \int_M \frac{n}{2}|S(g)|^{\frac{n}{2}-1}\left(-\frac{d}{dt}S(g)\right)\,dv_g \\
&\quad - \int_M |S(g)|^{\frac{n}{2}}(\delta S(g))\,dv_g \qquad (\text{as } S(g) < 0) \\
&= \int_M \frac{n}{2}|S(g)|^{\frac{n}{2}-1}\left(-\Delta S(g) - 2|\mathbf{z}|^2 - \frac{2\delta S(g)}{n}S(g)\right)\,dv_g \\
&\quad - \int_M |S(g)|^{\frac{n}{2}}(\delta S(g))\,dv_g \\
&= -\int_M \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)|S(g)|^{\frac{n}{2}-2}|\nabla|S(g)||^2 \\
&\quad -2\int_M \frac{n}{2}|S(g)|^{\frac{n}{2}-1}|\mathbf{z}|^2\,dv_g \le 0,
\end{aligned}$$

as $-\Delta S(g) = \Delta|S(g)|$. □

THEOREM 3.5 [4]. *For $n \geq 4$, let (M, h) be a compact Einstein n-manifold of strictly negative sectional curvature. Then there exists an open neighborhood of h in the space of smooth metrics on M with C^∞-norm such that for any metric g in the open neighborhood,*

$$\int_M |S(g)|^{\frac{n}{2}}\,dv_g \geq \int_M |S(h)|^{\frac{n}{2}}\,dv_h.$$

Proof. The existence of such an open neighborhood of h for which the the Ricci curvature flow (3.1) converges to h is shown in [17]. Furthermore, we may choose the open neighborhood such that the scalar curvature remains negative during the Ricci curvature flow. Then we can apply Lemma 3.2.

THEOREM 3.6. *For $n \geq 4$, let (M, h) be a compact hyperbolic n-manifold. Then there exists an open neighborhood of h in the space of smooth metrics on M with C^∞-norm such that for any metric g_o in the open neighborhood, if g is a solution to the Ricci curvature flow* (3.1) *with initial condition g_o, then*

$$\frac{d}{dt}\int_M |\operatorname{Ric}(g)|^{\frac{n}{2}}\,dv_g \le 0.$$

Proof. As $|\operatorname{Ric}(g)|^2 = |\mathbf{z}(g)|^2 + S(g)^2/n$, we have

$$\frac{d}{dt}(|\operatorname{Ric}(g)|^{\frac{n}{2}}) = \frac{d}{dt}((|\operatorname{Ric}(g)|^2)^{\frac{n}{4}})$$

$$
\begin{aligned}
&= \frac{n}{4}(|\operatorname{Ric}(g)|^2)^{(\frac{n}{4}-1)}\frac{d}{dt}|\operatorname{Ric}(g)|^2 \\
&= \frac{n}{4}(|\operatorname{Ric}(g)|^2)^{(\frac{n}{4}-1)}\frac{d}{dt}\left(|\mathbf{z}(g)|^2 + \frac{S(g)^2}{n}\right).
\end{aligned}
$$

We have (see [17])

$$
\frac{\partial}{\partial t}|\mathbf{z}(g)|^2 = \Delta|\mathbf{z}(g)|^2 - 2|\nabla\mathbf{z}(g)|^2 + 4Rm(\mathbf{z}(g))\cdot\mathbf{z}(g) + \frac{4}{n}\delta S(g)|\mathbf{z}(g)|^2,
$$

where $Rm(\mathbf{z}(g))\cdot\mathbf{z}(g) = g^{ii'}g^{jj'}g^{kk'}g^{ll'}R(g)_{ijkl}\mathbf{z}(g)_{i'k'}\mathbf{z}(g)_{j'l'}$. From (3.3) we have

$$
\begin{aligned}
\frac{\partial}{\partial t}|S|^2 &= 2S(g)\Delta S(g) + 4S(g)|\mathbf{z}(g)|^2 + \frac{4}{n}\delta S(g)S(g)^2 \\
&= \Delta|S(g)|^2 - 2|\nabla|S(g)||^2 + 4S(g)|\mathbf{z}(g)|^2 + \frac{4}{n}\delta S(g)S(g)^2,
\end{aligned}
$$

as $S(g) < 0$ and $\Delta u^2 = 2u\Delta u + 2|\nabla u|^2$. Therefore

$$
\begin{aligned}
\frac{d}{dt}\int_M |\operatorname{Ric}(g)|^{\frac{n}{2}}\,dv_g &= \int_M \frac{n}{4}[(|\operatorname{Ric}(g)|^2)^{(\frac{n}{4}-1)}(\Delta|\mathbf{z}(g)|^2 - 2|\nabla\mathbf{z}(g)|^2 \\
&\quad + 4Rm(\mathbf{z}(g))\cdot\mathbf{z}(g) + \frac{4}{n}\delta S(g)|\mathbf{z}(g)|^2 \\
&\quad + \frac{1}{n}(\Delta|S(g)|^2 - 2|\nabla|S(g)||^2) + \frac{4}{n}S(g)|\mathbf{z}(g)|^2 \\
&\quad + \frac{4}{n}\delta S(g)\frac{S(g)^2}{n}]\,dv_g - \int_M |\operatorname{Ric}(g)|^{\frac{n}{2}}\delta S(g)\,dv_g \\
&= \int_M \frac{n}{4}\left(\frac{n}{4}-1\right)(|\operatorname{Ric}(g)|^2)^{(\frac{n}{4}-2)}(-|\nabla\operatorname{Ric}(g)|^2 \\
&\quad +4Rm(\mathbf{z}(g))\cdot\mathbf{z}(g) - \frac{2}{n}|\nabla|S(g)||^2 + \frac{4}{n}S(g)|\mathbf{z}(g)|^2)\,dv_g,
\end{aligned}
$$

as $\Delta|\mathbf{z}(g)|^2 + \frac{1}{n}\Delta|S(g)|^2 = \Delta|\operatorname{Ric}(g)|^2$. Therefore $\frac{d}{dt}\int_M |\operatorname{Ric}(g)|^{\frac{n}{2}}\,dv_g \le 0$ if we can show that

$$
Rm(\mathbf{z}(g))\cdot\mathbf{z}(g) + \frac{1}{n}S(g)|\mathbf{z}(g)|^2 \le 0.
$$

LEMMA 3.7. *There exists a positive constant ϵ which depends on n only ($n \ge 4$) such that if (M, g) is a compact Riemannian n-manifold with sectional curvature K satisfying $-1-\epsilon \le K \le -1+\epsilon$, then*

$$
nRm(\mathbf{z}(g))\cdot\mathbf{z}(g) + S(g)|\mathbf{z}(g)|^2 \le 0.
$$

Proof. We show the case $n = 4$ first. Let $x \in M$. Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for the tangent space above x such that, at the point x,

$$g_{ij} = \delta_{ij} \qquad \text{and} \qquad \mathbf{z}(g)_{ij} = \lambda_i \delta_{ij} \qquad \text{for } 1 \le i, j \le 4.$$

Let σ_{ij} be the sectional curvature of the plane spanned by e_i and e_j. Then, at the point $x \in M$,

$$\begin{aligned} Rm(\mathbf{z}(g)) \cdot \mathbf{z}(g) &= R(g)_{ijkl}\mathbf{z}(g)_{i'k'}\mathbf{z}(g)_{j'l'}g^{ii'}g^{jj'}g^{kk'}g^{ll'} \\ &= \sum_{i \neq j} R(g)_{ijij}\mathbf{z}(g)_{ii}\mathbf{z}(g)_{jj} \\ &= \sum_{i \neq j} \sigma_{ij}\lambda_i\lambda_j. \end{aligned}$$

Therefore

$$\begin{aligned} 4Rm(\mathbf{z}(g)) \cdot \mathbf{z}(g) + S(g)|\mathbf{z}(g)|^2) &= 4\sum_{i \neq j} \sigma_{ij}\lambda_i\lambda_j + \sum_{i \neq j} \sigma_{ij}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \\ &= \sum_{i \neq j} \sigma_{ij}(4\lambda_i\lambda_j + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2). \end{aligned}$$

We need to show that

$$\sum_{i \neq j} \sigma_{ij}(4\lambda_i\lambda_j + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \le 0. \tag{3.8}$$

Assume that $-1 - \epsilon \le \sigma_{ij} \le -1 + \epsilon$ for $1 \le i, j \le 4$. Then

$$\begin{aligned} &\sigma_{12}(4\lambda_1\lambda_2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) + \sigma_{34}(4\lambda_3\lambda_4 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \\ &\quad = -2[(\lambda_1 + \lambda_2)^2 + (\lambda_3 + \lambda_4)^2] \\ &\qquad + O(\epsilon)[4(\lambda_1\lambda_2 + \lambda_3\lambda_4) + 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)]. \end{aligned} \tag{3.9}$$

And

$$\begin{aligned} &\sigma_{13}(4\lambda_1\lambda_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) + \sigma_{14}(4\lambda_1\lambda_4 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \\ &\quad +\sigma_{23}(4\lambda_2\lambda_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) + \sigma_{24}(4\lambda_2\lambda_4 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \\ &= -[2\lambda_1\lambda_3 + (\lambda_1 + \lambda_3)^2 + \lambda_2^2 + \lambda_4^2 + 2\lambda_1\lambda_4 + (\lambda_1 + \lambda_4)^2 + \lambda_2^2 + \lambda_3^2 \\ &\quad +2\lambda_2\lambda_3 + (\lambda_2 + \lambda_3)^2 + \lambda_1^2 + \lambda_4^2 + 2\lambda_2\lambda_4(\lambda_2 + \lambda_4)^2 + \lambda_1^2 + \lambda_3^2] \\ &\quad +O(\epsilon)[4(\lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)]. \end{aligned} \tag{3.10}$$

Since

$$-[(\lambda_1+\lambda_2)^2+(\lambda_3+\lambda_4)^2+2(\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4)] = -[(\lambda_1+\lambda_2)+(\lambda_3+\lambda_4)]^2,$$

we add (3.9) and (3.10) together to obtain

$$
\begin{aligned}
(3.11)\quad &\sigma_{12}(4\lambda_1\lambda_2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) + \sigma_{34}(4\lambda_3\lambda_4 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \\
&+\sigma_{13}(4\lambda_1\lambda_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) + \sigma_{14}(4\lambda_1\lambda_4 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \\
&+\sigma_{23}(4\lambda_2\lambda_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) + \sigma_{24}(4\lambda_2\lambda_4 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \\
&= -\ [(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2 + (\lambda_1 + \lambda_2)^2 + (\lambda_3 + \lambda_4)^2 + (\lambda_1 + \lambda_3)^2 \\
&+ (\lambda_1 + \lambda_4)^2 + (\lambda_2 + \lambda_4)^2 + (\lambda_2 + \lambda_4)^2 + 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)] \\
&+ O(\epsilon)[4(\lambda_1\lambda_2 + \lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4) \\
&+ 6(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)] \le 0.
\end{aligned}
$$

The last inequality holds if we choose ϵ to be small, as the term $(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)$ will dominate all the terms with ϵ. We can explicitly choose $\epsilon = 1/4$. As $\sigma_{ij} = \sigma_{ji}$, the remaining six terms in (3.8) is in fact the same as in (3.11). Hence

$$4Rm(\mathbf{z}(g)) \cdot \mathbf{z}(g) + S(g)|\mathbf{z}(g)|^2 \le 0.$$

For $n > 4$, the proof is similar but more complicated. Choose an orthonormal basis for the tangent space above $x \in M$ such that $\mathbf{z}(g)_{ij} = \lambda_i\delta_{ij}$ for $1 \le i, j \le n$. We need to show that

$$\sum_{i<j,1\le i,j\le n} \sigma_{ij}(n\lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2) \le 0.$$

By induction, we may assume that there exists a positive number c_{n-1} such that

$$
\begin{aligned}
&\sum_{i<j,1\le i,j\le n-1} \sigma_{ij}[(n-1)\lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2] \\
&\quad \le -c_{n-1}(\lambda_1^2 + \cdots + \lambda_{n-1}^2) \\
&\quad O(\epsilon)\left(\sum_{i<j,1\le i,j\le n-1} \lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{i<j,1\le i,j\le n-1} \sigma_{ij}[n\lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2] \\
&\quad \le -c_{n-1}(\lambda_1^2 + \cdots + \lambda_{n-1}^2) - \sum_{i<j,1\le i,j\le n-1} \lambda_i\lambda_j - \frac{(n-1)(n-2)}{2}\lambda_n^2 \\
&\quad + O(\epsilon)\left(\sum_{i<j,1\le i,j\le n-1} \lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2\right)
\end{aligned}
$$

In the sum $\sum_{i<j,1\le i,j\le n-1} \lambda_i\lambda_j$, each λ_i appears $n-2$ times for $1 \le i \le n-1$. We have

$$\sum_{i<j,1\le i,j\le n} \sigma_{ij}(n\lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2)$$

$$
\begin{aligned}
&\le -c_{n-1}(\lambda_1^2+\cdots+\lambda_{n-1}^2) - \sum_{i<j,1\le i,j\le n-1} \lambda_i\lambda_j - \frac{(n-1)(n-2)}{2}\lambda_n^2 \\
&\quad - (n\lambda_1\lambda_n + \lambda_1^2+\cdots+\lambda_{n-1}^2+\lambda_n^2) \\
&\qquad \vdots \\
&\quad - (n\lambda_{n-1}\lambda_n + \lambda_1^2+\cdots+\lambda_{n-1}^2+\lambda_n^2) \\
&\quad + O(\epsilon)\Big(\sum_{i<j,1\le i,j\le n} \lambda_i\lambda_j + \lambda_1^2+\cdots+\lambda_{n-1}^2+\lambda_n^2\Big) \\
&= -c_{n-1}(\lambda_1^2+\cdots+\lambda_{n-1}^2) - \sum_{i<j,1\le i,j\le n-1} \lambda_i\lambda_j - n(\lambda_1\lambda_n+\cdots+\lambda_{n-1}\lambda_n) \\
&\quad - (n-1)(\lambda_1^2+\cdots+\lambda_{n-1}^2) - \left[\frac{(n-1)(n-2)}{2} + (n-1)\right]\lambda_n^2 \\
&\quad + O(\epsilon)\Big(\sum_{i<j,1\le i,j\le n} \lambda_i\lambda_j + \lambda_1^2+\cdots+\lambda_{n-1}^2+\lambda_n^2\Big) \\
&= -c_{n-1}(\lambda_1^2+\cdots+\lambda_{n-1}^2) - \sum_{i<j,1\le i,j\le n-1} \left(\frac{1}{2}\lambda_i + \lambda_i\lambda_j + \frac{1}{2}\lambda_j\right) \\
&\quad - \frac{n}{2}(\lambda_1^2+\cdots+\lambda_{n-1}^2) - n(\lambda_1\lambda_n+\cdots+\lambda_{n-1}\lambda_n) - \left(\frac{n}{2}\right)(n-1)\lambda_n^2 \\
&\quad + O(\epsilon)\left(\sum_{i<j,1\le i,j\le n} \lambda_i\lambda_j + \lambda_1^2+\cdots+\lambda_{n-1}^2+\lambda_n^2\right) \\
&= -c_{n-1}(\lambda_1^2+\cdots+\lambda_{n-1}^2) - \frac{1}{2}(\lambda_1+\lambda_2)^2 - (\lambda_3+\lambda_n)^2 \\
&\quad - \sum_{i<j,1\le i,j\le n-1,(i,j)\ne(1,2)} \left(\frac{1}{2}\lambda_i^2 + \lambda_i\lambda_j + \frac{1}{2}\lambda_j^2\right) \\
&\quad - \left(\frac{n}{2}\lambda_1^2 + \frac{n}{2}\lambda_2^2 + \left(\frac{n}{2}-1\right)\lambda_3^2 + \frac{n}{2}\lambda_4^2 + \cdots + \frac{n}{2}\lambda_{n-1}^2\right) \\
&\quad - (n\lambda_1\lambda_n + n\lambda_2\lambda_n + (n-2)\lambda_3\lambda_n + n\lambda_4\lambda_n + \cdots + n\lambda_{n-1}\lambda_n) \\
&\quad - \left[\left(\frac{n}{2}\right)(n-1) - 1\right]\lambda_n^2 \\
&\quad + O(\epsilon)\left(\sum_{i<j,1\le i,j\le n} \lambda_i\lambda_j + \lambda_1^2+\cdots+\lambda_{n-1}^2+\lambda_n^2\right).
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{1}{2}(\lambda_1+\lambda_2)^2 + (\lambda_3+\lambda_n)^2 + \sqrt{2}(\lambda_1\lambda_3 + \lambda_1\lambda_n + \lambda_2\lambda_3 + \lambda_2\lambda_n) \\
&\qquad = \left(\frac{1}{\sqrt{2}}\lambda_1 + \frac{1}{\sqrt{2}}\lambda_2 + \lambda_3 + \lambda_n\right)^2.
\end{aligned}
$$

Therefore

$$\sum_{i<j,1\le i,j\le n} \sigma_{ij}(n\lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2)$$
$$\le -c_{n-1}(\lambda_1^2 + \cdots + \lambda_{n-1}^2) - \left(\frac{1}{\sqrt{2}}\lambda_1 + \frac{1}{\sqrt{2}}\lambda_2 + \lambda_3 + \lambda_n\right)^2$$
$$- \sum_{i<j,1\le i,j\le n-1,(i,j)\ne(1,2),(1,3),(2,3)} \frac{1}{2}(\lambda_i + \lambda_j)^2$$
$$- \left(\frac{1}{2}\lambda_1^2 - (\sqrt{2}-1)\lambda_1\lambda_3 + \frac{1}{2}\lambda_3^2\right)$$
$$- \left(\frac{1}{2}\lambda_2^2 - (\sqrt{2}-1)\lambda_2\lambda_3 + \frac{1}{2}\lambda_3^2\right)$$
$$- \left(\frac{n}{2}\lambda_1^2 + \frac{n}{2}\lambda_2^2 + (\frac{n}{2} - 1\right)\lambda_3^2 + \frac{n}{2}\lambda_4^2 + \cdots + \frac{n}{2}\lambda_{n-1}^2)$$
$$- \left[\left(n - \sqrt{2}\right)\lambda_1\lambda_n + (n-\sqrt{2})\lambda_2\lambda_n + (n-2)\lambda_3\lambda_n + n\lambda_4\lambda_n + \cdots + n\lambda_{n-1}\lambda_n)\right.$$
$$- \left[\left(\frac{n}{2}\right)(n-1) - 1\right]\lambda_n^2 + O(\epsilon)\left(\sum_{i<j,1\le i,j\le n} \lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2\right)$$
$$\le -c_n(\lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2) - \left(\frac{1}{\sqrt{2}}\lambda_1 + \frac{1}{\sqrt{2}}\lambda_2 + \lambda_3 + \lambda_4\right)^2$$
$$- \sum_{i<j,1\le i,j\le n-1,(i,j)\ne(1,2),(1,3),(2,3)} \frac{1}{2}(\lambda_i + \lambda_j)^2$$
$$- \frac{\sqrt{2}-1}{2}[(\lambda_1 - \lambda_3)^2 + (\lambda_2 + \lambda_3)^2]$$
$$- \frac{n}{2}[(\lambda_3 + \lambda_n)^2 + \cdots + (\lambda_{n-1} + \lambda_n)^2] - \frac{n-\sqrt{2}}{2}[(\lambda_1 + \lambda_n)^2 + (\lambda_2 + \lambda_n)^2]$$
$$+ O(\epsilon)\left(\sum_{i<j,1\le i,j\le n} \lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2\right),$$

where in the last inequality c_n is a positive constant. Therefore

$$\sum_{i<j,1\le i,j\le n} \sigma_{ij}[n\lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2]$$
$$\le -c_n(\lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n)$$
$$+ O(\epsilon)\left(\sum_{i<j,1\le i,j\le n} \lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2\right).$$

Hence we can choose ϵ sufficiently small so that

$$\sum_{i<j,1\leq i,j\leq n} \sigma_{ij}(n\lambda_i\lambda_j + \lambda_1^2 + \cdots + \lambda_{n-1}^2 + \lambda_n^2) \leq 0.$$

By induction, we have finished the proof for all $n \geq 4$. □

Proof of Theorem 3.6 *continued.* We may choose an open neighborhood of h such that the sectional curvatures of all the metrics in the open neighborhood is sufficiently pinched. As shown in [17], curvature pinching is preserved during the Ricci curvature flow. Therefore we can apply Lemma 3.7 to finish the proof.

Remark. We may apply Lemma 3.7 to show theorem 3 in the introduction.

4. Conformal changes of metrics

We begin with the following lemma (cf. [4]), which says that among all conformal metrics, the ones with constant nonpositive scalar curvatures have minimal $L^{\frac{n}{2}}$-norms of scalar curvatures. The result has been proved in [4]. For the sake of completeness we present a proof here, using a different scalar curvature equation.

LEMMA 4.1. *Let M be a compact n-manifold with $n \geq 3$ and g be a Riemannian metric on M with constant nonpositive scalar curvature. Then for any metric g' that is conformal to g, we have*

$$\int_M |S(g')|^{\frac{n}{2}}\, dv_{g'} \geq \int_M |S(g)|^{\frac{n}{2}}\, dv_g,$$

where equality holds if and only if $g' = cg$ for some positive constant c

Proof. Let $g = u^{\frac{4}{n-2}} g'$ with $u > 0$. If $S(g')$ is the scalar curvature of the metric g', then

$$C_n \Delta' u - S(g')u = -S(g)u^{\frac{n+2}{n-2}}, \tag{4.2}$$

where $C_n = 4(n-1)/(n-2)$ and Δ' is the Laplacian for the metric g'. Multiplying (4.2) by u and then integrating by parts we have

$$\begin{aligned} -C_n \int_M |\nabla u|_{g'}^2\, dv_{g'} - \int_M S(g')u^2\, dv_{g'} &= |S(g)| \int_M u^{\frac{2n}{n-2}}\, dv_{g'} \\ &= |S(g)|\, \mathrm{Vol}(M, g), \end{aligned}$$

as $S(g)$ is a nonpositive constant. Therefore

$$-\int_M S(g')u^2\, dv_{g'} \geq |S(g)|\, \mathrm{Vol}(M, g), \tag{4.3}$$

and equality holds if and only if u is a constant. Using Hölder's inequality we obtain

$$\left(\int_M |S(g')|^{\frac{n}{2}}\, dv_{g'}\right)^{\frac{2}{n}} \left(\int_M u^{\frac{2n}{n-2}}\, dv_{g'}\right)^{\frac{n-2}{n}} \ge -\int_M S(g')u^2\, dv_{g'}.$$

Combine with (4.3) to obtain

$$\left(\int_M |S(g')|^{\frac{n}{2}}\, dv_{g'}\right)^{\frac{2}{n}} (\mathrm{Vol}(M, g))^{\frac{n-2}{n}} \ge |S(g)|\,\mathrm{Vol}(M, g).$$

That is,

$$\int_M |S(g')|^{\frac{n}{2}}\, dv_{g'} \ge |S(g)|^{\frac{n}{2}}\,\mathrm{Vol}(M, g) = \int_M |S(g)|^{\frac{n}{2}}\, dv_g. \qquad \square$$

For a Riemannian metric g on a compact manifold M, the Yamabe invariant is defined as

$$(4.4)\quad Q(M, g) = \inf\left\{ \frac{\frac{4(n-1)}{n-2}\int_M |\nabla u|^2\, dv_g + \int_M R_g u^2\, dv_g}{(\int_M |u|^{\frac{2n}{n-2}}\, dv_g)^{\frac{n-2}{n}}} \,\middle|\, u \in C^\infty(M), u \not\equiv 0 \right\}.$$

It is known that the Yamabe invariant for the standard unit sphere is equal to the best constant for the Sobolev inequality on $\mathbf{R}^n$ (Theorem 3.3 of [13]); i.e.,

$$Q(S^n, g_o) = n(n-1)\omega_n^{\frac{2}{n}},$$

where ω_n is the volume of the unit n-sphere.

Lemma (4.1) does not hold in general for constant positive scalar curvature. However, for Einstein metrics with positive scalar curvature we have the following result.

LEMMA 4.5. *For $n \ge 3$, let (M, g_o) be a compact Einstein manifold with positive scalar curvature. Then for any metric g that is conformal to g_o, we have*

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \ge \int_M |S(g_o)|^{\frac{n}{2}}\, dv_{g_o}.$$

Proof. As the scalar curvature of (M, g_o) is positive, we have $Q(M, g_o) > 0$. If $Q(M, g_o) < n(n-1)\omega_n^{\frac{2}{n}}$, then there is a smooth positive function u such that

$$Q(M, g_o) = \frac{\frac{4(n-1)}{n-2}\int_M |\nabla u|^2\, dv_g + \int_M R_g u^2\, dv_g}{(\int_M |u|^{\frac{2n}{n-2}}\, dv_g)^{\frac{n-2}{n}}},$$

and the metric $u^{4/(n-2)}g_o$ has constant positive scalar curvature. Obata's theorem A implies that u is a positive constant and

$$Q(M, g_o) = n(n-1)\,\mathrm{Vol}(M, g_o)^{\frac{2}{n}}.$$

The same relation holds of the standard n-sphere. (4.4) gives the inequality

$$\text{(4.6)}\quad n(n-1)\,\mathrm{Vol}(M, g_o)^{\frac{2}{n}} \left(\int_M |u|^{\frac{2n}{n-2}}\, dv_{g_o}\right)^{\frac{n-2}{n}} \le 4\frac{n-1}{n-2}\int_M |\nabla u|^2\, dv_{g_o} + \int_M R_{g_o} u^2\, dv_{g_o},$$

for $u \in C^\infty(M)$. Let $g = u^{\frac{4}{n-2}} g_o$, $u > 0$. We have

$$\text{(4.7)}\qquad 4\frac{n-1}{n-2}\Delta_o u - S(g_o)u = -S(g)u^{\frac{n+2}{n-2}},$$

where Δ_o is the Laplacian for (S^n, g_o). Multiplying (4.7) by u and then integrating by parts we obtain

$$\text{(4.8)}\qquad 4\frac{n-1}{n-2}\int_M |\nabla u|^2\, dv_{g_o} + \int_M S(g_o)u^2\, dv_{g_o} = \int_M S(g)u^{\frac{2n}{n-2}}\, dv_{g_o}.$$

Applying the Hölder's inequality and the inequality (4.6) we have

$$\begin{aligned}\int_M S(g)u^{\frac{2n}{n-2}}\, dv_{g_o} &\le \left(\int_M |S(g)|^{\frac{n}{2}} u^{\frac{2n}{n-2}}\, dv_{g_o}\right)^{\frac{2}{n}} \left(\int_M u^{\frac{2n}{n-2}}\, dv_{g_o}\right)^{\frac{n-2}{n}}\\ &\le [n(n-1)\,\mathrm{Vol}(M, g_o)^{\frac{2}{n}}]^{-1} \left(\int_M |S(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{2}{n}}\\ &\quad\times \left(4\frac{n-1}{n-2}\int_M |\nabla u|^2\, dv_{g_o} + \int_M u^2\, dv_{g_o}\right).\end{aligned}$$

So from (4.8) we obtain

$$\begin{aligned}&4\frac{n-1}{n-2}\int_M |\nabla u|^2\, dv_{g_o} + \int_M S(g)u^2\, dv_{g_o}\\ &\quad\le [n(n-1)\,\mathrm{Vol}(M, g_o)^{\frac{2}{n}}]^{-1} \left(\int_M |S(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{2}{n}}\\ &\quad\times \left(4\frac{n-1}{n-2}\int_M |\nabla u|^2\, dv_{g_o} + \int_M u^2\, dv_{g_o}\right).\end{aligned}$$

We must have

$$[n(n-1)\,\mathrm{Vol}(M, g_o)^{\frac{2}{n}}]^{-1} \left(\int_M |S(g)|^{\frac{n}{2}}\, dv_g\right)^{\frac{2}{n}} \ge 1,$$

or

$$\int_M |S(g)|^{\frac{n}{2}}\, dv_g \ge [n(n-1]^{\frac{n}{2}}\,\mathrm{Vol}(M, g_o) = \int_M |S(g_o)|^{\frac{n}{2}}\, dv_{g_o},$$

as $S(g_o) = n(n-1)$. □

COROLLARY 4.9. *For any metric g on S^n that is conformal to g_o and with $S(g) \leq n(n-1)$, we have* $\mathrm{Vol}\,(S^n, g) \geq \mathrm{Vol}\,(S^n, g_o)$

PROPOSITION 4.10. *Let (M, g) be an n-manifold with $b^2 g \geq \mathrm{Ric}\,(g) \geq a^2 g$ for some positive numbers a and b. Then for any metric $g' = u^{\frac{4}{n-2}} g$, $u > 0$, we have*

$$\int_M |S(g')|^{\frac{n}{2}}\, dv_{g'} \geq c_n \int_M |S(g)|^{\frac{n}{2}}\, dv_g,$$

where c_n is a positive constant that depends on a, b and n only.

Proof. For the smooth positive function u, the Sobolev inequality on (M, g) [1] gives

$$(4.11)\qquad \left(\int_M u^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{2n}} \leq (\mathrm{Vol}(M, g))^{-\frac{1}{n}} \left[\tau\sigma_n \left(\int_M |\nabla u|^2\, dv_g\right)^{\frac{1}{2}} + \left(\int_M u^2\, dv_g\right)^{\frac{1}{2}}\right],$$

where $\tau = \mathrm{Diam}\,(M, g)/\alpha_n$ and σ_n, α_n are positive constants that depend on n only. As $\mathrm{Ric}(g) \geq a^2 g$, Myers' theorem gives $\mathrm{Diam}\,(M, g) \leq \pi\sqrt{n-1}/a$. Therefore there exists a positive constant $C(n, a)$, which depends on n and a only, such that

$$(4.12)\qquad \left(\int_M u^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}} \leq C(n, a)\, (\mathrm{Vol}(M, g))^{-\frac{2}{n}} \left(\int_M |\nabla u|^2\, dv_g + \int_M u^2\, dv_g\right).$$

In the proof of Lemma (4.5), if we use the inequality (4.12) instead of (4.6), we obtain

$$4\frac{n-1}{n-2}\int_M |\nabla u|^2\, dv_g + \int_M S(g) u^2\, dv_g$$
$$\leq C(n, a)\left(\int_M |S(g')|^{\frac{n}{2}}\, dv_g\right)^{\frac{2}{n}} (\mathrm{Vol}(M, g))^{-\frac{2}{n}} \left(\int_M |\nabla u|^2\, dv_g + \int_M u^2\, dv_g\right).$$

As $S(g) \geq na^2$, we must have

$$C(n, a)\left(\int_M |S(g')|^{\frac{n}{2}}\, dv_{g'}\right)^{\frac{2}{n}} (\mathrm{Vol}(M, g))^{-\frac{2}{n}} \geq \min\left\{\frac{4(n-1)}{(n-2)}, na^2\right\},$$

or

$$\int_M |S(g')|^{\frac{n}{2}}\, dv_{g'} \geq \int_M |S(g)|^{\frac{n}{2}}\, dv_g,$$

where

$$C(n, a, b) = \frac{\min\{\frac{4(n-1)}{(n-2)}, na^2\}}{C(n, a) n b^2}.$$

We have made use of the fact that $S(g) \le nb^2$. $C(n, a, b)$ is a positive constant that depends on n, a and b only. □

Hamilton has introduced the following normalized Yamabe flow (scalar curvature flow), similar to the Ricci curvature flow:

$$\frac{\partial g_t}{\partial t} = (\bar{s}(g_t) - S(g_t))g_t, \tag{4.13}$$

where $\bar{s}(g_t) = \int_M S(g_t)\, dv_{g_t} / \operatorname{Vol}(M, g_t)$. The Yamabe flow has been used by Hamilton, B. Chow [8], and R. Ye [18] to obtain constant scalar curvature metrics on various situations. As in Section 3, we consider the change of the $L^{\frac{n}{2}}$-norm on scalar curvatures along the Yamabe flow.

LEMMA 4.14. *Let (M, g_o) be a compact Riemannian n-manifold with $n \ge 4$. Assume that (M, g_o) has positive scalar curvature. If g_t is a solution to the Yamabe flow* (4.13) *with initial metric g_o, then*

$$\frac{d}{dt}\int_M |S(g_t)|^{\frac{n}{2}}\, dv_{g_t} \le 0,$$

and equality holds at time t if and only if g_t has constant scalar curvature.

Proof. It is more convenient to consider the unnormalized Yamabe flow

$$\frac{\partial g_t}{\partial t} = -S(g_t)g_t. \tag{4.15}$$

One can rescale in time for the solutions of (4.15) to obtain corresponding solutions of (4.13) [8], [17]. Under the flow (4.13), the evolution equation for the scalar curvature [8] is

$$\frac{\partial}{\partial t}S(g_t) = (n-1)\Delta S(g_t) + S(g_t)^2.$$

It follows from the maximal principle that if g_o has positive scalar curvature, then $S(g_t) > 0$ for all $t \ge 0$. Under the normalized Yamabe flow (4.13), the evolution equation for the scalar curvature [18] is

$$\frac{\partial}{\partial t}S(g_t) = (n-1)\Delta S(g_t) + S(g_t)(S(g_t) - \bar{s}(g_t)), \tag{4.16}$$

and

$$(dv_g)' = \frac{1}{2}\operatorname{tr}_g\left(\frac{dg}{dt}\right)dv_g = \frac{n}{2}(\bar{s}(g_t) - S(g_t)). \tag{4.17}$$

Therefore we have

$$\begin{aligned}\frac{d}{dt}\int_M |S(g_t)|^{\frac{n}{2}}\,dv_{g_t} &= \int_M \frac{n}{2}S(g_t)^{\frac{n}{2}-1}\frac{\partial}{\partial t}S(g_t)\,dv_{g_t}\\ &+\int_M \frac{n}{2}S^{\frac{n}{2}}(\bar{s}(g_t)-S(g_t))\,dv_{g_t} \qquad (\text{as}\quad S(g)>0)\\ &= \int_M \frac{n}{2}S(g_t)^{\frac{n}{2}-1}[(n-1)\Delta S(g_t)+S(g_t)(S(g_t)-\bar{s}(g_t))]\,dv_{g_t}\\ &+\int_M \frac{n}{2}S^{\frac{n}{2}}(\bar{s}(g_t)-S(g_t))\,dv_{g_t}\\ &= -\int_M \frac{n}{2}\left(\frac{n}{2}-1\right)S(g_t)^{\frac{n}{2}-2}|\nabla S(g)|^2\,dv_{g_t}\le 0,\end{aligned}$$

and equality holds if and only if $S(g_t)$ is a constant. □

Let (M, g) be a compact conformally flat manifold with positive Ricci curvature. The Yamabe flow (4.6) with initial metric g is known to converge to a constant curvature metric g_o as $t \to \infty$ [8]. Applying the above lemma we have the following.

THEOREM 4.18. *Let (M, g) be a compact conformally flat manifold with positive Ricci curvature. Then*

$$\int_M |S(g)|^{\frac{n}{2}}\,dv_g \ge \int_M |S(g_o)|^{\frac{n}{2}}\,dv_{g_o}, \tag{4.19}$$

where g_o has constant positive sectional curvature.

Remark. As the Ricci curvature of (M, g) is positive, it is bounded from below by a positive constant. Hence the fundamental group is finite by Myer's theorem. The universal covering of M is then conformally equivalent to the standard n-sphere S^n under the development map. Because a finite group of conformal transformations of the S^n is conjugate to a group of isometrics of S^n, we see that the metric g is conformal to a metric of g_o of constant positive sectional curvature. Proposition (4.10) provides a not so sharp lower bound on the $L^{\frac{n}{2}}$-norm on $S(g)$.

We note that there exists a family of metrics on S^n for $n \ge 3$ with $L^{\frac{n}{2}}$-norms on the scalar curvatures concentrate around one point. For any $\epsilon > 0$, the family of functions

$$u_\epsilon(x) = \left(\frac{\epsilon}{\epsilon^2+|x|^2}\right)^{\frac{n-2}{2}}, \quad x \in \mathbf{R}^n,$$

satisfy the equation

$$\Delta_o u_\epsilon + n(n-2)u_\epsilon^{\frac{n+2}{n-2}} = 0,$$

where Δ_o is the Laplacian for $\mathbf{R}^n$ with the standard flat metric δ_{ij}. That is, the metric $g_{o,\epsilon} = u_\epsilon^{\frac{4}{n-2}}\delta_{ij}$ has scalar curvature equal to $n(n-2)$. Let $\Phi : S^n \to \mathbf{R}^n$ be the sterographic projection which sends the north pole to infinity. Using the fact that $d((0,0,\dots 0,1), y) \sim 1/|\Phi(y)|$, where $(0,0,..,0,1)$ is the north pole of S^n, $y \in S^n \setminus (0,0,\dots,0,1)$ and d is the distance on S^n, the pull back of the family of metrics $g_{o,\epsilon}$ by Φ, denoted by g_ϵ, on S^n, is a family of nonsingular metrics on S^n. Then $\Phi : (S^n \setminus (0,0,..0,1), g_\epsilon) \to (\mathbf{R}^n, g_{o,\epsilon})$ is an isometry. The scalar curvature of (S^n, g_ϵ) equals $n(n-2)$. And

$$\begin{aligned}\int_{S^n} |S(g_\epsilon)|^{\frac{n}{2}}\, dv_{g_\epsilon} &= \int_{\mathbf{R}^n} [n(n-2)]^{\frac{n}{2}}\, dv_{g_{o,\epsilon}}\\ &= \int_{\mathbf{R}^n} [n(n-2)]^{\frac{n}{2}} u_\epsilon^{\frac{2n}{n-2}}\, dv_o\\ &= \int_{\mathbf{R}^n} [n(n-2)]^{\frac{n}{2}} \left(\frac{\epsilon}{\epsilon^2+|x|^2}\right)^n dv_o\\ &= c_n \int_0^\infty \left(\frac{1}{1+r^2}\right)^n r^{n-1}\, dr,\end{aligned}$$

where $c_n = [n(n-2)]^{\frac{n}{2}}\,\mathrm{Vol}(S^{n-1})$ and $r = |x|/\epsilon, x \in \mathbf{R}^n$. As $\epsilon \to 0$, $L^{\frac{n}{2}}$-norms on the scalar curvatures concentrate around the south pole; i.e., there exist a positive constant C_n such that

$$\int_{S^n} |S(g_\epsilon)|^{\frac{n}{2}}\, dv_{g_\epsilon} \geq C_n$$

for all $1 > \epsilon > 0$ while if O is any open neighborhood of the south pole, then

$$\int_{S^n\setminus O} |S(g_\epsilon)|^{\frac{n}{2}}\, dv_{g_\epsilon} \to 0 \qquad \text{as} \quad \epsilon \to 0.$$

While as $\epsilon \to \infty$, the integral concentrates around the north pole.

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