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NORM INEQUALITIES FOR VECTOR VALUED RANDOM SERIES

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1. Introduction

It is well known that Rademacher functions, $r_n(t)$, which are defined by

$$r_0(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \end{cases}, \qquad r_0(t+1) = r_0(t), \quad r_n(t) = r_0(2^n t),\ n \geq 1,$$

form a sequence of independent, symmetric and identically distributed random variables. Rademacher series $\sum r_j(t)u_j$ where u_j belong to a Banach space have been investigated extensively; see [1], [5], [9], [12], [13].

An important result for Rademacher series is the Khinchin-Kahane inequality: for any $0 < q < p < \infty$, there exists constant $b(p, q)$ such that for any $N > 1$,

$$\left\| \sum_{j=1}^{N} r_{j-1} u_j \right\|_p \leq b(p, q) \left\| \sum_{j=1}^{N} r_{j-1} u_j \right\|_q$$

holds in any Banach space.

This inequality holds for a large class of zero–mean random variables; see [2], [4], [5], [8], [14]. We extend the inequality to a class of nonzero-mean random variables and we show that a constant vector can be added to both sides of the inequality. The latter enables us to study vector valued versions of the Burkholder local distribution estimates which Stein used in the proof of his celebrated theorem on limits of sequences of operators [13]. In a subsequent paper we will give a vector valued version of Stein's theorem [13] by using this local distribution estimate.

We prove vector valued local norm inequalities in L^p as well as in some Orlicz spaces for certain independent random variables which satisfy the Khinchin-Kahane inequality. We show that the local behavior of the tail series is the same as the global behavior of the series itself.

2. An extension of the Khinchin-Kahane inequality

Throughout the paper, for a sequence of independent random variables $\{X_j\}$, we will denote $X = \{X_j\}$.

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For a given sequence X and Banach space $\mathcal{B}$, if

$$\left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_p \leq b(q, p) \quad \left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_q \tag{1}$$

where $0 < q < p < \infty$, $N \geq 1$ arbitrary, $b(q, p) = b(q, p, \mathcal{B})$ and arbitrary $u_0, u_j \in \mathcal{B}$, then we say X satisfies the Khinchin-Kahane inequality in $\mathcal{B}$. By a result in [6], see also Theorem 5 below, if (1) holds for one $q \in (0, p)$ then it holds for every $q \in (0, p)$.

The treatment of this inequality in general Banach space in the literature is mainly for zero mean random variables and $u_0 = 0$; see [5], [7], [8], [14]. We show that the inequality (1) holds for a class of nonzero mean random variables.

DEFINITION 1. *Let $(\Omega, \sum, \mu)$ be a measure space, and $\mathcal{B}$ be a Banach space. Let $(F_B^{\Omega}, \| \ \|_F)$ be a normed subspace of the space of strongly measurable functions on $(\Omega, \sum, \mu)$:*

$$F_B^{\Omega} = \{f \mid f\colon \Omega \to \mathcal{B}, f \text{ measurable}, \ \|f\|_F < \infty\}.$$

Denote by f_* the distribution function of $\|f\|_B$:

$$f_*(\alpha) = \mu\{w\colon \|f(w)\|_B > \alpha\}, \quad \forall \alpha \geq 0.$$

If $f \in F_B^{\Omega}$, and $f_*(\alpha) = g_*(\alpha), \forall \alpha > 0$, implies $g \in F_B^{\Omega}$ and $\|f\|_F = \|g\|_F$, then we say that $(F_B^{\Omega}, \| \ \|_F)$ is a rearrangement invariant Banach space.

The principle of contraction, proved in the next lemma, is proved in [5], [12] for L^p spaces with $u_0 = 0$.

LEMMA 1. *Let $(F_B^{\Omega}, \| \ \|_F)$ be a rearrangement invariant Banach space. Let X be a sequence of independent symmetric random variables such that $X_j u \in (F_B^{\Omega}, \| \ \|_F), \forall u \in \mathcal{B}$. Let $\lambda_j \in R, j = 0, 1, \ldots, N$. Then for any $N \geq 1$ and any vectors $\{u_j, 0 \leq j \leq N\}$,*

$$\left\| \lambda_0 u_0 + \sum_{j=1}^{N} \lambda_j X_j u_j \right\|_F \leq \max_j(|\lambda_j|) \left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_F .$$

Proof. We may assume $|\lambda_j| \leq 1, j = 0, 1, \ldots, N$. Let V_0 be a Bernoulli random variable which is independent of $(X_1, \ldots, X_N)$. Define $V_j = V_0 X_j$, $1 \leq j \leq N$. Then $(V_0, V_1, \ldots, V_N)$ is a sequence of independent symmetric random variables. Since

$$\left\| \lambda_0 u_0 + \sum_{j=1}^{N} \lambda_j X_j u_j \right\| = \left\| \lambda_0 V_0 u_0 + \sum_{j=1}^{N} \lambda_j V_0 X_j u_j \right\| = \left\| \sum_{j=0}^{N} \lambda_j V_j u_j \right\|,$$

and since, if $\theta_j = \pm 1$, the sums $\sum_{j=0}^N \lambda_j V_j u_j$ and $\sum_{j=0}^N \theta_j \lambda_j V_j u_j$ are equidistributed, it suffices to consider $0 \le \lambda_j \le 1$, $\quad j = 0, \ldots, N$.

The technique we use below is due to Kahane [5]. Let us first consider the case that λ_j is either 0 or 1, $0 \le j \le N$. Define $\theta_j = 2\lambda_j - 1$, $j = 0, 1, \ldots, N$; thus $\theta_j = \pm 1$.

Since $\sum_{j=0}^N \theta_j V_j u_j$ is equidistributed with $\sum_{j=0}^N V_j u_j$, we have

$$\begin{aligned} \left\| \sum_{j=0}^N \lambda_j V_j u_j \right\|_F &= \frac{1}{2} \left\| \sum_{j=0}^N V_j u_j + \sum_{j=0}^N \theta_j V_j u_j \right\|_F \\ &\le \frac{1}{2} \left\| \sum_{j=0}^N V_j u_j \right\|_F + \frac{1}{2} \left\| \sum_{j=0}^N \theta_j V_j u_j \right\|_F = \left\| \sum_{j=0}^N V_j u_j \right\|_F \\ &= \left\| V_0 u_0 + \sum_{j=1}^N V_0 X_j u_j \right\|_F = \left\| u_0 + \sum_{j=1}^N X_j u_j \right\|_F . \end{aligned}$$

If $0 \le \lambda_j \le 1$, $j = 0, 1, \ldots, N$, let

$$\lambda_j = \sum_{k=1}^\infty 2^{-k} \lambda_{jk}, \quad \lambda_{jk} = 0, 1.$$

Denoting $X_0 = 1$, we have $\sum_{j=0}^N \lambda_j X_j u_j = \sum_{k=1}^\infty 2^{-k} \sum_{j=0}^N \lambda_{jk} X_j u_j$. Hence by the first part of the proof,

$$\left\| \sum_{j=0}^N \lambda_j X_j u_j \right\|_F \le \sum_{k=1}^\infty 2^{-k} \left\| \sum_{j=0}^N \lambda_{jk} X_j u_j \right\|_F \le \left\| \sum_{j=0}^N X_j u_j \right\|_F . \qquad \square$$

For a sequence of random variables $X = \{X_j\}$ with $EX_j = m_j$, we will let $Z_j = X_j - m_j$ and $Z = \{Z_j\}$ throughout the paper.

THEOREM 1. *Let $\mathcal{B}$ be a Banach space. For $p > 0$, if X satisfies the Khinchin-Kahane inequality then so does Z. Conversely, for $p > 1$, if Z satisfies the Khinchin-Kahane inequality with $u_0 = 0$ then X satisfies the inequality for any $u_0 \in \mathcal{B}$.*

Proof. Assume that X satisfies (1). Fix an $N \ge 1$ and $u_0, u_j \in \mathcal{B}$. Let $v = \sum_{j=1}^N m_j u_j$. Then

$$\begin{aligned} \left\| u_0 + \sum_{j=1}^N Z_j u_j \right\|_p &= \left\| u_0 - v + \sum_{j=1}^N X_j u_j \right\|_p \\ &\le b(q, p) \left\| u_0 - v + \sum_{j=1}^N X_j u_j \right\|_q \end{aligned}$$

$$= b(q, p) \left\| u_0 + \sum_{j=1}^{N} Z_j u_j \right\|_q .$$

Assume that Z satisfies (1) with $u_0 = 0$. Let us first show that the symmetrization X^s of X satisfies (1). Let $\{X_j'\}$ be an independent copy of $\{X_j\}$. Let $X_j^s = X_j - X_j'$, $Z_j' = X_j' - m_j$. Then $X_j^s = Z_j^s$. It suffices to show the inequality for one $q \in (1, p)$. Since $(u_0 + \sum_{j=1}^N Z_j u_j,\ u_0 + \sum_{j=1}^N Z_j u_j - \sum_{j=1}^N Z_j' u_j)$ is a two-term martingale, we have, for all $1 \le r \le p$ and any $u_0 \in \mathcal{B}$,

$$\left\| u_0 + \sum_{j=1}^{N} Z_j u_j \right\|_r \le \left\| u_0 + \sum_{j=1}^{N} Z_j^s u_j \right\|_r = \left\| u_0 + \sum_{j=1}^{N} X_j^s u_j \right\|_r .$$

Also since

$$\|u_0\| \le \left\| u_0 + \sum_{j=1}^{N} Z_j u_j \right\|_1 \le \left\| u_0 + \sum_{j=1}^{N} Z_j u_j \right\|_q ,$$

we get

$$\begin{aligned}
\left\| u_0 + \sum_{j=1}^{N} X_j^s u_j \right\|_p &\le \|u_0\| + 2 \left\| \sum_{j=1}^{N} Z_j u_j \right\|_p \\
&\le \left\| u_0 + \sum_{j=1}^{N} Z_j u_j \right\|_q + 2b(q, p) \left\| \sum_{j=1}^{N} Z_j u_j \right\|_q \\
&\le \left\| u_0 + \sum_{j=1}^{N} X_j^s u_j \right\|_q + 2b(q, p) \left\| \sum_{j=1}^{N} X_j^s u_j \right\|_q \\
&\le (1 + 2b(q, p)) \left\| u_0 + \sum_{j=1}^{N} X_j^s u_j \right\|_q ,
\end{aligned}$$

where in the last inequality we apply Lemma 1. By the previous argument we have

$$\begin{aligned}
\left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_p &\le \left\| u_0 + v + \sum_{j=1}^{N} X_j^s u_j \right\|_p \\
&\le (1 + 2b(q, p)) \left\| u_0 + v + \sum_{j=1}^{N} X_j^s u_j \right\|_q \\
&\le (1 + 2b(q, p)) \left(\left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_q + \left\| \sum_{j=1}^{N} Z_j u_j \right\|_q \right)
\end{aligned}$$

$$\begin{aligned}
&\le (1+2b(q,p))\left(\left\|u_0+\sum_{j=1}^N X_ju_j\right\|_q+\left\|\sum_{j=1}^N X_j^su_j\right\|_q\right)\\
&= (1+2b(q,p))\left(\left\|u_0+\sum_{j=1}^N X_ju_j\right\|_q\right.\\
&\qquad\left.+\left\|u_0+\sum_{j=1}^N X_ju_j-\left(u_0+\sum_{j=1}^N X_j'u_j\right)\right\|_q\right)\\
&\le 3(1+2b(q,p))\left\|u_0+\sum_{j=1}^N X_ju_j\right\|_q. \qquad \square
\end{aligned}$$

THEOREM 2. *Let $\mathcal{B}$ be a Banach space and X^s be the symmetrization of X. For $p \ge 1$, if X satisfies the Khinchin-Kahane inequality then so does X^s. Conversely, for $p > 1$, if X^s satisfies the Khinchin-Kahane inequality then so does X.*

Proof. Assume that X satisfies (1). By Theorem 1, Z also satisfies (1), and so for $0 < q < 1$, we have

$$\begin{aligned}
\left\|u_0+\sum_{j=1}^N Z_ju_j\right\|_q^q &= \left\|u_0+\sum_{j=1}^N Z_ju_j-\int\sum_{j=1}^N Z_j(w')u_j\,d\mu(w')\right\|_q^q\\
&\le \left\|\int\left\|u_0+\sum_{j=1}^N Z_ju_j-\sum_{j=1}^N Z_j(w')u_j\right\|d\mu(w')\right\|_q^q\\
&\le b^q(q,1)\left\|\int\left\|u_0+\sum_{j=1}^N Z_ju_j-\sum_{j=1}^N Z_j(w')u_j\right\|^q d\mu(w')\right\|_1\\
&\le b^q(q,1)\left\|u_0+\sum_{j=1}^N X_j^su_j\right\|_q^q.
\end{aligned}$$

Since for $p \geq 1$, $b(q, 1) \leq b(q, p)$,

$$
\begin{aligned}
\left\| u_0 + \sum_{j=1}^{N} X_j^s u_j \right\|_p &\leq \left\| u_0 + \sum_{j=1}^{N} Z_j u_j \right\|_p + \left\| \sum_{j=1}^{N} Z_j' u_j \right\|_p \\
&\leq \left\| u_0 + \sum_{j=1}^{N} Z_j u_j \right\|_p + \left\| \sum_{j=1}^{N} X_j^s u_j \right\|_p \leq 3 \left\| u_0 + \sum_{j=1}^{N} Z_j u_j \right\|_p \\
&\leq 3b(q, p) \left\| u_0 + \sum_{j=1}^{N} Z_j u_j \right\|_q \leq 3b^2(q, p) \left\| u_0 + \sum_{j=1}^{N} X_j^s u_j \right\|_q .
\end{aligned}
$$

The proof of the other part is contained in the proof of Theorem 1. □

Let us recall [7], [8]:

DEFINITION 2. *Let Y be an $L^p(\mu)$ random variable defined on $(\Omega, \sum, \nu)$ and $\mathcal{B}$ a Banach space. If for some $0 < q < p < \infty$, there is a constant $c = c_{qp}(\mathcal{B})$ such that*

$$\|u_0 + cYu_1\|_p \leq \|u_0 + Yu_1\|_q$$

holds for any u_j in $\mathcal{B}$, $j = 0, 1$, then we say Y is a hypercontractive random variable, and write $Y \in \mathcal{HC}(p, q, c, \mathcal{B})$.

It is well known, see [8], that a sequence of independent symmetric hypercontractive random variables satisfies (1) with $u_0 = 0$. In the next theorem we show that (1) holds without the condition that Y_j are symmetric.

THEOREM 3. *Let $\mathcal{B}$ be a Banach space. Let $\{X_j\}$ be independent and $Z_j \in \mathcal{HC}(p, q, c_j, \mathcal{B})$ with $p \geq 1$, and $c = \inf\{c_j\} > 0$. Then X satisfies the Khinchin-Kahane inequality.*

Proof. Since $Z_j \in \mathcal{HC}(p, q, c_j, \mathcal{B})$, we have, by Lemma 1, for $0 < q < 1$,

$$
\begin{aligned}
&\int \left\| u_0 + \sum_{j=1}^{N} Z_j(w) u_j \right\|^q d\mu(w) \\
&= \int \left\| u_0 + \sum_{j=1}^{N} Z_j(w) u_j - \int \sum_{j=1}^{N} c_j Z_j(w') u_j \, d\mu(w') \right\|^q d\mu(w) \\
&\leq \int \left(\int \left\| u_0 + \sum_{j=1}^{N} Z_j(w) u_j - \sum_{j=1}^{N} c_j Z_j(w') u_j \right\| d\mu(w') \right)^q d\mu(w)
\end{aligned}
$$

$$\leq \int \left(\int \left\| u_0 + \sum_{j=1}^N Z_j(w)u_j - \sum_{j=1}^N Z_j(w')u_j \right\|^q d\mu(w') \right) d\mu(w)$$

$$= \left\| u_0 + \sum_{j=1}^N Z_j^s u_j \right\|_q^q = \left\| u_0 + \sum_{j=1}^N X_j^s u_j \right\|_q^q .$$

We also have from the proof of Theorem 1, that the same inequality holds for $1 < q < p$. Applying Lemma 1, we thus get

$$\begin{aligned}
\left\| cu_0 + c\sum_{j=1}^N X_j^s u_j \right\|_p &\leq \|u_0\| + \left\| \sum_{j=1}^N c_j X_j^s u_j \right\|_p \\
&= \|u_0\| + \left\| u_0 + \sum_{j=1}^N c_j Z_j u_j - \left(u_0 + \sum_{j=1}^N c_j Z_j' u_j \right) \right\|_p \\
&\leq \|u_0\| + 2\left\| u_0 + \sum_{j=1}^N c_j Z_j u_j \right\|_p \leq 3 \left\| u_0 + \sum_{j=1}^N c_j Z_j u_j \right\|_p \\
&\leq 3 \left\| u_0 + \sum_{j=1}^N Z_j u_j \right\|_q \leq 3 \left\| u_0 + \sum_{j=1}^N X_j^s u_j \right\|_q .
\end{aligned}$$

Let $v = \sum_{j=1}^N m_j u_j$. First, assume that $0 < q < 1$. Then

$$\begin{aligned}
\left\| u_0 + \sum_{j=1}^N X_j u_j \right\|_p &\leq \left\| u_0 + v + \sum_{j=1}^N X_j^s u_j \right\|_p \\
&\leq 3c^{-1} \left\| u_0 + v + \sum_{j=1}^N X_j^s u_j \right\|_q \\
&\leq 3c^{-1} \left(\left\| u_0 + \sum_{j=1}^N X_j u_j \right\|_q^q + \left\| \sum_{j=1}^N Z_j u_j \right\|_q^q \right)^{1/q} \\
&\leq 3c^{-1} \left(\left\| u_0 + \sum_{j=1}^N X_j u_j \right\|_q^q + \left\| \sum_{j=1}^N X_j^s u_j \right\|_q^q \right)^{1/q} \\
&= 3c^{-1} \left(\left\| u_0 + \sum_{j=1}^N X_j u_j \right\|_q^q \right.
\end{aligned}$$

$$+ \left\| u_0 + \sum_{j=1}^{N} X_j u_j - \left(u_0 + \sum_{j=1}^{N} X'_j u_j \right) \right\|_q^q \Bigg)^{1/q}$$

$$\leq 3^{1+\frac{1}{q}} \, c^{-1} \left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_q .$$

If $q \geq 1$, the above proof gives

$$\left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_p \leq 9c^{-1} \left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_q . \qquad \square$$

Since any L^2 random variable X_j with mean zero is hypercontractive in any Hilbert space (see [7]), applying Theorem 3 to the independent sequence X, we have:

COROLLARY 1. *Let $\mathcal{B}$ be a Hilbert space. Then any sequence of independent L^2 random variables satisfies the Khinchin-Kahane inequality in $\mathcal{B}$.*

We extend a result of Kwapień and Szulga on the connection between the Khinchin-Kahane inequality and the hypercontractivity.

THEOREM 4. *Let $\mathcal{B}$ be a Banach space and X be i.i.d. random variables. Let X^s be the symmetrization of X. Then for $p > 1$, X satisfies the Khinchin-Kahane inequality iff $X_j^s \in \mathcal{HC}(p, q_0, c, \mathcal{B}), \forall j \geq 1$ for any $q_0 \in (1, p)$.*

Proof. From Theorem 3, if $X_j^s \in \mathcal{HC}(p, q_0, c, \mathcal{B}), \forall j \geq 1$, then $\{X_j^s\}$ satisfies (1). By Theorem 2, $\{X_j\}$ satisfies (1). Conversely, by Theorem 2, if X satisfies (1), then so does X^s. The latter, by a theorem of Kwapień and Szulga [8], is equivalent to $X_j^s \in \mathcal{HC}(p, q_0, c, \mathcal{B})$ for any $q_0 \in (1, p)$. $\square$

3. Distribution estimates

Let us recall the Marcinkiewicz–Paley–Zygmund property [6].

DEFINITION 3. *Let $\mathcal{B}$ be a Banach space. Let X be a sequence of independent random variables. If for some $0 < p < \infty$ there exist positive constants $\alpha = \alpha(p, \mathcal{B}, X)$, $\beta_p = \beta(p, \mathcal{B}, X)$ such that for any $N > 0$ and any vectors $\{u_j\} \subset \mathcal{B}$, we have the inequality*

$$\mu \left\{ w \colon \left\| u_0 + \sum_{j=1}^{N} X_j(w) u_j \right\| \geq \beta_p \left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_p \right\} \geq \alpha,$$

then we say X has the $MPZ(p, \mathcal{B})$ property, and write $X \in MPZ(p, \mathcal{B})$.

The following result is proved in [6].

THEOREM 5. *Let $\mathcal{B}$ be a Banach space. If X satisfies the Khinchin-Kahane inequality for some $0 < q < p$ then $X \in MPZ(p, \mathcal{B})$. If $X \in MPZ(p, \mathcal{B})$ then X satisfies the Khinchin-Kahane inequality for all $0 < q < p$:*

$$\beta_p \alpha^{\frac{1}{q}} \left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_p \leq \left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_q . \tag{2}$$

LEMMA 2. *If $X \in MPZ(p, \mathcal{B})$, then $\sum_{j=1}^{\infty} X_j u_j$ converges a.e. in $\mathcal{B}$ iff $\sum_{j=1}^{\infty} X_j u_j$ converges in $L^p(\mu)$.*

Proof. For a proof that L^p convergence of a series of vector valued independent random variables implies its a.e. convergence, see [4] for example. To show the converse, we apply the $MPZ(p, \mathcal{B})$ property and get for any $N > 0$ and any $\{u_j\} \subset \mathcal{B}$,

$$\mu \left\{ w \colon \left\| \sum_{j=1}^{N} X_j(w) u_j \right\| \geq \beta_p \left\| \sum_{j=1}^{N} X_j(w) u_j \right\|_p \right\} \geq \alpha.$$

If the series does not converge in L^p, then there exists an $\epsilon > 0$ and an increasing sequence of integers N_k, such that

$$\left\| \sum_{j=N_k+1}^{N_{k+1}} X_j u_j \right\|_p > \epsilon, \quad k = 1, 2, \ldots$$

This implies

$$\mu \left\{ w \colon \left\| \sum_{j=N_k+1}^{N_{k+1}} X_j(w) u_j \right\| \geq \beta_p \epsilon \right\} \geq \alpha, \quad k = 1, 2, \ldots$$

Let us define

$$A_k = \left\{ w \colon \left\| \sum_{j=N_k+1}^{N_{k+1}} X_j(w) u_j \right\| \geq \beta_p \epsilon \right\}, k = 1, 2, \ldots .$$

Then at all $w \in \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k$, the series diverges. Since

$$\mu \left\{ \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k \right\} \geq \alpha,$$

we have a contradiction. □

We therefore have:

THEOREM 6. *If $X \in MPZ(p, \mathcal{B})$ and $u_j \in \mathcal{B}$, then for every $0 < q < p$, there is a constant $B(q, p) = B(q, p, \mathcal{B}, X)$ such that if $\sum_{j=1}^{\infty} X_j u_j$ converges a.e., then*

$$\left\| u_0 + \sum_{j=1}^{\infty} X_j u_j \right\|_p \le B(q, p) \left\| u_0 + \sum_{j=1}^{\infty} X_j u_j \right\|_q .$$

THEOREM 7. *Let $X \in MPZ(p, \mathcal{B})$ and $u_j \in \mathcal{B}$. If $\sum_{j=1}^{\infty} X_j u_j$ converges to zero a.e. on a set of measure greater than $1 - \alpha$, where α is the constant appearing in the $MPZ(p, \mathcal{B})$ property, then $u_j = 0$, $j \ge 1$.*

Proof. If $\left\|\sum_{j=1}^{\infty} X_j u_j\right\|_p \neq 0$, then, since $X \in MPZ(p, \mathcal{B})$,

$$\mu\left\{w\colon \left\|\sum_{j=1}^{\infty} X_j u_j\right\| > 0\right\} \ge \alpha,$$

a contradiction. Thus we have $\| \sum_{j=1}^{\infty} X_j(w) u_j \|_p = 0$. Let X_j^s's be the symmetrizations of X_j's. Since $\sum_{j=1}^{\infty} X_j(w) u_j = 0$ a.e., we also have

$$\sum_{j=1}^{\infty} X_j^s(w) u_j = 0 \text{ a.e.}$$

By the Paul Lévy inequality, for any $t > 0$,

$$\mu\left\{w\colon \sup_{1 \le j} \|X_j^s(w) u_j\| > t\right\} \le 2\mu\left\{w\colon \left\|\sum_{j=1}^{\infty} X_j^s(w) u_j\right\| > t\right\} = 0.$$

This shows that $X_j^s u_j = 0$ a.e. which implies that $u_j = 0$, $j \ge 1$. □

The Stein property, a local version of the MPZ property, was originally defined in the scalar case by Burkholder in [3]. We give a somewhat different definition in the vector valued case:

DEFINITION 4. *Let $\mathcal{B}$ be a Banach space. Let X be a sequence of independent random variables. If for some $p > 0$ there exist positive constants $\alpha = \alpha(p, \mathcal{B}, X)$, $\beta = \beta(p, \mathcal{B}, X)$ such that for any $E \in \sum$, $\mu(E) > 0$, there is an $n = n(E)$ such that for any $N > n$ and any $u_j \in \mathcal{B}$,*

$$\mu\left\{w \in E\colon \left\|u_0 + \sum_{j=1}^{N} X_j u_j\right\| \ge \beta \left\|\sum_{j=n+1}^{N} X_j u_j\right\|_p\right\} \ge \alpha\ \mu(E),$$

then we say X has the p-Stein property (in $\mathcal{B}$) and write $X \in S(p, \mathcal{B})$.

With exactly the same proof as that in [3], we obtain:

THEOREM 8 [3]. *Let $\mathcal{B}$ be a Banach space. Then $X \in S(p, \mathcal{B})$ iff there exists a positive constant $\beta = \beta(p, \mathcal{B}, X)$ such that for any $E \in \sum$, $\mu(E) > 0$, there is an $n = n(E)$ such that for any $N > n$ and any $\{u_j\} \subset \mathcal{B}$,*

$$\operatorname{esssup}_{w \in E} \left\| u_0 + \sum_{j=1}^{N} X_j(w) u_j \right\| \geq \beta \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_p .$$

THEOREM 9. *If $X \in MPZ(p, \mathcal{B})$ for $p \geq 1$ and $EX_j = 0$, $j \geq 1$, then $X \in S(p, \mathcal{B})$. Moreover, if $\sum_{j=1}^{\infty} X_j u_j$ converges a.e. for some vectors $\{u_j\} \subset \mathcal{B}$, then $\forall E \in \sum$, $\mu(E) > 0$, $\exists n = n(E)$ such that*

$$\mu \left\{ w \in E\colon \left\| u_0 + \sum_{j=1}^{\infty} X_j(w) u_j \right\| \geq \beta \left\| \sum_{j=n+1}^{\infty} X_j u_j \right\|_p \right\} \geq \alpha \cdot \mu(E).$$

Proof. Since X satisfies the Khinchin-Kahane inequality, by Theorem 2, the symmetrization X^s of X also satisfies the Khinchin-Kahane inequality, which implies that the $X^s \in MPZ(p, \mathcal{B})$. Thus, there are some constants β' and α' such that for any $N > 0$,

$$\mu \left\{ w\colon \left\| u_0 + \sum_{j=1}^{N} X_j^s(w) u_j \right\| \geq \beta' \left\| u_0 + \sum_{j=1}^{N} X_j^s u_j \right\|_p \right\} \geq \alpha' > 0.$$

The following argument is similar to one used by Burkholder in [3].

Let $A \in \sum$ and $\mu(A) > 0$. Let $\mathcal{F}_n = \sigma(X_1, \dots X_n)$, $\mathcal{F}_\infty = \sigma(F_n, n \geq 1)$. Let $U = E(\chi_A | \mathcal{F}_\infty)$ and $V_n = E(\chi_A | \mathcal{F}_n)$. Choose $0 < \delta < \frac{1}{8} \cdot \alpha' \cdot \mu(A)$. Then there exists an n such that

$$E|U - V_n| < \delta.$$

Define

$$B = \left\{ w\colon \left\| u_0 + \sum_{j=1}^{N} X_j(w) u_j \right\| < \frac{\beta'}{2} \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_p \right\}.$$

By independence,

$$E(\chi_B | \mathcal{F}_n)(w') = \mu \left\{ w\colon \left\| v(w') + \sum_{j=n+1}^{N} X_j(w) u_j \right\| < \frac{\beta'}{2} \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_p \right\}$$

where $v(w') = u_0 + \sum_{j=1}^{n} X_j(w')u_j$ and $w' \in \Omega$. Since

$$\begin{aligned}
&\mu^2 \left\{ w\colon \left\| v(w') + \sum_{j=n+1}^{N} X_j(w)u_j \right\| < \frac{\beta'}{2} \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_p \right\} \\
&= \mu \left\{ w\colon \left\| v(w') + \sum_{j=n+1}^{N} X_j(w)u_j \right\| < \frac{\beta'}{2} \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_p, \right. \\
&\qquad \left. \left\| v(w') + \sum_{j=n+1}^{N} X_j'(w)u_j \right\| < \frac{\beta'}{2} \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_p \right\} \\
&\le \mu \left\{ w\colon \left\| \sum_{j=n+1}^{N} X_j^s(w)u_j \right\| < \beta' \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_p \right\} \\
&\le \mu \left\{ w\colon \left\| \sum_{j=n+1}^{N} X_j^s(w)u_j \right\| < \beta' \left\| \sum_{j=n+1}^{N} X_j^s u_j \right\|_p \right\} < 1 - \alpha',
\end{aligned}$$

we have $E(\chi_B|\mathcal{F}_n)(w') < \sqrt{1-\alpha'}$. Therefore

$$\begin{aligned}
\mu(A \cap B) &= E(U \cdot \chi_B) < E(V_n \cdot \chi_B) + \delta \\
&= E[V_n \cdot E(\chi_B|\mathcal{F}_n)] + \delta < E(V_n) \cdot \sqrt{1-\alpha'} + \delta \\
&\le (EU + \delta) \cdot \sqrt{1-\alpha'} + \delta \le \mu(A) \cdot \sqrt{1-\alpha'} + 2\delta. \\
&< \mu(A) \cdot \left(1 - \frac{\alpha'}{2}\right) + \mu(A) \cdot \frac{\alpha'}{4} = \mu(A) \cdot \left(1 - \frac{\alpha'}{4}\right).
\end{aligned}$$

Thus we can take $\alpha = \frac{\alpha'}{4}$, $\beta = \frac{\beta'}{2}$ and get

$$\mu \left\{ w \in A\colon \left\| u_0 + \sum_{j=1}^{N} X_j(w)u_j \right\| \ge \beta \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_p \right\} \ge \alpha\mu(A).$$

Finally,

$$\begin{aligned}
&\mu \left\{ w \in A\colon \left\| u_0 + \sum_{j=1}^{\infty} X_j(w)u_j \right\| < \beta \left\| \sum_{j=n+1}^{\infty} X_j u_j \right\|_q \right\} \\
&\le \mu \left\{ w \in A\colon \lim_N \left\| u_0 + \sum_{j=1}^{N} X_j(w)u_j \right\| < \beta \lim_N \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_q \right\}
\end{aligned}$$

$$= \mu\left(\liminf_N \left\{ w \in A\colon \left\| u_0 + \sum_{j=1}^{N} X_j(w) u_j \right\| < \beta \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_q \right\}\right)$$

$$\leq \liminf_N \mu \left\{ w \in A\colon \left\| u_0 + \sum_{j=1}^{N} X_j(w) u_j \right\| < \beta \left\| \sum_{j=n+1}^{N} X_j u_j \right\|_q \right\}$$

$$\leq (1-\alpha)\ \mu(A). \qquad \square$$

Repeating the proof of Theorem 7, we have:

THEOREM 10. *Let B be a Banach space. Let $X \in S(p, \mathcal{B})$ and $u_j \in \mathcal{B}$. If $u_0 + \sum_{j=1}^{\infty} X_j u_j$ converges to zero on any subset $A \in \sum$ of positive measure, then all but finitely many u_j are zero.*

4. Local L^p-norm inequalities

We show that behavior of the tail series of sequences of independent random variables on subset $E \in \mathcal{F}_\infty$ mimics their global behavior.

Let $E \in \sum$, $\mu(E) > 0$. We let

$$\|f\|_{L^p_E} = \left(\frac{1}{\mu(E)} \int_E \|f\|^p d\mu_E\right)^{1/p}.$$

THEOREM 11. *Let $X \in MPZ(p, \mathcal{B})$. Then $\forall 0 < \eta < 1$, $\forall 0 < \epsilon < p$ and $\forall E \in \mathcal{F}_\infty$, $\mu(E) > 0$, $\exists n = n(E, \epsilon, \eta, p)$ such that for all $0 < q \leq p - \epsilon$,*

$$\left| \frac{\|u_0 + \sum_{j=n+1}^{\infty} X_j u_j\|^q_{L^q_E}}{\|u_0 + \sum_{j=n+1}^{\infty} X_j u_j\|^q_{L^q}} - 1 \right| \leq \eta.$$

Proof. Define $s = \frac{p}{\epsilon}$, $s' = \frac{p}{p-\epsilon}$. For given $E \in \mathcal{F}_\infty$, there exist $n = n(E, \epsilon, \eta, p)$ and $E_n \in \mathcal{F}_n$ such that

$$\mu^{1/s}(E_n \Delta E) < \mu(E) \cdot \eta \cdot \beta_p^p \alpha \cdot (1 + \beta_p^p \alpha)^{-1}.$$

Let $f_n = \|u_0 + \sum_{j=n+1}^{\infty} X_j u_j\|$. By (2), we have

$$\int_{E_n \Delta E} f_n^q d\mu \leq \mu^{1/s}(E_n \Delta E) \left(\int_\Omega f_n^{qs'} d\mu\right)^{1/s'}$$

$$\leq \mu^{1/s}(E_n \Delta E) \|f_n\|_p^q \leq \mu^{1/s}(E_n \Delta E) (\beta_p \alpha^{1/q})^{-q}\ \|f_n\|_q^q.$$

Since $\beta_p \leq 1$, we have $\beta_p^{-q}\alpha^{-1}(1+\beta_p^q\alpha) \leq \beta_p^{-p}\alpha^{-1}(1+\beta_p^p\alpha)$ and so

$$\begin{aligned}
\left|\frac{1}{\mu(E)}\int_E f_n^q d\mu - \int_\Omega f_n^q d\mu\right| &\leq \left|\frac{1}{\mu(E)}\int_E f_n^q d\mu - \frac{1}{\mu(E)}\int_{E_n} f_n^q d\mu\right| \\
&\quad + \left|1-\frac{\mu(E_n)}{\mu(E)}\right|\int_\Omega f_n^q d\mu \\
&\leq \frac{1}{\mu(E)}\int_{E_n\Delta E} f_n^q d\mu + \frac{\mu(E_n\Delta E)}{\mu(E)}\int_\Omega f_n^q d\mu \\
&\leq \left[\frac{\mu^{\frac{1}{s}}(E_n\Delta E)}{\mu(E)}\beta_p^{-q}\alpha^{-1} + \frac{\mu^{\frac{1}{s}}(E_n\Delta E)}{\mu(E)}\right]\|f_n\|_q^q \\
&\leq \frac{\mu^{\frac{1}{s}}(E_n\Delta E)}{\mu(E)}\beta_p^{-q}\alpha^{-1}\,(1+\beta_p^q\alpha)\,\|f_n\|_q^q \\
&\leq \eta\,\|f_n\|_q^q. \qquad \square
\end{aligned}$$

COROLLARY 2. *Let $X \in MPZ(p,\mathcal{B})$. Then $\forall 0<\epsilon<p, \forall E \in \mathcal{F}_\infty, \mu(E)>0, \exists n = n(E,\epsilon,p)$ such that for all $0<q\leq p-\epsilon$,*

$$\frac{1}{2}\left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L_E^q} \leq \left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L^q} \leq \frac{3}{2}\left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L_E^q}.$$

A similar argument shows:

THEOREM 12. *Let $X \in MPZ(p,\mathcal{B})$ for all $0<p<\infty$. There are constants $a_p = a(p,\mathcal{B}), b_p = b(p,\mathcal{B})$ such that $\forall E \in \mathcal{F}_\infty, \mu(E)>0, \exists n = n(E)$ such that for all $0<p<\infty$,*

$$a_p\left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L_E^p} \leq \left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L^p} \leq b_p\left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L_E^p}.$$

The last result and (2) prove:

COROLLARY 3 [11]. *Let $X \in MPZ(p,\mathcal{B})$ for all $0<p<\infty$. There are constants $c_p = c(p,\mathcal{B}), d_p = d(p,\mathcal{B})$ such that $\forall E \in \mathcal{F}_\infty, \mu(E)>0, \exists n = n(E)$ such that for all $0<p<\infty$,*

$$c_p\left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L_E^2} \leq \left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L_E^p} \leq d_p\left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L_E^2}.$$

5. Some Orlicz-norm inequalities

We now consider some Orlicz spaces. Recall the definition of the norm in an Orlicz space L^{ϕ}: Let ϕ be a Young function defined on $[0, \infty)$, and let f be a measurable function on a measure space $(\Omega, \sum, \mu)$. Then

$$\|f\|_{L^{\phi}} = \inf\left\{\lambda > 0\colon \int_{\Omega} \phi\left(\frac{|f|}{\lambda}\right) d\mu < 1\right\}.$$

In what follows, we denote by L^{ψ_α} the Orlicz space with respect to the Young function: $\psi_\alpha(t) = \exp(t^{1/\alpha}) - 1,\ \ 0 < \alpha \leq 1$.

We consider sequences X such that for some $0 < \alpha \leq 1$, all $p > 2$ and any $N \geq 1$,

$$\left\|\sum_{j=1}^{N} X_j u_j\right\|_p \leq B(p, \mathcal{B}) \left\|\sum_{j=1}^{N} X_j u_j\right\|_2, \qquad B(p, \mathcal{B}) = O(p^{\alpha}). \tag{3}$$

THEOREM 13. *Let X satisfy condition* (3), *and $\{u_j, 0 \leq j < \infty\}$ be vectors in $\mathcal{B}$. If $\sum_{j=1}^{\infty} X_j u_j$ converges a.e., then $\sum_{j=1}^{\infty} X_j u_j$ converges in L^{ψ_α}. Moreover, there are constants $A_\alpha = A(\alpha, X)$, $B_\alpha = B(\alpha, X)$ such that*

$$A_\alpha \left\|u_0 + \sum_{j=1}^{\infty} X_j u_j\right\|_2 \leq \left\|u_0 + \sum_{j=1}^{\infty} X_j u_j\right\|_{L^{\psi_\alpha}} \leq B_\alpha \left\|u_0 + \sum_{j=1}^{\infty} X_j u_j\right\|_2 .$$

Proof. By Lemma 2, we have $\|u_0 + \sum_{j=1}^{\infty} X_j u_j\| \in L^2$. Let $d = \|u_0 + \sum_{j=1}^{\infty} X_j u_j\|_2$. The proof of the inequality is along the same lines as the proof in [15] for $\alpha = \frac{1}{2}$. We may assume that in (3), $B(p, \mathcal{B}) \leq r\, p^{\alpha}$ for some $r > 0$. Set $\gamma_0 = r^{-1}\, \alpha^{\alpha}\, (2e)^{-\alpha}$. Taking $p = k/\alpha, k = 1, 2, \ldots$ in (3), we have

$$\begin{aligned}
&\int_{\Omega} \exp\left[\left(\gamma_0 \frac{\|u_0 + \sum_{j=1}^{\infty} X_j u_j\|}{d}\right)^{1/\alpha}\right] d\mu \\
&\quad = \sum_{k=0}^{\infty} \gamma_0^{k/\alpha}\, d^{-k/\alpha}\, \frac{1}{k!} \int \left\|u_0 + \sum_{j=1}^{\infty} X_j u_j\right\|^{k/\alpha} d\mu \\
&\quad \leq \sum_{k=0}^{\infty} \alpha^{-k}\, \gamma_0^{k/\alpha}\, k^k\, r^{k/\alpha}\, \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{(2e)^k}\, \frac{k^k}{k!} < 2.
\end{aligned}$$

Hence

$$\left\|u_0 + \sum_{j=1}^{\infty} X_j u_j\right\|_{L^{\psi_\alpha}} \leq \frac{1}{\gamma_0} d = r\, \alpha^{-\alpha}\, (2e)^{\alpha} \left\|u_0 + \sum_{j=1}^{\infty} X_j u_j\right\|_2 ,$$

and the L^2 convergence of $\sum_{j=1}^{\infty} X_j u_j$ implies its convergence in L^{ψ_α}. $\square$

Similarly one can also prove:

THEOREM 14. *Let X satisfy condition* (3). *Assume that* $\exp(C|X_j|^{1/\alpha}) \in L^1(\mu)$, $\forall C > 0$, $j \geq 1$. *Let* $\{u_j\}$ *be vectors in* $\mathcal{B}$. *If* $\sum_{j=1}^{\infty} X_j u_j$ *converges a.e., then for all* $C > 0$,

$$\exp\left(C\left\|u_0 + \sum_{j=1}^{\infty} X_j u_j\right\|^{1/\alpha}\right) \in L^1(\mu).$$

We can also prove a local version of the Orlicz norm inequalities. Let

$$\|f\|_{L_E^{\phi}} = \inf\left\{\lambda > 0\colon \frac{1}{\mu(E)}\int_E \phi\left(\frac{|f|}{\lambda}\right) d\mu < 1\right\}.$$

THEOREM 15. *Let X satisfies condition* (3). *There are constants* $C_\alpha = C(\alpha, X)$, $D_\alpha = D(\alpha, X)$ *such that for* $\{u_j\} \subset \mathcal{B}$, *if* $\sum_{j=1}^{\infty} X_j u_j$ *converges a.e., then for any* $E \in \mathcal{F}_\infty$, $\mu(E) > 0$, $\exists n = n(E, \alpha)$, *such that*

$$C_\alpha \left\|u_0 + \sum_{j=n+1}^{\infty} X_j u_j\right\|_{L_E^{\psi_\alpha}} \leq \left\|u_0 + \sum_{j=n+1}^{\infty} X_j u_j\right\|_{L^{\psi_\alpha}} \leq D_\alpha \left\|u_0 + \sum_{j=n+1}^{\infty} X_j u_j\right\|_{L_E^{\psi_\alpha}}.$$

Proof. In view of Theorem 13 and Theorem 12, it suffices to prove that there are constants $C'_\alpha = C'(\alpha, X)$, $D'_\alpha = D'(\alpha, X)$ such that for any $E \in \mathcal{F}_\infty$, $\mu(E) > 0$, $\exists n = n(E, \alpha)$ such that

$$C'_\alpha \left\|u_0 + \sum_{j=n+1}^{\infty} X_j u_j\right\|_{L_E^2} \leq \left\|u_0 + \sum_{j=n+1}^{\infty} X_j u_j\right\|_{L_E^{\psi_\alpha}} \leq D'_\alpha \left\|u_0 + \sum_{j=n+1}^{\infty} X_j u_j\right\|_{L_E^2}.$$

Let $d_{k+1} = \left\|u_0 + \sum_{j=k+1}^{\infty} X_j u_j\right\|_2$, $k \geq 1$. We have shown that

$$\int \exp\left[\left(\gamma_0 \frac{\|u_0 + \sum_{j=k+1}^{\infty} X_j u_j\|}{d_{k+1}}\right)^{\frac{1}{\alpha}}\right] d\mu < 2.$$

For $E \in F_\infty$, $\exists n = n(E)$ and $E_n \in \mathcal{F}_n$ such that both Theorem 12 and

$$\frac{\mu^{1/2}(E_n \Delta E)}{\mu(E)} < 1$$

hold, which implies

$$\frac{1}{2} < \frac{\mu(E_n)}{\mu(E)} < 2.$$

Thus, applying Hölder's inequality, we get

$$\frac{1}{\mu(E)}\int_E \exp\left[\frac{1}{2}\left(\gamma_0\,\frac{\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\|}{d_{n+1}}\right)^{1/\alpha}\right]d\mu$$

$$\leq \frac{1}{\mu(E)}\int_{E_n} \exp\left[\frac{1}{2}\left(\gamma_0\,\frac{\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\|}{d_{n+1}}\right)^{1/\alpha}\right]d\mu$$

$$+\ \frac{1}{\mu(E)}\int_{E\setminus E_n} \exp\left[\frac{1}{2}\left(\gamma_0\,\frac{\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\|}{d_{n+1}}\right)^{1/\alpha}\right]d\mu$$

$$\leq \frac{\mu(E_n)}{\mu(E)}\int \exp\left[\frac{1}{2}\left(\gamma_0\,\frac{\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\|}{d_{n+1}}\right)^{1/\alpha}\right]d\mu$$

$$+\ \frac{\mu^{\frac{1}{2}}(E\setminus E_n)}{\mu(E)}\left(\int \exp\left(\gamma_0\,\frac{\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\|}{d_{n+1}}\right)^{1/\alpha}d\mu\right)^{1/2}$$

$$\leq 3\sqrt{2}.$$

Let us denote $\gamma = (\frac{1}{6})^{\alpha}\gamma_0$.

$$\frac{1}{\mu(E)}\int_E \exp\left[\left(\gamma\,\frac{\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\|}{d_{n+1}}\right)^{1/\alpha}\right]d\mu$$

$$\leq \Big(\frac{1}{\mu(E)}\int_E \exp\left[3\left(\gamma\,\frac{\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\|}{d_{n+1}}\right)^{1/\alpha}\right]d\mu\Big)^{1/3}$$

$$= \Big(\frac{1}{\mu(E)}\int_E \exp\left[\frac{1}{2}\left(\gamma_0\,\frac{\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\|}{d_{n+1}}\right)^{1/\alpha}\right]d\mu\Big)^{1/3} < 2.$$

Applying Theorem 12, we get that for $n \geq n(E)$,

$$\left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_2 \leq b_2\left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L_E^2}.$$

Thus

$$\left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L_E^{\psi_\alpha}} \leq b_2\gamma^{-1}\left\|u_0+\sum_{j=n+1}^{\infty}X_ju_j\right\|_{L_E^2}. \qquad \square$$

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