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PARITY OF FOURIER COEFFICIENTS OF MODULAR FORMS

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1. Introduction

A partition of a non-negative integer n is a non-increasing sequence of positive integers whose sum is n. It is of interest to examine the number of partitions of n under some additional restriction on the summands. Various partition functions arise in the representation theory of permutation groups (see [2]). For example, if p is prime, then let $b_p(n)$ denote the number of partitions of a non-negative integer n where the summands are not multiples of p. If n is a positive integer, then $b_p(n)$ denotes the number of irreducible representations of the symmetric group S_n over the finite field with p elements [2, Lemma 6.1.2].

For $b_k(n)$, the number of partitions of n into parts none of which is a multiple of k, the generating function is given by the infinite product

$$\sum_{n=0}^{\infty} b_k(n)q^n = \prod_{n=1}^{\infty} \frac{1-q^{kn}}{1-q^n}. \tag{1}$$

There are other important examples of partition generating functions which contain similar infinite products. In particular we shall consider certain partition generating functions which contain infinite products of the form

$$\prod_{1\leq n\equiv g \pmod{\delta}} (1-q^n) \prod_{1\leq n\equiv -g \pmod{\delta}} (1-q^n)$$

where $0 \leq g \leq \delta$. For example the two Rogers-Ramanujan identities (see [1]),

$$\sum_{n=0}^{\infty} \frac{q^{n^2+an}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{\left(1-q^{5n+a+1}\right)\left(1-q^{5n+4-a}\right)},$$

where $a = 0$ or 1, involve such products.

For $r_{g,\delta}(n)$ the number of partitions of n into parts that are congruent to $\pm g \pmod{\delta}$ where $0 < g < \lfloor \frac{\delta+1}{2} \rfloor$, the generating function for $r_{g,\delta}(n)$ is given by the infinite product

$$\sum_{n=0}^{\infty} r_{g,\delta}(n)q^n = \prod_{1\leq n\equiv g \pmod{\delta}} \frac{1}{(1-q^n)} \prod_{1\leq n\equiv -g \pmod{\delta}} \frac{1}{(1-q^n)}. \tag{2}$$

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We shall also examine the coefficients $c(n)$ of Klein's modular function $j(z)$. Its Fourier expansion is given by

$$j(z) = \frac{(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n)^3}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} = \sum_{n=-1}^{\infty} c(n)q^n, \tag{3}$$

where $\sigma_3(n) := \sum_{d|n} d^3$.

In this paper we consider the parity of the Fourier coefficients of certain modular forms which include the arithmetic functions $b_k(n)$, $r_{g,\delta}(n)$, and $c(n)$. It is conjectured (see [6]), that the number of non-negative integers $n \leq x$ for which $p(n)$ is even is $\sim \frac{1}{2}x$. Very little is known about this specific conjecture; however there are weaker conjectures regarding the parity of the partition function which are more easily attacked. In [12], Subbarao conjectured that in an arithmetic progression $r \pmod{t}$ there are infinitely many integers $N \equiv r \pmod{t}$ for which $p(N)$ is even, and that there are infinitely many integers $M \equiv r \pmod{t}$ for which $p(M)$ is odd.

Using the theory of modular forms, the first author proved that in any arithmetic progression $r \pmod{t}$ there are infinitely many $N \equiv r \pmod{t}$ for which $p(N)$ is even, and there are infinitely many $M \equiv r \pmod{t}$ for which $p(M)$ is odd provided that there is at least one such M. Moreover the smallest such M (if there are any) is less than $10^{10}t^7$. Using these results and a fair bit of machine computation, the conjecture has now been verified for every arithmetic progression (mod t) where $t \leq 100,000$.

In [9], Serre pointed out that the argument in [3] and [4] could be generalized to a broader family of modular forms. We carry out these suggestions and show that the same parity properties also hold for any meromorphic half-integral or integral weight modular forms with respect to $\Gamma_1(N)$ possessing integer coefficients, provided that all of its poles are at cusps.

2. Facts about modular forms

If N is a positive integer, define the following level N *congruence subgroups* of $SL_2(\mathbb{Z})$ by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc=1,\ c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc=1,\ a \equiv d \equiv 1 \pmod{N},\ c \equiv 0 \pmod{N} \right\}.$$

These subgroups of $SL_2(\mathbb{Z})$ act on $\mathfrak{H}$, the upper half of the complex plane, as follows: if $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and z is in $\mathfrak{H}$, define Az by $Az = \frac{az+b}{cz+d}$. If k is an integer and $f(z)$ is a meromorphic function on $\mathfrak{H}$ then $f(z)$ is a *modular form of weight k with respect to* Γ if

$$f(Az) = (cz+d)^k f(z)$$

for all $A \in \Gamma \subseteq SL_2(\mathbb{Z})$ and all $z \in \mathfrak{H}$. If $f(z)$ is holomorphic on $\mathfrak{H}$ as well as at the cusps of Γ (i.e., the rationals), then $f(z)$ is called a *holomorphic modular form.* Of particular interest are those holomorphic modular forms which vanish at cusps, the *cusp forms.*

Note that any modular form of weight k with respect to $\Gamma_0(N)$ is automatically one with respect to $\Gamma_1(N)$ since $\Gamma_1(N) \subseteq \Gamma_0(N)$. A weight k modular form with respect to $\Gamma_1(N)$ has *Nebentypus character* χ if

$$f(Az) = \chi(d)(cz+d)^k f(z) \tag{4}$$

for all $A \in \Gamma_0(N)$ where χ is a Dirichlet character modulo N. The finite-dimensional $\mathbb{C}$-vector space of holomorphic modular forms of weight k and Nebentypus χ is denoted $M_k(N, \chi)$; its subspace of cusp forms is denoted $S_k(N, \chi)$. If $N|N'$ then $M_k(N) \subseteq M_k(N')$ (resp. $S_k(N) \subseteq S_k(N')$) and for fixed N the $M_k(N)$ form a graded algebra; i.e., if f is of weight k and g is of weight k' then fg is of weight $k+k'$.

In the variable $q = e^{2\pi i z}$, these modular forms have the Fourier expansion

$$f(z) = \sum_{n \geq N_0}^{\infty} a(n) q^n$$

where the Fourier coefficients $a(n)$ are complex numbers. In [8], Serre proved that if $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ is a holomorphic modular form with integer weight k with respect to some congruence subgroup of $SL_2(\mathbb{Z})$ where the coefficients $a(n)$ are in the integer ring O_K of some number field K, then for any positive integer m the number of $n \leq x$ such that $a(n) \not\equiv 0 \pmod{m}$ is $O(\frac{x}{\log^{\alpha} x})$ for some $\alpha > 0$; i.e., if m is a positive integer, then

$$a(n) \equiv 0 \pmod{m}$$

for almost all n. In particular $a(N)$ is a multiple of m for almost all $N \equiv r \pmod{t}$.

If m is a positive integer and $g(z) = \sum_{n=0}^{\infty} a(n)q^n$ is a holomorphic modular form of integer weight k with respect to $\Gamma \supseteq \Gamma_1(N)$ for some positive integer N with algebraic integer Fourier coefficients from a fixed number field, let $\mathrm{Ord}_m(g(z))$ be the smallest integer n such that $a(n) \not\equiv 0 \pmod{m}$. Sturm [11] proved if

$$\mathrm{Ord}_m(g(z)) > \frac{k}{12}[SL_2(\mathbb{Z})\colon \Gamma],$$

then $\mathrm{Ord}_m(g(z)) = \infty$. (i.e., $a(n) \equiv 0 \pmod{m}$ for all n).

Shimura [10] developed a theory of half-integer weight modular forms which satisfy an analogue of (4) with some auxillary characters. An important point in Shimura's theory is that the level N of a half-integer weight form is necessarily a multiple of 4.

The classical theta function $\Theta(z) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$ is a holomorphic modular form of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$. We note that $\Theta(z) \equiv 1 \pmod{2}$. Another example is the Dedekind Eta-function, a weight $\frac{1}{2}$ cusp form on $\Gamma_0(576)$ defined by

$$\eta(24z) = q\prod_{n=1}^{\infty}(1-q^{24n}). \tag{5}$$

Many modular forms are products of the Dedekind Eta-function; for example Ramanujan's Δ-function, the unique normalized weight 12 cusp form with respect to $SL_2(\mathbb{Z})$, and $\Theta(z)$ are given by

$$\begin{aligned} \Delta(z) &= \eta^{24}(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \\ \Theta(z) &= \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}. \end{aligned} \tag{6}$$

It is well known that

$$\Delta(z) \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod 2.$$

The generalized Dedekind Eta-products are also fundamental modular forms. If $0 \leq g < \delta$ are non-negative integers, then the generalized Dedekind Eta-product $\eta_{g,\delta}(z)$ is defined by

$$\eta_{g,\delta}(z) := e^{\pi i P_2(\frac{g}{\delta})\delta z} \prod_{1\leq n\equiv g \pmod \delta}(1-q^n) \prod_{1\leq n\equiv -g \pmod \delta}(1-q^n). \tag{7}$$

Here $P_2(t)$ is defined by $P_2(t) := \{t\}^2 - \{t\} + \frac{1}{6}$ where $\{t\}$ is the fractional part of t. If $g = 0$ (resp. $g = \frac{1}{2}\delta$), then $\eta_{g,\delta}(z)$ is $\eta^2(\delta z)$ (resp. $\frac{\eta^2(\delta z/2)}{\eta^2(\delta z)}$). If $g \neq 0, \frac{1}{2}\delta$, then $\eta_{g,\delta}(z)$ is a weight 0 meromorphic modular form that does not vanish on the upper half of the complex plane. For more on the arithmetic of these modular forms see [7]. Hence we see the generating functions for $r_{g,\delta}(n)$ in (2) are, up to a power of q, the Fourier expansions of $\frac{1}{\eta_{g,\delta}(z)}$.

3. The general theorem

THEOREM 1. *Suppose that $f(z) = \sum_{n\geq N_0} a(n)q^n$ is a modular form of half integer or integer weight k with respect to $\Gamma_1(N)$ for some positive integer N. If $f(z)$ is holomorphic on the upper half of the complex plane and the coefficients $a(n)$ are integers, then in any arithmetic progression r (mod t) there are infinitely many $N \equiv r$ (mod t) for which $a(N)$ is even, and there are infinitely many $M \equiv r$ (mod t) for which $a(M)$ is odd, provided there is at least one such non-zero M.*

Proof. First suppose that $f(z)$ is a half integer weight form, then

$$f(z) \equiv f(z)\cdot\Theta(z) \pmod 2$$

where $f(z)\cdot\Theta(z)$ is a modular form with integer weight $k+\frac{1}{2}$ with respect to $\Gamma_1(N)$. Hence if $f(z)$ is a half integer weight modular form with respect to $\Gamma_1(N)$, then there exists an integer weight modular form with the same Fourier expansion modulo 2. So we may assume that $f(z)$ is an integer weight k form.

Since $f(z)$ is holomorphic on $\mathfrak{H}$, its only poles (if there are any) occur at cusps. Since $\Delta(z)$ is a cusp form, there is a minimal non-negative integer j for which $F_t(z) := f(z)\cdot\Delta^{2^j}(tz)$ is holomorphic at the cusps. Hence $F_t(z)$ is in $M_{2^j\cdot 12+k}(Nt)$ since $\Delta(tz)$ is in $S_{12}(t)$.

Since

$$(8)\qquad \Delta^{2^j}(tz)\equiv\Delta(2^j tz)\equiv\sum_{n=0}^{\infty}q^{2^j\cdot t(2n+1)^2}\pmod 2,$$

the modular form $F_t(z)$ has the convenient (mod 2) factorization

$$(9)\qquad F_t(z)=\sum_{n=0}^{\infty}c_t(n)q^n\equiv\left(\sum_{n\geq N_0}a(n)q^n\right)\cdot\left(\sum_{n=0}^{\infty}q^{2^j\cdot t(2n+1)^2}\right)\pmod 2.$$

We now prove there are infinitely many integers $N\equiv r$ (mod t) for which $a(N)$ is even. Suppose $a(N)$ is odd for all but finitely many $N\equiv r$ (mod t); in particular that $a(n)$ is odd for all $n\geq n_0$ with $n\equiv r$ (mod t). Without loss of generality we may assume that $j\geq 1$. Comparing the coefficient of $q^{2^j tk^2+n}$ on both sides of (9) we find that

$$c_t(2^j tk^2+n)\equiv\sum_{i\geq 1,\ i\text{ odd}}a(2^j t(k^2-i^2)+n)\pmod 2.$$

Note that each $2^j t(k^2-i^2)+n\equiv n\equiv r$ (mod t). Now if $i\leq k$ then $2^j t(k^2-i^2)+n\geq n\geq n_0$ so that $a(2^j t(k^2-i^2)+n)$ is odd. If k is odd and $i>k>\frac{-N_0+n}{2^{j+2}t}-1$ then $2^j t(k^2-i^2)+n<N_0$ so that $a(2^j t(k^2-i^2)+n)=0$. Therefore, for such k, we have $c_t(2^j tk^2+n)\equiv\frac{k+1}{2}\pmod 2$. We have now proved that for all sufficiently large $k\equiv 1\pmod 4$ we have $c_t(n)$ odd for all $n\equiv r$ (mod t) in the interval $[2^j tk^2+n_0, 2^j t(k+2)^2+r-t]$ (assuming, without loss of generality that $0\leq r\leq t-1$). By taking all such intervals into account we have a positive proportion of $c_t(n)$ with $n\equiv r$ (mod t) which are odd, contradicting Serre's Theorem [8] since $F_t(z)$ is in $M_{2^j\cdot 12+k}(Nt)$. Therefore there are infinitely many integers $N\equiv r$ (mod t) for which $a(N)$ is even.

We now establish the existence of infinitely many $M\equiv r$ (mod t) for which $a(M)$ is odd provided that there is at least one such M. To study the Fourier coefficients attached to those exponents that are in the arithmetic progression r (mod t), we define $F_{r,t}(z)$ by

$$F_{r,t}(z):=\sum_{n\equiv r\ (\mathrm{mod}\ t)}c_t(n)q^n.$$

By [4, Lemma 2], $F_{r,t}(z)$ is in $M_{2^j\cdot 12+k}(\frac{Nt^3}{d})$ where $d:=\gcd(r,t)$.

Suppose there are only finitely many $M \equiv r \pmod{t}$ for which $a(M)$ is odd. In particular suppose $a(tm+r)$ is even if $m > m_0$. Then from (8) we find

$$F_{r,t}(z) \equiv \left(\sum_{m \le m_0} a(tm+r)q^{tm+r}\right)\left(\sum_{n=0}^{\infty} q^{2^j t(2n+1)^2}\right) \pmod{2}. \tag{10}$$

This means

$$F_{r,t}(z) \equiv \sum_{1 \le i \le s} \sum_{n=0}^{\infty} q^{2^j t(2n+1)^2 + b_i} \pmod{2} \tag{11}$$

where $b_1, b_2, \ldots b_s$ are the only integers for which $b_i \equiv r \pmod{t}$ and $a(b_i)$ are odd. If $a(0)$ is odd and $0 \equiv r \pmod{t}$, then replace $F_{r,t}(z)$ by $F_{r,t}(z) - \Delta^{2^j}(tz)\Theta^{2k}(z)$. Therefore without loss of generality we may assume that $a(0)$ is even, and that

$$F_{r,t}(z) \equiv \sum_{1 \le i \le s} \sum_{n=0}^{\infty} q^{2^j t(2n+1)^2 + b_i} \pmod{2}$$

is in $M_{2^j \cdot 12+k}(\frac{4Nt^3}{d})$ where the b_i are distinct non-zero integers. By [4, Lemma 1], it is known that there is no such integer weight holomorphic modular form unless $F_{t,r}(z) \equiv 0 \pmod{2}$. However this is not the case if there is at least one non-zero $M \equiv r \pmod{t}$ for which $a(M)$ is odd. □

4. Applications

In this section we apply the main theorem to certain *well poised* modular forms.

COROLLARY 1. *Let $b(n)$ be $b_k(n)$, $r_{g,\delta}(n)$, or $c(n)$ for any $k \ge 2$ or $0 < g < \lfloor\frac{\delta+1}{2}\rfloor$, then there are infinitely many $N \equiv r \pmod{t}$ for which $b(N)$ is even. There are infinitely many $M \equiv r \pmod{t}$ for which $b(M)$ is odd provided there is at least one such M.*

Proof. By Theorem 1 it is enough to find a modular form whose Fourier coefficients are, up to change of variable, congruent modulo 2 to $b_k(n)$, $r_{g,\delta}(n)$, and $c(n)$. After change of variables, (1) gives $b_k(n)$ as an Eta-product, (2) and (7) give $r_{g,\delta}(n)$ as coefficients of $\frac{1}{\eta_{g,\delta}(z)}$. (3) gives $c(n)$ as the coefficients of the modular function $j(z)$. □

COROLLARY 2. *If $2 \le k \le 25$, then for every arithmetic progression $r \pmod{t}$ where $0 \le r < t < 10$ there are infinitely many $M \equiv r \pmod{t}$ for which $b_k(M)$ is odd except for $r \in R$ where (k, R, t) is any of the following:*

$$\begin{gathered}(2,\{3,4\},5),(2,\{3,4,6\},7),(4,2,3),(4,\{2,4\},5),(4,\{2,5\},6),\\ (4,\{2,4,5\},7),(4,\{2,4,5,7,8\},9),(5,2,4),(5,\{2,6\},8),\\ (13,2,6),(16,\{2,8\},9),(17,2,8).\end{gathered} \tag{12}$$

For these cases,

$$b_k(tn+r) \equiv 0 \pmod 2$$

for all n.

Proof. By Corollary 1, it is enough to find a single $M \equiv r$ (mod t) for which $b_k(M)$ is odd. Computations using recurrences for $b_k(n)$ from [5] find an M for each case not listed in (12).

The congruences for $k = 2, 4$, and 16 follow directly from well known q-series infinite product identities. The congruences for $k = 5, 13, 17$ were verified by machine computation using Sturm's theorm. For instance to prove that

$$b_{13}(6n+2) \equiv 0 \pmod 2$$

we examine the modular form $f(z)$ defined by

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n = \frac{\eta(13z)\eta^6(6z)\eta^8(78z)\eta^4(z)}{\eta(z)\eta^2(2z)}.$$

This is a weight 8 holomorphic modular form on $\Gamma_0(234)$ with coefficients given by

$$\sum_{n=0}^{\infty} a(n)q^{n-28} = \left(\sum_{n=0}^{\infty} b_{13}(n)q^n\right)\prod_{n=1}^{\infty}(1-q^{6n})^6\prod_{n=1}^{\infty}(1-q^{78n})^8\prod_{n=1}^{\infty}\frac{1-q^{4n}}{(1-q^{2n})^2}.$$

The final factor doesn't affect parity questions since

$$\prod_{n=1}^{\infty}\frac{1-q^{4n}}{(1-q^{2n})^2} \equiv 1 \pmod 2.$$

All powers of q in $\prod_{n=1}^{\infty}(1-q^{78n})^8$ and $\prod_{n=1}^{\infty}(1-q^{6n})^6$ are multiples of 6 so if there is a minimal n' such that $b_{13}(6n'+2) \equiv 1 \pmod 2$ then $a(6n'+30) \equiv 1 \pmod 2$; i.e., to prove $b_{13}(6n+2)$ is always even it is enough to show $a(6n)$ is always even. Acting by the Hecke operator $T(6)$ we get the weight 8 holomorphic modular form on $\Gamma_0(234)$:

$$f(z)|T(6) = \sum_{n=0}^{\infty} a(6m)q^m.$$

By Sturm's theorem, to prove $a(6m) \equiv 0 \pmod 2$ for all m it suffices to check all $m \le 336$, since

$$\frac{k}{12}[SL_2(\mathbb{Z})\colon \Gamma_0(234)] = \frac{8}{12}\cdot 234\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{14}{13} = 336.$$

Computations verify $b_{13}(6n+2) \equiv 0 \pmod 2$ for all $n \le 400$ so for all n. □

COROLLARY 3. *If* $3 \le \delta \le 20$, $0 < g < \lfloor \frac{\delta+1}{2} \rfloor$, $\gcd(\delta, g) = 1$ *and* $0 \le r < t \le 75$, *then there are infinitely many* $M \equiv r \pmod t$ *for which* $r_{g,\delta}(M)$ *is odd except when* (g, δ) *is* $(1, 4)$.

Proof. By Corollary 1 it is enough to produce a single $M \equiv r \pmod t$ for which $r_{g,\delta}(M)$ is odd. This is easily done with a computer search.

For $(g, \delta) = (1, 4)$ we get legitimate congruences since

$$\sum_{n=0}^{\infty} r_{1,4}(n)q^n = \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^n} = \prod_{n=1}^{\infty}(1+q^n) \equiv \sum_{n=-\infty}^{\infty} q^{\frac{3n^2+n}{2}} \pmod 2$$

by Euler's Pentagonal Number Formula. □

As a final application we consider the coefficients of $j(z)$. By (3) we see

$$j(z) = \sum_{n=-1}^{\infty} c(n)q^n \equiv q^{-1} \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n}\right)^{24} \equiv q^{-1} \prod_{n=1}^{\infty} \frac{1}{(1-q^{8n})^3} \pmod 2.$$

In particular $c(n)$ is even for all $n \not\equiv 7 \pmod 8$. By machine computation, we obtain:

COROLLARY 4. *If* $0 \le r < t \le 1000$, *then there exist infinitely many integers* $M \equiv r \pmod t$ *for which* $c(M)$ *is odd provided that the arithmetic progression* $r \pmod t$ *has a non-empty intersection with the progression* $7 \pmod 8$.

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