ILLINOIS JOURNAL OF MATHEMATICS
Volume 41, Number 1, Spring 1997

ALGEBRAIC FIBERINGS OF GRASSMANN VARIETIES

R. J. D. FERDINANDS AND R. E. SCHULTZ

Standard results from calculus show that one can use relatively elementary methods to study a surface if it is a surface of revolution, and classical results in differential and algebraic geometry show the usefulness of recognizing the ruled surfaces that are essentially given as parametrized families of parallel submanifolds. Of course, more general notions known as *fiberings* have also played important roles in geometry and topology for a long time, and in this context one of the most natural questions is whether a given space can be realized by a fibering with suitable properties. Topological questions of this sort have been studied intermittently for about six decades (e.g., see [BoS], [St], [Br], [CG], [Sch], [Got], [Fe1–2]), and in some important cases one has a fairly good understanding of the types of fiberings a space can support. For example, if the space in question is a sphere, then every smooth fibering with compact connected fibers is loosely related to one of the so-called Hopf fiberings whose fibers are spheres and whose parameter spaces are projective spaces over the complex numbers, quaternions or Cayley numbers (cf. [Br, §6]), and if the space in question is the coordinate space $\mathbb{R}^n$, then no fiberings of this type exist if one insists that neither the fibers nor the base consist of a single point [BoS].

Projective spaces form another class of cases in which the fiberability question has been analyzed systematically (see [BG1], [CG], [Sch]); in particular, for even dimensional projective spaces there are no smooth fiberings with compact connected fibers such that both the fibers and the base consist of more than one point. Partial generalizations of these results to Grassmann manifolds were obtained in previous papers of the authors [Fe1–2], [Sch]. On the other hand, in [J, §5], R. Joshua obtained an analogous result for even dimensional projective spaces in a suitable category of algebraic varieties over algebraically closed fields of characteristic $\neq 2$. The purpose of this paper is to combine the methods of [Fe1–2] and [Sch] with those of [J] to obtain the corresponding nonfibering results for Grassmann varieties in suitable categories of algebraic varieties (see [GH, §1.5] or [Hrs, Lecture 6] for background information on these objects). In particular, our result applies to the Grassmann varieties $\mathbf{G}_{n,2}(\mathbb{F})$ of 2-planes in $\mathbb{F}^n$ where $n \equiv 2 \bmod 4$ and $\mathbb{F}$ is a field of characteristic $\neq 2$. In such cases *there are no nonconstant smooth maps from* $\mathbf{G}_{n,2}(\mathbb{F})$ *to a nonsingular projective variety* V *over* $\mathbb{F}$ *with connected positive dimensional fibers* (see Theorem 2.1). It

Received July 3, 1995.
1991 Mathematics Subject Classification. Primary 14D99, 14F20, 55R12, 55R20, 55S10.

seems likely that the same conclusion holds if $n \equiv 3 \bmod 4$, $n \geq 5$, and $n+1$ is not a power of 2; as noted in Observation 2.12, this is true if we also assume $n \equiv 2 \bmod 3$.

One clear purpose of this work is to study the effectiveness with which specific topological techniques can be applied to algebraic geometry, but another motivation is that fiberings of Grassmann varieties in algebraic geometry could lead to the construction of topological fiberings for complex Grassmann manifolds; results of E. Friedlander show that information about algebraic groups over fields of prime characteristic can be used effectively to obtain information on the classifying spaces of compact Lie groups (cf. [Fr2, §9]), and at least some of the ideas from this work would apply to any exceptional algebraic fiberings of the Grassmann varieties $\mathbf{G}_{n,k}(\mathbb{F})$ that might exist.

Since one cannot avoid using ideas from both algebraic geometry and algebraic topology in studying the sorts of problems considered here, we have attempted to provide references for definitions and concepts that are standard in one subject but not necessarily the other, but we do not claim that our treatment is complete. The standard texts by Hartshorne [Hrt] and Spanier [Sp] on algebraic geometry and topology should contain any additional background information that is needed on these subjects, and more specialized background on profinite completions and etale homotopy can be found in Sullivan's paper [Su1] and the monographs of Artin-Mazur [AM], Friedlander [Fr2], and Milne [Mi].

1. Background information

As the title indicates, this section contains some general observations that are needed in our study of the algebraic fibering problem for Grassmann varieties.

DEFAULT HYPOTHESES. *Unless and until indicated otherwise, ℓ will denote a fixed prime and all spaces are assumed to be* 1*-connected, $\mathbb{Z}/\ell$-complete in the sense of Bousfield and Kan* [BK] *or Sullivan* [Su1], *with finite mod ℓ cohomology and ℓ-profinite homotopy in each dimension.*

Of course the main examples of interest in this paper are the simply connected $\mathbb{Z}/\ell$-completions of the etale homotopy types of complete smooth varieties over an algebraically closed field (cf. [Fr2, discussion on p. 56 following Thm. 6.6 and Thm 7.3, p. 65]; see also [AM, §11]).

The conditions in the Default Hypotheses immediately yield a finiteness property for homotopy groups; although this is elementary and surely known to others, adequate citations in the literature seem to be somewhat elusive.

PROPOSITION 1.1. Let X be a simply connected space with finite mod ℓ cohomology and ℓ-profinite homotopy in each dimension. Then the homotopy groups of X are ℓ-profinite completions of finitely generated abelian groups.

Proof. If $\{X_k\}$ is the system of k-connective fiberings of X, it suffices to show that each group $\pi_{k+1}(X_k)$ is the ℓ-profinite completion of a finitely generated abelian group. We shall proceed by induction on k and and prove this finiteness result together with the finiteness of $H^i(X_k; \mathbb{Z}/\ell)$ for all $i \geq 0$. If $k = 0$ the statement is true by the simple connectivity of X and the finiteness of its mod ℓ homology groups. Assume the statement is true for X_{k-1} and consider the principal fibration $X_k \to X_{k-1} \to K(\pi_{k-1}(X_{k-1}), k)$; note that the base of this fibration is simply connected even if $k = 1$ because $X = X_0$ is simply connected. Since $\pi_{k-1}, (X_{k-1}) = \pi_{k-1}(X)$ is the ℓ-profinite completion of a finitely generated abelian group by the inductive hypothesis, it follows that its mod ℓ cohomology is finite in every dimension. Therefore a Serre spectral sequence argument shows that the mod ℓ cohomology of X_k is also finite in every dimension. In particular, if one specializes to dimension k and combines this with the Hurewicz Theorem, it follows that

$$H^k(X_k; \mathbb{Z}/\ell) \approx \mathrm{Hom}(\pi_k(X_k), \mathbb{Z}/\ell) \approx \mathrm{Hom}(\pi_k(X), \mathbb{Z}/\ell)$$

is also finite. On the other hand, the group $\pi_k(X)$ is ℓ-profinite by hypothesis, and an elementary argument shows that a ℓ-profinite abelian group A is the ℓ-profinite completion of a finitely generated abelian group if and only if $\mathrm{Hom}(A, \mathbb{Z}/\ell)$ is finite (for example, this is a straightforward consequence of [Fr2, Prop. 7.5, p. 67]). Therefore $\pi_k(X)$ is the ℓ-profinite completion of a finitely generated abelian group and the proof of the inductive argument is complete. ■

At one point we shall need a version of the Künneth formula for spaces that satisfy the conditions of Proposition 1.1; in order to avoid additional digressions we shall only prove a result that suffices for our purposes.

PROPOSITION 1.1A. *Let X and Y be simply connected spaces with with finite mod ℓ cohomology and ℓ-profinite homotopy in each dimension. Then the cohomology groups of X and Y with coefficients in the ℓ-adic integers $\widehat{\mathbb{Z}}_\ell$ are finitely generated $\widehat{\mathbb{Z}}_\ell$ modules (hence direct sums of cyclic modules) and the $\mathbb{Q}$ localized external cohomology product*

$$\left(H^*(X; \widehat{\mathbb{Z}}_\ell) \otimes \mathbb{Q}\right) \otimes_{\widehat{\mathbb{Z}}_\ell} \left(H^*(Y; \widehat{\mathbb{Z}}_\ell) \otimes \mathbb{Q}\right) \to H^*(X \times Y; \widehat{\mathbb{Z}}_\ell) \otimes \mathbb{Q}$$

is an isomorphism of $\widehat{\mathbb{Q}_\ell}$ vector spaces.

Warning. If W satisfies the conditions of Proposition 1.1A then the groups $H^*(W; \widehat{\mathbb{Z}}_\ell) \otimes \mathbb{Q}$ and $H^*(W; \widehat{\mathbb{Q}_\ell})$ are isomorphic only if W has finite homotopy groups in every dimension, in which case W is rationally acylic. In other cases the groups in question differ by an uncountable dimensional rational vector space.

Proof. By Proposition 1.1 we know that the homotopy groups of X and Y are ℓ-profinite completions of finitely generated abelian groups and hence are direct sums of cyclic $\widehat{\mathbb{Z}_\ell}$ modules. If π is a finitely generated abelian group and $n \geq 2$ it follows immediately that $H^*(K(\pi, n); \widehat{\mathbb{Z}_\ell})$ is a finitely generated $\widehat{\mathbb{Z}_\ell}$ module in each dimension, and by the results of [Su1, §3] the analogous statement is true if π is replaced by its ℓ-profinite completion. Let W be either X or Y, and let $\{W_m\}$ be a Postnikov system for W where $W \to W_m$ is $(m+1)$-connected and W_m has vanishing homotopy in dimensions $\geq m+1$. The preceding discussion of Eilenberg-MacLane spaces shows that W_2 has finitely generated $\widehat{\mathbb{Z}_\ell}$ cohomology in every dimension; if the finite generation property is true for the cohomology of W_{m-1}, then one can combine this with the analogous result for Eilenberg-MacLane spaces and a Serre spectral sequence argument to prove the finite generation property for the cohomology of W_m. Thus the finite generation property for $\widehat{\mathbb{Z}_\ell}$ cohomology holds for all the spaces W_m; since $W \to W_m$ is $(m+1)$-connected and m can be taken arbitrarily large, it follows that the finite generation property for $\widehat{\mathbb{Z}_\ell}$ cohomology also holds for W.

Suppose now that X and Y are Eilenberg-MacLane spaces satisfying the conditions of the proposition. Then X and Y are ℓ-profinite completions of Eilenberg-MacLane spaces X' and Y' with finitely generated homotopy groups, and by the preceding paragraph we know that their cohomology groups with coefficients in $\widehat{\mathbb{Z}_\ell}$ are finitely generated in each dimension. Similarly, since $X \times Y$ is the ℓ-profinite completion of $X' \times Y'$ it follows that the cohomology of $X \times Y$ with coefficients in $\widehat{\mathbb{Z}_\ell}$ is also finitely generated in each dimension. Furthermore, if W' denotes one of X', Y' or $X' \times Y'$ and W denotes the corresponding ℓ-profinite completion, then finite generation implies that $H^*(W; \widehat{\mathbb{Z}_\ell})$ is naturally isomorphic to $H^*(W'; \mathbb{Z}) \otimes \widehat{\mathbb{Z}_\ell}$; specifically, the completion map $W' \to W$ determines an isomorphism in $\widehat{\mathbb{Z}_\ell}$ cohomology and the canonical map from $H^*(W'; \mathbb{Z}) \otimes \widehat{\mathbb{Z}_\ell}$ to $H^*(W'; \widehat{\mathbb{Z}_\ell})$ is also an isomorphism. These isomorphisms are compatible with the external cohomology product, and therefore the proof of the proposition when X and Y are Eilenberg-MacLane spaces reduces to knowing that the map

$$\left(H^*(X'; \mathbb{Z}) \otimes \widehat{\mathbb{Z}_\ell}\right) \otimes_{\widehat{\mathbb{Z}_\ell}} \left(H^*(Y'; \mathbb{Z}) \otimes \widehat{\mathbb{Z}_\ell}\right) \to H^*(X' \times Y'; \mathbb{Z}) \otimes \widehat{\mathbb{Z}_\ell}$$

becomes an isomorphism after tensoring with $\mathbb{Q}$. But this follows immediately from the usual Künneth formula, the universal coefficient theorems for rational coefficients, and the finite generation of the cohomology groups of X' and Y'. This completes the proof of the proposition when X and Y are Eilenberg-MacLane spaces.

Now suppose that X is an Eilenberg-MacLane space and Y is an arbitrary space satisfying the conditions of the proposition. Let $\{Y_m\}$ be a Postnikov system for Y as usual. Then Y_2 is a $K(\pi, 2)$, and therefore the conclusion of the proposition holds for X and Y_2. If the conclusion holds for the pair (X, Y_{m-1}) and $Y_m \to Y_{m-1} \to BK_m$ is the fibration defined by the appropriate k-invariant of the Postnikov system, then validity of the proposition for (X, BK_m) and a spectral sequence comparison argument involving the Serre spectral sequences of the fibrations $Y_m \to Y_{m-1} \to BK_m$ and $X \times Y_m \to X \times Y_{m-1} \to X \times BK_m$ yields the conclusion of the proposition for (X, Y_m). Thus the proposition is true for all pairs of this type, and as in the first paragraph of this proof it follows that the proposition is true for all pairs (X, Y) where

X is an Eilenberg-MacLane space and Y is essentially unrestricted. Finally, if X and Y are arbitrary and $\{X_m\}$ is a Postnikov system for X with corresponding Eilenberg-MacLane spaces BK'_m, then one can argue similarly to show that the validity of the proposition for the pairs (BK'_m, Y) inductively implies the result for the pairs (X_m, Y) and thus also for the pair (X, Y). ■

We shall also need the following analog of an elementary result in rational homotopy theory:

Proposition 1.1*B*. *Let X be a space satisfying the conditions of Proposition* 1.1, *and assume in addition that all odd dimensional homotopy groups of X are finite. Then $H^*(X; \widehat{\mathbb{Z}}_\ell) \otimes \mathbb{Q}$ is a graded polynomial algebra on the graded $\widehat{\mathbb{Q}}_\ell$ vector space* $\pi_*(X) \otimes \mathbb{Q}$.

Proof. Let $\{X_n\}$ be a Postnikov system for X. As before it suffices to prove the proposition for each X_n. Since X_2 is an Eilenberg-MacLane space and $\pi_2(X_2) \approx \pi_2(X)$ is a direct sum of a finite abelian ℓ-group and copies of $\widehat{\mathbb{Z}}_\ell$, the Künneth formula in Proposition 1.1A reduces the proof in this case to a verification for $K(\widehat{\mathbb{Z}}_\ell, 2)$. Since the $\widehat{\mathbb{Z}}_\ell$ cohomology of this space is in fact a polynomial algebra on a single 2-dimensional generator, the truth of the proposition for $n = 2$ follows. More generally, if X is an Eilenberg-MacLane space $K(\pi, n)$ such that π is finite if n is odd, then a similar argument shows that the rationalized $\widehat{\mathbb{Z}}_\ell$ cohomology of X has the prescribed form (but of course this probably will not hold for the $\widehat{\mathbb{Z}}_\ell$ cohomology itself).

Assume now that the result is true for X_{n-1}, where $n \geq 3$, and consider the principal fibration

$$K(\pi_n(X), n) \to X_n \to X_{n-1}$$

and let $\theta \in H^{n+1}(X_{n-1}; \pi_n(X))$ denote the cohomology class corresponding to a classifying map $X_{n-1} \to K(\pi_n(X), n+1)$ for this principal fibration. We claim that θ has finite order; if n is even this follows from the induction hypothesis because $H^{odd}(X_{n-1}; \widehat{\mathbb{Z}}_\ell) \otimes \mathbb{Q} = 0$ implies that $H^{odd}(X_{n-1}; \widehat{\mathbb{Z}}_\ell)$ is finite (we know the group is a finitely generated $\widehat{\mathbb{Z}}_\ell$ module), and if n is odd the assertion follows because the odd dimensional homotopy groups of X are finite. If q is the order of θ, let A_q be a self map of $K(\pi_n(X), n+1)$ that induces multiplication by q on homotopy groups, and let $k(\theta)$: $X_{n-1} \to K(\pi_n(X), n+1)$ be a representative for θ (cf. [Sp]). It follows that the map $A_q \circ k(\theta)$ is nullhomotopic, and the latter in turn implies the existence of the commutative diagram below, in which all horizontal sequences are extended fibration sequences (*exact* in the sense of [Sp, p. 366]) and $K_m = K(\pi_n(X), m)$:

$$\begin{array}{ccccccc}
K_n & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \xrightarrow{\theta} & K_{n+1} \\
\big\downarrow{\scriptstyle \Omega(A_q)} & & \big\downarrow & & \big\downarrow & & {\scriptstyle A_q}\big\downarrow \\
K_n & \xrightarrow{=} & K_n & \longrightarrow & \mathbf{P}(K_{n+1}) & \longrightarrow & K_{n+1}
\end{array}$$

Here $\mathbf{P}(Y)$ denotes the contractible space of basepoint preserving paths on a space Y and $\Omega(A_q)$ is the self map that induces multiplication by q on homotopy. We claim that the map $\Omega(A_q)$ induces an automorphism of $H^*(K(\pi_n(X), n); \widehat{\mathbb{Z}_\ell}) \otimes \mathbb{Q}$. If $\pi_n(X)$ is a finitely generated abelian group then it is well known that the map corresponding to $\Omega(A_q)$ induces endomorphisms of integral cohomology with finite kernels and cokernels, and since $\pi_n(X)$ is the ℓ-profinite completion of a finitely generated abelian group the corresponding result for $\Omega(A_q)$ follows because profinite completion induces isomorphisms of $\widehat{\mathbb{Z}_\ell}$ cohomology. If we combine this information with the left hand square of the commutative diagram, it follows that the restriction map induces a split surjection from $H^*(X_n; \widehat{\mathbb{Z}_\ell}) \otimes \mathbb{Q}$ to $H^*(K(\pi_n(X, n); \widehat{\mathbb{Z}_\ell}) \otimes \mathbb{Q}$. Therefore the Serre spectral sequence in rationalized cohomology for the original principal fibration collapses. Since the cohomology groups of the fiber and base both have the form given by the conclusion of the proposition and the rational homotopy of X_n is a special type of direct sum of the rational homotopy of X_{n-1} and $K(\pi_n(X), n)$, the truth of the proposition for X_n follows. ∎

Algebraic topology and algebraic fibrations. As indicated in [J], the algebraic analog of a submersion for nonsingular varieties over a field $\Bbbk$ is a proper smooth map over $\Bbbk$ (see [Hrt, §III.10, pp. 268–276] or [Hrs, Lecture 14, pp. 174–185]). The results of [Fr1] show that such maps induce fibrations in etale homotopy theory under suitable hypotheses, and our next objective is to show that Friedlander's hypotheses are valid for the sorts of examples considered in this paper. Specifically, we are interested in simply connected, nonsingular projective algebraic varieties over the complex numbers that are given by complexifications of nonsingular noetherian schemes defined over Spec $\mathbb{Z}$. If V is a complex Grassmann variety, then V has a well known algebraic embedding into projective space by means of Plücker coordinates (cf. [GH, pp. 209–211] for the complex case); the image of this embedding is a variety defined by polynomials with integral coefficients, and the corresponding projective varieties over other fields are merely the analogous Grassmann varieties over these fields. An elaboration of these observations and standard results in topology show that complex Grassmann manifolds satisfy the abstract conditions we have imposed.

Throughout this discussion V will denote the complex variety under consideration and X will denote the associated scheme over Spec $\mathbb{Z}$ such that V is given by the complexification of X. If R is an arbitrary noetherian commutative ring with unit then X_R will denote the associated scheme obtained by change of rings, and if $\overline{\Bbbk_p}$ is the algebraic closure of a finite field of characteristic p then we shall often simplify the notation to X_p.

Since we are ultimately interested in cases where V is a Grassmann manifold, we shall also assume that V is simply connected, the group $H^2(V; \mathbb{Z})$ is infinite cyclic (equivalently, $\pi_2(V)$ is infinite cyclic), and $H^3(V, \mathbb{Z}) = 0$. In addition to this we shall assume that the Euler characteristic $\chi(V)$ is nonzero. Finally, ℓ will denote a prime that does not divide $\chi(V)$ and p will denote a prime that is distinct from ℓ; in this section it will not matter whether or not p divides $\chi(V)$. The crucial link between topology and algebraic geometry in this paper is *the existence of a homotopy equivalence between the $\mathbb{Z}/\ell$ completion of V (viewed as a topological space) and the $\mathbb{Z}/\ell$ completion of $(X_p)_{\mathrm{et}}$*, where $(\ldots)_{\mathrm{et}}$ represents the associated etale homotopy type (see [AM, §12], [Fr2, §8]).

Now suppose that we have a smooth map of schemes over $\overline{\mathbb{k}_p}$ of the form $f\colon X_p \to Y$ where Y is nonsingular, the fibers are connected, and $0 < \dim Y < \dim X_p$. Let X_y denote some fiber of f and consider the associated etale homotopy diagram $(X_y)_{\mathrm{et}} \to X_{\mathrm{et}} \to Y_{\mathrm{et}}$. The result below shows that the assumptions of the preceding paragraphs combine with the previously mentioned results of Friedlander to yield a strong conclusion. In fact, with a little effort one also obtains a lower bound on the connectivity of the $\mathbb{Z}/\ell$ completion of Y_{et}.

PROPOSITION 1.2. *In the setting above, the ℓ-profinite completion of the fundamental group of $(X_y)_{\mathrm{et}}$ is trivial and the etale homotopy sequence becomes a fibration sequence after $\mathbb{Z}/\ell$ completion. Furthermore, in this situation the $\mathbb{Z}/\ell$ completion of Y_{et} is 2-connected.*

Proof. Since the fundamental groups of the spaces in question are ℓ-profinite, the assertions about fundamental groups reduce to showing that $H^1(Y;\mathbb{Z}/\ell)$ and $H^1(X_y;\mathbb{Z}/\ell)$ both vanish; in this context cohomology can be interpreted either as etale cohomology or as the singular cohomology of the $\mathbb{Z}/\ell$ completions of the respective etale homotopy types. Since ℓ is prime to the Euler characteristic of V, results of R. Joshua on etale transfers [J, Thm. 1.2, p. 455] show that the morphisms in mod ℓ etale cohomology associated to the smooth map $X_p \to Y$ are split injections. Since V is simply connected, it follows that $H^1(X_p;\mathbb{Z}/\ell) = 0$ and hence that $H^1(Y;\mathbb{Z}/\ell)$ must also be trivial. Therefore the results of [Fr1] yield a Leray-Serre spectral sequence

$$H^s(Y; H^t(X_y;\mathbb{Z}/\ell) \Rightarrow H^{s+t}(X_p;\mathbb{Z}/\ell)$$

(with untwisted coefficients), and since $H^2(Y;\mathbb{Z}/\ell) \to H^2(X_p;\mathbb{Z}/\ell) \approx \mathbb{Z}/\ell$ is split injective, it follows that $H^2(Y;\mathbb{Z}/\ell)$ is either trivial or isomorphic to $\mathbb{Z}/\ell$. In the first case we may argue as before to say that the second homotopy group of the $\mathbb{Z}/\ell$ completion of Y_{et} is trivial, so we need only show that the second alternative cannot hold. To see this, consider the commutative diagram below, in which $\widehat{\mathbb{Z}}_\ell$ denotes the ℓ-adic integers and the horizontal maps are given by the coefficient homomorphisms associated to the canonical projection from $\widehat{\mathbb{Z}}_\ell$ to $\mathbb{Z}/\ell$.

$$\begin{array}{ccc} H^2(Y;\widehat{\mathbb{Z}}_\ell) & \longrightarrow & H^2(Y;\mathbb{Z}/\ell) \\ \downarrow & & \downarrow \\ H^2(X_p;\widehat{\mathbb{Z}}_\ell) & \longrightarrow & H^2(X_p;\mathbb{Z}/\ell) \end{array}$$

It follows that either $H^2(Y;\widehat{\mathbb{Z}}_\ell) \to H^2(X_p;\widehat{\mathbb{Z}}_\ell)$ is bijective or else $H^2(Y;\widehat{\mathbb{Z}}_\ell) = 0$. The first case can be excluded as follows: Let $0 \neq c \in H^*(X_p;\widehat{\mathbb{Z}}_\ell)$. Since V is a Kähler manifold, it follows that $c^n \neq 0$ where $n = \dim_{\mathbb{C}} V$. On the other hand, the bijectivity condition in 2-dimensional cohomology implies that c^n lies in the image of $H^{2n}(Y; \widehat{\mathbb{Z}_\ell})$. But the dimension of Y is less than n by hypothesis, and this implies that the cohomology group in question is trivial [Mi, Thm. 1.1, p. 221]; this contradiction shows that the first alternative does not hold. But if the second alternative holds then one can use a Bockstein argument to show that $H^3(Y; \widehat{\mathbb{Z}_\ell}) \neq 0$, and this contradicts the split injectivity of $H^*(Y; \widehat{\mathbb{Z}_\ell}) \to H^*(X_p; \widehat{\mathbb{Z}_\ell})$ and the triviality of $H^{\text{odd}}(X_p; \widehat{\mathbb{Z}_\ell}) \approx H^{\text{odd}}(V; \widehat{\mathbb{Z}_\ell})$. It follows that the $\mathbb{Z}/\ell$ completion of Y_{et} is 2-connected as claimed. ■

Corollary 1.3. *In the setting described above, the $\mathbb{Z}/\ell$ completion of $(X_y)_{\text{et}}$ is the homotopy fiber of the induced map of $\mathbb{Z}/\ell$ completions from $X_{\text{et}}^{\widehat{\{\ell\}}}$ to $Y_{\text{et}}^{\widehat{\{\ell\}}}$.*

The corollary follows by combining Proposition 1.2 with Friedlander's exact homotopy sequence [Fr1, Cor. 4.8, p. 34]]. ■

Corollary 1.3 combines with the standard finiteness theorem of etale cohomology ([Mi, §VI.2] or [Fr2, §7]) to yield an algebraic version of a standard result on topological fiberings. Following standard practice, if ℓ is a prime and X is a space whose $\mathbb{Z}/\ell$ cohomology groups are finite in each dimension and trivial in all but finitely many dimensions, then the mod ℓ Euler characteristic $\chi(X; \mathbb{Z}/\ell)$ is the alternating sum of the dimensions of the $\mathbb{Z}/\ell$ cohomology groups of X.

PROPOSITION 1.4. (Compare [J, p. 491, line 6].) *In the setting above, the $\mathbb{Z}/\ell$ cohomology groups of the spaces X_y, X_p, Y are finite in each dimension and trivial for almost all dimensions and the mod ℓ Euler characteristics of these objects satisfy $\chi(X_p; \mathbb{Z}/\ell) = \chi(Y; \mathbb{Z}/\ell) \cdot \chi(X_y; \mathbb{Z}/\ell)$.*

The results of Proposition 1.2 reduce the conclusion of (1.4) to a special case of [McL, Cor. XI.2.1, p. 323]. ■

Since X_p and Y are complete nonsingular varieties, basic results of etale theory imply that both satisfy Poincaré duality for suitably twisted $\mathbb{Z}/\ell$ coefficients (see [Mi, §6.11] or [Fr2, Thm. 17.6, p. 180, and the second part of Prop. 7.6, pp. 68–69]); specifically, the cohomology in a given dimension with ordinary $\mathbb{Z}/\ell$ coefficients is isomorphic to the cohomology with coefficients in a locally constant sheaf with stalks isomorphic to $\mathbb{Z}/\ell$ (i.e., the *Tate twist* as defined in [Mi, p. 163] or [Fr2, pp. 170–171]). The simply connectivity properties of Proposition 1.2 imply that these locally constant sheaves are in fact constant over X_p and Y, and therefore one has, in fact, Poincaré duality with respect to untwisted coefficients. If we combine this and Proposition 1.4 with the previous observations, we obtain useful algebraic analogs of some standard facts about the mod 2 cohomology of Poincaré duality spaces.

PROPOSITION 1.5. *In the setting of Proposition* 1.2, *assume further that* $p > 2$, $\ell = 2$, *and* $\chi(X_p; \mathbb{Z}/2)$ *is odd. Then the cohomological dimensions of* X_y, X_p, *and* Y *are all divisible by* 4, *the dimension of the middle dimensional* $\mathbb{Z}/2$ *cohomology group is odd, and for each object there is a class in the middle dimensional cohomology group whose square* (*with respect to the usual cup product*) *generates the top dimensional cohomology.*

Proof. The mod 2 Euler characteristics of X_y and Y are both odd by Proposition 1.4, and therefore it suffices to show that the conclusions of the proposition are all true if W is complete and nonsingular over an algebraically closed field of characteristic $\neq 2$ and the 2-adic completion of its Grothendieck (equivalently, its etale) fundamental group is trivial. Under these conditions W satisfies Poincaré duality with respect to untwisted $\mathbb{Z}/2$ coefficients and the mod 2 Euler characteristic is defined; for the sake of definiteness let m denote the algebraic dimension of W, so that $2m$ is its cohomological dimension. Duality implies that the mod 2 Euler characteristic is congruent to $\dim H^m(W; \mathbb{Z}/2)$ mod 2, and hence the latter is also odd. Since the cup product form on the latter cohomology group defines a nondegenerate symmetric blinear form on $\dim H^m(W; \mathbb{Z}/2)$, it follows that some element of the latter must have nonzero cup square. This proves everything except the evenness of m. Choose $v \in H^m(W; \mathbb{Z}/2)$ so that $\mathrm{Sq}^m v = v^2 \neq 0$. Assume that m is odd and write $m = 2j + 1$. Then the Adem relation $\mathrm{Sq}^{2s+1} = \mathrm{Sq}^1 \mathrm{Sq}^{2s}$ implies that v^2 lies in $\mathrm{Sq}^1\left(H^{2m-1}(W; \mathbb{Z}/2)\right)$ (*Note*: The Adem relations for Steenrod squares are discussed explicitly in [ES]). On the other hand, by Poincaré duality we have $H^{2m-1}(W; \mathbb{Z}/2) \approx H^1(W; \mathbb{Z}/2)$, and the latter group is trivial by the fundamental group condition. This implies that $v^2 = 0$, contradicting our choice of v and thereby showing that m must be even as claimed. ■

The operation of Sq^2 *on Grassmann varieties.* For the remainder of this section we shall no longer assume the default hypotheses.

We shall conclude this section with a result that is useful in analyzing the mod 2 cohomology of the base space for an algebraic fibering of a Grassmann varieties. If X is a space whose mod 2 cohomology is concentrated in even dimensions, then the Adem relations imply that $\mathrm{Sq}^2 \mathrm{Sq}^2 = 0$ on $H^*(X; \mathbb{Z}/2)$; more generally, given an arbitrary space X for which $\mathrm{Sq}^2 \mathrm{Sq}^2 = 0$ one can form the Sq^2-homology of $H^*(X; \mathbb{Z}/2)$ as in, say, [ABP]. Even more generally, if m is a positive integer such that $\mathrm{Sq}^m \mathrm{Sq}^m = 0$ on $H^*(X; \mathbb{Z}/2)$ then one can form the Sq^m-homology of $H^*(X; \mathbb{Z}/2)$ and we shall denote these groups by

$$\mathcal{H}\left(H^*(X; \mathbb{Z}/2); \mathrm{Sq}^m\right)$$

(cf. [ABP, p. 275, second paragraph following Thm. 3.1]); by construction this object is a graded $\mathbb{Z}/2$ vector space. Since $\mathrm{Sq}^1 \mathrm{Sq}^1 = 0$ it follows that Sq^1 homology is defined for all spaces; under the standard identification of Sq^1 with the Bockstein for the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

the Sq^1 homology corresponds to the E^2 term in the 2-primary Bockstein spectral sequence (the latter is discussed in [Z, §3.3]).

PROPOSITION 1.7. *Let $n \geq 5$. In dimensions $\leq 2n-5$ the graded Sq^2 homology group of the Grasssmann varieties $\mathcal{H}(H^*(\mathbf{G}_{n,2}(\mathbb{C}); \mathbb{Z}/2); \mathrm{Sq}^2)$ is isomorphic to $\mathbb{Z}/2$ in all dimensions divisible by 8 and zero in all other dimensions. For each $k > 0$ the nonzero class in dimension $8k$ is represented by b^{2k} where $b \in H^4(\mathbf{G}_{n,2}(\mathbb{C}); \mathbb{Z}/2)$ is indecomposable. If n is even the same conclusion also holds in dimensions $\leq 2n-3$.*

Reminder. The results of etale homotopy theory imply that one can replace $\mathbb{C}$ by an arbitrary algebraically closed field $\mathbb{F}$ of characteristic $\neq 2$.

Proof. Standard results on the mod 2 cohomology of homogeneous spaces imply that $H^{2*}(\mathbf{G}_{n,2}(\mathbb{C}); \mathbb{Z}/2) \approx H^*(\mathbf{G}_{n,2}(\mathbb{R}); \mathbb{Z}/2)$ such that the action of Sq^{2j} in the former corresponds to the action of Sq^j in the latter. Thus it suffices to determine $\mathcal{H}(H^*(\mathbf{G}_{n,2}(\mathbb{R}); \mathbb{Z}/2); \mathrm{Sq}^1)$ in dimensions $\leq n-3$ and to double all dimensions afterwards. Since the standard map from $\mathbf{G}_{n,2}(\mathbb{R})$ to BO_2 induces an isomorphism of mod 2 cohomology in dimensions $\leq n-2$, the problem reduces to computing $\mathcal{H}(H^*(BO_2; \mathbb{Z}/2); \mathrm{Sq}^1)$ in dimensions $\leq n-3$. It is known that all torsion in $H^*(BO_2; \mathbb{Z})$ is 2-torsion (e.g., see [Bor2, §24]), and therefore the identification of the Sq^1 homology with the E^2 term of the Bockstein spectral sequence implies that

$$\mathcal{H}(H^*(\mathbf{G}_{n,2}(\mathbb{R}); \mathbb{Z}/2); \mathrm{Sq}^1) \cong \left(H^*(\mathbf{G}_{n,2}(\mathbb{R}); \mathbb{Z})/\mathrm{Torsion}\right) \otimes \mathbb{Z}_2.$$

On the other hand, it is also known that $H^*(\mathbf{G}_{n,2}(\mathbb{R}); \mathbb{Z})/\mathrm{Torsion}$ is a polynomial ring on a 4-dimensional generator in dimensions $\leq n-3$ such that the image of the generator under mod 2 reduction is the square of the indecomposable class in $H^2(\mathbf{G}_{n,2}(\mathbb{R}); \mathbb{Z}/2)$. Therefore through dimension $n-2$ the Sq^1 homology of $H^*(\mathbf{G}_{n,2}(\mathbb{R}); \mathbb{Z}/2)$ is $\mathbb{Z}/2$ in all nonnegative dimensions divisible by 4 and zero otherwise, with the generators of the nonzero groups given by even powers of an indecomposable class in $H^2(\mathbf{G}_{n,2}(\mathbb{R}); \mathbb{Z}/2)$. As indicated in the first sentence of this paragraph, the statement regarding the Sq^2 cohomology of $H^*(\mathbf{G}_{n,2}(\mathbb{C}); \mathbb{Z}/2)$ in dimensions $\leq 2n-5$ follows directly from this.

If n is even then the sharper conclusion in the last sentence of the proposition will follow if we know that all torsion in $H^*(\mathbf{G}_{n,2}(\mathbb{R}); \mathbb{Z})$ is 2-torsion and the quotient $H^*(\mathbf{G}_{n,2}(\mathbb{R}); \mathbb{Z})/\mathrm{Torsion}$ is a polynomial algebra on one generator through dimension $n-1$. To verify these, consider the double covering of $\mathbf{G}_{n,2}(\mathbb{R})$ by the oriented Grassmann varieties $\mathbf{G}^+_{n,2}(\mathbb{R})$ of oriented 2-planes in $\mathbb{R}^n$. Since this is a 2-sheeted covering, the assertion regarding torsion will follow if $H^*(\mathbf{G}^+_{n,2}(\mathbb{R}); \mathbb{Z})$ is torsion free and the assertion on the multiplicative structure mod torsion will follow if $H^*(\mathbf{G}^+_{n,2}(\mathbb{R}); \mathbb{Z})$ is a polynomial algebra on one generator through dimension $n-1$. By the results of [Lai], the integral cohomology of $\mathbf{G}^+_{n,2}(\mathbb{R})$ satisfies both of these conditions. ■

2. Applications to Grassmann varieties

In [J, §5], R. Joshua used the etale transfer and the methods of [BG1] to prove that a projective space $\mathbb{F}\mathbb{P}^{2n}$ admits no smooth maps to a nonsingular projective variety V over $\mathbb{F}$ such that the fibers are connected and $0 < \dim V < 2n$ provided the characteristic of $\mathbb{F}$ is odd. Here is the corresponding result for Grassmann manifolds of 2-planes.

THEOREM 2.1. *Let $n \geq 6$ be an integer congruent to 2 mod 4, and let $\mathbb{F}$ be a field of characteristic $\neq 2$. Then there are no smooth maps from the Grassmann varieties $\mathbf{G}_{n,2}(\mathbb{F})$ to a nonsingular projective variety V over $\mathbb{F}$ such that the fibers are connected and $0 < \dim V < \dim \mathbf{G}_{n,2}(\mathbb{F})$.*

Proof. In [J], Joshua proves his nonfibering theorem for even dimensional projective spaces by a straightforward modification of the argument in [BG1]. However, it is not possible to find a similar extension of the proof in [Sch] for Grassmann varieties of 2-planes over the complex numbers because the proof in [Sch] uses the Hodge Signature Theorem for Kähler manifolds and signatures are not definable for cohomology over the ℓ-adic numbers. Therefore it is necessary to give a different argument. Since the Euler characteristic of $\mathbf{G}_{n,2}(\mathbb{C})$ is the binomial coefficient $\binom{n}{2}$ and the latter is odd if $n \equiv 2, 3 \bmod 4$, we can apply the machinery of the preceding section if $\ell = 2$.

Assume that there is a smooth map $\Pi\colon \mathbf{G}_{n,2}(\mathbb{F}) \to V$, where V is a nonsingular projective variety over $\mathbb{F}$, the fibers are connected, and

$$0 < \dim_{\mathbb{F}} V < \dim_{\mathbb{F}} \mathbf{G}_{n,2}(\mathbb{F}) = 2n - 4.$$

Let L be an arbitrary fiber of this map; if L' is another fiber, then L and L' might not be isomorphic, but they determine the same etale homotopy type (cf. [J, p. 455, four lines preceding Thm. 1.2]). As in the topological setting, the existence of a transfer for Π implies that the induced map Π^* in mod 2 etale cohomology is injective; in fact, Π^* is split injective both as a map of modules over the Steenrod algebra $\mathcal{a}_2$ and as a map of modules over the graded ring $H^*(V, \mathbb{Z}/2)$, where the module structure over $H^*(\mathbf{G}_{n,2}(\mathbb{F}); \mathbb{Z}/2)$ is defined by the graded ring homomorphism Π^*. The topological version of this is stated in [Sch, (5.2)–(5.3), p. 200]; the algebraic analog of [Sch, (5.2)] follows because the etale transfer is a map in the stable category, and the algebraic analog of [Sch, (5.3)] follows by the multiplicative properties of the etale homotopy transfer [J, Thm. 3.2, p. 474]. If $d = \dim_{\mathbb{F}} V$, then the nonvanishing of $H^{2d}(V; \mathbb{Z}/2)$ implies that V_{et} is not contractible; since the injectivity of Π^* implies that the first nonzero cohomology of V_{et} appears in an even dimension, it follows that the connectivity is an odd integer, say $2r-1$. By Proposition 1.1 we know that $r \geq 2$.

ASSERTION 2.2. *If V and r are given as in the previous paragraph, then* $\dim_{\mathbb{F}} L \geq 2r-2$, $2r \leq n-1$ *and* Sq^2 *is injective on* $H^{2r}(V; \mathbb{Z}/2)$.

Proof of Assertion 2.2. The assumptions imply that the restriction map from $H^j(\mathbf{G}_{n,2}(\mathbb{F}); \mathbb{Z}/2)$ to $H^j(L; \mathbb{Z}/2)$ is bijective in dimensions $\leq 2r-2$. We shall first note that $r \leq n-2$. To see this, note that $2r \leq \dim_{\mathbb{F}} V$ by Proposition 1.5 and Poincaré duality, and combine this with the elementary inequality $\dim_{\mathbb{F}} V < \dim_{\mathbb{F}} \mathbf{G}_{n,2}(\mathbb{F}) = 2n-4$. Since the mod 2 etale cohomology groups of $\mathbf{G}_{n,2}(\mathbb{F})$ are concentrated in even dimensions and the dimensions of the even dimensional groups are nondecreasing through dimension $2n-4$, it follows that the analogous statement holds for the mod 2 etale cohomology of L through dimension $2r-2$. If we let d be the dimension of L as before, then by Poincaré duality the mod 2 etale cohomology groups of L from dimension $2d-2r+2$ to $2d$ are also concentrated in even dimensions and the dimensions of the even dimensional groups are nonincreasing. These facts and the evenness of d (by Prop. 1.5) imply the inequality $\dim_{\mathbb{F}} L \geq 2r-2$.

The inequality $2r \leq n-1$ is an immediate consequence of $\dim_{\mathbb{F}} L \geq 2r-2$, $\dim_{\mathbb{F}} V \geq 2r$ and

$$\dim_{\mathbb{F}} L + \dim_{\mathbb{F}} V = \dim_{\mathbb{F}} \mathbf{G}_{n,2}(\mathbb{F}) = 2n-4.$$

Let $\mathcal{M}^* \subset H^*(\mathbf{G}_{n,2}(\mathbb{F}); \mathbb{Z}/2)$ be the direct summand that is complementary to the image of Π^*; as noted before, $\mathcal{M}^*$ is closed with respect to the operations of $\mathcal{A}_2$ and $H^*(V; \mathbb{Z}/2)$. If there is a class $u \in H^{2r}(V; \mathbb{Z}/2)$ with $\mathrm{Sq}^2 u = 0$, then the inequalities $2r \leq n-1$ and $n \geq 6$ imply $2r+2 < 2n-4$, and therefore by Proposition 1.7 we know that $u = Sq^2 u_1 + K c_2^{[r/2]}$ for some cohomology class u_1 and some constant K, where we assume $K = 0$ if r is not divisible by 4. Since $\mathcal{M}^* = H^*(\mathbf{G}_{n,2}(\mathbb{F}); \mathbb{Z}/2)$ below dimension $2r$, it follows that $u_1 \in \mathcal{M}^*$ and if $r = 4t$ then $c_2^{[r/2]} = c_2^{2t} = \mathrm{Sq}^t c_2^t$ also lies in $\mathcal{M}^*$; therefore $u \in \mathcal{M}^*$, and since $\mathcal{M}^*$ is complementary to the image of Π^* it follows that $u = 0$. Consequently Sq^2 must be injective on the image of Π^*. ■

Our next objective is to compare the dimensions of L and V.

Assertion 2.3. *The dimensions of V and L satisfy* $\dim_{\mathbb{F}} L \leq n-4$ *and* $\dim_{\mathbb{F}} B \geq n$.

Proof of Assertion 2.3. Suppose that $\dim_{\mathbb{F}} V \leq n-1$. Since $H^*(\mathbf{G}_{n,2}(\mathbb{F}); \mathbb{Z}/2)$ is a polynomial ring below dimension $2n-2$, it follows from Assertion 2.2 that the mod

2 etale cohomology of V is nonzero in all dimensions of the form $2q = 2(s+t)r + 2t$ where $s, t \geq 0$, $s+t > 0$, and $q \leq n-2$. In particular, if the mod 2 etale cohomology of V is zero in all dimensions from i through j, where $i < j$ and $2r \leq j < 2n-2$, then $j - i \leq 2r - 4$ and equality can hold only if $i = 2r + 2$. On the other hand, if $r > 2$ and $e = \dim_{\mathbb{F}} V$ then $2e \leq 2n - 2$ and by Poincaré duality the mod 2 etale cohomology of V is zero in all dimensions from $2e - 2r + 1$ through $2e - 1$, and by assumption we have $2e - 1 < 2n - 2$. Since $(2e-1) - (2e - 2r + 1) = 2r - 2$, we obtain a contradiction to the inequality $j - i \leq 2r - 4$. ■

Recall (e.g., from [Sp, pp. 34–35] or [Z, p. 9]) that a topological space X with basepoint $\mathbf{e}$ is said to be an H-space (or Hopf space) if there is a continuous map $\mu\colon X \times X \to X$ such that the restrictions of μ to $X \times \{\mathbf{e}\}$ and $\{\mathbf{e}\} \times X$ are homotopic to the identity. We shall say that X is a *finite* H-space if in addition X is homotopy equivalent to a finite complex (recall also that the most basic examples of finite H-spaces are compact Lie groups and S^7).

In the proof of the nonfibering theorems for even dimensional projective spaces that is outlined in [Sch, Remark 5.5, p. 200], one of the steps involved taking the composite of a smooth map on the projective space with the standard homogeneous coordinate map from $\mathbb{F}^{2n+1} - \{0\}$ to $\mathbb{FP}^{2n}$. The benefit of doing this is that one obtains a new smooth map for which the 2-adic completion of the etale homotopy type is the 2-adic completion of a finite H-space, which by the results of [Br] must be a sphere of dimension 1, 3, or 7. We would like to do something similar for Grassmann varieties of 2-planes. More generally, if $\mathbb{F}$ is an arbitrary algebraically closed field and $\mathbf{W}_{n,k}(\mathbb{F})$ denotes the set of $n \times k$ matrices over $\mathbb{F}$ with rank k, where $k \leq n$, then the Plücker coordinate map defines a smooth map Φ from $\mathbf{W}_{n,k}(\mathbb{F})$ to $\mathbb{FP}^N$, where $N = \binom{n}{k}$, whose image is $\mathbf{G}_{n,k}(\mathbb{F})$; since $\mathbf{W}_{n,k}(\mathbb{F})$ is defined to be the set of matrices such that some $k \times k$ submatrix has a nontrivial determinant, this set of matrices determines a Zariski open subset of $\mathbb{F}^{nk}$ and hence is isomorphic to an affine variety; by construction the equations defining this variety have integral coefficients. Specializing now to the case $k = 2$ where n satisfies the conditions of Theorem 2.1, we shall now consider the etale homotopy types of the fibers of the composite map $\Pi\Phi\colon \mathbf{W}_{n,k}(\mathbb{F}) \to V$. ■

ASSERTION 2.4. *In the setting of the previous paragraph, let Λ' be the homotopy fiber of $\Pi^{\widehat{\{2\}}}_{\mathrm{et}} \circ \Phi^{\widehat{\{2\}}}_{\mathrm{et}}$. Then Λ' is an H-space whose $\mathbb{Z}/2$ cohomology is finite in each dimension and zero in all but finitely many dimensions; a similar conclusion holds for the $\widehat{\mathbb{Z}}_2$ cohomology of Λ' with "finitely generated over $\widehat{\mathbb{Z}}_2$" replacing "finite".*

This assertion and straightforward analogs of standard results on finite H-spaces yield the following additional information.

COMPLEMENT 2.5. *If Λ' is given as before, then there is a space P and a map ρ from P to Λ' such that*

(*i*) *the space P is a product of 2-profinite completions of odd-dimensional spheres,*
(*ii*) *the map ρ induces isomorphisms of homotopy groups tensored with the rationals and of cohomology with 2-adic integer coefficients tensored with the rationals.*

Proof of Assertion 2.4. First of all, the standard comparison theorems for homotopy types over different fields imply that $\mathbf{W}_{n,2}(\mathbb{F})^{\widehat{\{2\}}}_{\mathrm{et}}$ is homotopy equivalent to $\mathbf{W}_{n,2}(\mathbb{C})^{\widehat{\{2\}}}$, which in turn is homotopy equivalent to the complex Stiefel manifold $\mathbf{V}_{n,2}(\mathbb{C})^{\widehat{\{2\}}}$. Furthermore, the naturality properties of these comparisons imply that the associated map $\Phi^{\widehat{\{2\}}}_{\mathrm{et}}$ corresponds to the canonical map from $\mathbf{V}_{n,2}(\mathbb{C})^{\widehat{\{2\}}}$ to $\mathbf{G}_{n,2}(\mathbb{C})^{\widehat{\{2\}}}$ under these homotopy equivalences. Since this canonical map is the homotopy fiber of the map from $\mathbf{G}_{n,2}(\mathbb{C})^{\widehat{\{2\}}}$ to $BU_2^{\widehat{\{2\}}}$ given by the 2-profinite completion of the classifying map for the standard 2-plane bundle, standard results on pullback fibrations and interlocking exact sequences imply that Λ' is also equivalent to the homotopy fiber of the following composite:

$$L^{\widehat{\{2\}}}_{\mathrm{et}} \to \mathbf{G}_{n,2}(\mathbb{F})^{\widehat{\{2\}}}_{\mathrm{et}} \simeq \mathbf{G}_{n,2}(\mathbb{C})^{\widehat{\{2\}}} \to BU_2^{\widehat{\{2\}}}$$

In particular, it follows that Λ' fibers homotopically over $L^{\widehat{\{2\}}}_{\mathrm{et}}$ with homotopy fiber $U_2^{\widehat{\{2\}}}$, so that the cohomology of Λ' vanishes above dimension $2n-4$ (since the mod 2 etale cohomology of L vanishes above dimension $2n-8$ and $\dim U_2 = 4$). Since the Stiefel manifold $\mathbf{V}_{n,2}(\mathbb{C})$ is $(2n-4)$-connected it follows that $\mathbf{W}_{n,2}(\mathbb{F})^{\widehat{\{2\}}}_{\mathrm{et}}$ is also $(2n-4)$-connected, and therefore every map from Λ' to $\mathbf{W}_{n,2}(\mathbb{F})^{\widehat{\{2\}}}_{\mathrm{et}}$ is nullhomotopic. On the other hand, a standard result in algebraic topology states that a homotopy fiber of a fibration is an H-space if the fiber inclusion is nullhomotopic (cf. [SW]), and therefore Λ' must be an H-space. The fibration $U_2^{\widehat{\{2\}}} \to \Lambda' \to L^{\widehat{\{2\}}}_{\mathrm{et}}$ also shows that the groups $H^*(\Lambda'; \mathbb{Z}/2)$ and $H^*(\Lambda'; \widehat{\mathbb{Z}}_2)$ are finitely generated in each dimension, and the definition of Λ as the homotopy fiber of a map between 2-connected spaces also shows that Λ' is 1-connected. ■

Proof of Complement 2.5 It is well known that the rational cohomology of a finite H-space is isomorphic to an exterior algebra on a finite set of odd dimensional generators (cf. the appendix to [MM]); we claim that the analogous statement holds for $H^*(\Lambda'; \widehat{\mathbb{Z}}_2) \otimes \mathbb{Q}$. Assertion 2.4 and the Künneth formula in Proposition 1.1A imply that $H^*(\Lambda'; \widehat{\mathbb{Z}}_2) \otimes \mathbb{Q}$ is a connected cocommutative Hopf algebra over $\widehat{\mathbb{Q}}_2$ that is finite dimensional in each grading and zero in all but finitely many dimensions, and because of this the standard arguments for finite H-spaces extend directly to the type of example under consideration (e.g., see [MM, Thm. 7.5, p. 252, and remarks on p. 253 following the proof]).

The remainder of the proof is similar to the argument establishing the corresponding result for simply connected finite H-spaces. By the Künneth formula there is a map ρ_1 from Λ' to a product of Eilenberg-MacLane spaces

$$P_1 = K(\widehat{\mathbb{Z}}_2, n_1) \times \cdots \times K(\widehat{\mathbb{Z}}_2, n_r)$$

(where each n_j is odd) such that ρ_1 induces an isomorphism on rationalized cohomology with $\widehat{\mathbb{Z}}_2$ coefficients. Define Γ to be the homotopy fiber of ρ_1; by construction the homotopy groups of Γ are finite 2-groups in all dimensions. Let P be the product of the corresponding 2-profinite completions of spheres

$$P = (S^{n_1})^{\widehat{\{2\}}} \times \cdots (S^{n_r})^{\widehat{\{2\}}}$$

and let J be the product of the maps $(S^{n_j})^{\widehat{\{2\}}} \to K(\widehat{\mathbb{Z}}_2, n_j)$ that are surjections in $\widehat{\mathbb{Z}/2}$ cohomology (and bijections in rationalized $\widehat{\mathbb{Z}/2}$ cohomology). Furthermore, let $\Psi\colon P \to P$ be the product of the maps of degree 2 on the factors $(S^{n_j})^{\widehat{\{2\}}}$.

The proof of Complement 2.5 reduces to showing that there is a self map ω of P inducing an isomorphism of rationalized $\widehat{\mathbb{Z}/2}$ cohomology such that the composite $J \circ \omega$ lifts tc Λ'. Such a lifting obviously exists on the 0-skeleton of a CW complex representing P, so suppose by induction that a lifting of this sort has been constructed on the k-skeleton. If $k \geq \sum n_j$ then $H^{k+1}(P; \pi) = 0$ for all finite abelian 2-groups π, and since $\pi_{k+1}(\Gamma)$ is a finite abelian 2-group there are no obstructions to extending the lifting to the $(k+1)$-skeleton. Suppose therefore that $k < \sum n_j$. Then the obstruction to lifing the map $J \circ \omega$ over the $(k+1)$-skeleton lies in the group $H^{k+1}(P; \pi_{k+1}(\Gamma))$. Let q be the exponent of this group; by construction $q = 2^t$ for some nonnegative integer t. If $\Psi^{[t]}$ denotes the t-fold iterate of Ψ, then image of the induced self map $\Psi^{[t]}$ of $H^{k+1}(P; \pi)$ consists of elements divisble by $q = 2^t$ for an arbitrary abelian group π. The naturality properties of obstructions then show that the obstruction to lifting $J \circ \omega \circ \Psi^{[t]}$ are trivial. Replace ω with $\omega \circ \Psi^{[t]}$ yields the inductive step needed to proceed to the next dimension if $k + 1 \leq \sum n_j$. Since the self map ω only changes at finitely many steps, this yields a lifting of some map of the form $J \circ \omega$ as required. ■

COROLLARY 2.6. *For each $j > 1$ the rationalized homotopy group $\pi_j(\Lambda') \otimes \mathbb{Q}$ is a finite dimensional vector space over $\widehat{\mathbb{Q}}_2$. These groups are zero if j is even or $j \geq 2n - 5$.*

Proof. The groups $\pi_j(\Lambda')$ are 2-profinite completions of finitely generated abelian groups by Proposition 1.1 and Assertion 2.4, and the conclusion of the first sentence of the corollary follows immediately from these facts. To prove the statement in the second sentence, consider the map $P \to \Lambda'$ in Complement 2.5. By construction the induced morphisms of $\widehat{\mathbb{Z}}_2$ cohomology groups have finite kernels and cokernels. Since the homotopy groups of P and Λ' are 2-profinite, a straightforward analog of the usual Serre $\mathcal{C}$-theoretic arguments shows that the morphism of homotopy groups induced by the map $P \to \Lambda'$ also has finite kernel and cokernel. In particular, $\pi_*(\Lambda') \otimes \widehat{\mathbb{Q}_2}$ is isomorphic to $\pi_*(P) \otimes \widehat{\mathbb{Q}_2}$; since the latter groups vanish in even dimensions, the corresponding statement also holds for the rational homotopy groups of Λ'. On the other hand, if $d = \dim_{\mathbb{F}} L$, then the Serre spectral sequence of the fibration $\widehat{U_2^{\{2\}}} \to \Lambda' \to \widehat{L_{\text{et}}^{\{2\}}}$ shows that the $\widehat{\mathbb{Z}}_2$ cohomology groups of Λ' vanish above dimension $2d + 4$, and therefore the same is true for the rationalized cohomology of P; in fact, since the cohomology of P is torsion free the same conclusion holds for $\widehat{\mathbb{Z}}_2$ cohomology. Since P is a product of 2-profinite completions of odd dimensional spheres, it follows that $\pi_j(P) \otimes \mathbb{Q} \approx \pi_j(\Lambda') \otimes \mathbb{Q}$ must be trivial if $j > 2d + 4$, and since $d \leq n - 4$ this implies the vanishing of the rationalized homotopy groups if $j > 2n - 4$. ■

The conclusions of 2.4–2.6 have strong implications for the rationalized homotopy and $\widehat{\mathbb{Z}}_2$ cohomology groups of the base space $\widehat{V_{\text{et}}^{\{2\}}}$.

ASSERTION 2.7. *Let $2r$ be the first positive integer for which $H^{2r}(V; \mathbb{Z}/2) \neq 0$. Then the rational homotopy groups of $\widehat{V_{\text{et}}^{\{2\}}}$ are 1-dimensional $\widehat{\mathbb{Q}_2}$ vector spaces in degrees $2r$, $2r + 2$, $2n - 3$ and $2n - 1$, and all remaining rational homotopy groups vanish. Furthermore, the Poincaré polynomial for the cohomology of $\widehat{V_{\text{et}}^{\{2\}}}$ with $\mathbb{Z}/2$ or $\widehat{\mathbb{Z}}_2$ coefficients is given by*

$$\frac{(1 - t^{2n-2})(1 - t^{2n})}{(1 - t^{2r})(1 - t^{2r+2})}.$$

As in [HLP1–2] (and many other references) if A^* is a graded vector space over a field $\Bbbk$ that is finite dimensional in each grading, then its *Poincaré series* $\mathbf{p}(A; \Bbbk)$ is given by

$$\sum_{j=-\infty}^{\infty} \dim A^j \cdot t^j$$

and similarly if the grading is a subscript rather than a superscript. If $A^j = 0$ for $j < 0$ and $j > M$ for some positive integer M, this series is often called the Poincaré polynomial.

Before proving Assertion 2.7 we shall derive some consequences. If we were working with topological fiberings these would follow quickly from the results of [HLP 2] on rational fibrations for which the fiber and base have the rational cohomology of finite complexes.

COROLLARY 2.8. *The dimension of the fiber L over the field $\mathbb{F}$ (in the sense of algebraic geometry) is equal to $2r-2$, the Poincaré polynomial for the cohomology of $\widehat{L_{\mathrm{et}}^{\{2\}}}$ with $\mathbb{Z}/2$ or $\widehat{\mathbb{Z}}_2$ coefficients is given by*

$$\frac{(1-t^{2r+2})(1-t^{2r})}{(1-t^2)(1-t^4)}$$

and the morphisms in $\mathbb{Z}/2$ or $\widehat{\mathbb{Z}}_2$ cohomology associated to the inclusion of L in $\mathbf{G}_{n,2}(\mathbb{F})$ are surjective (hence the Serre spectral sequences collapse over these coefficients).

Recall that the formal cohomological dimension of L (= the top dimension with nonzero cohomology) is equal to twice the dimension of L as a variety over $\mathbb{F}$.

Proof that Assertion 2.7 *implies Corollary* 2.8. The Poincaré polynomial formula in Assertion 2.7 implies that $\dim_{\mathbb{F}} V = 2(n-r)-2$, and the formula for $\dim_{\mathbb{F}} L$ follows from this because the sum of the fiber and base dimensions is $2n-4$ (= the dimension of the Grassmann variety). The argument in the first paragraph of the proof of Assertion 2.2 then shows that $H^*(L;\mathbb{Z}/2)$ and $H^*(L;\widehat{\mathbb{Z}}_2)$ vanish in all odd dimensions and completely determine the dimensions of the cohomology groups in all even dimensions. Since these cohomology groups are isomorphic to the corresponding cohomology groups in dimensions $\leq 2r-2$, it follows that the Poincaré polynomial must be given by the formula given above. The collapsing of the Serre spectral sequences now follows because the cohomology groups of the fiber and base are both concentrated in even dimensions, and the surjectivity of the restriction maps in cohomology follows immediately from this.

Proof of Assertion 2.7. Let M be the Postnikov approximation to $\widehat{V_{\mathrm{et}}^{\{2\}}}$ such that the map $\widehat{V_{\mathrm{et}}^{\{2\}}} \to M$ is $(2n-3)$-connected and the homotopy groups of M vanish in dimensions $\geq 2n-3$. This map fits into the commutative diagram displayed below, in which the horizontal sequences represent fibrations:

$$\begin{array}{ccccc}
\Lambda' & \longrightarrow & \widehat{\mathbf{W}_{n,2}(\mathbb{F})_{\mathrm{et}}^{\{2\}}} & \longrightarrow & \widehat{V_{\mathrm{et}}^{\{2\}}} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega(M) & \longrightarrow & \mathbf{P}(M) & \longrightarrow & M
\end{array}$$

In this diagram $\Omega(M)$ denotes the based loop space of M and $\mathbf{P}(M)$ denotes the contractible space of paths in M that begin at a prescribed basepoint. By construction the map from $\widehat{\mathbf{W}_{n,2}(\mathbb{F})_{\mathrm{et}}^{\{2\}}}$ to M is nullhomotopic, and therefore the map from $\mathbf{W}_{n,2}(\mathbb{F})$ to $\widehat{V_{\mathrm{et}}^{\{2\}}}$ factors through the homotopy fiber $\mathbb{T}$ of the map from $\widehat{V_{\mathrm{et}}^{\{2\}}} \to M$. Since $\mathbf{W}_{n,2}(\mathbb{F}) \simeq \mathbf{V}_{n,2}(\mathbb{C})^{\{2\}}$ is $(2n-4)$-connected, a chase of the diagram implies that the map $\Lambda' \to \Omega(M)$ is $(2n-4)$ connected and induces an isomorphism of rational homotopy groups; these in turn imply that the map from $\mathbf{W}_{n,2}(\mathbb{F})_{\mathrm{et}}^{\widehat{\{2\}}}$ to $\mathbb{T}$ induces isomorphisms of rational homotopy groups and rationalized $\widehat{\mathbb{Z}}_2$ cohomology (note that the space $\mathbb{T}$ satisfies the conditions of Proposition 1.1 for $\ell = 2$ so that the appropriate finite generation conditions hold for its homotopy and cohomology).

Since the $\widehat{\mathbb{Z}}_2$ cohomology groups of all spaces under consideration are finitely generated $\widehat{\mathbb{Z}}_2$ modules and the spaces are also simply connected, it follows that the rationalized Serre spectral sequence for the fibration $\mathbb{T} \to V_{\mathrm{et}}^{\widehat{\{2\}}} \to M$ has the form

$$E_2^{s,t} = \left(H^s(M; \widehat{\mathbb{Z}}_2) \otimes \mathbb{Q}\right) \otimes \left(H^t(\mathbb{T}; \widehat{\mathbb{Z}}_2) \otimes \mathbb{Q}\right) \Rightarrow H^{s+t}(V_{\mathrm{et}}^{\widehat{\{2\}}}; \widehat{\mathbb{Z}}_2) \otimes \mathbb{Q}$$

(note that the right hand term is isomorphic to the 2-adic cohomology of the variety V as described in [Hrt]). Our next order of business is to determine the Poincaré series $\mathbf{p}(E_k)$ for this spectral sequence. If $f(t)$ and $g(t)$ are the Poincaré series for the rationalized $\widehat{\mathbb{Z}}_2$ cohomology of M and $V_{\mathrm{et}}^{\widehat{\{2\}}}$ respectively, then general considerations imply that $\mathbf{p}(E_2) = f(t)(1+t^{2n-3})(1+t^{2n-1})$ (because $\mathbf{p}(\mathbb{T}) = (1+t^{2n-3})(1+t^{2n-1})$) and $\mathbf{p}(E_\infty) = g(t)$, where the latter is in fact a polynomial in t with no odd degree terms because the cohomology of V is concentrated in even dimensions).

We begin by describing $f(t)$. By Corollary 2.6 we know that Λ' has the same rational homotopy groups as some product

$$P = (S^{2m_1-1})^{\widehat{\{2\}}} \times \cdots \times (S^{2m_q-1})^{\widehat{\{2\}}}$$

and by the preceding discussion we know that the rational homotopy groups of $\Omega(M)$ are isomorphic to those of P. Proposition 1.1B implies that $H^*(M; \widehat{\mathbb{Z}}_2) \otimes \mathbb{Q}$ must be a polynomial algebra over $\widehat{\mathbb{Q}}_2$ on generators in dimensions $2m_1, \cdots 2m_q$, and therefore we have

$$f(t) = \frac{1}{(1-t^{2m_1})\cdots(1-t^{2m_q})}.$$

The connectivity assumptions imply that at one of the integers m_j is equal to r; furthermore, the 3-connectivity of M and Assertion 2.2 imply that at least one of the integers $m_{j'}$ is equal to $r+1$. In particular, this implies that $q \geq 2$.

Since $E_2^{s,t} = 0$ if $0 < s < 4$, it follows that the first nontrivial differentials appear in E_{2n-2}. Since nothing in odd total degrees can survive to E_∞ and $f(t)$ has only even degree terms, this forces the differentials $d_{2n-2}^{s,t}$ to be injective if $s = 2n-3$ or $4n-4$. The latter in turn implies that

$$\mathbf{p}(E_{2n-1}) = (1-t^{2n-2})(1+t^{2n-1})f(t).$$

Dimensional considerations imply that $E_2n = E_{2n-1}$ and $E_{2n+1} = E_\infty$, and the triviality of E_∞ in odd total degrees implies that

$$g(t) = \mathbf{p}(E_\infty) = \mathbf{p}(E_{2n+1}) = (1-t^{2n-2})(1-t^{2n})f(t).$$

If $q \geq 3$ then the right hand side has a pole of order $q - 2 \geq 1$ at $t = 1$; since the left hand side is a polynomial in t this is impossible. Therefore $q = 2$ and the Poincaré polynomial us given by the formula in the assertion. ■

In the topological category the next result would also follow directly from the theory of rational fibrations in [HLP2] (see also [Fe2]). It uses the standard description of $H^*(\mathbf{G}_{n,2}(\mathbb{C}); \mathbb{Z})$ as the quotient of the polynomial algebra on c_1 and c_2 modulo two relations R_{n-1} and R_n in dimensions $2n - 2$ and $2n$ respectively (e.g., this follows immediately from [Bor1, Prop. 31.1, p. 202]; see also [BT, Prop. 23.2, pp. 293–294]).

ASSERTION 2.9. *If $\alpha_r \in H^{2r}(\mathbf{G}_{n,2}(\mathbb{C}); \widehat{\mathbb{Z}_2})$ and $\alpha_{r+1} \in H^{2r+2}(\mathbf{G}_{n,2}(\mathbb{C}); \widehat{\mathbb{Z}_2})$ correspond to the image of $H^{2j}(V; \widehat{\mathbb{Z}_2})$ in $H^{2j}(\mathbf{G}_{n,2}(\mathbb{F}); \widehat{\mathbb{Z}_2})$ for $j = r, r+1$, then one can choose the relations R_{n-1} and R_n to be polynomials in α_r and α_{r+1}.*

Proof. Since the Serre spectral sequence collapses it follows that $H^*(\mathbf{G}_{n,2}(\mathbb{F}); \widehat{\mathbb{Z}_2})$ is a free $H^*(V; \widehat{\mathbb{Z}_2})$ module on $\chi(L; \widehat{\mathbb{Z}_2})$ generators that project to a set of free generators for $H^*(L; \widehat{\mathbb{Z}_2})$. If R_{n-1} could not be chosen to be a polynomial in α_r and α_{r+1}, then the rank of $H^*(V; \widehat{\mathbb{Z}_2})$ would be equal to the rank of the polynomial ring $\widehat{\mathbb{Z}_2}[\alpha_r, \alpha_{r+1}]$ through dimension $2n - 2$, and this in turn would imply that the rank of $H^*(\mathbf{G}_{n,2}(\mathbb{F}); \widehat{\mathbb{Z}_2})$ would equal the rank of $\widehat{\mathbb{Z}_2}[c_1, c_2]$ through the same range of dimensions. Since the latter does not hold, it follows that R_{n-1} must be a polynomial in α_r and α_{r+1}. Similarly, if R_n can not be chosen to be a polynomial in these classes then the ranks of $H^*(V; \widehat{\mathbb{Z}_2})$ and $\widehat{\mathbb{Z}_2}[\alpha_r, \alpha_{r+1}]$ would be equal through dimension $2n$ except for dimension $2n - 2$, and this would likewise yield an incorrect value for the rank of $H^*(\mathbf{G}_{n,2}(\mathbb{F}); \widehat{\mathbb{Z}_2})$. ■

Proof of Theorem 2.1. Let α_r be given as in the preceding assertion. If $2r < n - 1$ it follows immediately that the image of α_r^2 in $\mathbb{Z}/2$ cohomology is nonzero, and $2r = n - 1$ is impossible since n is even. Therefore the mod 2 reduction $\overline{\alpha_r}$ maps nontrivially under Sq^{2r}.

Now the mod 2 Euler characteristic of $L_{\mathrm{et}}^{\widehat{\{2\}}}$ is odd by Proposition 1.5, and by Corollary 2.8 its value is $\binom{r+1}{2}$. This implies that r is congruent to either 1 or 2 mod 4. Therefore the Adem relations imply that Sq^{2r} is decomposable unless $r = 2$. If in fact $r \geq 3$, then the structure of $H^*(V; \mathbb{Z}/2)$ combines with this decomposability property to show that $\mathrm{Sq}^{2r-2}\,\mathrm{Sq}^2\,\overline{\alpha_r} = \overline{\alpha_r}^2 \neq 0$. Since $H^*(V; \mathbb{Z}/2)$ has no cohomology between dimensions $2r + 2$ and $4r$ it follows that Sq^{2r-2} must be indecomposable, so that $r - 1$ is a power of 2, and in fact the results of Adams on factorizations of Steenrod squares by secondary operations imply that $r - 1$ must be either 2 or 4. These cases can be eliminated as follows: As in the proof of Assertion 2.2 let $\mathcal{M}^* \subset H^*(\mathbf{G}_{n,2}(\mathbb{F}); \mathbb{Z}/2)$ be the complementary direct summand to the image of Π^* with respect to the action of $\mathcal{Q}_2$. If $r = 3$ or 5 then direct computation in $H^*(\mathbf{G}_{n,2}(\mathbb{C}); \mathbb{Z}/2)$ shows that $\mathrm{Sq}^2\,\overline{\alpha_r} = y^2 = \mathrm{Sq}^{r+1}\,y$ for some $(r+1)$-dimensional class y. Since $r+1<2r$ we must have $y \in \mathcal{M}^{r+1}$ and hence $\mathrm{Sq}^2\,\overline{\alpha_r} = \mathrm{Sq}^{r+1}\,y \in \mathcal{M}^{2r+2}$; on the other hand, $\overline{\alpha_r} \in \mathrm{Image}\,\Pi^*$ implies $\mathrm{Sq}^2\,\overline{\alpha_r} \in \mathrm{Image}\,\Pi^*$, so that $\mathrm{Sq}^2\,\overline{\alpha_r} = 0$. The last equation contradicts Assertion 2.2, and therefore one cannot have $r = 3$ or 5.

We have now reduced the proof of Theorem 2.1 to eliminating the case where $r = 2$, so for the balance of this section we shall make this assumption.

If polynomials ρ_k are defined recursively by the formulas $\rho_{-1} = 0$, $\rho_1 = 1$, and $\rho_k + \rho_{k-1}c_1 + \rho_{k-2}c_2 = 0$ then it is well known that ρ_{n-1} and ρ_n generate the ideal of relations for $H^*(\mathbf{G}_{n,2}(\mathbb{C}); \mathbb{Z}/2)$. We shall need some explicit properties of these relations:

ASSERTION 2.10. *If ρ_{n-1} and ρ_n are given as above and*

$$c_1, c_2 \in H^*(\mathbf{G}_{n,2}(\mathbb{F}); \mathbb{Z}/2) \cong H^*(\mathbf{G}_{n,2}(\mathbb{C}); \mathbb{Z}/2)$$

are the standard generators, then ρ_{n-1} is equal to c_1^{n-1} plus terms divisible by c_2. Furthermore, if $2s$ is the unique even integer in the set $\{n-1, n\}$, then $\rho_{2s} = c_2^s$ plus terms divisible by c_1.

This can be verified by an inductive argument.

ASSERTION 2.11. *If α_2 and α_3 are given as in Assertion 2.9, then $\alpha_2 = c_2 + c_1^2$ and $\alpha_3 = c_1c_2$.*

Proof. Let $\mathcal{M}^*$ be defined as in the proof of Assertion 2.2, and let

$$c_1, c_2 \in H^*(\mathbf{G}_{n,2}(\mathbb{F}); \mathbb{Z}/2) \cong H^*(\mathbf{G}_{n,2}(\mathbb{C}); \mathbb{Z}/2)$$

be the standard generators. Since $c_1 \in \mathcal{M}^*$ by the connectivity assumption on V, it follows that $c_1^2 = \mathrm{Sq}^2\,c_1 \in \mathcal{M}^*$ too, and the submodule properties of $\mathcal{M}^*$ imply that $H^4(V; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and a generator is given by $c_2 + c_1^2$. Since $\mathrm{Sq}^2(c_2 + c_1^2) = c_1c_2$ the right hand side lies in the image of Π^*, and the additional relations given by

$$\mathrm{Sq}^2\,c_1^3 = c_1^4 = \mathrm{Sq}^4(c_1^2) \in \mathrm{Sq}^4(\mathcal{M}^4) \subset \mathcal{M}^8$$

imply that c_1c_2 generates the image of Π^* in dimension 6 as claimed.

Since $n \equiv 2 \bmod 4$, we have $n = 4k+2$ for some integer $k \geq 2$. By Assertion 2.10 we know that $\rho_{4k+1} = c_1^{4k+1}$ modulo terms divisible by c_2. But no polynomial in α_2 and α_3 can be equal to an odd power of c_1 modulo terms divisible by c_2, so in particular ρ_{4k+1} cannot be such a polynomial; since this contradicts Assertion 2.9, the conclusion of Theorem 1 follows. ■

As noted at the beginning of this paper, it seems likely that Theorem 2.1 extends to cases where $n \equiv 3 \bmod 4$ but $n+1$ is not a power of 2; in analogy with [Sch, Remark

6.19, p. 607], one reason for optimism is that one cannot choose the polynomial relations in

$$H^*(\mathbf{G}_{n,2}(\mathbb{F}); \mathbb{Z}/2) \cong \mathbb{Z}/2[c_1, c_2]/(r_{n-1}, r_n)$$

to be polynomials in $c_1^2 + c_2$ and c_1c_2 if $n+1$ is not a power of 2, and a condition of this sort is usually necessary for the existence of compact fiberings in the topological category of finite CW complexes (cf. [Fe2]). In contrast, one can choose the relations to be polynomials of this sort if $n = 7$, 15 or 31. The following result shows that Theorem 2.1 does extend in a substantial number of cases:

OBSERVATION 2.12. *The conclusion of Theorem* 2.1 *also holds if* $n \geq 11$ *is congruent to* 11 mod 12.

Note that the hypothesis on n is equivalent to $n \equiv 3 \bmod 4$ and $n \equiv 2 \bmod 3$.

Proof. Since the Euler characteristic of the Grassmann manifold of 2-planes in n-space is $\binom{n}{2}$ and the latter is odd if $n \equiv 3 \bmod 4$, it follows that the entire discussion of this section through the proof of Assertion 2.11 goes through in such cases.

To complete the proof, assume the conclusion of the theorem does not hold for some value of n satisfying the given conditions, and note that by Corollary 2.8 the mod 2 Euler characteristic of the etale fiber $\widehat{L_{\text{et}}^{\{2\}}}$ is equal to 3 because $r = 2$ (see the second paragraph preceding the statement of Assertion 2.10). By Proposition 1.4 this implies that the mod 2 Euler characteristic of $\widehat{\mathbf{G}_{n,2}(\mathbb{F})_{\text{et}}^{\{2\}}}$, which is equal to the mod 2 Euler characteristic of $\mathbf{G}_{n,2}(\mathbb{C})$, is divisible by 3. On the other hand, the binomial coefficient $\binom{n}{2}$ is divisible by 3 if and only if n is not congruent to 2 mod 3, and therefore we have a contradiction if $n \equiv 3 \bmod 4$ and $n \equiv 2 \bmod 3$. Therefore the conclusion of Theorem 2.1 holds whenever n satisfies the latter conditions. ■

Final remarks. Several steps in the proof of Theorem 2.1 involve ideas from rational homotopy theory. However, one cannot apply the standard machinery of [Su2] or [HLP1–2] because the rational homotopy groups of an ℓ-profinite completion are either trivial or uncountable dimensional rational vector spaces. Given a space X that is 1-connected and ℓ-profinitely complete with finitely generated $\widehat{\mathbb{Z}_\ell}$ cohomology in each dimension, it would be extremely useful to have a well behaved ℓ-adic minimal model for the ℓ-adic cohomology groups $H^*(X; \widehat{\mathbb{Z}_\ell}) \otimes \mathbb{Q}$, with all their multiplicative structure (including higher order Massey products), such that the ℓ-adic minimal model somehow refines the usual rational minimal model for $H^*(X; \mathbb{Q})$. If such a theory of ℓ-adic minimal models existed, several arguments in this paper could be shortened or eliminated, and algebraic versions of many nonfibering theorems in [Fe1] for Grassmann varieties of 3-planes would follow directly by applying the techniques of [Fe1] and this paper to the ℓ-adic completions of etale homotopy types and their ℓ-adic minimal models. Standard methods of rational homotopy (e.g., the techniques of [GM]) yield candidates for the ℓ-adic minimal models of spaces satisfying the conditions described above, but it is not obvious that these objects have the naturality properties that are needed to work with them effectively.

References

[ABP] D. W. Anderson, E. H. Brown, and F. P. Peterson, *The structure of the Spin cobordism ring*, Ann. of Math. **86** (1967), 271–298.

[AM] M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Mathematics, Vol. 100, Springer, New York, 1969.

[BG1] J. C. Becker and D. H. Gottlieb, *Applications of the evaluation map and transfer map theorems*, Math. Ann. **211** (1974), 277–288.

[BG2] ———, *The transfer map and fiber bundles*, Topology **14** (1975), 1–12.

[BG3] ———, *Transfer maps for fibrations and duality*, Comp. Math. **33** (1976), 107–133.

[Bor1] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–208.

[Bor2] ———, *Topics in the homology theory of fiber bundles* (Lectures, Univ. of Chicago, 1954. notes by E. Halperin), Lecture Notes in Mathematics, Vol. 36, Springer, New York, 1967.

[BoS] A. Borel and J.-P. Serre, *Impossibilité de fibrer un espace euclidien par des fibres compactes*, C. R. Acad. Sci. Paris **230** (1950), 2258–2260.

[BT] R. Bott and L. Tu, *Differential forms in algebraic topology*, Graduate Texts in Math., Vol. 78, Springer, New York, 1984.

[BK] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions, and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer, New York, 1972.

[Br] W. Browder, *Higher torsion in H-spaces*, Trans. Amer. Math. Soc. **108** (1963), 353–375.

[CG] A. Casson and D. H. Gottlieb, *Fibrations with compact fibres*, Amer. J. Math. **99** (1977), 159–189.

[ES] D. B. A. Epstein and N. E. Steenrod, *Cohomology operations*, Annals of Mathematics Studies, Vol. 50, Princeton Univ. Press, Princeton, N. J., 1962.

[Fe1] R. J. D. Ferdinands, *Some complex Grassmann manifolds that do not fiber nontrivially*, Topology Appl. **40** (1991), 221–231.

[Fe2] ———, *$G_{n,3}(\mathbb{C})$ is connectedwise prime for $n \geq 5$*, preprint, Calvin College, 1994.

[Fr1] E. Friedlander, *Fibrations in etale homotopy theory*, I. H. E. S. Publ. Math. **42** (1972), 5–46.

[Fr2] ———, *The etale homotopy of simplicial schemes*, Ann. of Math. Studies, Vol. 104, Princeton Univ. Press, Princeton, 1982.

[Got] D. H. Gottlieb, *Robots and fiber bundles*, Bull. Soc. Math. Belg. Sér. A **38** (1986), 219–223.

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley/Interscience Series in Pure and Applied Mathematics, Wiley, New York, 1978.

[GM] P. Griffiths and J. Morgan, *Rational homotopy theory and differential forms*, Progress in Mathematics, Vol. 16, Birkhäuser, Boston, 1981.

[Hlp1] S. Halperin, *Finiteness in the miminal models of Sullivan*, Trans. Amer. Math. Soc. **230** (1977), 173–199.

[Hlp2] ———, *Rational fibrations, minimal models and fiberings of homogeneous spaces*, Trans. Amer. Math. Soc. **244** (1978), 199–224.

[Hrs] J. Harris, *Algebraic geometry, a first course*, Graduate Texts in Mathematics, Vol. 133, Springer, New York-*etc.*, 1992.

[Hrt] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Vol. 52, Springer, New York, 1977.

[J] R. Joshua, *Becker-Gottlieb transfers in etale homotopy*, Amer. J. Math. **109** (1987), 453–497.

[Lai] H. F. Lai, *On the topology of even-dimensional complex quadrics*, Proc. Amer. Math. Soc. **46** (1974), 419–425.

[McL] S. MacLane, *Homology*, Grundlehren der Math. Wissenschaften, Bd. 114, Springer, New York, 1963.

[Mi] W. S. Milne, *Étale cohomology*, Princeton Math. Series, Vol. 33, Princeton Univ. Press, Princeton, N.J., 1980.

[MM] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. **81** (1965), 211–264.

[Ray] M. Raynaud, *Géometrie algébrique et géometrie analytique*, Séminaire de Géométrie Algébrique 1: Revêtements Étales et Groupe Fondemantale, Lecture Notes in Mathematics, Vol. 224, Springer, New York, 1971.

[Sch] R. Schultz, *Compact fiberings of homogeneous spaces*, Comp. Math. **43** (1981), 181–215, *Correction*, 419–421.

[Sp] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York (1967).

[SW] E. H. Spanier and J. H. C. Whitehead, *On fibre spaces in which the fibre is contractible*, Comment. Math. Helv. **29** (1955), 1–8.

[St] J. Stallings, "On fibering certain 3-manifolds" in *Topology of 3-manifolds and related topics*, Proc. Univ. of Georgia Inst., 1961, Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 95–100.

[Su1] D. Sullivan, *Genetics of homotopy theory and the Adams Conjecture*, Ann. of Math. **100** (1974), 1–79.

[Su2] ———, *Infinitesimal computations in topology*, I. H. E. S. Publ. Math. **47** (1977), 269–331.

[Z] A. Zabrodsky, *Hopf spaces*, North-Holland Mathematics Studies, No. 22, North-Holland, Amsterdam (NL), 1976.

R. J. D. Ferdinands, Dept. of Math and Comp. Sci., Calvin College, Grand Rapids, Michigan

R. E. Schultz, Dept. of Math., Purdue University, West Lafayette, Indiana 47907