THE CENTRALIZER OF INVARIANT FUNCTIONS AND DIVISION PROPERTIES OF THE MOMENT MAP

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1. Introduction

Let $\Phi\colon M \longrightarrow \mathfrak{g}^*$ be a moment map associated to a Hamiltonian action of a compact connected Lie group G on a compact [1] connected symplectic manifold (M, ω). Pullbacks by Φ of smooth functions on $\mathfrak{g}^*$ are called *collective functions*. They form a Poisson subalgebra of the algebra of smooth functions on M. Its centralizer is the algebra of invariant functions; i.e., a smooth function f on M is invariant if and only if $\{f, h\} = 0$ for every collective function h, where $\{\ ,\ \}$ denotes the Poisson bracket corresponding to the symplectic form ω.

Motivated by a study of completely integrable systems in [GS1], Guillemin and Sternberg conjectured in [GS3] that the centralizer of the algebra of invariant functions is the algebra of collective functions. They proved this conjecture for neighborhoods of generic points in M.

A collective function is clearly constant on the level sets of the moment map. The converse need not be true. For example, the standard linear action of the group $G = SU(2)$ on $\mathbb{C}^2$ has a moment map $\Phi(u, v) = (\bar{u}v, \frac{1}{2}|u|^2 - \frac{1}{2}|v|^2)$ when we identify the vector space $\mathfrak{g}^*$ with $\mathbb{R} \times \mathbb{C}$. The function $f(u, v) = |u|^2 + |v|^2$ is constant on the level sets of Φ because it is equal to $(|\bar{u}v|^2 + (\frac{1}{2}|u|^2 - \frac{1}{2}|v|^2)^2)^{\frac{1}{2}} = 2\|\Phi\|$. It is not collective because the function $\|x\|$ is not smooth on $\mathbb{R} \times \mathbb{C}$.

In Section 2 of this paper we show that the centralizer of the algebra of invariant functions is the algebra of functions that are constant on the level sets of the moment map. In fact, these two algebras are mutual centralizers in the Poisson algebra $C^\infty(M)$. See Theorem 1 and Corollary 2.12. This was already shown in the thesis of the first author [K], but our current proof is shorter.

This result raises the following question: what is the obstruction for a function that is constant on the level sets of the moment map to be collective? In Section 3, Theorem 2, we express this obstruction as a condition on the Taylor series of the function. The proof uses theorems of Bierstone and Milman and of Marle, Guillemin, and Sternberg. Theorem 2 essentially reduces the identification of the centralizer of the invariant functions to an algebraic question. Based on this, F. Knop recently

announced a complete description of the centralizer of the invariant functions on a Hamiltonian space in terms of the little Weyl group (defined in [KN]) of the space.

If the Lie group G is abelian, every function which is constant on the level sets of the moment map is a collective function. So the conjecture of Guillemin and Sternberg is true for torus actions. The history of the proof is as follows. In their paper [GS3], Guillemin and Sternberg proved that for a linear torus action on a symplectic vector space the centralizer of the invariants consists of functions that are constant on the level sets of the moment map. They claimed that these functions are collective. This claim is not obvious; we prove it in Section 4 of this paper. In [L2], the second author showed that this claim implies that the conjecture of Guillemin and Sternberg is true for torus actions on compact manifolds, and, more generally, for actions of compact Lie groups on compact manifolds, provided that the image of the moment map does not intersect the walls of the Weyl chambers. We recall (slightly stronger versions of) these results in Section 4.

The first counterexample to the conjecture of Guillemin and Sternberg was given by the second author in [L2]. This is the standard action of $SU(2)$ on $\mathbb{C}^2$ with moment map $\Phi(u, v) = (\overline{u}v, \frac{1}{2}|u|^2 - \frac{1}{2}|v|^2)$. This action of $SU(2)$ extends to the standard action of $U(2)$ on $\mathbb{C}^2$ with the same orbits (spheres) and with a moment map $\tilde{\Phi}(u, v) = (\overline{u}v, |u|^2, |v|^2)$. The centralizer of the invariants (for either $SU(2)$ or $U(2)$) consists of the $U(2)$-collective functions. For instance, the function $f(u, v) = |u|^2 + |v|^2$ on $\mathbb{C}^2$ is in the centralizer of the invariants, is not $SU(2)$-collective, but is $U(2)$-collective.

A similar phenomenon happens more generally: for a Hamiltonian action of $SU(2)$ with a proper moment map, either the centralizer of the invariant functions consists of the collective functions, or the action of $SU(2)$ extends to an action of $U(2)$ with the same orbits and for which the centralizer of the invariant functions consists of the $U(2)$-collective functions. This we show in Section 5.

In the rest of this introduction we describe the context in which Guillemin and Sternberg posed their conjecture. The notion of mutually centralizing subgroups in the symplectic group originated in physics. It was studied by Sternberg and Wolf, by Howe, by Kashiwara and Vergne, and by Jakobsen and Vergne, in [SW], [H], [KV], and [JV] respectively. In the classical analogue of this notion one considers two connected Lie groups, G and H, that act on a symplectic manifold M with moment maps

$$
\begin{array}{ccc}
 & M & \\
{\scriptstyle F}\swarrow & & \searrow{\scriptstyle \Phi} \\
\mathfrak{h}^* & & \mathfrak{g}^*
\end{array}
\qquad (1.1)
$$

such that the respective algebras of collective functions, $F^*C^\infty(\mathfrak{h}^*)$ and $\Phi^*C^\infty(\mathfrak{g}^*)$, are mutual centralizers in the Poisson algebra $C^\infty(M)$. The G-moment map, Φ, then becomes an orbit map for the action of H. This means that the H-invariants are exactly the pull-backs via Φ of smooth functions on $\mathfrak{g}^*$. Similarly, the G-invariants

are the H-collective functions. This has several consequences. First, generically, the G-reduced spaces are coverings of coadjoint orbits of H and vice versa. Moreover, we get a correspondence between the coadjoint orbits of G and those of H which occur in the images of the respective moment maps. This phenomenon, which is the classical analogue of Howe's dual pairs in representation theory [H], was observed and explained by Kazhdan, Kostant and Sternberg in [KKS] for the case that the G-orbits form a foliation.

More generally, A. Weinstein [W] defined a *dual pair* to be a pair of Poisson maps $f: M \longrightarrow A$ and $g: M \longrightarrow B$ from a symplectic manifold M to Poisson manifolds A and B such that the algebras $f^*C^\infty(A)$ and $g^*C^\infty(B)$ are mutual centralizers in the Poisson algebra $C^\infty(M)$. Dual pairs and their infinite dimensional analogs occur in the study of tops, compressible fluids, elasticity, Maxwell-Vlasov equations [MRW], etc., and have led to the notion of Morita equivalence of Poisson manifolds (cf. [GL]).

The conjecture of Guillemin and Sternberg is equivalent to the maps

$$
\begin{array}{ccc}
& M & \\
\pi \swarrow & & \searrow \Phi \\
M/G & & \mathfrak{g}^*
\end{array}
$$

forming a "dual pair", where Φ is the moment map and π is the quotient map, and where we interpret the Poisson algebra $C^\infty(M/G)$ as the algebra of functions on M/G whose pullback to M is smooth. Note that the quotient M/G need *not* be a manifold hence the quotes around the expression "dual pair".

Given a Hamiltonian action of a Lie group G, one may wonder whether there exists a Hamiltonian action of another Lie group, H, such that the corresponding moment maps form a dual pair (1.1). A necessary condition for the existence of this other action is that the centralizer in $C^\infty(M)$ of the algebra of G-invariant functions be equal to the algebra of G-collective functions.

An example to keep in mind is the standard action of the group $G = U(2)$ on the symplectic vector space $M = \mathbb{C}^2$. The G-orbit map is $F(z, w) = |z|^2 + |w|^2$, which generates the diagonal action of $H = S^1$. Another interesting example is the natural action of the orthogonal group $O(k)$ on $(T^*\mathbb{R}^k)^n$, the n-fold product of the cotangent bundle of $\mathbb{R}^k$. The space $(T^*\mathbb{R}^k)^n$ is the phase space of the n-body problem, and $O(k)$ is its natural symmetry group. The group H in this case is the symplectic group $Sp(\mathbb{R}^{2n})$ (cf. [LMS]).

2. The centralizer of invariant functions

Let $\Phi: M \longrightarrow \mathfrak{g}^*$ be a moment map associated to a Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω). Recall, this means that for any element ξ of the Lie algebra $\mathfrak{g}$ of G we have $d\Phi^\xi = -\iota(\xi_M)\omega$, where $\Phi^\xi = \langle \Phi, \xi \rangle$ is the ξ-component of the moment map and ξ_M is the vector field on M that generates the action of the one parameter subgroup $\{\exp(t\xi), \ t \in \mathbb{R}\}$ of G. We also require

that Φ be equivariant with respect to the given action of G on M and the coadjoint action on $\mathfrak{g}^*$. The main result of this section reads:

THEOREM 1. *The centralizer of the algebra of G-invariant functions in the Poisson algebra of smooth functions on M is the set of smooth functions that are locally constant on the level sets of the moment map.*

Proof. Since the Hamiltonian flow of an invariant function preserves the level sets of the moment map, the Poisson bracket of an invariant function and a function that is locally constant on the level sets of the moment map is zero. This shows that the centralizer of the invariant functions contains the functions that are locally constant on the level sets of the moment map. We would like to show that there is nothing else in the centralizer.

Let h be a function in the centralizer of the invariant functions. Let $\gamma(t)$ be any smooth curve contained in a level set of the moment map Φ. Since any two points in a connected component of a level set of Φ can be connected by a piece-wise smooth curve (see Lemma A.4), we are done if we can prove that the derivative of $h(\gamma(t))$ is zero for all t. This derivative is equal to $\omega(\dot{\gamma}, X_h)$ where X_h is the Hamiltonian vector field of h.

For any vector ξ in the Lie algebra $\mathfrak{g}$ we have $0 = \langle \dot{\gamma}, d\Phi^\xi \rangle = \omega(\dot{\gamma}, \xi_M)$. Hence if $\gamma(t)$ is a smooth curve contained in a level set of the moment map, the tangent vectors $\dot{\gamma}$ lie in the symplectic perpendiculars to the G-orbits.

To finish the argument it suffices to show that the Hamiltonian vector field, X_h, of a function h in the centralizer of the invariant functions is tangent to the G-orbits. Let $\sigma(t)$ be an integral curve of the vector field X_h. Then for any G-invariant function, f, we have $\frac{d}{dt}(f(\sigma(t)) = (X_h f)(\sigma(t)) = 0$; i.e., f is constant along $\sigma(t)$. Since, G being compact, the G-invariant functions separate orbits, the integral curve $\sigma(t)$ is contained in a single G-orbit. Hence the vector field X_h is tangent to G-orbits. This proves Theorem 1. $\square$

The rest of this section contains corollaries of Theorem 1. We assume throughout that Φ is the moment map for a Hamiltonian action of a compact Lie group G on a connected symplectic manifold (M, ω). We do not assume that G is connected or that Φ is proper unless we explicitly say so.

COROLLARY 2.1. *The set of functions that are locally constant on the level sets of Φ is a Poisson algebra.*

Proof. The Jacobi identity implies that for any subset, R, of $C^\infty(M)$ the centralizer, $R^c := \{f \in C^\infty(M) \mid \{f, h\} = 0 \text{ for all } h \in R\}$, is a Poisson algebra. Apply this when R is the set of invariant functions. $\square$

The conjecture of Guillemin and Sternberg in [GS3] is stated differently than the way we quoted it in the introduction. Namely, it is stated in terms of the the double

centralizer ("double commutator") of the set of functions $\{\Phi_1, \ldots, \Phi_{\dim G}\}$, where Φ_i are the coordinates of the moment map with respect to some basis of the vector space $\mathfrak{g}^*$. However,

Lemma 2.2. *The double centralizer of the set of coordinate functions of the moment map is equal to the centralizer of the set of invariant functions.*

The proof uses two lemmas:

Lemma 2.3. *A function Poisson commutes with all the coordinates of the moment map if and only if the function is invariant under the identity component of the group. Consequently, the centralizer of the collective functions is equal to the functions invariant under the identity component of the group.*

Proof. This follows easily from the definition of the moment map. □

Lemma 2.4. *Every smooth function on M which is invariant under the identity component of the compact group G is locally equal to a G-invariant function.*

Proof. This is an easy consequence of the slice theorem for actions of compact groups. □

Proof of Lemma 2.2. If G is connected, the lemma follows immediately from Lemma 2.3. If G is disconnected, the centralizer of the G-invariant functions is the same as the centralizer of the G_0-invariant functions where G_0 is the identity component; this follows from Lemma 2.4. □

Notation 2.5. Let $C^\infty(M)^G$ denote the G-invariant smooth functions on M. Let G_0 be the identity component of G, and let $C^\infty(M)^{G_0}$ denote the G_0-invariant smooth functions; equivalently, these are the smooth functions on M that are locally constant on the G-orbits.

Let $C^\infty(M)^\Phi$ denote the smooth functions on M that are constant on the level sets of the moment map Φ, and $C^\infty(M)^\Phi_{\mathrm{loc}}$ the functions that are *locally* constant on these level sets. Let $\Phi^* C^\infty(\mathfrak{g}^*)$ denote the collective functions, i.e., pullbacks by Φ of smooth functions on $\mathfrak{g}^*$.

In this notation, Theorem 1 says that the centralizer of $C^\infty(M)^G$ is $C^\infty(M)^\Phi_{\mathrm{loc}}$.

COROLLARY 2.6. *The algebra $C^\infty(M)^\Phi_{\mathrm{loc}}$ and the algebra $C^\infty(M)^{G_0}$ are mutual centralizers in the Poisson algebra $C^\infty(M)$.*

Moreover, the centralizers of the algebras $\Phi^* C^\infty(M) \subseteq C^\infty(M)^\Phi \subseteq C^\infty(M)^\Phi_{\mathrm{loc}}$ are all equal to $C^\infty(M)^{G_0}$, and the centralizers of the algebras $C^\infty(M)^G \subseteq C^\infty(M)^{G_0}$ are both equal to $C^\infty(M)^\Phi_{\mathrm{loc}}$.

Proof. Theorem 1 applied to the group G_0 implies that the centralizer of $C^\infty(M)^{G_0}$ is equal to $C^\infty(M)^\Phi_{\text{loc}}$. Conversely, since the algebra $C^\infty(M)^\Phi_{\text{loc}}$ contains the collective functions, its centralizer is contained in the centralizer of the collective functions, which is equal to the G_0-invariant functions by Lemma 2.3.   □

COROLLARY 2.7.   *Let Φ be a moment map associated to a Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω). If G is connected, the following properties of Φ are equivalent:*

1. *The algebra of collective functions and the algebra of invariant functions are mutual centralizers in the Poisson algebra $C^\infty(M)$.*
2. *The double centralizer of the set of coordinate functions of the moment map is the algebra of collective functions.*
3. *Every smooth function on M that is locally constant on the level sets of the moment map is collective.*

Proof.   Since the centralizer of the collective functions is the invariant functions (Lemma 2.3), condition 1 is equivalent to

1′. *The centralizer of the algebra of invariant functions is the algebra of collective functions.*

The equivalence of conditions 1′ and 2 is immediate from Lemma 2.2; the equivalence of conditions 1′ and 3 is immediate from Theorem 1.   □

Definition 2.8.   A smooth map $\Phi: M \longrightarrow N$ between two smooth manifolds has the *division property* if any smooth function on M that is locally constant on the level sets of Φ is the pullback via Φ of a smooth function on N.

The hard part of Corollary 2.7 can be rephrased as follows:

COROLLARY 2.9.   *Let $\Phi: M \longrightarrow \mathfrak{g}^*$ be a moment map associated to an action of a compact connected Lie group G on a symplectic manifold (M, ω). The algebra of collective functions and the algebra of invariant functions are mutual centralizers in the Poisson algebra $C^\infty(M)$ if and only if the moment map Φ has the division property.*

One obstruction for a map to have the division property is topological—the connectedness of the level sets of the map. Another obstruction has to do with analytic properties of the map. The following examples illustrate these ideas: (1) The map $x \mapsto x^3$ on $\mathbb{R}$ does not have the division property because of the singularity at $x = 0$; the function $f(x) = x$ is not a pullback of a smooth function. (2) The map $e^{i\theta} \mapsto e^{2i\theta}$

on the unit circle does not have the division property because the level sets are not connected; the function $f(e^{i\theta}) = \cos(\theta)$ is not a pullback. (3) The map $(x, y) \mapsto x^2 + y^2$ from $\mathbb{R}^2$ to $\mathbb{R}$ has the division property; every rotationally invariant smooth function is a smooth function of $x^2 + y^2$. This is a special case of a theorem of G. Schwarz [Sch].

Two global properties of a moment map will be relevant to us: properness of the map and connectedness of its level sets. The first of these often implies the second:

PROPOSITION 2.10. *Let $\Phi: M \longrightarrow \mathfrak{g}^*$ be a moment map associated to an action of a compact Lie group, G, on a connected symplectic manifold, (M, ω). If this map is proper, its level sets are connected.*

Proof. For torus actions this proposition was proved by Atiyah [A]. The general case was proved by Kirwan [Ki1, Ki2]. $\square$

This result has recently been generalized:

Proposition 2.11. *Let $\Phi: M \longrightarrow \mathfrak{g}^*$ be a moment map associated to an action of a compact Lie group G on a connected symplectic orbifold (M, ω). Let $\mathfrak{t}_+^*$ be a Weyl chamber, identified with a subset of $\mathfrak{g}^*$ via a choice of an Ad-invariant metric. Suppose that there exists a G-invariant open subset N of $\mathfrak{g}^*$ containing the image $\Phi(M)$, such that the intersection $N \cap \mathfrak{t}_+^*$ is convex and the map Φ is proper as a map from M to N. Then the level sets of Φ are connected.*

Proof. See [LMTW]. $\square$

We proceed with consequences of Theorem 1.

Corollary 2.12. *Let $\Phi: M \longrightarrow \mathfrak{g}^*$ be a moment map associated to a Hamiltonian action of a compact Lie group, G, on a connected symplectic manifold, (M, ω). If the group G is connected and the moment map, Φ, is proper, the algebras $C^\infty(M)^\Phi$ and $C^\infty(M)^G$ are mutual centralizers in the Poisson algebra $C^\infty(M)$. (See Notation 2.5.)*

Proof. If G is connected, $C^\infty(M)^{G_0} = C^\infty(M)^G$. If Φ is proper, its level sets are connected, so $C^\infty(M)_{\text{loc}}^\Phi = C^\infty(M)^\Phi$. The rest is immediate from Corollary 2.6. $\square$

Corollary 2.13. *If the moment map, $\Phi: M \longrightarrow \mathfrak{g}^*$, is proper, the centralizer of the invariant functions consists of those smooth functions on M that are pullbacks by Φ of continuous functions on $\mathfrak{g}^*$:*

$$(C^\infty(M)^G)^c = C^\infty(M) \cap \Phi^* C^0(\mathfrak{g}^*). \tag{2.1}$$

Let $N \subset \mathfrak{g}^*$ be an open set containing the moment image, $\Phi(M)$. If the moment map, Φ, is proper as a map from M to N and has connected level sets, the centralizer of the invariant functions consists of those smooth functions on M that are pullbacks by Φ of continuous functions on N:

$$(C^\infty(M)^G)^c = C^\infty(M) \cap \Phi^*C^0(N).$$

Proof of Corollary 2.13.   The first part follows from the second, because the level sets of a proper moment map are connected (Proposition 2.10). A function in the centralizer of the invariants is the pullback of a function on N (Theorem 1). The function on N is continuous because the moment map is proper.   □

Remark 2.14.   If the moment map is not proper, (2.1) may fail to hold. See Examples 3.9 and 3.10.

3. Division property can be detected formally

Let $\Phi: M \longrightarrow \mathfrak{g}^*$ be a proper moment map associated to an action of a compact Lie group G on a symplectic manifold (M, ω). In this section we show that the difference between the algebra of functions that are constant on the level sets of the moment map and the algebra of collective functions can already be detected on the level of power series of these functions. This result is stated in Theorem 2. The main idea of the proof is to apply a theorem of Bierstone and Milman to the Marle-Guillemin-Sternberg local normal form of the moment map.

We first recall a definition of Bierstone and Milman. Recall that a function is *flat* at a point if its Taylor series at that point vanishes.

Definition 3.1.   Let $\Phi: M \longrightarrow N$ be a smooth map between two smooth manifolds. A smooth function f on M is a *formal pullback with respect to* Φ if for every point y in the image $\Phi(M)$ there exists a function, φ, on N such that $f - \Phi^*\varphi$ is flat at all the points of $\Phi^{-1}(y)$.

Remark 3.2.   Every formal pullback with respect to Φ is constant on the level sets of Φ; if $f - \Phi^*\varphi$ is flat, $f(x) = \varphi(y)$ for all $x \in \Phi^{-1}(y)$.

Remark 3.3.   In the notation of Bierstone and Milman, the set of formal pullbacks with respect to Φ is $(\Phi^*C^\infty(N))\hat{)}$; they have no term to describe the elements of this algebra. In a more recent paper [BMP], with W. Pawlucki, they use the term "formal composite with Φ" to describe a formal pullback.

Remark 3.4.   Recall, the Taylor series at a point p of a smooth function $f: M \longrightarrow \mathbb{R}$ is an element of the algebra $\prod_i S^i(T_p^*M)$ of (formal) power series at p. Conversely,

by Borel's theorem, every power series at p is a Taylor series of some smooth function on M. Therefore, one can think of a power series at a point p as an equivalence class of functions: two functions are equivalent if and only if their difference is flat at p. Since the pullback of functions induces a well defined pullback of Taylor series, being a formal pullback is a condition on Taylor series: a smooth function $f: M \longrightarrow \mathbb{R}$ is a formal pullback with respect to a smooth map $\Phi: M \longrightarrow N$ if and only if for every $y \in N$ there exists a power series φ on N, centered at y, such that for all x in the level set $\Phi^{-1}(y)$, the power series of f at x is the pullback of the power series φ.

We will now state the main result of this section. Recall, a continuous map $\psi: A \longrightarrow B$ is *semi-proper* if for every compact set $L \subset B$ there is a compact set $K \subset A$ such that $\psi(K) = L \cap \psi(A)$.

THEOREM 2.   *Let $\Phi: M \longrightarrow \mathfrak{g}^*$ be a moment map associated to a Hamiltonian action of a compact Lie group G on a connected symplectic manifold (M, ω). If this map, Φ, is proper, every formal pullback with respect to Φ is a collective function, i.e., is in $\Phi^* C^\infty(\mathfrak{g}^*)$.*

Let N be an open subset of $\mathfrak{g}^*$ containing the moment image, $\Phi(M)$. If the moment map, Φ, is semi-proper as a map from M to N and has connected level sets, every formal pullback with respect to Φ is a pullback of a smooth function on N, i.e., is in $\Phi^* C^\infty(N)$.

COROLLARY 3.5.   *A proper moment map Φ has the division property (see Definition 2.8) if and only if every smooth function on M that is locally constant on the level sets of Φ is a formal pullback with respect to Φ.*

The proof of Theorem 2 relies upon the following theorem of Bierstone and Milman. Recall that a *semi-analytic* subset of an analytic manifold is a subset that is locally defined by inequalities involving analytic functions.

THEOREM 3 [Bierstone-Milman [BM2, Theorem 0.1]] .   *Let M and N be real analytic manifolds. Let $\Phi: M \longrightarrow N$ be a real analytic mapping that is semi-proper and whose image, $\Phi(M)$, is semi-analytic. Then a function f is a formal pullback with respect to Φ if and only if it is the pullback by Φ of a smooth function on N.*

Remark 3.6.   Theorem 0.1 in [BM2] requires the image $\Phi(M)$ to be Nash subanalytic. Bierstone and Milman point out in [BM1] that every semi-analytic set is Nash subanalytic.

A priori, our manifolds and maps are only smooth and not real analytic, so we cannot apply Theorem 3 directly. We will use the following variant:

PROPOSITION 3.7. *Let M and N be smooth manifolds, and let $\Phi: M \longrightarrow N$ be a smooth map that satisfies the following conditions.*

1. *The image $\Phi(M)$ is closed.*
2. *For every point x in M there exist neighborhoods U_x of x in M and W_x of $\Phi(x)$ in N such that:*

   (a) *$\Phi(U_x) = \Phi(M) \cap W_x$;*

   (b) *the restriction $\Phi|_{U_x}: U_x \longrightarrow W_x$ is semi-proper;*

   (c) *there exist real analytic structures on U_x and on W_x compatible with their smooth structures such that the restriction $\Phi|_{U_x}: U_x \longrightarrow W_x$ is a real analytic map whose image is a semi-analytic subset of W_x.*

Then the set of pullbacks by the map Φ coincides with the set of formal pullbacks with respect to Φ.

Proof. Clearly, every pullback is a formal pullback. Conversely, let $f \in C^\infty(M)$ be a formal pullback with respect to Φ. Let x be a point in M, and let U_x and W_x be as in Condition 2 above. Since f is a formal pullback with respect to Φ, its restriction $f|_{U_x}$ is a formal pullback with respect to the map $\Phi|_{U_x}: U_x \longrightarrow W_x$. Theorem 3 applies to this map because of Conditions 2(b) and 2(c). Hence there exists a smooth function φ_x on W_x such that $f = \varphi_x \circ \Phi$ on U_x. This equality holds on all of $\Phi^{-1}(\Phi(U_x))$ because f, being a formal pullback with respect to Φ, is constant on the level sets of Φ (see Remark 3.2). Condition 2(a) implies that $\Phi^{-1}(\Phi(U_x)) = \Phi^{-1}(W_x)$, so $f = \varphi_x \circ \Phi$ on all of $\Phi^{-1}(W_x)$. The open sets W_x together with the complement of the image $\Phi(M)$ form an open cover of the target manifold, N. Using a partition of unity subordinate to this cover we piece together the functions φ_x to form a function φ on N such that $f = \varphi \circ \Phi$. $\square$

Thus to prove Theorem 2 it is enough to verify that every proper moment map on a connected symplectic manifold satisfies the hypotheses of Proposition 3.7.

PROPOSITION 3.8. *Let $\Phi: M \longrightarrow \mathfrak{g}^*$ be a moment map associated to an action of a compact connected Lie group G on a connected symplectic manifold (M, ω). Assume that the map Φ is semi-proper as a map into an open subset N of $\mathfrak{g}^*$ and that its level sets are connected. Let x be a point in M, and let G_α be the stabilizer of its image, $\alpha = \Phi(x)$, under the coadjoint action. Then there exist a neighborhood U_x of the orbit $G_\alpha \cdot x$ in M and a neighborhood W_x of the point $\Phi(x)$ in $\mathfrak{g}^*$ with the following properties.*

1. *$\Phi(U_x) = \Phi(M) \cap W_x$.*
2. *The restriction $\Phi|_{U_x}: U_x \longrightarrow W_x$ is semi-proper.*

3. *There exist real analytic structures on U_x and on W_x, compatible with their smooth structures, such that*:

   (a) *the restriction $\Phi|_{U_x}: U_x \longrightarrow W_x$ is a real analytic map*;

   (b) *the image $\Phi(U_x)$ is a semi-analytic subset of W_x.*

Moreover, the neighborhoods U_x and W_x can be chosen to be arbitrarily small, i.e., can be chosen to be contained in any given neighborhoods U' of $G_\alpha \cdot x$ and W' of $\Phi(x)$.

Note that only property 1 is global. It is only to prove this property that we assume that the moment map is semi-proper and its level sets are connected.

Proof of Proposition 3.8.   Let us first prove properties 1–3 when the orbit $G \cdot x$ is isotropic, equivalently, when α is fixed under the coadjoint action of G. Since α is fixed, the translation $\Phi - \alpha$ of the moment map by $-\alpha$ is still a moment map. So, without loss of generality, we can assume that $\alpha = 0$.

In the appendix (Theorem 5) we describe a local model for a neighborhood of an isotropic orbit $G \cdot x$,

$$Y = G \times_{G_x} (\mathfrak{g}_x^0 \oplus V),$$

where G_x is the stabilizer of x, $\mathfrak{g}_x$ is its Lie algebra, $\mathfrak{g}_x^0$ is the annihilator of $\mathfrak{g}_x$ in $\mathfrak{g}^*$, and the vector space V is the symplectic slice at x. The action of G on Y is Hamiltonian with a moment map $\Phi_Y: Y \longrightarrow \mathfrak{g}^*$ given by the formula

$$\Phi_Y([g, \eta, v]) = \mathrm{Ad}^\dagger(g)(\eta + i(\Phi_V(v))),$$

where Φ_V is a quadratic map from V to $\mathfrak{g}_x^*$ and i is a G_x-equivariant embedding of $\mathfrak{g}_x^*$ in $\mathfrak{g}^*$. Moreover, by Theorem 5 there exists a neighborhood U_x of $G \cdot x$ in M and an equivariant embedding, $\iota: U_x \longrightarrow Y$, of U_x onto a neighborhood of the zero section in the model Y, such that $\Phi = \Phi_Y \circ \iota$.

By Lemma A.5, the image under the moment map of a small neighborhood of an orbit $G \cdot x$ does not change as x varies along a connected component of the level set $\Phi^{-1}(\Phi(x))$. By Remark A.6, this image is the intersection of the cone $\Phi_Y(Y)$ with a neighborhood of the origin in $\mathfrak{g}^*$. Lemma A.5 together with the facts that Φ is semi-proper and its level sets are connected implies that we can choose a neighborhood W_x of the origin in $\mathfrak{g}^*$ and shrink the neighborhood U_x of $G \cdot x$ so that $\Phi(U_x) = \Phi(M) \cap W_x = \Phi_Y(Y) \cap W_x$, i.e., so that property 1 holds.

The map Φ_Y is analytic with respect to the natural real analytic structures of the model Y and of the vector space $\mathfrak{g}^*$. If we endow U_x with the real analytic structure induced by its embedding, ι, into Y, property 3(a) holds.

Consider the action of $\mathbb{R}_+$ on Y given by

$$\lambda \cdot [g, \eta, v] = [g, \lambda\eta, \sqrt{\lambda}v].$$

The map $\Phi_Y\colon Y \longrightarrow \mathfrak{g}^*$ is homogeneous of degree one with respect to this action of $\mathbb{R}_+$. After possibly shrinking U_x and W_x further, we can assume that the open set $\iota(U_x) \subseteq Y$ is preserved under multiplication by any $\lambda < 1$; for such λ we define $\lambda\colon U_x \longrightarrow U_x$ by $\iota(\lambda \cdot m) = \lambda \cdot \iota(m)$. Let K be a compact subset of the open set W_x. Then there exists a positive number $\lambda < 1$ such that K is contained in λW_x. By homogeneity, $K \cap \Phi(U_x)$ is contained in $\Phi(\lambda \cdot U_x)$. Then $L := \mathrm{closure}(\lambda \cdot U_x) \cap \Phi^{-1}(K)$ is a compact subset of U_x whose image is $K \cap \Phi(U_x)$. This proves property 2.

Since the map Φ_V is algebraic, its image, $\Phi_V(V)$, is a semi-algebraic subset of $\mathfrak{g}_x^*$, by the Tarski-Seidenberg theorem (see, for example, [BR, Theorem 2.3.4]). Furthermore, since $\mathrm{Ad}^\dagger(G) \subseteq \mathrm{GL}(\mathfrak{g}^*)$ is algebraic, the set $\Phi_Y(Y) = \mathrm{Ad}^\dagger(G)(\mathfrak{g}_x^0 \times \Phi_V(V))$ is a semi-algebraic subset of $\mathfrak{g}^*$. Restricting to the open subset W_x, we see that $\Phi(U_x) = \Phi_Y(Y) \cap W_x$ is a semi-analytic subset of W_x. This proves property 3(b).

We now remove the assumption that the orbit is isotropic. Let $\alpha = \Phi(x)$, and let G_α denote its stabilizer under the coadjoint action. Let S be a slice at α for the action of G and $R = \Phi^{-1}(S)$ the corresponding symplectic cross-section (cf. Theorem 5). By Corollary A.2, up to a composition with diffeomorphisms, the moment map Φ is a fiber bundle map

$$G \times_{G_\alpha} R \longrightarrow G \times_{G_\alpha} S, \qquad [g, r] \mapsto [g, \Phi_R(r)],$$

and $\Phi_R = \Phi|_R$ is the G_α moment map.

The subgroup G_α acts on G by right multiplication. Since $G \longrightarrow G/G_\alpha$ is a locally trivial fibration, there exists a section on a neighborhood $\mathcal{V}$ of the identity coset in G/G_α. This section simultaneously trivializes the bundles $G \times_{G_\alpha} R \longrightarrow G/G_\alpha$ and $G \times_{G_\alpha} S \longrightarrow G/G_\alpha$ over the set $\mathcal{V}$. With respect to these trivializations, the moment map Φ is the map

$$\mathrm{id} \times \Phi_R\colon \mathcal{V} \times R \longrightarrow \mathcal{V} \times S \qquad (n, r) \mapsto (n, \Phi_R(r)).$$

Since $G_\alpha \cdot x$ is isotropic in R (Remark A.3) and Φ_R is a G_α moment map, properties 1–3 are satisfied by the map $\Phi_R\colon U_x \longrightarrow W_x$ where U_x is a neighborhood of $G_\alpha \cdot x$ in R and W_x is a neighborhood of α in S. Properties 1–3 for Φ_R immediately imply properties 1–3 for $\mathrm{id} \times \Phi_R\colon \mathcal{V} \times U_x \longrightarrow \mathcal{V} \times W_x$. Hence the moment map Φ has properties 1–3. $\quad\square$

To prove our main result, Theorem 2, it suffices to verify that the hypotheses of Proposition 3.7 are satisfied by the moment map:

Proof of Theorem 2. The first part of the theorem follows from the second by setting $N = \mathfrak{g}^*$.

Let N be an open subset of $\mathfrak{g}^*$ containing the moment image, $\Phi(M)$, with the property the moment map $\Phi\colon M \longrightarrow N$ is semi-proper. Condition 1 in Proposition 3.7, that the image $\Phi(M)$ is a closed subset of N, is satisfied because the image of any semi-proper map is closed. Conditions 2(a)–2(c) hold by Proposition 3.8. Theorem 2 then follows from Proposition 3.7. $\quad\square$

We conclude this section with two examples which show that the properness condition in Theorem 2 is necessary.

Example 3.9.   Let M be the cotangent bundle of the two dimensional torus, T^2, minus the zero section. This manifold is the product $M = T^2 \times (\mathbb{R}^2 \smallsetminus \{0\})$. The moment map Φ is the projection onto the second factor; it is not proper. The function $f(x, y) = y/\sqrt{x^2 + y^2}$ does not extend to a smooth function on $\mathbb{R}^2$ but it does pull back to a smooth function on M which is a formal pullback.

Example 3.10.   In this example, a formal pullback f does not even descend to a continuous function on the image of the moment map.

We construct a Hamiltonian T^2-space by gluing two spaces. The first space, M_1, is the product $T^2 \times U$ where U is the subset of $\mathbb{R}^2$ obtained by removing the origin and the positive x-axis. We can view M_1 as an open subset of the cotangent bundle of T^2 and take the induced symplectic form. The moment map is the obvious projection onto U. Its image is $\mathbb{R}^2$ minus the origin and the positive x-axis.

The second space, M_2, consists of the points of $\mathbb{C}^2$ whose first coordinate is nonzero. This space inherits a symplectic form and an action of T^2 from $\mathbb{C}^2$. The moment map sends $(z, w) \mapsto (|z|^2, |w|^2)$. The image of the moment map is the set $\{(x, y) \in \mathbb{R}^2 : x > 0, \ y \geq 0\}$.

We glue the two spaces along the pre-images of the open positive quadrant by sending (z, w) to $(\frac{z}{|z|}, \frac{w}{|w|}, |z|^2, |w|^2)$. This gluing map is an equivariant symplecto-morphism.

We obtain a space M with a symplectic form, a T^2-action and a moment map. The image of the moment map is $\mathbb{R}^2$ minus the origin. The branch of $\arctan(y/x)$ which is discontinuous along the positive x-axis pulls back to a smooth function on M.

4. Division property of a toral moment map

Consider a Hamiltonian action of a torus, T, on a symplectic manifold, (M, ω). Recall that the conjecture of Guillemin and Sternberg, asserting that the algebra of invariant functions and the algebra of collective functions are mutual centralizers in the Poisson algebra $C^\infty(M)$, holds if and only if the moment map, $\Phi \colon M \longrightarrow \mathfrak{t}^*$, has the division property (Definition 2.8 and Corollary 2.9).

Theorem 2, proved in Section 3, provides a criterion for determining that a moment map has the division property; see Corollary 3.5. In this section we use this criterion to prove that moment maps arising from torus actions have the division property.

The results presented here were believed to be known for some time. By (the easy part of) Corollary 2.7, our Proposition 4.1 is equivalent to Proposition 4.1 in [GS3], which asserts that for a symplectic linear action of a torus on a symplectic vector space, the double centralizer of the set of coordinate functions of the moment map is the algebra of collective functions. Unfortunately, the arguments presented in [GS3]

only show that this double centralizer consists of functions that are constant on the level sets of the moment map. The missing arguments are not trivial (we don't see a way of avoiding the theorem of Bierstone and Milman); we provide them in our proof of Proposition 4.1.

In [L2], Lerman used Proposition 4.1 of [GS3] and the Marle-Guillemin-Sternberg normal form to deduce a set of sufficient conditions for the Guillemin-Sternberg conjecture to hold, equivalently, for the moment map to have the division property. We recall (slight generalizations of) these conditions in Corollaries 4.2 and 4.4. Our proof of Proposition 4.1 thus closes the gap in the proof of these corollaries.

PROPOSITION 4.1.   *Let T be a torus acting linearly and symplectically on a symplectic vector space V, and let $\Phi\colon V \longrightarrow \mathfrak{t}^*$ be a corresponding moment map. Then for any $l \geq 0$, the map*

$$I \times \Phi \colon \mathbb{R}^l \times V \longrightarrow \mathbb{R}^l \times \mathfrak{t}^*, \quad (u, v) \mapsto (u, \Phi(v)) \tag{4.1}$$

has the division property (see Definition 2.8).

Proof.   We may identify V with $\mathbb{C}^n$ (where $n = \frac{1}{2}\dim V$) in such a way that T acts as a subtorus of the standard maximal torus $\mathbb{T}^n$ of $U(n)$. The T-moment map, Φ, is the composition of the $\mathbb{T}^n$-moment map,

$$F\colon (z_1, \ldots, z_n) \mapsto (|z_1|^2, \ldots, |z_n|^2),$$

with a linear projection $\pi\colon \mathrm{Lie}(\mathbb{T}^n)^* = \mathbb{R}^n \longrightarrow \mathfrak{t}^*$. The diagram

$$
\begin{array}{ccc}
\mathbb{R}^l \times \mathbb{C}^n & \xrightarrow{\;I \times F\;} & \mathbb{R}^l \times \mathbb{R}^n \\
& {\scriptstyle I \times \Phi}\searrow & \downarrow{\scriptstyle I \times \pi} \\
& & \mathbb{R}^l \times \mathfrak{t}^*
\end{array}
$$

commutes. Consider the action of $\mathbb{T}^n$ on $\mathbb{R}^l \times \mathbb{C}^n$ which is trivial on $\mathbb{R}^l$ and standard on $\mathbb{C}^n$. It is easy to see that the $\mathbb{T}^n$-invariant polynomials on $\mathbb{R}^l \times \mathbb{C}^n$ are the polynomials in $u_1, \ldots, u_l, |z_1|^2, \ldots, |z_n|^2$ where $u_1, \ldots, u_l$ are coordinates on $\mathbb{R}^l$. That is, the $\mathbb{T}^n$-invariant polynomials on $\mathbb{R}^l \times \mathbb{C}^n$ are the pullbacks by $I \times F$ of polynomials on $\mathbb{R}^l \times \mathbb{R}^n$. It follows by a theorem of G. Schwarz [Sch] that the $\mathbb{T}^n$-invariant smooth functions on $\mathbb{R}^l \times \mathbb{C}^n$ are the pullbacks by $I \times F$ of smooth functions on $\mathbb{R}^l \times \mathbb{R}^n$.

In particular, if $f \in C^\infty(\mathbb{R}^l \times \mathbb{C}^n)$ is constant on the level sets of $I \times \Phi$, it is $\mathbb{T}^n$-invariant (because it is also constant on the level sets of $I \times F$), so there exists a smooth function, $g \in C^\infty(\mathbb{R}^l \times \mathbb{R}^n)$, with $g \circ (I \times F) = f$. Since the image of $I \times F$ is the closed positive orthant $\mathbb{R}^l \times \mathbb{R}^n_+$, the function g is constant on the intersections of the level sets of $I \times \pi$ with this orthant. It is not clear *a priori* that g can be chosen to be constant on the level sets of $I \times \pi$.

We would like to show that f is a formal pullback with respect to Φ. It is enough to find, for every value α in the image of $I \times \Phi$, a Taylor series on $\mathbb{R}^l \times \mathfrak{t}^*$, centered

at α, whose pullback is equal to the Taylor series of f at all the points in the level set $(I \times \Phi)^{-1}(\alpha)$.

Since f can be written as a composition $f = g \circ (I \times F)$ for $g \in C^{\infty}(\mathbb{R}^l \times \mathbb{R}^n)$, it is enough to find a formal power series on $\mathbb{R}^l \times \mathfrak{t}^*$, centered at α, whose pullback to $\mathbb{R}^l \times \mathbb{R}^n$ is equal to the Taylor series of g at every point in the level set $(\mathbb{R}^l \times (\mathbb{R}_+)^n) \cap (I \times \pi)^{-1}(\alpha)$.

We can choose new coordinates on $\mathbb{R}^l \times \mathbb{R}^n$ and on $\mathbb{R}^l \times \mathfrak{t}^*$ such that the map $I \times \pi$ becomes the projection $(x_1, \ldots, x_k, y_1, \ldots, y_m) \mapsto (x_1, \ldots, x_k)$, where $k = \dim T$ and $k + m = l + n$. It is sufficient to show that for every point $p \in (I \times \pi)^{-1}(\alpha) \cap (\mathbb{R}^l \times (\mathbb{R}_+)^n)$, the mixed partial derivatives

$$\frac{\partial^{a+b} g}{\partial y^a \partial x^b}(p)$$

are

 (1) independent of p and
 (2) equal to zero whenever the multi-index a is not zero.

Condition 2 implies condition 1 because any two such points p_1 and p_2 can be connected by a smooth path which lies entirely inside the closed positive orthant and on which the y_i-coordinates are constant.

Since in the interior of the positive orthant the function g is constant on level sets of the projection, $\frac{\partial^{a+b} g}{\partial y^a \partial x^b}(q) = 0$ for every q in the interior of the positive orthant provided $a \neq 0$. By continuity, this also holds for all q in the closed orthant. This proves condition 2.

We have shown that if a function f is (locally) constant on the level sets of Φ then it is a formal pullback with respect to Φ. By Corollary 3.5 of Theorem 2, the map Φ has the division property. $\square$

COROLLARY 4.2.   *A proper moment map for a Hamiltonian action of a torus on a connected symplectic manifold has the division property.*

Proof.   This was proved by Lerman in [L2]. The essential ingredients are the facts that the level sets of Φ are connected (cf. Proposition 2.11) and that, by the Marle-Guillemin-Sternberg local normal form, the moment map locally looks like the map (4.1). $\square$

Notation 4.3.   Denote by $\mathfrak{g}^*_{reg}$ the elements of $\mathfrak{g}^*$ whose stabilizers under the coadjoint action of G are tori.

COROLLARY 4.4.   *Let G be a compact Lie group acting on a symplectic manifold M with a proper moment map $\Phi \colon M \longrightarrow \mathfrak{g}^*$. Suppose that the image $\Phi(M)$ is contained in $\mathfrak{g}^*_{reg}$. Then Φ has the division property.*

Proof. This too was proved in [L2]. Like Corollary 4.2, it follows from the fact that the level sets of a proper moment map are connected and from the fact that, by the local normal form, the moment map on a neighborhood of a point x locally looks like the map (4.1), with the torus T being the stabilizer of $\Phi(x)$.   □

In Section 5 we will use the following, somewhat stronger, statement.

COROLLARY 4.5.   *Let* $\Phi\colon M \longrightarrow \mathfrak{g}^*$ *be a proper moment map associated to a Hamiltonian action of a compact connected Lie group G on a connected symplectic manifold M. Then the restriction* $\Phi|_{\Phi^{-1}(\mathfrak{g}^*_{\mathrm{reg}})}\colon \Phi^{-1}(\mathfrak{g}^*_{\mathrm{reg}}) \longrightarrow \mathfrak{g}^*_{\mathrm{reg}}$ *has the division property.*

Proof. The proof, *mutatis mutandis*, is the same as the proof of Corollary 4.4.
   □

Note that the hypothesis that the moment map is proper can be replaced by the hypotheses that it is semi-proper as a map into some open subset of $\mathfrak{g}^*$ and that its level sets are connected.

Remark 4.6.   For Lie groups of rank 1 or 2 we can prove Corollary 4.5 without appeal to the theorem of Bierstone and Milman. The key point is:

LEMMA 4.7.   *If V is a symplectic vector space and $\Phi\colon V \longrightarrow \mathbb{R}$ is a moment map for a linear circle action, Φ has the division property.*

Proof. Assume, without loss of generality, that $\Phi(0) = 0$. We can identify V with $\mathbb{C}^n$ in such a way that the circle action becomes $\lambda \cdot (z_1, \ldots, z_n) = (\lambda^{m_1} z_1, \ldots, \lambda^{m_n} z_n)$ and the moment map becomes $\Phi(z_1, \ldots, z_n) = \sum m_i |z_i|^2$ for some integers $m_1, \ldots, m_n$ which are not all zero. Note that the level sets of Φ are connected. Let f be a smooth function on V that is constant on these level sets. We need to find a smooth function, φ, on $\mathbb{R}$ such that $f = \varphi \circ \Phi$.

If image$(\Phi) = \mathbb{R}$, the exponents m_i cannot all have the same sign. There exists then a unique function φ on $\mathbb{R}$ such that $f = \varphi \circ \Phi$. Since the image of the regular points of Φ is then all of $\mathbb{R}$, the function φ is smooth.

Otherwise, image(Φ) is a ray and all the m_i's have the same sign. The moment map Φ is a composition of the map $J(z_1, \ldots, z_n) = (|z_1|^2, \ldots, |z_n|^2)$ from V to $\mathbb{R}^n$ and of the linear projection $\pi(x_1, \ldots, x_n) = \sum m_i x_i$ from $\mathbb{R}^n$ to $\mathbb{R}$. By a theorem of G. Schwarz, there exists a smooth function $\tilde{\varphi}$ on $\mathbb{R}^n$ such that $f = \tilde{\varphi} \circ J$. The diagonal line $\{(t, \ldots, t)\}$ in $\mathbb{R}^n$ is transverse to the kernel of the projection π; this follows from the fact that all the m_i's have the same sign. Therefore there exists a linear map $s\colon \mathbb{R} \longrightarrow \mathbb{R}^n$ whose image is the diagonal line and such that $\pi \circ s$ is the identity. The function $\varphi = \tilde{\varphi} \circ s$ is smooth on $\mathbb{R}$ and satisfies $f = \varphi \circ \pi$; the reason is that if x is in $\Phi(V)$ then $s(x)$ is in $J(V)$.   □

A similar proof works for actions of tori of rank 2 but not for higher dimensional tori. This is because if the image of the positive orthant via a linear projection from $\mathbb{R}^n$ to $\mathbb{R}^k$ has more than k extremal rays, this image is identified via π with the intersection of an affine subspace with the positive orthant in $\mathbb{R}^n$.

5. The centralizer of $SU(2)$-invariant functions

Suppose that the group $SU(2)$ acts on a connected symplectic manifold (M, ω) in a Hamiltonian fashion and that the moment map, $\Phi: M \longrightarrow su(2)^*$, is proper. In this section we completely characterize the centralizer of the $SU(2)$-invariant functions in $C^\infty(M)$.

THEOREM 4.   1. *If the zero level set $\Phi^{-1}(0)$ is empty, or if there is a point in the zero level set which is not fixed by $SU(2)$, the moment map has the division property; equivalently, the centralizer of the invariant functions consists of the algebra of collective functions.*
2. *If the zero level set $\Phi^{-1}(0)$ is nonempty, and if all the points in this level set are fixed by $SU(2)$, the centralizer of the invariant functions is strictly larger than the algebra of $SU(2)$-collective functions. The action of $SU(2)$ then extends to an action of $U(2)$ with the same orbits and hence the same algebra of invariant functions:*

$$C^\infty(M)^{SU(2)} = C^\infty(M)^{U(2)}.$$

The centralizer of the algebra of invariant functions consists of the $U(2)$-collective functions:

$$(C^\infty(M)^{SU(2)})^c = \tilde{\Phi}^* C^\infty(u(2)^*),$$

where $\tilde{\Phi}$ is the $U(2)$-moment map.

   Proof.   Let f be in the centralizer of the invariant functions. By Corollary 2.13, f pushes forward to a continuous function, $\Phi_* f$, on the image, $\Phi(M)$. Recall that the function f is collective if and only if there exists a smooth function φ on $su(2)^*$ with $f = \varphi \circ \Phi$, i.e., if and only if $\Phi_* f$ is smooth on the interior of $\Phi(M)$ and extends to a smooth function on all of $su(2)^*$.

   Since the coadjoint action of $SU(2)$ factors through the standard representation of $SO(3)$ on $su(2)^* \cong \mathbb{R}^3$, the origin is the only point which has a nonabelian stabilizer. By Corollary 4.5 there exists a smooth function φ on $\mathbb{R}^3 \setminus \{0\}$ such that $\varphi \circ \Phi = f$ on $\Phi^{-1}(\mathbb{R}^3 \setminus \{0\})$. The function φ may or may not extend to a function on $\mathbb{R}^3$ which is smooth at the origin.

   If the zero level set $\Phi^{-1}(0)$ is empty then, since the moment map is proper, the image $\Phi(M)$ avoids a whole neighborhood of 0. The push-forward, $\Phi_* f$, then extends to a smooth function on $\mathbb{R}^3$.

If the zero level set is nonempty, the image $\Phi(M)$ contains a neighborhood of 0 (unless M is a single point).

Assume that the zero level set contains a point p whose stabilizer is not all of $SU(2)$. This stabilizer can be either zero or one dimensional. If it is zero dimensional, the moment map Φ is a submersion from a neighborhood of p to a neighborhood of 0, hence the push-forward $\Phi_* f$ is smooth at 0.

If the stabilizer of p is one dimensional, it is either a maximal torus in $SU(2)$ or the normalizer of a maximal torus. Denote the stabilizer by H. The orbit of p can be identified with the quotient $SU(2)/H$, which is either a sphere, S^2, or a real projective plane, $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$. It is no loss of generality to assume that the orbit is S^2. By Theorem 5, a neighborhood of the orbit is equivariantly diffeomorphic to a neighborhood of the zero section in the bundle $SU(2) \times_H (\mathfrak{h}^0 \oplus V)$ over $SU(2)/H$. Here V is a symplectic vector space and $\mathfrak{h}^0$ is the annihilator of $\mathfrak{h}$ in $su(2)^*$. The cotangent bundle of the orbit is the sub-bundle $SU(2) \times_H \mathfrak{h}^0$. The moment map on $SU(2) \times_H \mathfrak{h}^0$ restricted to the fiber over $eH \in SU(2)/H$ is the inclusion of $\mathfrak{h}^0$ in $su(2)^*$. The image of $SU(2) \times_H \mathfrak{h}^0$ under the moment map is $SU(2) \cdot \mathfrak{h}^0 = su(2)^*$ (the annihilator $\mathfrak{h}^0$ is a plane through the origin in $su(2)^* \cong \mathbb{R}^3$ and $SU(2)$ acts by rotations). We are thus reduced to proving the following:

PROPOSITION 5.1. *Consider the bundle* $TS^2 = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |x|^2 = 1, x \cdot y = 0\}$ *and consider the map from it to* $\mathbb{R}^3$ *given by* $(x, y) \mapsto y$. *Suppose that* φ *is a continuous function on* $\mathbb{R}^3$ *that is smooth on* $\mathbb{R}^3 \setminus \{0\}$ *and whose pullback to* TS^2 *via the map* $(x, y) \mapsto y$ *is smooth. Then* φ *is smooth on* $\mathbb{R}^3$.

The restriction of the bundle TS^2 to the equator $S^1 \subset S^2$ can be identified with the cylinder

$$S^1 \times \mathbb{R} \times \mathbb{R}$$

with coordinates $\theta \mod 2\pi$, r, and u. In these coordinates the embedding $S^1 \times \mathbb{R} \times \mathbb{R} \hookrightarrow TS^2$ is given by

$$(\theta, r, u) \mapsto (\sin\theta, -\cos\theta, 0, r\cos\theta, r\sin\theta, u).$$

On this cylinder the map to $\mathbb{R}^3$ is

$$J\colon S^1 \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^3, \qquad J(\theta, r, u) = (r\cos\theta, r\sin\theta, u). \qquad (5.1)$$

Its image, $J(S^1 \times \mathbb{R} \times \mathbb{R})$, is all of $\mathbb{R}^3$. We are done if we can show

PROPOSITION 5.2. *Suppose that* φ *is a continuous function on* $\mathbb{R}^3$ *that is smooth on* $\mathbb{R}^3 \setminus \{0\}$ *and whose pullback,* $J^*\varphi$, *is smooth on* $S^1 \times \mathbb{R} \times \mathbb{R}$. *Then* φ *is smooth on* $\mathbb{R}^3$.

We prove Proposition 5.2 in a string of lemmas. We keep the notation of the Proposition.

LEMMA 5.3.   *Suppose φ is a smooth function on $\mathbb{R}^3 \smallsetminus \{0\}$ whose pullback, $J^*\varphi$, extends to a smooth function on $S^1 \times \mathbb{R} \times \mathbb{R}$. Then for every multi-index α the function $J^* \frac{\partial^\alpha \varphi}{\partial y^\alpha}$ extends to a smooth function on $S^1 \times \mathbb{R} \times \mathbb{R}$.*

Proof.   Let $f \in C^\infty(S^1 \times \mathbb{R} \times \mathbb{R})$ be a smooth extension of $J^*\varphi$. We can write

$$f(\theta, r, u) = f(\theta, 0, u) + r\psi(\theta, r, u)$$

where ψ is smooth on $S^1 \times \mathbb{R} \times \mathbb{R}$. For all θ we have $f(\theta, 0, u) = f(0, 0, u)$; for $u \neq 0$ this holds because f is then a pullback, and for $u = 0$ this holds by continuity. So we can write

$$f(\theta, r, u) = f(0, 0, u) + r\psi(\theta, r, u). \tag{5.2}$$

Since

$$\frac{\partial}{\partial y_1} = J_* \left(\cos\theta \frac{\partial}{\partial r} - \frac{1}{r} \sin\theta \frac{\partial}{\partial \theta} \right),$$

since f is an extension of $J^*\varphi$, and since (5.2) holds, we have

$$J^* \frac{\partial \varphi}{\partial y_1} = \cos\theta \frac{\partial(r\psi)}{\partial r} - \sin\theta \frac{\partial \psi}{\partial \theta} \tag{5.3}$$

on $J^{-1}(\mathbb{R}^3 \smallsetminus \{0\})$. The right hand side of (5.3) provides a smooth extension of $J^* \frac{\partial \varphi}{\partial y_1}$ to all of $S^1 \times \mathbb{R} \times \mathbb{R}$. By a similar argument $J^* \frac{\partial \varphi}{\partial y_2}$ and $J^* \frac{\partial \varphi}{\partial y_3}$ also extend to functions in $C^\infty(S^1 \times \mathbb{R} \times \mathbb{R})$.

We have shown that all the first partials, $J^* \frac{\partial \varphi}{\partial y_i}$, extend to smooth functions on $S^1 \times \mathbb{R} \times \mathbb{R}$. The lemma follows by a successive application of this argument to the partial derivatives of φ.   □

LEMMA 5.4.   *Suppose that φ is a continuous function on $\mathbb{R}^3$ which is twice continuously differentiable on $\mathbb{R}^3 \smallsetminus \{0\}$. Suppose that the first and second partial derivatives of φ extend to continuous functions on $\mathbb{R}^3$. Then φ is continuously differentiable at the origin.*

Proof.   Let $g_i \in C^0(\mathbb{R}^3)$ be the continuous extension of $\frac{\partial \varphi}{\partial y_i}$. We need to show that the partial derivative $\frac{\partial \varphi}{\partial y_i}$ exists at the origin and is equal to $g_i(0)$. By restricting attention to the appropriate line in $\mathbb{R}^3$ the problem becomes one dimensional.

An easy estimate shows that a function φ in $C^0(\mathbb{R}) \cap C^2(\mathbb{R} \smallsetminus \{0\})$ whose first derivative extends to a continuous function on $\mathbb{R}$ and whose second derivative is bounded near 0 is continuously differentiable at zero.   □

LEMMA 5.5.   *Suppose that φ is a continuous function on $\mathbb{R}^3 \smallsetminus \{0\}$ and that its pullback $J^*\varphi$ extends to a continuous function on the cylinder $S^1 \times \mathbb{R} \times \mathbb{R}$. Then φ extends to a continuous function on $\mathbb{R}^3$.*

Proof. Let $f \in C^0(S^1 \times \mathbb{R} \times \mathbb{R})$ be the continuous extension of $J^*\varphi$. For all θ we have

$$f(\theta, \dot{0}, u) = f(0, 0, u) \tag{5.4}$$

for all $u \neq 0$, because f is then a pullback. By continuity (5.4) also holds when $u = 0$. Hence f is constant on the level sets of J and descends to a function on $\mathbb{R}^3$. This function coincides with φ on $\mathbb{R}^3 \smallsetminus 0$. It is continuous on $\mathbb{R}^3$ because the map J is proper. $\square$

Proof of Proposition 5.2. By Lemma 5.3, for every multi-index α the pullback $J^* \frac{\partial^\alpha \varphi}{\partial y^\alpha}$ extends to a smooth function on $S^1 \times \mathbb{R} \times \mathbb{R}$. Therefore, by Lemma 5.5, all partial derivatives $\frac{\partial^\alpha \varphi}{\partial y^\alpha}$ extend to continuous functions on $\mathbb{R}^3$. In particular, the first and second partial derivatives of φ extend to continuous functions on $\mathbb{R}^3$. Hence, by Lemma 5.4, $\varphi \in C^1(\mathbb{R}^3)$.

This proves that if $\varphi \in C^\infty(\mathbb{R}^3) \cap C^0(\mathbb{R}^3 \smallsetminus \{0\})$ and $J^*\varphi \in C^\infty(S^1 \times \mathbb{R} \times \mathbb{R})$, $\varphi \in C^1(\mathbb{R}^3)$.

Now consider a first partial, $\frac{\partial \varphi}{\partial y_i}$. We know that it is in $C^\infty(\mathbb{R}^3 \smallsetminus \{0\}) \cap C^0(\mathbb{R}^3)$ and that $J^*(\frac{\partial \varphi}{\partial y_i}|_{\mathbb{R}^3 \smallsetminus \{0\}})$ extends to a smooth function on $S^1 \times \mathbb{R} \times \mathbb{R}$ (hence, by continuity, $J^*(\frac{\partial \varphi}{\partial y_i}) \in C^\infty(S^1 \times \mathbb{R} \times \mathbb{R})$.) Therefore, by the above argument, $\frac{\partial \varphi}{\partial y_i}$ is in $C^1(\mathbb{R}^3)$.

The argument proceeds by induction on the length of the multi-index α in the partial derivative $\frac{\partial^\alpha \varphi}{\partial y^\alpha}$. $\square$

This completes the proof of Proposition 5.1 and hence the proof of part 1 of Theorem 4. Let us now prove part 2 of the theorem. As before, suppose that $\Phi\colon M \longrightarrow su(2)^*$ is a proper moment map for a Hamiltonian action of $SU(2)$ on a symplectic manifold (M, ω). Let

$$Z = \Phi^{-1}(0)$$

be the zero level set of the moment map. Every $SU(2)$-fixed point must lie on Z, because the moment map is equivariant. We assume that Z is nonempty and coincides with the set of fixed points.

The normal bundle of Z in M is a symplectic vector bundle, hence can be given the structure of a complex Hermitian vector bundle. The key to the proof is that the representation of $SU(2)$ on a fiber of this normal bundle must be the standard representation of $SU(2)$ on $\mathbb{C}^2$.

By the equivariant symplectic embedding theorem (see [AG] or [M]) there exists an $SU(2)$-equivariant diffeomorphism from a neighborhood of the zero section in the normal bundle to a neighborhood of Z in M such that on a fiber of the normal bundle, the pull-back of the moment map coincides with the moment map for the linear action of $SU(2)$ on that fiber.

Recall that for every non-negative integer m there exists exactly one irreducible representation of $SU(2)$ of complex dimension $m + 1$. The moment map for the

representation of $SU(2)$ on $\mathbb{C}^{m+1}$, as a map from $\mathbb{C}^{m+1}$ into $\mathbb{C} \times \mathbb{R} \cong su(2)^*$, can be easily computed to be

$$(u_0, \ldots, u_m) \mapsto \left(\overline{u_0} u_1 + \overline{u_1} u_2 + \cdots + \overline{u_{m-1}} u_m , \right.$$
$$\left. \frac{m}{2} |u_0|^2 + \frac{m-2}{2} |u_1|^2 + \ldots + \frac{-m}{2} |u_m|^2 \right).$$

If m is greater than one then the zero level set of this moment map contains the vector $v = (1, 0, \ldots, 0, 1)$, which is not fixed by the action of $SU(2)$. Therefore, in the representation of $SU(2)$ on the fibers of the normal bundle of Z in M, all the irreducible components are the standard representation on $\mathbb{C}^2$. Moreover, this representation can occur only once: on $\mathbb{C}^2 \oplus \mathbb{C}^2$, the moment map is

$$((u_0, u_1), (v_0, v_1)) \mapsto \left(\overline{u_0} u_1 + \overline{v_0} v_1, \frac{1}{2} |u_0|^2 - \frac{1}{2} |u_1|^2 + \frac{1}{2} |v_0|^2 - \frac{1}{2} |v_1|^2 \right),$$

and its zero level sets contains a vector, $((1, 0), (0, 1))$, which is not fixed by $SU(2)$.

We have shown that the representation of $SU(2)$ on the fibers of the normal bundle of Z in M is the irreducible representation on $\mathbb{C}^2$ with the moment map

$$\Phi(u_0, u_1) = \left(\overline{u_0} u_1, \frac{1}{2} |u_0|^2 - \frac{1}{2} |u_1|^2 \right).$$

Note that $||\Phi|| = (|\overline{u_0} u_1|^2 + \frac{1}{4} (|u_0|^2 - |u_1|^2)^2)^{1/2} = \frac{1}{2} (|u_0|^2 + |u_1|^2) = \frac{1}{2} ||u||^2$. Since the square of the norm on a Hermitian vector bundle is a smooth function, the function $||\Phi||$ is smooth on a neighborhood of Z in M. In particular, its Hamiltonian vector field is well defined. We will show that this vector field is tangent to the $SU(2)$-orbits, that it generates a circle action on M which commutes with the action of $SU(2)$, and that these actions fit together to an effective action of $SU(2) \times_{Z_2} S^1 = U(2)$.

At points of Z itself, $d||\Phi|| = 0$, so the Hamiltonian vector field of $||\Phi||$ vanishes. Hence it is sufficient to prove the required properties on the complement, $M \smallsetminus Z$.

Since the function $||\Phi||$ is $SU(2)$-invariant, its Hamiltonian flow preserves Φ. In particular, it preserves the cross section, $\Phi^{-1}(\ell)$, where ℓ is an open ray in $su(2)^* \cong \mathbb{R}^3$ emanating from the origin. On the cross section, the function $||\Phi||$ coincides with the Hamiltonian generator of the action of the circle subgroup which stabilizes the ray ℓ. Hence, along the cross section, the Hamiltonian flow of the function $||\Phi||$ coincides with the action of this circle subgroup. The intersection of all these circles is the center, Z_2, of $SU(2)$. Hence the action of $SU(2) \times S^1$ descends to an effective action of $SU(2) \times_{Z_2} S^1 \cong U(2)$.

To prove that the centralizer of the invariants consists of the $U(2)$-collective functions we need, by Theorem 1, to prove that $C^\infty(M)^\Phi = \tilde{\Phi}^* C^\infty(u(2)^*)$. Note that the image under Φ of the normal bundle of Z in M coincides with the image of one fiber, and that the level sets of the $SU(2)$-moment map, Φ, coincide with the level sets of

the $U(2)$-moment map, $\tilde{\Phi}$. Hence it is sufficient to show that the $U(2)$ moment map on $\mathbb{C}^2$,

$$F \colon \mathbb{C}^2 \longrightarrow \mathbb{C} \times \mathbb{R}^2, \quad (u_0, u_1) \mapsto (\overline{u_0}u_1, |u_0|^2, |u_1|^2),$$

has the division property, i.e., that it satisfies

$$C^\infty(\mathbb{C}^2)^F = F^*C^\infty(\mathbb{C} \times \mathbb{R}^2). \tag{5.5}$$

The functions in $C^\infty(\mathbb{C}^2)^F$ are exactly the S^1-invariant functions on $\mathbb{C}^2$. The coordinates of F generate the ring of S^1-invariant polynomials on $\mathbb{C}^2$. Schwarz's theorem [Sch] then gives (5.5). This completes the proof of Theorem 4. $\quad\square$

Appendix A. Local normal form for the moment map and implications

In this section we describe a version of the Marle-Guillemin-Sternberg normal form of the moment map (Theorems 5 and 6, Corollary A.2) and some of its implications (Lemmas A.4 and A.5, Remark A.6) which were use in the proofs of Theorems 1 and 2. This version of the Marle-Guillemin-Sternberg normal form describes the moment map for a Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω) in terms of nice "coordinates" on a neighborhood of an orbit, $G \cdot x$, in M. The construction of this normal form is carried out in two steps: the construction of the symplectic cross-section (Theorem 5 and Corollary A.2), and the construction of the normal form near an isotropic orbit (Theorem 6).

THOEREM 5 (The symplectic cross-section). *Let* $\Phi \colon M \longrightarrow \mathfrak{g}^*$ *be a moment map associated to a Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω). Let α be a point in $\mathfrak{g}^*$ and let G_α denote its stabilizer under the coadjoint action.*

Then for a sufficiently small slice S at α for the coadjoint action of G, the preimage $R := \Phi^{-1}(S)$ is a symplectic G_α-invariant submanifold of M. Moreover, the action of G_α on R is Hamiltonian and the restriction of the moment map Φ to R followed by the natural projection $\mathfrak{g}^* \longrightarrow \mathfrak{g}_\alpha^*$ is a corresponding moment map.

Proof. Theorem 26.7 in [GS5]. $\quad\square$

Remark A.1. The submanifold R is called a *symplectic cross-section*.

Since S is a slice, the open invariant neighborhood $G \cdot S \subset \mathfrak{g}^*$ of the coadjoint orbit of α is equivariantly diffeomorphic to the associated bundle $G \times_{G_\alpha} S$. Since the moment map Φ is equivariant, $G \cdot R$ is an open invariant subset of the manifold M and it is equivariantly diffeomorphic to the associated bundle $G \times_{G_\alpha} R$. Up to these identifications, the moment map $\Phi \colon G \cdot R \longrightarrow G \cdot S$ is the equivariant bundle map

$G \times_{G_\alpha} R \longrightarrow G \times_{G_\alpha} S, [g, r] \mapsto [g, \Phi_R(r)]$, where Φ_R is the restriction of Φ to the cross-section R, and $[g, r]$ and $[g, \Phi_R(r)]$ denote the G_α-orbits of $(g, r) \in G \times R$ and $(g, \Phi_R(r)) \in G \times S$ respectively. If the slice S is sufficiently small or, more generally, is well-chosen, then the restriction to S of the projection $\mathfrak{g}^* \longrightarrow \mathfrak{g}_\alpha^*$ is a diffeomorphism onto the image. This proves the following:

COROLLARY A.2.   *Let* Φ: $M \longrightarrow \mathfrak{g}^*$ *be as in Theorem 5, x be a point in M, $\alpha = \Phi(x)$, S a small slice at α and $R = \Phi^{-1}(S)$ the corresponding cross-section. Then, up to composition with diffeomorphisms, the moment map Φ in an invariant neighborhood of x is a bundle map*

$$G \times_{G_\alpha} R \longrightarrow G \times_{G_\alpha} S, \qquad [g, r] \mapsto [g, \Phi_R(r)],$$

where Φ_R is the G_α moment map (up to a composition with a diffeomorphism).

Remark A.3.   Since the point α is fixed by the action of G_α, the G_α-orbit through any point $x \in \Phi^{-1}(\alpha)$ is *isotropic* in the symplectic cross-section, R. Since the orbit $G_\alpha \cdot x$ is isotropic, the tangent space $T_x(G_\alpha \cdot x)$ is contained in its symplectic perpendicular $T_x(G_\alpha \cdot x)^{\omega_R}$. Hence the quotient $V = T_x(G_\alpha \cdot x)^{\omega_R}/T_x(G_\alpha \cdot x)$ is a symplectic vector space. It is called the *symplectic slice* at x. Note that V is a natural symplectic representation of the isotropy group G_x of x.

THEOREM 6 [Local normal form near an isotropic orbit].   Let Ψ: $N \longrightarrow \mathfrak{h}^*$ be a moment map associated to a Hamiltonian action of a compact Lie group H on a symplectic manifold (N, σ). Suppose the orbit $H \cdot x$ is *isotropic* in N. Let H_x denote the stabilizer of x in H, let $\mathfrak{h}_x^0$ denote the annihilator of its Lie algebra in $\mathfrak{h}^*$, and let $H_x \longrightarrow Sp(V)$ denote the symplectic slice representation.

Given an H_x-equivariant embedding, i: $\mathfrak{h}_x^* \longrightarrow \mathfrak{h}^*$, there exists an H-invariant closed two-form, ω_Y, on the manifold $Y = H \times_{H_x} (\mathfrak{h}_x^\circ \times V)$, such that

1. the form ω_Y is nondegenerate near the zero section of the bundle $Y \longrightarrow H/H_x$,
2. a neighborhood U_x of the orbit of x in N is equivariantly symplectomorphic to a neighborhood of the zero section in Y, and
3. the action of H on (Y, ω_Y) is Hamiltonian with a moment map Φ_Y: $Y \longrightarrow \mathfrak{h}^*$ given by

$$\Phi_Y([g, \eta, v]) = Ad^\dagger(g)(\eta + i(\Phi_V(v)))$$

where $Ad^\dagger$ is the coadjoint action, and Φ_V: $V \longrightarrow \mathfrak{h}_x^*$ is the moment map for the slice representation of H_x.

Consequently, the equivariant embedding ι: $U_x \hookrightarrow Y$ intertwines the two moment maps, up to translation: $\Psi|_{U_x} = \Phi_Y \circ \iota + \Psi(x)$.

Proof. Theorem 5 is essentially an equivariant version of Weinstein's isotropic embedding theorem. See [GS4] or [GS5]. $\square$

One easy consequence of the local normal form is the following technical lemma, which was used in the proof of Theorem 1.

LEMMA A.4. *Let Φ: $M \longrightarrow \mathfrak{g}^*$ be a moment map associated to a Hamiltonian action of a compact Lie group, G, on a symplectic manifold, (M, ω). Then any two points in a connected component of a level set of Φ can be joined by a piecewise smooth curve that lies in the level set.*

Proof. It is sufficient to prove the theorem for a neighborhood of a point x in M. By Theorem 5, Corollary A.2, and Remark A.3 we may assume that the orbit $G \cdot x$ is isotropic. By Theorem 5 it is sufficient to prove the lemma for the zero level set of the map Φ_Y of the local model. This level set is $G \times_{G_x} (\{0\} \times \Phi_V^{-1}(0))$. The moment map on the symplectic vector space V is homogeneous, therefore $\Phi_V^{-1}(0)$ is a cone in V. Since any point in the cone can be connected to the vertex by a straight line, any two points in the level set can be connected by a piece-wise smooth curve. $\square$

A non-trivial consequence of the local normal form theorem are the following lemma and remark, which were used in the proof of Theorem 2. They essentially say that the image under the moment map of a small invariant neighborhood of an orbit $G \cdot x$ does not change as x varies along a connected component of the level set $\Phi^{-1}(\Phi(x))$:

LEMMA A.5. *Let Φ: $M \longrightarrow \mathfrak{g}^*$ be a moment map associated to an action of a compact connected Lie group G on a connected symplectic manifold (M, ω). Let x be a point in M.*

Let T be a maximal torus of the isotropy group of $\Phi(x)$ (hence of G), $\mathfrak{t}$ its Lie algebra, identified with its dual $\mathfrak{t}^*$ and embedded in $\mathfrak{g}^*$ via a choice of an Ad-invariant inner product, and let $\mathfrak{t}_+^* \subset \mathfrak{g}^*$ be the Weyl chamber containing $\Phi(x)$.

There exists a rational polyhedral cone C_x in $\mathfrak{t}^*$ with vertex at $\Phi(x)$, such that for every G-invariant neighborhood U of x in M there exists an $\mathrm{Ad}^\dagger(G)$-invariant neighborhood $\mathcal{V}$ of $\Phi(x)$ in $\mathfrak{g}^*$ such that $\Phi(U) \cap \mathfrak{t}_+^*$ contains $C_x \cap \mathcal{V}$. Hence $\Phi(U \cap \Phi^{-1}(\mathcal{V})) \cap \mathfrak{t}_+^* = C_x \cap \mathcal{V}$. Moreover, if the neighborhood U is sufficiently small then for any point y in $U \cap \Phi^{-1}(\Phi(x))$, the cone C_y is equal to the cone C_x.

If the group G is abelian, the result is easy. The general case is due to Sjamaar [Sj]. Lerman, Meinrenken, Tolman, and Woodward later found a more "elementary" proof: the lemma follows from the proofs of Theorems 6.1 and 6.2 of [LMTW].

Remark A.6. If $\Phi(x) = 0$, the cone C_x is $\Phi_Y(Y) \cap \mathfrak{t}_+^*$ where Y is the local model described in Theorem 5.

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