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NEST ALGEBRAS ARE HYPERFINITE

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In [12], Paulsen, Power and Ward show that nest algebras are semidiscrete. This is an important tool in developing a good dilation theory for representations of nest algebras (see also [13]). These ideas have been extended to establish semidiscreteness and dilation theorems for larger classes of nonself-adjoint operator algebras [6], [4]. Paulsen and Power have asked whether nest algebras actually have the stronger property of hyperfiniteness. In this paper, we establish this via a refinement of the techniques used in [12] and [5].

A weakly closed operator algebra in a category $\mathcal{C}$ is *hyperfinite* if it is the increasing union of finite dimensional subalgebras which are completely isometrically isomorphic to (finite dimensional) members of $\mathcal{C}$. For von Neumann algebras, deep results of Connes, Haagerup, Choi, Effros and others have shown that hyperfiniteness is equivalent to various other properties including semidiscreteness and amenability. Moreover hyperfiniteness is a stronger condition in the sense that it readily implies the others for elementary reasons.

Paulsen, Power and Ward show that for any nest algebra $\mathcal{T}(\mathcal{N})$ on a separable Hilbert space, there is a sequence $\mathcal{A}_n$ of finite dimensional nest algebras together with completely isometric homomorphisms Φ_n of $\mathcal{A}_n$ into $\mathcal{T}(\mathcal{N})$ and completely contractive weak-$*$ continuous maps E_n of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{A}_n$ such that $\Psi_n = \Phi_n E_n$ are idempotent maps converging point–weak-$*$ to the identity on $\mathcal{T}(\mathcal{N})$ and converging in norm on $\mathcal{T}(\mathcal{N}) \cap \mathcal{K}$, where $\mathcal{K}$ is the ideal of compact operators. In our argument, we achieve this but in addition arrange that the algebras $\mathcal{B}_n = \Phi_n(\mathcal{A}_n)$ are nested unital algebras.

An even stronger form of hyperfiniteness would require the imbeddings α_n of $\mathcal{A}_n$ into $\mathcal{A}_{n+1}$ induced by the containment of $\mathcal{B}_n$ in $\mathcal{B}_{n+1}$ to be nice maps. Recent interest has been focussed on imbeddings which extend to $*$-endomorphisms of the enveloping matrix algebras $\mathfrak{A}_n$ (isomorphic to the $k \times k$ matrices $\mathfrak{M}_k$ for some k) which are regular in the following sense. The algebras $\mathcal{A}_n$ each contain a masa $\mathcal{D}_n$ of $\mathfrak{A}_n$ which form an increasing sequence. They determine a set of matrix units for each matrix algebra $\mathfrak{A}_n$; and $\mathcal{A}_n$ are block upper triangular with respect to this basis. The imbedding is *regular* if each matrix unit of $\mathfrak{A}_n$ is sent to a sum of matrix units in $\mathfrak{A}_{n+1}$. The direct limit of the sequence $(\mathcal{A}_n, \alpha_n)$ is a subalgebra $\mathcal{A}$ of the AF C*-algebra

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$\mathfrak{A}$ which is the limit of the $\mathfrak{A}_n$'s. Such limits are characterized by the fact that they contain a Cartan masa of $\mathfrak{A}$ (roughly speaking, one which is obtained by an increasing sequence of $\mathcal{D}_n$'s as above).

Working from the other direction, Orr and Peters [11] considered a subalgebra $\mathcal{A}$ of an AF C*-algebra $\mathfrak{A}$ which is *triangular* (meaning that $\mathcal{A} \cap \mathcal{A}^*$ is a Cartan masa) and *Dirichlet* (meaning that $\mathcal{A} + \mathcal{A}^*$ is dense in $\mathfrak{A}$). Such algebras are called *strongly maximal triangular*. They showed that when $\mathfrak{A}$ is primitive, there is an irreducible representation of $\mathfrak{A}$ which carries $\mathcal{A}$ onto a weak-$*$ dense subalgebra of a triangular nest algebra. An easy example shows that the Volterra nest, which has uniform multiplicity one, can be achieved in this way. Thus any continuous nest of uniform multiplicity can be obtained in this way if the condition of triangularity is replaced by the more general class of limits of nest algebras.

We will show that for any continuous nest, there is a direct limit $\mathcal{A}$ of nest subalgebras of full matrix algebras with regular $*$-extendible imbeddings (so that the C*-algebra $\mathfrak{A}$ is matroid) and a $*$-representation Φ of $\mathfrak{A}$ on $\mathcal{H}$ such that $\Phi(\mathcal{A})$ is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$. A suitable modification of this argument works for any nest with no finite rank atoms. We also show that any nest with finite rank atoms which is not atomic cannot be obtained in this way.

Our argument and that of [12] rely in an essential way on the spectral theory for unitary invariants of nests developed by Erdos [7] that is based on the Hellinger–Hahn classification of abelian von Neumann algebras. Given a continuous nest $\mathcal{N}$ and a parametrization of $\mathcal{N}$ by $[0, 1]$ as $\{N_t : 0 \leq t \leq 1\}$, there is a spectral measure $E_{\mathcal{N}}$ on $[0, 1]$ such that $E_{\mathcal{N}}[0, t] = N_t$. Since any non-atomic regular Borel measure on $[0, 1]$ may be converted to Lebesgue measure by a reparametrization of the interval, we may work only with spectral measures $E_{\mathcal{N}}$ equivalent to Lebesgue measure. Moreover, there is a Borel multiplicity function m of $[0, 1]$ into $\mathbb{N}_0 \cup \{\infty\}$. The sets $A_k = m^{-1}(k)$ have the property that the nest restricted to $E_{\mathcal{N}}(A_k)$ is unitarily equivalent to the k-fold ampliation of multiplication on $L^2(A_k)$ by the characteristic functions of $[0, t]$. We do not know of any way to avoid confronting these measure theoretic issues head on.

The proof of hyperfiniteness makes a careful decomposition of the interval into the disjoint union of Cantor sets of uniform multiplicity. The image algebras will contain lots of rank one elements. This makes it easy to establish that every rank one operator of the nest algebra is a limit in norm of such elements. Thus the Erdos Density Theorem [8] can be used to show weak-$*$ density. The construction also yields semidiscreteness at the same time, which is not surprising since this is a modification of the argument in [12].

In order to obtain regular $*$-extendible embeddings in the case of continuous nests, one must abandon finite rank operators because the diagonal matrix units are necessarily mapped to infinite rank projections. The partial isometries that we construct are built from matrix units of (partial) homeomorphisms between Cantor sets. This idea was used in [5] to construct unitarily implemented outer automorphisms of continuous nest algebras. The construction is quite delicate, and the proof of density is more subtle. Density relies on a result of Arveson [1] stating that a subalgebra of a nest algebra containing a masa and having the nest as its only invariant subspaces is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$.

It is an easy fact that if $\mathcal{A}$ is a Dirichlet subalgebra of $\mathfrak{A}$ and Φ is a representation of $\mathfrak{A}$ which carries $\mathcal{A}$ into a maximal nest algebra and contains a masa in its weak-$*$ closure, then $\Phi(\mathcal{A})$ is weak-$*$ dense in the nest algebra if and only if Φ is irreducible. Muhly and Solel [9] prove similar results for analytic algebras associated to C*-dynamical systems on the real line. Surprisingly, it is possible for $\Phi(\mathcal{A})$ to contain such a masa in its weak-$*$ closure when the representation is not irreducible. We establish this at the end of the paper with some careful modifications of our arguments.

The necessary background on nest algebras including most of the results quoted here are contained in our textbook [3]. See Power's monograph [14] for details on the class of limit algebras.

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1. Hyperfiniteness

To set the stage, we prove an easy result about atomic nest algebras.

THEOREM 1.1. *Let $\mathcal{N}$ be an atomic nest. There is a sequence of finite dimensional nest algebras $\mathcal{T}(\mathcal{M}_n)$ and regular multiplicity one $*$-extendible imbeddings α_n of $\mathcal{T}(\mathcal{M}_n)$ into $\mathcal{T}(\mathcal{M}_{n+1})$ such that there is a $*$-extendible isomorphism Φ of the direct limit $\mathcal{A}$ onto $\mathcal{T}(\mathcal{N})\cap\mathcal{K}$. The nests $\mathcal{M}_n$ may be chosen so that there are also completely isometric isomorphisms Φ'_n of $\mathcal{T}(\mathcal{M}_n)$ onto an increasing sequence $\mathcal{A}'_n$ of* unital *finite dimensional subalgebras with union weak-$*$ dense in $\mathcal{T}(\mathcal{N})$. Furthermore, there are completely contractive idempotent maps (expectations) Ψ_n and Ψ'_n of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{A}_n = \Phi(\mathcal{T}(\mathcal{M}_n))$ and $\mathcal{A}'_n$ which converge point–weak-$*$ on $\mathcal{T}(\mathcal{N})$ to the identity map.*

Proof. This result is easy, and only needs a small device to obtain unital subalgebras. First suppose that $\mathcal{N}$ is infinite. Enumerate its atoms as $\{A_m : m \geq 1\}$. Choose orthonormal bases $\{e_{m,i} : 0 \leq i < d_m\}$ for each $A_m\mathcal{H}$, where $d_m = \operatorname{rank} A_m$.

For each integer n, consider the subspace

$$\mathcal{H}_n = \operatorname{span}\big\{e_{m,i} \colon\ 1 \leq m \leq n,\ 0 \leq i < \min\{d_m, n\}\big\} \cup \{e_{n+1,0}\}.$$

Let $\mathcal{M}_n$ be the nest on $\mathcal{H}_n$ obtained by compressing $\mathcal{N}$ to this subspace; and let $\mathcal{T}(\mathcal{M}_n)$ be the corresponding nest algebra.

The injection of $\mathcal{T}(\mathcal{M}_n)$ into $\mathcal{T}(\mathcal{M}_{n+1})$ determined by the natural inclusion of $\mathcal{H}_n$ into $\mathcal{H}_{n+1}$ will be denoted by α_n, and Φ_n will denote the injection of $\mathcal{T}(\mathcal{M}_n)$ into $\mathcal{T}(\mathcal{N})$ determined by the natural inclusion of $\mathcal{H}_n$ into $\mathcal{H}$. It is evident that these maps are $*$-extendible, multiplicity one maps. Thus there is a $*$-extendible isomorphism

$\Phi = \varinjlim \Phi_n$ of the direct limit $\mathcal{A}$ onto the closed union of $\mathcal{A}_n = \Phi_n(\mathcal{T}(\mathcal{M}_n))$, which is evidently $\mathcal{T}(\mathcal{N}) \cap \mathcal{K}$.

To obtain unital algebras, define maps $\Phi'_n = \Phi_n + \Delta_n$ of $\mathcal{T}(\mathcal{M}_n)$ by setting

$$\Delta_n(A) = (Ae_{n+1,0}, e_{n+1,0})\Phi_n(I_n)^\perp,$$

where I_n is the identity of $\mathcal{T}(\mathcal{M}_n)$. It is easy to verify that these maps are completely isometric unital algebra isomorphisms because the compression of $\mathcal{T}(\mathcal{M}_n)$ to the one dimensional atom $\mathbb{C}e_{n+1,0}$ is multiplicative. However, these maps are no longer $*$-extendible, nor do they commute with the inclusions α_n. Let $\mathcal{A}'_n = \Phi'_n(\mathcal{T}(\mathcal{M}_n))$. Moreover, since $e_{n+1,0}e^*_{n+1,0}$ belongs to $\mathcal{A}'_{n+1}$, it is clear that $\mathcal{A}'_n$ is contained in $\mathcal{A}'_{n+1}$.

Finally, define expectations E_n onto $\mathcal{T}(\mathcal{M}_n)$ by compression to $\mathcal{H}_n$. Then $\Psi_n = \Phi_n E_n$ and $\Psi'_n = \Phi'_n E'_n$ are completely contractive idempotent maps which converge point–weak-$*$ to the identity map on $\mathcal{T}(\mathcal{N})$. Moreover, the latter maps are unital.

If $\mathcal{N}$ is a finite nest, let the infinite rank atoms be A_m for $1 \le m \le M$ and the finite rank atoms A_m for $M < m \le N$. Define the bases as before and set

$$\mathcal{H}_n = \operatorname{span}\{e_{m,i}\colon 1 \le m \le M,\ 0 \le i < n\} \cup \{A_m\mathcal{H}\colon M < m \le N\}.$$

Define $\mathcal{M}_n$ and Φ_n as before.

To obtain the unital imbeddings, let $\mathcal{K}_{m,n} = \operatorname{span}\{e_{m,i}\colon 0 \le i < n\}$ for $1 \le m \le M$. Let $\delta_{m,n}(A)$ denote the homomorphism of $\mathcal{T}(\mathcal{M}_n)$ onto $\mathcal{B}(\mathcal{K}_{m,n})$ obtained by compression. Define an isometry $W_{m,n}$ of $\mathcal{K}^{(\infty)}_{m,n}$ onto $A_m\mathcal{H} \ominus \mathcal{K}_{m,n}$. Then define $\Phi'_n = \Phi_n + \Delta_n$ where

$$\Delta_n(A) = \sum_{m=1}^{M} W_{m,n}\delta_{m,n}(A)^{(\infty)}W^*_{m,n}.$$

The remaining details are left to the reader. □

Next we establish hyperfiniteness in complete generality.

THEOREM 1.2. *Every nest algebra is hyperfinite. Indeed, given $\mathcal{T}(\mathcal{N})$, there is an increasing sequence $\mathcal{A}_n$ of unital finite dimensional subalgebras which are each completely isometrically isomorphic to nest algebras $\mathcal{T}(\mathcal{M}_n)$ such that their closed union contains $\mathcal{T}(\mathcal{N}) \cap \mathcal{K}$, and thus is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$. Moreover, there are completely contractive idempotent weak-$*$ continuous maps Ψ_n of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{A}_n$ which converge point–weak-$*$ on $\mathcal{T}(\mathcal{N})$ to the identity map.*

Proof. We have already seen a proof for the case of atomic nests. We will prove it now for continuous nests. The extension to arbitrary nests will be routine.

Parameterize $\mathcal{N}$ by $[0, 1]$ in such a way that the spectral measure $E_{\mathcal{N}}$ is equivalent to Lebesgue measure. Choose pairwise disjoint measurable sets A_k for $k \ge 0$ such that the multiplicity of $\mathcal{N}_k := E_{\mathcal{N}}(A_k)\mathcal{N}$ is $m(k) := k$ for $k \geq 1$ and $m(0) := \aleph_0$ for $k = 0$, and so that

$$\cup_{k\geq 0} A_k \subseteq [0,1] \setminus \mathbb{Q}.$$

For each $k \geq 0$, choose a complete set of matrix units in the diagonal algebra $\mathcal{D}(\mathcal{N}_k)$ for the multiplicity $m(k)$ nest. That is, for each $k \geq 1$, choose partial isometries $W^k_{pq} \in \mathcal{D}(\mathcal{N}_k)$ for $1 \leq p, q \leq k$ which are a set of $k \times k$ matrix units and $\sum_{p=1}^k W^k_{pp} = E_{\mathcal{N}}(A_k)$. And for $k = 0$, choose a corresponding set of matrix units for $p, q \geq 1$ so that

$$\text{SOT-}\sum_{p\geq 1} W^0_{pp} = E_{\mathcal{N}}(A_0).$$

Also choose a cyclic vector x_k for the multiplicity free nest $W^k_{11}\mathcal{N}$. This acts on a Hilbert space which is naturally identified with $L^2(A_k)$.

Now use the fact that Lebesgue measure is regular to find pairwise disjoint compact sets $A_{k,n}$ of A_k so that

$$A_k \setminus \cup_{n\geq 1} A_{k,n}$$

is a null set for all $k \geq 0$. There is some integer k_0 so that A_{k_0} has positive measure. We may suppose that $A_{k_0,n}$ has positive measure for all $n \geq 1$. Let $F_0 = \{0, 1\}$. At the n-th stage, we will construct a finite subset F_n of $[0,1] \cap \mathbb{Q}$ which contains F_{n-1} and all points of the form $j2^{-n}$ for $0 \leq j \leq 2^n$.

Proceed as follows. Consider the finite collection $A_{k,j}$ for $j, k \leq n$ and $A_{k_0,n+1}$ of disjoint compact sets. Choose pairwise disjoint open sets containing them. By the compactness of each $A_{k,j}$, we may assume that each open set is a finite union of disjoint intervals with rational endpoints. Moreover, we may expand this set of endpoints to include F_{n-1} and $j2^{-n}$ for $0 \leq j \leq 2^n$. Let this enlarged set be denoted by F_n. It partitions the unit interval into a number of smaller intervals J^n_i for $1 \leq i \leq N(n)$ in the usual order. Moreover each J^n_i intersects at most one of the sets $A_{k,j}$ in a set of positive measure. Let $\kappa_n(i) := (k_n(i), j_n(i)) = (k, j)$ and $X_{n,i} = A_{k,j} \cap J^n_i$ when this holds for $j, k \leq n$, and leave $\kappa_n(i)$ undefined otherwise. Pick one i_0 such that $X_{n,i_0} := J^n_{i_0} \cap A_{k_0,n+1}$ has positive measure.

Let $m_{n,i} = \min\{m(k(i)), n\}$ when $k(i) \leq n$, $m_{n,i_0} = 1$ and $m_{n,i} = 0$ otherwise. For each i with $\kappa_n(i) = (k, j)$ defined, let

$$U^{n,i}_{pq} = E_{\mathcal{N}}(X_{n,i}) W^k_{pq} \quad \text{for} \quad 1 \leq p, q \leq m_{n,i}.$$

and set

$$U^{n,i_0}_{11} = I - \sum_{i\neq i_0} \sum_{p=1}^{m_{n,i}} U^{n,i}_{pp}.$$

Then define vectors

$$x^{n,i}_p = U^{n,i}_{p1} x_k \quad \text{for} \quad 1 \leq i \leq N(n),\ 1 \leq p \leq m_{n,i},\ k(i) \leq n$$

and

$$x_1^{n,i_0} = E_{\mathcal{N}}(X_{n,i_0})x_{k_0}.$$

Then define subspaces

$$\mathcal{H}_n = \operatorname{span}\{x_p^{n,i} \colon 1 \le i \le N(n),\ 1 \le p \le m_{n,i}\}.$$

Define a finite dimensional algebra $\mathcal{A}_n$ to be the span of the following elements:

(i) $U_{pq}^{n,i}$ for $1 \le i \le N(n)$ and $1 \le p, q \le m_{n,i}$.
(ii) $x_p^{n,i} x_q^{n,j*}$ for $1 \le i < j \le N(n)$, $1 \le p \le m_{n,i}$ and $1 \le q \le m_{n,j}$.

It is easy to verify that $\mathcal{A}_n$ is contained in $\mathcal{T}(\mathcal{N})$.

Let $\mathcal{M}_n$ be the finite nest on $\mathcal{H}_n$ with atoms $A_{n,i}$ for $1 \le i \le N(n)$ spanned by $\{x_p^{n,i} \colon 1 \le p \le m_{n,i}\}$. The nest algebra $\mathcal{T}(\mathcal{M}_n)$ is spanned by matrix units

$$E_{pq}^{n,ij} = x_p^{n,i} x_q^{n,j*}$$

for $1 \le i \le j \le N(n)$, $1 \le p \le m_{n,i}$ and $1 \le q \le m_{n,j}$. The map Φ_n of $\mathcal{T}(\mathcal{M}_n)$ into $\mathcal{B}(\mathcal{H})$ given by

$$\Phi_n(E_{pq}^{n,ij}) = \begin{cases} E_{pq}^{n,ij} & \text{for } 1 \le i < j \le N(n) \\ U_{pq}^{n,i} & \text{for } 1 \le i = j \le N(n) \end{cases}$$

is readily seen to be a unital completely isometric isomorphism of $\mathcal{T}(\mathcal{M}_n)$ onto $\mathcal{A}_n$. This map is not $*$-extendible because the off-diagonal matrix units of $\mathcal{A}_n$ are rank one while the diagonal matrix units are infinite rank.

Next observe that $\mathcal{A}_{n+1}$ contains $\mathcal{A}_n$ because the partition F_{n+1} is a refinement of F_n. Hence each interval J_i^n is the union of certain $J_{i'}^{n+1}$'s. Let

$$\Sigma_{n,i} = \{i' \colon J_{i'}^{n+1} \subset J_i^n \quad \text{and} \quad \kappa_{n+1}(i') = \kappa_n(i)\}.$$

Then with $(k, j) = \kappa_n(i)$,

$$X_{n,i} = \bigcup_{i' \in \Sigma_{n,i}} X_{n+1,i'}.$$

Therefore for $i \ne i_0$,

$$U_{pq}^{n,i} = \sum_{i' \in \Sigma_{n,i}} U_{pq}^{n+1,i'}.$$

Consequently U_{11}^{n,i_0} also lies in $\mathcal{A}_{n+1}$ since both algebras are unital. Also

$$x_p^{n,i} = \sum_{i' \in \Sigma_{n,i}} x_p^{n+1,i'}.$$

Thus if $1 \le i < j \le N(n)$, then

$$x_p^{n,i} x_q^{n,j*} = \sum_{i' \in \Sigma_{n,i}} \sum_{j' \in \Sigma_{n,j}} x_p^{n,i'} x_q^{n,j'*}.$$

We claim that the union of the $\mathcal{A}_n$'s is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$. By the Erdos Density Theorem [8], it suffices to approximate each rank one element xy^* in $\mathcal{T}(\mathcal{N})$; and this may be done in norm. Indeed, there is a diadic rational $2^{-n}j$ so that $x' = E_{\mathcal{N}}[0, 2^{-n}j]x$ and $y' = E_{\mathcal{N}}[2^{-n}j, 1]y$ are as close in norm to x and y as desired. Since the vectors x_k are cyclic for the diagonal algebras of the nests $\mathcal{N}_k$, any vector may be approximated by a linear combination of terms of the form $E_{\mathcal{N}}(J) W_{p1}^k x_k$ where J is a diadic interval. These in turn are approximated by sums of $E_{\mathcal{N}}(J \cap X_{k,n}) W_{p1}^k x_k$ for $k \ge 0$, $n \ge 1$ and $p \ge 1$. Thus for sufficiently large n, enough of these terms will lie in $\mathcal{A}_n$ to approximate both x' and y' to any given accuracy by vectors x'' and y''. But then $x'' y''^*$ belongs to $\mathcal{A}_n$.

The compressions E_n of $\mathcal{T}(\mathcal{N})$ to $\mathcal{H}_n$ are completely contractive weak-$*$ continuous expectations that carry $\mathcal{T}(\mathcal{N})$ onto $\mathcal{T}(\mathcal{M}_n)$. It is easy to check that the maps $\Psi_n = \Phi_n E_n$ are unital idempotent maps that are completely contractive weak-$*$ continuous expectations of $\mathcal{T}(\mathcal{N})$ onto $\mathcal{A}_n$.

If $\mathcal{N}$ is an arbitrary nest, combine this argument with the easier atomic case. Include the first n atoms up to multiplicity n at the n-th stage, and chop up the continuous part on the intervals between these finitely many atoms. The partition set F_n used above will include the cuts from these atoms and the rationals will be replaced by a countable dense subset of the support of the continuous part along with the endpoints of all the atoms. With a bit of care, the result proceeds in the same manner. □

2. Dense representations of limit algebras

The maps constructed in the previous theorem are not $*$-extendible. However as we saw in Theorem 1.1, this more stringent form of hyperfiniteness is possible in the atomic case. This is not possible for arbitrary nests with both continuous and atomic parts. We establish this in Theorem 2.3. However, by refining the previous argument, we can achieve this stronger property for continuous nests.

THEOREM 2.1. *If $\mathcal{N}$ is a continuous nest, there is a regular limit $\mathcal{A}$ of finite dimensional nest algebras such that its enveloping C*-algebra is a matroid algebra and a $*$-extendible representation Φ of $\mathcal{A}$ such that $\Phi(\mathcal{A})$ is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$.*

Proof. We adopt the notation of the previous theorem. For each compact set $A_{k,n}$, we need to fix a good homeomorphism with the usual Cantor set. First we may assume that for each open set O, either $O \cap A_{k,n}$ is empty or it has positive measure. This is accomplished by taking the union of all open sets that meet $A_{k,n}$ in a set of measure zero. As the union of open sets of the line is the union of a countable subcollection, this is the largest open set meeting $A_{k,n}$ in a null set. Clearly we may replace $A_{k,n}$ by the its intersection with the complement of this open set.

Now $A_{k,n}$ is order homeomorphic to the Cantor set C. Let m_C denote the Cantor measure on C, which assigns measure 2^{-n} to each of the 2^n *diadic* subsets of C arising from the *middle thirds* construction, which we call the diadic subintervals of order n. We wish to choose such a map which has good control on the measure. Recursively partition $A_{k,n}$ into 2^n consecutive (in the order on the line) clopen subsets such that each piece has measure in the interval $\left(\frac{3}{4}2^{-n}m(A_{k,n}), \frac{4}{3}2^{-n}m(A_{k,n})\right)$. Then there is a unique map $\gamma_{k,n}$ of $A_{k,n}$ onto the Cantor set that carries this partition of $A_{k,n}$ into diadic intervals onto the corresponding diadic partition of the Cantor set. This map is an order preserving homeomorphism. Moreover, by construction, Lebesgue measure on $A_{k,n}$ is equivalent to $m_C \circ \gamma_{k,n}$, and the Radon–Nikodym derivative is bounded in the interval

$$\left(\frac{3}{4m(A_{k,n})}, \frac{4}{3m(A_{k,n})}\right).$$

We repeat the construction from Theorem 1.2 of the partitions F_n with an additional twist. The partition divides each Cantor set $A_{k,n}$ into clopen intervals. Such clopen intervals of a Cantor set are the disjoint union of certain consecutive diadic subintervals of order m for m sufficiently large. Taking the maximum $m(n)$ of all such m's needed for each element of the partition, we then further refine the partition to cut each $A_{k,n}$ into its $2^{m(n)}$ diadic subintervals of order $m(n)$. (Alternatively, just observe that given any finite set of disjoint Cantor sets, they can each be split into their 2^m diadic subintervals of order m for $m = m(n)$ sufficiently large so that the convex hulls of each subinterval is disjoint from all the others.)

Following the notation of the previous proof, F_n splits $[0, 1]$ into intervals J_i^n for $1 \le i \le N(n)$. They intersect at most one of the sets $A_{k,j}$ for $k, j \le n$. Set $\kappa_n(i) = (k_n(i), j_n(i)) = (k, j)$ and $X_{n,i} = A_{k,j} \cap J_i^n$. Notice that each $X_{n,i}$ is a diadic subinterval of order $m(n)$ of $A_{k,j}$. (By disregarding those intervals with empty intersection, we may suppose that each J_i^n intersects exactly one of the $A_{k,j}$. We do not use the set A_{n+1,i_0} in this construction because we are not making the maps unital.) Define a set of matrix units of homeomorphisms between these sets as follows. Note that $\gamma_{\kappa_n(i)}$ is a homeomorphism of $X_{n,i}$ onto an order $m(n)$ diadic subinterval $C_{n,i}$ of the Cantor set. For each $1 \le i, j \le N(n)$, let S_{ij}^n be the isometric translation of $C_{n,j}$ onto $C_{n,i}$. Then set

$$h_{ij}^n = \gamma_{\kappa_n(i)}^{-1} \circ S_{ij}^n \circ \gamma_{\kappa_n(j)}.$$

These formulae imply that:

(i) h_{ii}^n is the identity function on $X_{n,i}$ for $1 \le i \le N(n)$;
(ii) $h_{ij}^n \circ h_{jk}^n = h_{ik}^n$ for all i, j, k.

Thus these functions form a set of matrix units.

Because of the bounds on the derivatives of the γ's and the fact that the function S^n_{ij} is measure preserving, it follows that h^n_{ij} is absolutely continuous, and its derivative is bounded in the interval

$$\left(\frac{m(A_{k,i})}{2m(A_{k,j})}, \frac{2m(A_{k,i})}{m(A_{k,j})}\right) = \left(\frac{m(X_{n,i})}{2m(X_{n,j})}, \frac{2m(X_{n,i})}{m(X_{n,j})}\right).$$

(We use $(\frac{3}{4})^2 = \frac{9}{16} > \frac{1}{2}$ here.)

Define a partial isometry H^n_{ij} on $L^2(0,1)$ by

$$H^n_{ij} f(x) = \left(\frac{dh^n_{ji}}{dx}\right)^{1/2} \chi_{X_{n,i}} f(h_{ji}(x)).$$

It is clear that this has initial space $L^2(X_{n,j})$ and range space $L^2(X_{n,i})$. We identify $L^2(X_{n,i})$ with the range of $W^k_{11} E_{\mathcal{N}}(X_{n,i})$, where $k = k_n(i)$. Then we define a system of partial isometries on $\mathcal{H}$ by

$$V^{n,ij}_{pq} = U^{n,i}_{p1} H^n_{ij} U^{n,j}_{1q}.$$

for $1 \le i, j \le N(n)$, $1 \le p \le m_{n,i}$ and $1 \le q \le m_{n,j}$. The matrix unit relations on the homeomorphisms ensures that the $V^{n,ij}_{pq}$'s form a set of matrix units.

Define $\mathcal{H}_n$, $\mathcal{M}_n$ and $\mathcal{T}(\mathcal{M}_n)$ as before. Now we define a $*$-homomorphism Ψ_n of $\mathcal{B}(\mathcal{H}_n)$ into $\mathcal{B}(\mathcal{H})$ by

$$\Psi_n(E^{n,ij}_{pq}) = V^{n,ij}_{pq}$$

for $1 \le i, j \le N(n)$, $1 \le p \le m_{n,i}$ and $1 \le q \le m_{n,j}$. This is a (non-unital) $*$-homomorphism that carries $\mathcal{T}(\mathcal{M}_n)$ onto a subalgebra $\mathcal{B}_n$ of $\mathcal{T}(\mathcal{N})$.

Each $X_{n,i}$ is split by the partition F_{n+1} into $s_n = 2^{m(n+1)-m(n)}$ subsets $X_{n+1,i'}$. As before, set

$$\Sigma_{n,i} = \{i' \colon J^{n+1}_{i'} \subset J^n_i \quad \text{and} \quad \kappa_{n+1}(i') = \kappa_n(i)\}.$$

The homeomorphism h^n_{ij} carries subintervals of $X_{n,j}$ onto subintervals of $X_{n,i}$. Thus for each $j' \in \Sigma(n, j)$, there is a unique $i' = \tau^n_{ij}(j')$ such that $h^n_{ij}(X_{n+1,j'}) = X_{n+1,i'}$. It follows from the definition of these functions that

$$h^{n+1}_{i'j'} = h^n_{ij}|_{X_{n+1,j'}},$$

whence

$$h^n_{ij} = \sum_{j' \in \Sigma_{n,j}} h^{n+1}_{\tau^n_{ij}(j')j'}.$$

Now we can show that $\mathcal{B}_n$ is contained in $\mathcal{B}_{n+1}$. Indeed, it follows from these relations that for each $j' \in \Sigma_{n,j}$ and $i' = \tau^n_{ij}(j')$,

$$V^{n,ij}_{pq} E_{\mathcal{N}}(X_{n+1,j}) = V^{n+1,i'j'}_{pq}.$$

Thus

$$V_{pq}^{n,ij} = \sum_{j' \in \Sigma_{n,j}} V_{pq}^{n,ij} E_{\mathcal{N}}(X_{n+1,j}) = \sum_{j' \in \Sigma_{n,j}} V_{pq}^{n+1, \tau_{ij}^n(j')j'}.$$

The inclusions α_n of $\mathcal{B}(\mathcal{H}_n)$ into $\mathcal{B}(\mathcal{H}_{n+1})$ are induced by the formulae of the previous paragraph. Namely

$$\alpha_n(E_{pq}^{n,ij}) = \sum_{j' \in \Sigma_{n,j}} E_{pq}^{n+1, \tau_{ij}^n(j')j'}.$$

Evidently, this is a non-unital multiplicity s_n imbedding of the $d_n \times d_n$ matrices into the $d_{n+1} \times d_{n+1}$ matrices, where $d_n = \dim \mathcal{H}_n$. The limit is thus a matroid C*-algebra $\mathfrak{A}$. Evidently $\Psi_{n+1}\alpha_n = \Psi_n$. So the limit $\Psi = \varinjlim \Psi_n$ is a $*$-monomorphism of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$. It carries the limit algebra $\mathcal{A} = \varinjlim(\mathcal{T}(\mathcal{M}_n), \alpha_n)$ onto a subalgebra $\mathcal{B}$ of $\mathcal{T}(\mathcal{N})$.

We will show that $\overline{\mathcal{B}}^{\text{weak-}*}$ contains the diagonal algebra $\mathcal{D}(\mathcal{N})$. The diagonal has a weak-$*$ dense spanning set $W_{pq}^k E_{\mathcal{N}}(J \cap A_k)$ where J are diadic intervals and $0 \le p, q < m(k)$. These are in turn in the weak-$*$ closed span of the operators $W_{pq}^k E_{\mathcal{N}}(J \cap A_{k,j})$. These terms will be a sum of certain $V_{pq}^{n,ii}$'s in $\mathcal{B}_n$ provided that both k and j are at most n, and the endpoints of J are in $2^{-n}\mathbb{N}$.

Next we claim that $\operatorname{Lat}\mathcal{B} = \mathcal{N}$. As $\overline{\mathcal{B}}^{\text{weak-}*}$ contains the diagonal, any invariant subspace has the form $E_{\mathcal{N}}(X)\mathcal{H}$ for some measurable subset X of [0, 1]. It suffices to show that X is an initial segment modulo a null set. If it is not, then there are $0 < a < b < c < d < 1$ so that $(a, b) \cap X^c$ and $(c, d) \cap X$ have positive measure. Thus there are sets A_{n_1, j_1} and A_{n_2, j_2} which intersect these sets in positive measure respectively. Consequently, with $n = \max\{n_1, j_1, n_2, j_2\}$, there are $i < j$ so that $X^c \cap X_{n,i}$ and $X \cap X_{n,j}$ have positive measure. By the Lebesgue density theorem, for n' sufficiently large, there are diadic subintervals $X_{n',i'}$ of the Cantor set $X_{n,i}$ which intersect X^c in a set of measure at least $\frac{3}{4}m(X_{n',i'})$. Likewise, for the same (sufficiently large) n', X intersects some $X_{n',j'}$ in measure at least $\frac{3}{4}m(X_{n',j'})$. Thus $h_{i'j'}^{n'}$ maps $X \cap X_{n',j'}$ onto a subset of $X_{n',i'}$ of measure at least $\frac{3}{8}m(X_{n',i'})$. Since $\frac{3}{8} + \frac{3}{4} = \frac{9}{8}$, it follows that

$$m\big(X^c \cap X_{n',i'} \cap h_{i'j'}^{n'}(X \cap X_{n',j'})\big) > m(X_{n',i'})/8 > 0.$$

This implies that $E_{\mathcal{N}}(X)^{\perp} V_{11}^{n',i'j'} E_{\mathcal{N}}(X) \neq 0$, contradicting the invariance of $E_{\mathcal{N}}(X)\mathcal{H}$.

Finally we invoke a result of Arveson [1] (or Radjavi–Rosenthal [15] for the weak operator topology) that every weak-$*$ closed subalgebra of a nest algebra that contains a masa and has the same invariant subspaces is the whole nest algebra. So $\mathcal{B}$ is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$. $\square$

The existence of both finite rank atoms and some continuous part is an obstacle to doing the construction of the previous theorem. We will show, in fact, that such nest algebras are not the weak-$*$ closure of a representation of a regular limit of finite dimensional nest algebras. However, when the nest is infinite in the sense that all atoms have infinite rank, an ad hoc modification of our argument will suffice to establish the result.

COROLLARY 2.2. *Let $\mathcal{N}$ is nest with no finite rank atoms. Then there is a regular limit $\mathcal{A}$ of finite dimensional nest algebras such that its enveloping C*-algebra is a matroid algebra and a $*$-extendible representation Φ of $\mathcal{A}$ such that $\Phi(\mathcal{A})$ is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$.*

Proof. For each infinite atom A_i of $\mathcal{N}$, choose a non-atomic masa isomorphic to $L^\infty(0, 1)$ in $\mathcal{B}(A_i\mathcal{H})$. Chop this up into Cantor sets with union having full measure. Then proceed as in the proof of Theorem 2.1. At the n-th stage, include the first n Cantor sets from the first n infinite atoms and include cuts of these atoms into 2^n diadic intervals. Note that the atoms cut up the continuous part. So the introduction of finitely many atoms introduces finitely many more cuts in the infinite part.

The construction proceeds exactly as before except that the full matrix algebra is constructed on each atom, not just the upper triangular part. It is easy to verify that we construct finite dimensional nest algebras with regular imbeddings as before. To verify density, note that for each atom, the resulting algebra will be dense in the masa, will contain the Volterra nest by our previous argument and will be self-adjoint. So it is dense in $\mathcal{B}(A_i\mathcal{H})$ for each i. Thus the image of the diagonal is weak-$*$ dense in the diagonal algebra $\mathcal{D}(\mathcal{N})$. The same argument as above shows that the invariant subspace lattice is a nest. So the image algebra is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$ by Arveson's theorem. □

Orr and Peters [11] consider the problem of imbedding triangular limit algebras as weak-$*$ dense subalgebras of nest algebras from the other side of the fence, fixing the limit algebra and asking which nest algebras can be obtained as the weak-$*$ closure of various representations. Surprisingly, they are able to obtain nests with uniform multiplicity two from triangular algebras (which have multiplicity one). L. Zmarzly, a student of Orr's, has extended this to uniform multiplicity of arbitrary cardinality. While our techniques don't directly apply, they indicate that it may be possible to control mixed multiplicities in their context as well.

We end this section with an argument showing that the existence of a dense representation is limited to the two cases already handled, the atomic case and the case of infinite multiplicity. See [11, Prop. III.1.1] for a suggestive result along these lines with more stringent hypotheses.

THEOREM 2.3. *Suppose that there is a regular limit $\mathcal{A}$ of finite dimensional nest algebras such that its enveloping C*-algebra is a matroid algebra and a $*$-extendible representation Φ of $\mathcal{A}$ such that $\Phi(\mathcal{A})$ is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$. Then $\mathcal{N}$ is either atomic or has no finite rank atoms.*

Proof. Suppose to the contrary that $\mathcal{N}$ has an atom E of rank $k < \infty$ and has non-trivial continuous part. Let Φ_E denote the compression of Φ to the atom E. This is a completely contractive homomorphism onto $\mathcal{B}(E\mathcal{H})$ which is isomorphic to $\mathfrak{M}_k$. The limit algebra $\mathcal{A}$ is the limit of finite dimensional nest algebras $\mathcal{A}_n$. To simplify notation, we will consider the $\mathcal{A}_n$'s as a nested sequence of subalgebras of $\mathcal{A}$. Since $\mathfrak{M}_k$ is finite dimensional, the restriction to $\mathcal{A}_n$ is surjective for $n \geq n_0$, say. The only way for such an algebra to map onto a full matrix algebra is for it to factor through the compression to an atom A_n of $\mathcal{A}_n$ which has rank k.

(This is an easy fact, and is likely well known. The strictly upper triangular operators in a finite dimensional nest algebra is a nilpotent ideal, and thus is in the kernel of Φ_E. So the map factors through the diagonal, a sum of full matrix algebras. Now the structure of maps between matrix algebras (cf. [14]) shows that Φ_E is obtained by restriction to a summand isomorphic to $\mathfrak{M}_k$.)

Let E^n_{ij} for $1 \leq i, j \leq k$ be the standard matrix units for the $k \times k$ matrix algebra $A_n\mathcal{A}_nA_n$. Then $E_{ij} = \Phi_E(E^n_{ij})$ is a set of matrix units for $\mathcal{B}(E\mathcal{H})$ independent of $n \geq n_0$.

The atoms A_n must form a decreasing sequence of projections in $\mathcal{A}$. Reverting to the limit picture, the imbedding of $\mathcal{A}_n$ into $\mathcal{A}_{n+1}$ must send A_n onto A_{n+1} with multiplicity one (together with additional imbeddings into other blocks, possibly). The projections $P_n = \Phi(A_n)$ then form a decreasing sequence of projections in $\mathcal{D}(\mathcal{N})$ which converge strongly to a projection.

We claim that this limit is E. Indeed, there is a sequence X_n in $\mathcal{A}_n$ such that $\Phi(X_n)$ converges weak-$*$ to E. We may suppose that $X_n = A_nX_nA_n$. The compression $\Phi_E(X_n)$ must converge to the identity element in norm, which implies that

$$\lim_{n\to\infty} \|X_n - A_n\| = 0,$$

since $\Phi_E|_{A_n\mathcal{A}_nA_n}$ is a $*$-isomorphism and $\Phi_E(A_n) = I$. Hence we may replace X_n by A_n, which establishes the claim.

Now $\mathcal{N}$ also has some continuous part. For definiteness, let us assume that there is an interval F of $\mathcal{N}$ with $F \prec E$ in the order on $\mathcal{N}$ such that F dominates a non-atomic projection Q. Choose unit vectors $x = E_{11}x$ and $y = Qy$. Then $R = yx^*$ is a rank one operator in $\mathcal{T}(\mathcal{N})$. Choose elements $R_n \in \mathcal{A}_n$ such that $\Phi(R_n)$ converges weak-$*$ to R. For n sufficiently large, $QR_nx \neq 0$. Thus there is a matrix unit U_{n_0} of $\mathcal{A}_{n_0}$ such that $U_{n_0} = U_{n_0}E^{n_0}_{11}$ and $Q\Phi(U_{n_0})x \neq 0$.

Since U_{n_0} is a matrix unit, $V_{n_0} = \Phi(U_{n_0})$ is a partial isometry with initial projection $\Phi(E^{n_0}_{11}) \leq P_{n_0}$. Moreover the initial projection of U_{n_0} commutes with A_n for $n > n_0$. Thus $U_n = U_{n_0}A_n$ are matrix units in $\mathcal{A}_n$ for $n \geq n_0$ and $V_n = \Phi(U_n) = V_{n_0}P_n$ is a partial isometry. Consequently $QV_nx = QV_{n_0}x = z$ is a fixed vector supported on the non-atomic part of $\mathcal{N}$ for all $n \geq n_0$.

The range projections $B_n = U_nU_n^*$ form a decreasing sequence of minimal projections in $\mathcal{A}_n$. Thus $Q_n = \Phi(B_n)$ form a decreasing sequence of projections in $\mathcal{D}(\mathcal{N})$ such that $Q_nz = z$ for all n. Consequently Q_n decreases to a projection Q'

such that $Q'z = z$. Hence Q' supports a non-trivial non-atomic part. This is impossible. Indeed, there are many projections P' in $\mathcal{D}(\mathcal{N})$ commuting with Q' such that $0 < P'Q' < Q'$. Hence there must be minimal projections B' in the diagonal of $\mathcal{A}_n$ for n sufficiently large such that $\Phi(B')$ has this property. However the minimality of B_n ensures that $B'B_n$ is either 0 or B_n. This contradiction establishes our result. □

3. Non-dense representations of limit algebras

We conclude this paper with a related construction that indicates that our concern about the derivatives of the h_{ij}^n's was more than just a technical convenience. In fact, it is possible for the representations of limit algebras to contain the whole diagonal of the nest algebra and yet fail to be dense. Under hypotheses similar to what was achieved in Theorem 2.1, the density of $\Phi(\mathcal{A})$ in $\mathcal{T}(\mathcal{N})$ is equivalent to the irreducibility of the representation Φ. An application of direct integral theory provides a complete description of the possible weak-$*$ closures when the range contains the diagonal.

THEOREM 3.1. *Suppose that $\mathcal{A}$ is a Dirichlet subalgebra of a C^*-algebra $\mathfrak{A}$. Let Φ be a $*$-representation of $\mathfrak{A}$ on $\mathcal{H}$ such that $\Phi(\mathcal{A})$ is contained in a nest algebra $\mathcal{T}(\mathcal{N})$, and contains the diagonal $\mathcal{D}(\mathcal{N})$ in it weak-$*$ closure. Then $\Phi(\mathcal{A})$ is weak-$*$ dense in $\mathcal{T}(\mathcal{N}) \cap \Phi(\mathfrak{A})''$. In particular, $\Phi(\mathcal{A})$ is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$ if and only if Φ is irreducible.*

Proof. First suppose that Φ is irreducible. By the result of Arveson used in the last paragraph of the proof of Theorem 2.1, density will follow if we show that $\mathrm{Lat}(\Phi(\mathcal{A})) = \mathcal{N}$. Any invariant projection P is invariant for $\mathcal{D}(\mathcal{N})$, and thus lies in $\mathcal{D}(\mathcal{N})' = \mathcal{N}''$. Thus $P = E_{\mathcal{N}}(X)$ for some measurable set X. If P does not belong to the nest, then there is a nest projection N such that $P^\perp N \neq 0$ and $PN^\perp \neq 0$. Now $\Phi(\mathcal{A} + \mathcal{A}^*)$ is irreducible, and thus there are elements A and B in $\mathcal{A}$ such that

$$0 \neq P^\perp N \Phi(A + B^*) P N^\perp = P^\perp N \Phi(A) P N^\perp,$$

where we use the fact that $\Phi(B^*)$ lies in $\mathcal{T}(\mathcal{N})^*$ to conclude that $N\Phi(B^*)N^\perp = 0$. This contradicts the invariance of P. Hence $\Phi(\mathcal{A})$ is weak-$*$ dense in $\mathcal{T}(\mathcal{N})$.

The general case will follow from an application of the direct integral decomposition over the commutant $\mathfrak{M} = \Phi(\mathfrak{A})'$. This is a von Neumann subalgebra of $\mathcal{D}(\mathcal{N})'$, and thus is abelian. By the spectral theorem, this algebra is isomorphic and weak-$*$ homeomorphic to some $L^\infty(\Lambda, \lambda)$. By von Neumann's direct integral theory (see [16, Theorem 8.21]), $\Phi(\mathfrak{A})''$ decomposes as a direct integral over Λ of von Neumann algebras $\mathfrak{A}(t)$. Since $\mathfrak{M}$ is the full commutant, these algebras are irreducible for almost all t. By the corresponding direct integral theory for non-self-adjoint operator algebras [2, 10], $\overline{\Phi(\mathcal{A})}^{\text{weak-}*}$ decomposes as a direct integral of $\mathcal{A}(t)$, and $\mathcal{T}(\mathcal{N}) \cap \mathfrak{M}'$ decomposes as a direct integral of algebras $\mathcal{T}(t)$. Moreover $\mathcal{T}(t)$ will be a nest algebra for almost all t. The $\mathcal{A}(t)$ will contain the diagonal algebra $\mathcal{D}(t)$ obtained from the direct integral decomposition of $\mathcal{D}(\mathcal{N})$ and be contained in $\mathcal{T}(t)$. Thus by the first paragraph, $\mathcal{A}(t) = \mathcal{T}(t)$ for almost all t. Hence

$$\overline{\Phi(\mathcal{A})}^{\text{weak-}*} = \mathcal{T}(\mathcal{N}) \cap \mathfrak{M}' = \mathcal{T}(\mathcal{N}) \cap \Phi(\mathfrak{A})''. \qquad \square$$

This suggests trying to modify the construction of the previous proof to yield a representation which is reducible. Surprisingly, it turns out that this is possible with a modification of our argument.

THEOREM 3.2. *If $\mathcal{N}$ is a continuous nest, there is a regular limit $\mathcal{A}$ of finite dimensional nest algebras such that its enveloping C*-algebra is a matroid algebra and a $*$-extendible representation Φ of $\mathcal{A}$ such that $\Phi(\mathcal{A} \cap \mathcal{A}^*)$ is weak-$*$ dense in $\mathcal{D}(\mathcal{N})$, but $\Phi(\mathcal{A})$ is not dense in $\mathcal{T}(\mathcal{N})$ because Φ is reducible.*

We begin with some more delicate constructions of absolutely continuous homeomorphisms on the interval. We use the same notation as before. In addition, let P_s for $s \geq 1$ be a measurable partition of $[0, 1]$ with the property that whenever O is open and $O \cap A_{k,n}$ has positive measure, then $O \cap A_{k,n} \cap P_s$ also has positive measure. It is well known that $[0, 1]$ can be partitioned into sets P_s that all meet each open set in a set of positive measure. The same can thus be done on any Cantor set $A_{k,n}$, say $P_{k,n,s}$. Then one may define $P_s = \bigcup_{k,n} P_{k,n,s}$.

It is shown in [5, Theorem 2.4] that if P_s and Q_s are two partitions of $[0, 1]$ into measurable sets that intersect every interval in positive measure, then there is a homeomorphism h of the interval such that h and h^{-1} are absolutely continuous so that $h(P_s) = Q_s$ modulo null sets for all s. An examination of the argument shows the same holds for order preserving homeomorphisms of Cantor sets.

Our technique will be to build the matrix unit systems of absolutely continuous homeomorphisms so that they will leave invariant the partition P_s. This will be done at the expense of the bounds we had on the derivatives. This is necessary, as this control made it possible to prove irreducibility of the representation.

LEMMA 3.3. *Let X_j be compact subsets of $[0, 1] \setminus \mathbb{Q}$ which intersect each open set either in the empty set or a set of positive Lebesgue measure for $1 \leq j \leq n$. Let P_s^j be countable partitions of X_j respectively into measurable subsets which meet every open subset of X_j in a set of positive measure. Suppose that certain absolutely continuous homeomorphisms h_{ij} of $[0, 1]$ have been defined so that:*

(i) $h_{ii} = \mathrm{id}$ *for* $1 \leq i \leq n$;
(ii) *if* h_{ij} *is defined, then* h_{ji} *is defined*;
(iii) *if* h_{ij} *and* h_{jk} *are defined, then* h_{ik} *is defined and* $h_{ik} = h_{ij} \circ h_{jk}$;
(iv) h_{ij} *is absolutely continuous*;
(v) $h_{ij}(X_j) = X_i$;

(vi) $h_{ij}(X_j \cap P_s^j) \triangle (X_i \cap P_s^i)$ *has measure zero for all* $s \geq 1$;
(vii) $h_{ij}([0,1] \cap \mathbb{Q}) = [0,1] \cap \mathbb{Q}$.

Then this family may be extended to a complete family (matrix unit system) $\{h_{ij}\colon 1 \leq i, j \leq n\}$ *with the same properties.*

Proof. First we show how to construct one map h with properties (iv)–(vii). Using [5, Theorem 2.4], construct the order preserving homeomorphism h from X_j onto X_i with the desired properties including that h^{-1} is also absolutely continuous. Extend the definition of h to the whole interval by defining it to take each interval of $[0,1] \setminus X_j$ onto the corresponding interval of $[0,1] \setminus X_i$. This may be done with piecewise linear functions that carry rational points onto rational points (for example, by choosing a sequence of rationals in each interval converging to each endpoint, matching up these rational points, and making h piecewise linear in between). It is then evident that h and h^{-1} are absolutely continuous on the rest of $[0,1]$ as well.

It is easy to see that the intervals may be grouped into subsets with complete sets of matrix units defined and with no maps defined between intervals in distinct subsets. For convenience, we may suppose that there are exactly two such groups. Use the previous paragraph to define one homeomorphism between an interval in the first group to one in the second satisfying (iv)–(vii). This will uniquely determine a complete set of matrix units on the union which satisfy properties (i)–(iii). Properties (iv)–(vii) are easily verified for the new maps constructed in this way. □

Proof of Theorem 3.2. The proof follows exactly the lines of Theorem 2.1. However instead of using the matrix unit homeomorphisms constructed there, we use Lemma 3.3 to arrange that each partition P_s is preserved by every h_{ij}. The difference in the construction of these matrix units is in the induction step from the n-th partition to the $n+1$-st.

At the first stage, use Lemma 3.3 to construct a complete matrix unit system of absolutely continuous homeomorphisms h_{ij}^1 between the intervals J_i^1 matching up the sets $X_{1,i}$ and the partitions $P_s \cap X_{1,i}$. Now assume that at the nth stage, we also have a complete matrix unit system of absolutely continuous homeomorphisms h_{ij}^n between the intervals J_i^n matching up the sets $X_{n,i}$ and the partitions $P_s \cap X_{n,i}$ for $1 \leq i, j \leq N(n)$. Construct a new partition F_{n+1} as before, but then also include in F_{n+1} all point obtained by permuting points in F_{n+1} by the (partial) homeomorphisms h_{ij}^n. Clearly this still yields a finite set.

Given i, j and j' such that $J_{j'}^{n+1}$ intersects some $X_{n,j}$, then there will be an integer $i' = \tau_{ij}^n(j')$ such that $h_{ij}^n(X_{n+1,j'}) = X_{n+1,i'}$ in the subinterval $J_{i'}^{n+1}$ of J_i^n. This is because the h_{ij}^n carry intervals of the partition F_{n+1} onto other such intervals, and since $X_{n+1,j'}$ has positive measure, so does $X_{n+1,i'}$ by the absolute continuity. So define $h_{i'j'}^{n+1}$ to be the restriction of h_{ij} to $J_{j'}^{n+1}$ in this case. Otherwise, it is not yet defined.

Notice that if X_{n,i_0} is split by the partition F_{n+1} into s_n subsets $X_{n+1,i'}$ of positive measure, then each (non-trivial) $X_{n,i}$ is also split into exactly s_n such pieces because of the invariance of F_{n+1} under the system h^n_{ij}, say $\{X_{n+1,i'}\colon i' \in \Sigma_{n,i}\}$. Thus we see that we have defined h^{n+1}_{ij} on s_n subsets of the $n+1$st partition with $N(n)$ members each. We also define $h^{n+1}_{ii} = \mathrm{id}$ for all remaining intervals. By Lemma 3.3, we may extend this to a complete matrix unit system of absolutely continuous homeomorphisms that match up the $X_{n+1,j}$'s and the partitions P_s.

Using this system of matrix units, define matrix units of operators $V^{n,ij}_{pq}$ exactly as in Theorem 2.1. The proof proceeds verbatim to define the various maps. The argument showing that the image of the diagonal is weak-$*$ dense in the diagonal of $\mathcal{T}(\mathcal{N})$ is also the same. However the proof that Φ is irreducible fails. Indeed, we have arranged that $E_{\mathcal{N}}(P_s)\mathcal{H}$ are invariant for all $s \geq 1$ by construction because these sets are invariant for the homeomorphisms h^n_{ij}. So Φ is reducible. □

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