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# CAUCHY TRANSFORMS AND COMPOSITION OPERATORS

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**1.** A holomorphic function $f$ on the unit disk $\mathbb{D}$ is a Cauchy transform if it admits a representation

$$f(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \bar{\zeta} z}, \tag{1.1}$$

where $\mu$ is a finite Borel measure on the unit circle $\mathbb{T}$. The space $K$ of all Cauchy transforms becomes a Banach space under the norm $\|f\|_K = \inf \|\mu\|_M$, where the infimum is taken over all Borel measures $\mu$ satisfying (1.1). The Banach space $K$ is clearly the quotient of the Banach space $M$ of Borel measures by the subspace of measures with vanishing Cauchy transforms. It is an immediate consequence of the F. and M. Riesz theorem that a Borel measure $\mu$ has a vanishing Cauchy transform if and only if $\mu$ has the form $d\mu = f\,dm$, where $f \in \bar{H}_0^1$ and $m$ is normalized Lebesgue measure on $\mathbb{T}$. Here $\bar{H}_0^1$ is the subspace of $L^1$ consisting of functions with mean value 0 whose conjugates belong to the Hardy space $H^1$. Hence $K$ is isometrically isomorphic to $M/\bar{H}_0^1$. On the other hand, $M$ admits a decomposition $M = L^1 \oplus M_s$, where $M_s$ is the space of Borel measures which are singular with respect to Lebesgue measure, and $\bar{H}_0^1 \subset L^1$. Consequently $K$ is isometrically isomorphic to $L^1/\bar{H}_0^1 \oplus M_s$. In particular $K$ admits an analogous decomposition $K = K_a \oplus K_s$, where $K_a$ is isometrically isomorphic to $L^1/\bar{H}_0^1$ and $K_s$ to $M_s$.

Now let $\phi$ be a holomorphic map of the unit disk $\mathbb{D}$ into itself. The composition operator $C_\phi f = f \circ \phi$ acts on a variety of spaces of holomorphic functions, most notably the Hardy spaces $H^p$. It was established by Bourdon and Cima [2] that $C_\phi$ also acts on $K$, that is, $f \circ \phi \in K$ for all $f \in K$. It is an immediate consequence of the closed graph theorem that $C_\phi$ is a bounded operator on $K$. In fact Bourdon and Cima provide the estimate

$$\|C_\phi\|_K \leq \frac{2 + 2\sqrt{2}}{1 - |\phi(0)|} \tag{1.2}$$

for the norm of $C_\phi$ on $K$. A new proof of the boundedness of $C_\phi$ on $K$ will follow

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from the considerations in Section 2, and the norm estimate will be improved to

$$\|C_\phi\|_K \leq \frac{1+2|\phi(0)|}{1-|\phi(0)|}. \tag{1.3}$$

This new proof is conceptually similar to the proof of Bourdon and Cima.

The authors would like to thank the referee for pointing out an error in the norm estimate of the multiplication operator $M_z$ used in the submitted version of this paper, and for indicating how to correct it. The referee's comment shows that the multiplication operator has norm 2 on $K$. Indeed, $M_z$ is the adjoint of the backward shift on $A$ and the latter operator is readily seen to have norm 2. The referee's comment leads to an argument, to be presented in Section 4, showing that the estimate (1.3) is sharp in the sense that there are functions $\phi$, with $\phi(0) \neq 0$, for which equality is attained.

In Section 3 it will be shown that $\phi$ induces a compact composition operator on $K$ if and only if the corresponding composition operator on $H^2$ is compact. Thus J. H. Shapiro's characterization [11] of such $\phi$'s carries over to $K$. The argument will show, moreover, that weakly compact composition operators on $K$ must be compact. In Section 4 it will be shown that $C_\phi$ is weak*-weak*continuous on $K$, and hence is the adjoint of an operator on the disk algebra. This operator will be described in terms of Aleksandrov measures.

The approach taken here follows Sarason [10], who showed how to define composition operators on the space $M$ of complex measures on $\mathbb{T}$. The idea is simple. If $\mu$ is a positive Borel measure on $\mathbb{T}$, then the Poisson integral

$$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu(\zeta) \tag{1.4}$$

is a positive harmonic function. Since $\phi$ is holomorphic, $v(z) = u(\phi(z))$ is also a positive harmonic function. By Herglotz's theorem [7, p. 34], $v(z)$ is the Poisson integral of a unique positive measure $\nu$. Sarason defines $S_\phi \mu = \nu$. Since $\|\mu\|_M = u(0)$ and $\|\nu\|_M = v(0) = u(\phi(0))$, it follows from Harnack's inequality that

$$\|\nu\|_M \leq \frac{1+|\phi(0)|}{1-|\phi(0)|}\|\mu\|_M.$$

Applying this to the Jordan decomposition of each complex measure extends $S_\phi$ to all of $M$ with the norm estimate

$$\|S_\phi\|_M \leq \frac{1+|\phi(0)|}{1-|\phi(0)|}. \tag{1.5}$$

It is not difficult to see that the expression on the right of (1.5) is actually equal to the norm of $S_\phi$ on $M$. Indeed, if $\delta_\alpha$ is the unit point mass at $\alpha \in \mathbb{T}$ and if $\tau_\alpha = S_\phi \delta_\alpha$, then it is easy to see that the Poisson integral of $\tau_\alpha$ is the function $u_\alpha(z) = \frac{1-|\phi(z)|^2}{|\alpha - \phi(z)|^2}$. In particular, $\|\tau_\alpha\|_M = u_\alpha(0)$. Choosing $\alpha$ so that $|\alpha - \phi(0)| = 1 - |\phi(0)|$ produces the desired lower bound. Sarason showed that $S_\phi$ maps $L^1$ into itself, and that compactness on $M$ is equivalent to the absolute continuity of the measures $\tau_\alpha$ for all $\alpha \in \mathbb{T}$. This is called the absolute continuity condition. It is important to note that if $\tau_\alpha$ has the Lebesgue decomposition $d\tau_\alpha = h_\alpha\, dm + d\sigma_\alpha$, where $h_\alpha \in L^1$ and $\sigma_\alpha \perp m$, then

$$h_\alpha(\xi) = \lim_{r\to 1} u_\alpha(r\xi) = \frac{1 - |\phi(\xi)|^2}{|\alpha - \phi(\xi)|^2} \tag{1.6}$$

for almost every $\xi \in \mathbb{T}$. The measures $\tau_\alpha$ are the Aleksandrov measures of $\phi$.

It is a consequence of Littlewood's subordination principle [5], [8] that $C_\phi$ is a bounded operator on each Hardy space $H^p$, and it was shown by Shapiro and Taylor [13] that if $C_\phi$ is compact on $H^p$ for some $0 < p < \infty$, then $C_\phi$ is compact on $H^p$ for all $0 < p < \infty$. Later Shapiro [11] gave an expression for the essential norm of $C_\phi$ on $H^2$ in terms of the Nevanlinna counting function of $\phi$, thus providing a function theoretic characterization of compactness on the Hardy spaces $H^p$, $0 < p < \infty$. Since $H^1 \subset L^1$, Sarason's condition implies Shapiro's. Conversely, Shapiro and Sundberg [12] showed that Shapiro's condition implies Sarason's. A direct proof of this equivalence was found by Cima and Matheson [3], who showed that the essential norm of $C_\phi$ on $H^2$ is equal to $\sup_{\alpha\in\mathbb{T}} \sqrt{\|\sigma_\alpha\|_M}$.

**2.** The analysis of composition operators on $K$ begins with the following lemma, which relates composition on measures to composition on Cauchy transforms. It will be convenient to denote the Cauchy transform of $\mu \in M$ by $\hat{\mu}$, so that

$$\hat{\mu}(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \bar{\zeta} z}, \tag{2.1}$$

and the Poisson integral of $\mu$ by $U_\mu(z)$, so that

$$U_\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2}\, d\mu(\zeta). \tag{2.2}$$

LEMMA 1. *Let $\phi$ be a holomorphic map of the unit disk into itself satisfying $\phi(0) = 0$. Let $\mu \in M$ and let $\nu = S_\phi \mu$. Then*

$$\hat{\nu}(z) = \hat{\mu}(\phi(z)).$$

*Proof.* Let $v(z) = U_\mu(\phi(z))$. Then the measure $\nu = S_\phi\mu$ is obtained as the weak* limit of the measures $\frac{1}{2\pi}v(r\zeta)\,|d\zeta|$ as $r \to 1$. It follows that

$$\begin{aligned}
\hat{\nu}(z) &= \int_{\mathbb{T}} \frac{d\nu(\zeta)}{1-\bar{\zeta}z} \\
&= \lim_{r\to 1} \frac{1}{2\pi}\int_{\mathbb{T}} \frac{v(r\zeta)}{1-\bar{\zeta}z}\,|d\zeta| \\
&= \lim_{r\to 1} \frac{1}{2\pi}\int_{\mathbb{T}} \frac{|d\zeta|}{1-\bar{\zeta}z}\int_{\mathbb{T}} \frac{1-|\phi(r\zeta)|^2}{|\xi-\phi(r\zeta)|^2}\,d\mu(\xi) \\
&= \lim_{r\to 1} \int_{\mathbb{T}} d\mu(\xi)\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1}{1-\bar{\zeta}z}\,\frac{1-|\phi(r\zeta)|^2}{|\xi-\phi(r\zeta)|^2}\,|d\zeta|,
\end{aligned}$$

on applying Fubini's theorem. On the other hand the Poisson kernel admits the partial fraction decomposition

$$\frac{1-|w|^2}{|\xi-w|^2} = \frac{1}{1-\bar{\xi}w} - \frac{1}{1-\bar{\xi}/\bar{w}},$$

for $|\xi| = 1$. Moreover,

$$\int_{\mathbb{T}} \frac{1}{1-\bar{\zeta}z}\,\frac{1}{1-\bar{\xi}/\overline{\phi(r\zeta)}}\,|d\zeta| = \int_{\mathbb{T}} \frac{1}{1-\bar{\zeta}z}\,\frac{\overline{\phi(r\zeta)}}{\overline{\phi(r\zeta)}-\bar{\xi}}\,|d\zeta| = 0,$$

since the integrand is antiholomorphic in $\zeta$ and vanishes when $\zeta = 0$ because $\phi(0) = 0$. Hence, applying Cauchy's theorem,

$$\begin{aligned}
\hat{\nu}(z) &= \lim_{r\to 1} \int_{\mathbb{T}} d\mu(\xi)\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1}{1-\bar{\zeta}z}\,\frac{1}{1-\bar{\xi}\phi(r\zeta)}\,|d\zeta| \\
&= \lim_{r\to 1} \int_{\mathbb{T}} \frac{1}{1-\bar{\xi}\phi(rz)}\,d\mu(\xi) \\
&= \lim_{r\to 1} \hat{\mu}(\phi(rz)) \\
&= \hat{\mu}(\phi(z)),
\end{aligned}$$

completing the proof.

It is an immediate consequence of this lemma that composition takes Cauchy transforms to Cauchy transforms in case $\phi(0) = 0$, and that $\|C_\phi\|_K \le 1$ in this case. Indeed,

$$\|\hat{\nu}\|_K \le \|\nu\|_M \le \|\mu\|_M,$$

since $\|S_\phi\|_M = 1$, and

$$\|\hat{\mu}\|_K = \inf\{\, \|\sigma\|_M \mid \hat{\mu} = \hat{\sigma} \,\},$$

so that $\|\hat{\nu}\|_K \leq \|\hat{\mu}\|_K$. Furthermore, if $\pi\colon M \to K$ denotes the quotient operator taking $\mu$ to $\hat{\mu}$, then the lemma shows that the diagram

$$\begin{tikzcd} M \arrow[r, "S_\phi"] \arrow[d, "\pi"'] & M \arrow[d, "\pi"] \\ K \arrow[r, "C_\phi"'] & K \end{tikzcd}$$

commutes.

Since every holomorphic function $\phi$ from the disk into itself admits a factorization $\phi = \lambda_a \circ \psi$, where $\lambda_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation with $a = \phi(0)$, and $\psi = \lambda_a \circ \phi$ satisfies $\psi(0) = 0$, in order to establish a general result it remains only to analyze composition with the functions $\lambda_a$. This is provided by the following lemma.

LEMMA 2. *For every $a \in \mathbb{D}$, $f \circ \lambda_a \in K$ for every $f \in K$. Moreover*

$$\|f \circ \lambda_a\|_K \leq \frac{1+2|a|}{1-|a|}\|f\|_K$$

*for every $f \in K$.*

*Proof.* For each $\beta \in \bar{\mathbb{D}}$, let

$$f_\beta(z) = \frac{1}{1-\bar{\beta}z}.$$

Then $f_\beta \in K$ and $\|f_\beta\|_K = 1$. Indeed, if $|\beta| = 1$, $f_\beta$ is the Cauchy transform of the unit point mass $\delta_\beta$ at $\beta$. The general case was established by Hallenbeck, MacGregor and Samotij [6].

Now fix $a \in \mathbb{D}$ and $\beta \in \mathbb{T}$. Then

$$\begin{aligned} f_\beta \circ \lambda_a(z) &= \frac{1}{1-\bar{\beta}\left(\frac{a-z}{1-\bar{a}z}\right)} \\ &= (1-\bar{a}z)\frac{1}{1-\bar{a}z-\bar{\beta}(a-z)} \\ &= (1-\bar{a}z)\frac{1}{1-\bar{\beta}a-(\bar{a}-\bar{\beta})z} \\ &= \frac{1-\bar{a}z}{1-\bar{\beta}a}\frac{1}{1-\left(\frac{\bar{a}-\bar{\beta}}{1-\bar{\beta}a}\right)z}. \end{aligned}$$

By the observation above, since $\left|\frac{\bar{a}-\bar{\beta}}{1-\bar{\beta}a}\right| = 1$, the second factor has norm at most one in $K$. On the other hand, multiplication by $z$ has norm 2 on $K$, so multiplication by the first factor induces an operator of norm at most $\frac{1+2|a|}{1-|a|}$ on $K$. In particular, $\|f_\beta \circ \lambda_a\|_K \le \frac{1+2|a|}{1-|a|}$.

If $\mu = \sum_{i=1}^n \lambda_i \delta_{\beta_i}$, where $|\beta_i| = 1$, $i = 1, \ldots, n$, and $\sum_{i=1}^n |\lambda_i| = 1$, then

$$\hat{\mu}(z) = \sum_{i=1}^n \lambda_i f_{\beta_i}(z),$$

and

$$\hat{\mu} \circ \lambda_a(z) = \sum_{i=1}^n \lambda_i f_{\beta_i} \circ \lambda_a(z).$$

Hence $\|\hat{\mu} \circ \lambda_a\|_K \le \frac{1+2|a|}{1-|a|}$ once again.

In general, suppose $f \in K$ is the Cauchy transform of a measure $\mu$ with total variation one. Then $\mu$ is a weak* limit of a sequence $(\mu_i)$ of finitely supported measures, also of total variation one. Let $f_i = \hat{\mu}_i$. Clearly $f_i(z) \to f(z)$ for each $z \in \mathbb{D}$, and so $f_i \circ \lambda_a(z) \to f \circ \lambda_a(z)$ for each $z \in \mathbb{D}$. Hence $(f_i \circ \lambda_a)$ converges to $f \circ \lambda_a$ weak* where $K$ has been identified with the dual $A^*$ of the disk algebra $A$. Consequently,

$$\|f \circ \lambda_a\|_K \le \liminf_{i\to\infty} \|f_i \circ \lambda_a\|_K \le \frac{1+2|a|}{1-|a|},$$

and the lemma is proved.

The above analysis is summarized in the following theorem which improves upon the result of Bourdon and Cima. An example indicating that the estimate is sharp will be presented at the end of Section 4.

THEOREM 1. *Let $\phi$ be a holomorphic map of the unit disk into itself. Then $\phi$ induces a bounded composition operator $C_\phi$ on the space $K$ of Cauchy transforms. Moreover,*

$$\|C_\phi\|_K \le \frac{1+2|\phi(0)|}{1-|\phi(0)|},$$

*and this estimate is sharp in the sense that there are functions $\phi$ with $\phi(0) \neq 0$ for which equality is attained.*

**3.** Since composition with the Möbius transformation $\lambda_a$ is invertible on $K$, in order to investigate compactness of composition operators on $K$ it is enough to consider holomorphic maps $\phi$ which fix the origin. In this case composition commutes with the quotient map $\pi\colon M \to K$, and this leads to a sufficient condition for the compactness of $C_\phi$ on $K$. Indeed, suppose $\phi(0) = 0$ and $S_\phi$: $M \to M$ is compact. Since $\pi$ is onto, the open mapping principle produces an $\epsilon > 0$ such that $\pi(B_M(1)) \supset B_K(\epsilon)$, where $B_X(r)$ denotes the ball of radius $r$ centered at the origin in the Banach space $X$. Since $C_\phi$: $M \to M$ is compact, there is a compact set $E$ in $M$ containing $C_\phi(B_M(1))$. Clearly $\pi(E)$ is compact, and by commutativity, $\pi(E)$ contains $C_\phi(B_K(\epsilon))$. Hence $C_\phi$: $K \to K$ is compact. The converse, and a bit more, is also true.

THEOREM 2. *Let $\phi$ be a holomorphic map of the unit disk into itself. Then*:

(*i*) *$C_\phi$ is compact on $K$ if and only if $S_\phi$ is compact on $M$.*
(*ii*) *If $C_\phi$ is weakly compact on $K$, then $C_\phi$ is compact on $K$.*

*Proof.* From the above discussion it is evidently enough to prove that $S_\phi$ is compact on $M$ if $C_\phi$ is weakly compact on $K$ and $\phi(0) = 0$. For $0 < r < 1$ and $\alpha \in \mathbb{T}$, the function $\frac{1}{1-r\bar{\alpha}z}$ is the Cauchy transform of the Poisson probability measure $\frac{1-r^2}{|\xi-r\alpha|^2}\,dm(\xi)$. Since $\frac{\alpha+rz}{\alpha-rz} = \frac{1+r\bar{\alpha}z}{1-r\bar{\alpha}z} = \frac{2}{1-r\bar{\alpha}z} - 1$, the functions $\frac{\alpha+rz}{\alpha-rz}$ satisfy $\left\|\frac{\alpha+rz}{\alpha-rz}\right\|_K \leq 3$. Similarly, $\frac{1}{1-\bar{\alpha}z}$ is the Cauchy transform of the unit point mass $\delta_\alpha$ at $\alpha \in \mathbb{T}$, and again $\left\|\frac{\alpha+z}{\alpha-z}\right\|_K \leq 3$. Clearly $\frac{\alpha+rz}{\alpha-rz} \to \frac{\alpha+z}{\alpha-z}$ weak* (when $K$ is identified with the dual space $A^*$). Consequently, if $C_\phi$ is weakly compact on $K$, $\frac{\alpha+r\phi(z)}{\alpha-r\phi(z)} \to \frac{\alpha+\phi(z)}{\alpha-\phi(z)}$ weakly. Now consider the decomposition $K = K_a \oplus K_s$. Since $\frac{\alpha+r\phi(z)}{\alpha-r\phi(z)} \in H^\infty$ for all $\alpha \in \mathbb{T}$ and $0 < r < 1$, and since, as is well known, $H^\infty \subset K_a$, it follows that $\frac{\alpha+r\phi(z)}{\alpha-r\phi(z)} \in K_a$ for $\alpha \in \mathbb{T}$ and $0 < r < 1$. Since $K_a$ is closed in $K$, it is weakly closed by Mazur's theorem. Hence $\frac{\alpha+\phi(z)}{\alpha-\phi(z)} \in K_a$ for all $\alpha \in \mathbb{T}$. In particular, if $\mu_a$ is any measure whose Cauchy transform is $\frac{\alpha+\phi(z)}{\alpha-\phi(z)}$, then $\mu_\alpha$ is absolutely continuous.

As explained in Section 1, the harmonic function $\Re\frac{\alpha+\phi(z)}{\alpha-\phi(z)}$ is positive and so is the Poisson integral of some positive measure $\tau_\alpha$. Thus

$$\Re\frac{\alpha+\phi(z)}{\alpha-\phi(z)} = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\,d\tau_\alpha(\zeta).$$

Since $\Re\frac{\zeta+z}{\zeta-z} = \frac{1-|z|^2}{|\zeta-z|^2}$, $\Re\frac{\alpha+\phi(z)}{\alpha-\phi(z)}$ is the real part of the analytic function $\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,d\tau_\alpha(\zeta)$, and in fact

$$\frac{\alpha+\phi(z)}{\alpha-\phi(z)} = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,d\tau_\alpha(\zeta),$$

since $\phi(0) = 0$. But

$$\frac{\zeta+z}{\zeta-z} = \frac{1+\bar{\zeta}z}{1-\bar{\zeta}z} = \frac{2}{1-\bar{\zeta}z} - 1,$$

so

$$\frac{\alpha+\phi(z)}{\alpha-\phi(z)} = 2\int_{\mathbb{T}} \frac{1}{1-\bar{\zeta}z}\,d\tau_\alpha(\zeta) - \|\tau_\alpha\|_M.$$

Again, since $\phi(0) = 0$, $\|\tau_\alpha\|_M = 1$ for all $\alpha \in \mathbb{T}$. It follows that the Cauchy transform of $\mu_\alpha - 2\tau_\alpha$ is constant, so the measures $\mu_\alpha$ and $2\tau_\alpha$ have the same singular part. Since $\mu_\alpha$ is absolutely continuous for each $\alpha \in \mathbb{T}$, so is $\tau_\alpha$. Thus $\phi$ satisfies Sarason's absolute continuity condition and so $S_\phi$ is compact on $M$.

**4.** The preceding discussion indicates that $S_\phi$ might be weak*-weak* continuous as an operator on $M$. That this is indeed the case is a simple consequence of the following lemma, which ought to be known.

LEMMA 3. *Let $(\mu_n)$ be a sequence of positive measures with Poisson integrals $u_n$. If $(u_n)$ converges uniformly on compact subsets of the unit disk to the harmonic function $u$, then $u$ is the Poisson integral of a positive measure $\mu$, and the sequence $(\mu_n)$ converges weak*to $\mu$.*

*Proof.* Since $\|\mu_n\|_M = u_n(0)$, and $(u_n)$ converges uniformly on compact set to $u$, $(\mu_n)$ is a bounded sequence. Let $\mu$ be the positive measure whose Poisson integral is $u$, and let $\nu$ be any weak* limit point of the sequence $(\mu_n)$. To show that $\nu = \mu$ it will be enough to show that the Fourier-Stieltjes coefficients $\hat{\mu}_n(k)$ converge to $\mu(k)$ for each integer $k$. But

$$\begin{aligned} u(z) &= \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu(\zeta) \\ &= \int_{\mathbb{T}} \left\{ 1 + \sum_{k=1}^{\infty} \left( z^k \bar{\zeta}^k + \bar{z}^k \zeta^k \right) \right\} d\mu(\zeta) \\ &= \hat{\mu}(0) + \sum_{k=1}^{\infty} \left( \hat{\mu}(k) z^k + \hat{\mu}(-k) \bar{z}^k \right), \end{aligned}$$

since the series converges uniformly. Consequently

$$r^k \hat{\mu}(k) = \int_{\mathbb{T}} \bar{\zeta}^k u(r\zeta)\, dm(\zeta)$$

if $0 < r < 1$. Hence, since $u_n \to u$ uniformly on compact sets, $r^k \hat{\mu}_n(k) \to r^k \hat{\mu}(k)$ for each $k$, and the lemma follows.

This lemma also applies, *mutatis mutandis*, to bounded sequences of measures.

If $\phi$ is a holomorphic map of the disk $\mathbb{D}$ into itself and if $u_\alpha$ is the Poisson integral of the positive measure $\tau_\alpha$, then

$$u_\alpha(z) = \frac{1 - |\phi(z)|^2}{|\alpha - \phi(z)|^2},$$

and clearly $u_{\alpha_n}$ converges to $u_\alpha$ uniformly on compact sets if $\alpha_n \to \alpha$. Hence the function assigning $\tau_\alpha$ to $\alpha$ is weak* continuous. For $f \in C$, the space of continuous functions on $\mathbb{T}$, define

$$A_\phi f(\alpha) = \int_{\mathbb{T}} f(\zeta)\, d\tau_\alpha(\zeta), \qquad \alpha \in \mathbb{T}.$$

The operator $A_\phi$ was introduced by Aleksandrov [1] who showed, among other things, that $A_\phi f(\alpha)$ is defined almost everywhere if $f \in L^1$ and $A_\phi$ is a bounded operator on $L^p$ for $1 \le p \le \infty$. In the present context it is clear that

$$\begin{aligned} \|A_\phi\|_\infty &\le \|f\|_\infty \sup_\alpha \|\tau_\alpha\|_M \\ &= \|f\|_\infty \sup_\alpha \frac{1-|\phi(0)|^2}{|\alpha-\phi(0)|^2} \\ &= \|f\|_\infty \frac{1+|\phi(0)|}{1-|\phi(0)|} \end{aligned}$$

for $f \in C$, with equality for $f = 1$. That $A_\phi f \in C$ for each $f \in C$ is a consequence of the observation following Lemma 3 and the following theorem [4].

THEOREM. *Let $\tau\colon \mathbb{T} \to M$ be a function which assigns a Borel measure $\mu_\alpha$ to each $\alpha \in \mathbb{T}$. For each $f \in C$ define*

$$Tf(\alpha) = \int_{\mathbb{T}} f(\zeta)\, d\mu_\alpha(\zeta).$$

*Then*:

(*i*) *$T\colon C \to C$ is bounded if and only if $\tau$ is weak* continuous.*
(*ii*) *$T$ is weakly compact if and only if $\tau$ is weakly continuous.*
(*iii*) *$T$ is compact if and only if $\tau$ is norm continuous.*

Although this will not be pursued here, it is not too difficult to see that, in the present case, $A_\phi$ is compact if and only if each of the measures $\tau_\alpha$ is absolutely continuous. This depends ultimately on the fact that the singular parts of the $\tau_\alpha$ are supported on the pairwise disjoint Borel sets

$$E_\alpha = \{\, \zeta \in \mathbb{T} \mid \lim_{r\to 1} \phi(r\zeta) = \alpha \,\}, \qquad \alpha \in \mathbb{T}.$$

The next theorem indicates the relationship between composition operators and Aleksandrov operators. For $\phi$ inner this is essentially an observation of Lotto and McCarthy [9].

THEOREM 3. *Let $\phi$ be a holomorphic map of the unit disk into itself. Then the composition operator $S_\phi\colon M \to M$ is the adjoint of the Aleksandrov operator $A_\phi\colon C \to C$.*

*Proof.* This is almost obvious. Indeed, if $\alpha \in \mathbb{T}$, then $\tau_\alpha = S_\phi \delta_\alpha$, so, with $\langle f, \mu \rangle$ indicating the pairing between $f \in C$ and $\mu \in M$,

$$\begin{aligned} \langle A_\phi f, \delta_\alpha \rangle &= A_\phi f(\alpha) \\ &= \int_{\mathbb{T}} f(\zeta)\, d\tau_\alpha(\zeta) \\ &= \langle f, \tau_\alpha \rangle \\ &= \langle f, S_\phi \delta_\alpha \rangle \end{aligned}$$

for each $f \in C$ and $\alpha \in \mathbb{T}$. The general formula

$$\langle A_\phi f, \mu \rangle = \langle f, S_\phi \mu \rangle$$

follows from a weak* density argument as in Section 2.

Finally a similar result holds for $C_\phi$ acting on $K$. The difficulty in this case is that $A_\phi f$ will not be analytic unless $f(0) = 0$ or $\phi(0) = 0$. This can be rectified by introducing a rank one perturbation of $A_\phi$. This leads to the following theorem, the proof of which will be left to the reader.

THEOREM 4. *Let $\phi$ be a holomorphic map of the unit disk into itself. For $f \in C$ let*

$$\tilde{A}_\phi f(\alpha) = A_\phi f(\alpha) - f(0) \frac{\phi(0)}{\alpha - \phi(0)}.$$

*Then $\tilde{A}_\phi$ maps the disk algebra $A$ into itself and the composition operator $C_\phi$: $K \to K$ is the adjoint of the perturbed Aleksandrov operator $\tilde{A}_\phi$: $A \to A$.*

Theorem 4 leads to a quick proof of the estimate in Theorem 1. Since $C_\phi$ is the adjoint of $\tilde{A}_\phi$, it is enough to estimate $\|\tilde{A}_\phi\|_A$. Indeed $\|A_\phi\|_C \leq \frac{1+|\phi(0)|}{1-|\phi(0)|}$, while $\|\frac{\phi(0)}{\alpha-\phi(0)}\|_A = \frac{|\phi(0)|}{1-|\phi(0)|}$. Since $|f(0)| \leq \|f\|_A$, the estimate $\|\tilde{A}_\phi\|_A \leq \frac{1+2|\phi(0)|}{1-|\phi(0)|}$ follows.

An analysis of the disk automorphisms $\phi(z) = \lambda \frac{z+a}{1+\bar{a}z}$, $|\lambda| = 1$, $|a| < 1$, will show that the norm estimate is sharp. Indeed, for $\alpha \in \mathbb{T}$,

$$\begin{aligned} \frac{1 - |\phi(z)|^2}{|\alpha - \phi(z)|^2} &= \frac{1 - \left|\frac{z+a}{1+\bar{a}z}\right|^2}{\left|\alpha - \lambda \frac{z+a}{1+\bar{a}z}\right|^2} \\ &= \frac{(1-|a|^2)(1-|z|^2)}{|\alpha(1+\bar{a}z) - \lambda(z+a)|^2} \\ &= \frac{(1-|a|^2)(1-|z|^2)}{|(\bar{\lambda}\alpha - a) - (1 - \bar{\lambda}\alpha\bar{a})z|^2} \\ &= \frac{1-|a|^2}{|1-\bar{\lambda}\alpha\bar{a}|^2} \frac{1-|z|^2}{\left|\frac{\bar{\lambda}\alpha - a}{1-\bar{\lambda}\alpha\bar{a}} - z\right|^2}. \end{aligned}$$

Since $\left|\frac{\bar{\lambda}\alpha-a}{1-\bar{\lambda}\alpha\bar{a}}\right| = 1$, this function is the Poisson integral of the measure

$$\tau_\alpha = \frac{1-|a|^2}{|1-\bar{\lambda}\alpha\bar{a}|^2}\delta_{\left(\frac{\bar{\lambda}\alpha-a}{1-\bar{\lambda}\alpha\bar{a}}\right)}.$$

Hence, noting that $\phi(0) = \lambda a$,

$$\tilde{A}_\phi f(\alpha) = \frac{1-|a|^2}{|1-\bar{\lambda}\alpha\bar{a}|^2} f\left(\frac{\bar{\lambda}\alpha-a}{1-\bar{\lambda}\alpha\bar{a}}\right) - f(0)\frac{\lambda a}{\alpha-\lambda a}.$$

Choosing $\alpha$ so that $\alpha\bar{\lambda}\bar{a} = |a|$ gives

$$\begin{aligned}\tilde{A}_\phi f(\alpha) &= \frac{1+|a|}{1-|a|} f\left(\frac{\bar{\lambda}\alpha-a}{1-\bar{\lambda}\alpha\bar{a}}\right) - f(0)\frac{\bar{\alpha}\lambda a}{1-\bar{\alpha}\lambda a}\\ &= \frac{1+|a|}{1-|a|} f(\bar{\lambda}\alpha) - f(0)\frac{|a|}{1-|a|},\end{aligned}$$

since $\frac{\bar{\lambda}\alpha-a}{1-\bar{\lambda}\alpha\bar{a}} = \bar{\lambda}\alpha\frac{1-\lambda\bar{\alpha}a}{1-\bar{\lambda}\alpha\bar{a}} = \bar{\lambda}\alpha\frac{1-|a|}{1-|a|} = \bar{\lambda}\alpha$. If $f$ is a disk automorphism with $f(\bar{\lambda}\alpha) = 1$ and $f(0) = -1+\epsilon$, then

$$\tilde{A}_\phi f(\alpha) = \frac{1+2|a|}{1-|a|} - \epsilon\frac{|a|}{1-|a|}.$$

Letting $\epsilon \to 0$ shows that $\|\tilde{A}_\phi\|_A = \frac{1+2|a|}{1-|a|}$. Hence the norm estimate in theorem 2 is sharp for disk automorphisms.

## References


1. A. B. Aleksandrov, *Multiplicity of boundary values of inner functions*, Izv. Akad. Nauk Armyan. SSR ser. Mat. **22** (1987), 409–503 (Russian).
2. Paul Bourdon and Joseph A. Cima, *On integrals of Cauchy-Stieltjes type*, Houston J. Math. **14** (1988), 465–474.
3. Joseph A. Cima and Alec Matheson, *Essential norms of composition operators and Aleksandrov measures*, Pacific J. Math. (to appear).
4. N. Dunford and J. T. Schwartz, *Linear operators. Part I*, Interscience, New York, 1958.
5. Peter L. Duren, *Theory of $H^p$ spaces*, Academic Press, New York, 1970.
6. D. J. Hallenback, T. H. MacGregor and K. Samotij, *Fractional Cauchy transforms, inner functions and multipliers*, Proc. London Math. Soc. (3) **72** (1996), 157–187.
7. K. Hoffman, *Banach spaces of analytic functions*, Prentice Hall, Englewood Cliffs, N.J., 1962.
8. J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. **23** (1925), 481–519.
9. B. A. Lotto and J. E. McCarthy, *Composition preserves rigidity*, Bull. London Math. Soc. **25** (1993), 573–576.
10. D. Sarason, "Composition operators as integral operators" in *Analysis and partial differential equations*, Lecture Notes in Pure and Applied Math., vol. 122, edited by Cora Sadowsky, Marcel Dekker, New York, 1990, pp. 545–565.
11. J. H. Shapiro, *The essential norm of a composition operator*, Annals of Math. (2) **125** (1987), 375–404.

12. J. H. Shapiro and C. Sundberg, *Compact composition operators on* $L^1$, Proc. Amer. Math. Soc. **108** (1990), 443–449.
13. J. H. Shapiro and P. D. Taylor, *Compact, nuclear, and Hilbert-Schmidt composition operators on* $H^2$, Indiana Univ. Math. J. **125** (1973), 471–496.

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