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DISCRETIZATION OF LINEAR OPERATORS ON $L^p(\mathbb{R}^N)$

MARIA J. CARRO

1. Introduction

We say that the boundedness of an operator $T\colon L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ can be discretized if we can characterize it by the boundedness of a collection of operators T_n on ℓ^p. Throughout this paper, we shall work under the restriction $1 < p < \infty$. There are many results of this type in the literature:

(a) Using simple estimates and the density of the simple functions on $L^p(\mathbb{R}^N)$, one can obtain that the boundedness of a linear operator on $L^p(\mathbb{R})$ is equivalent to the boundedness on ℓ^p of the operators associated to the matrices

$$\left(\left\langle 2^{kN/p}T\left(\chi_{(0,1)}(2^k\cdot -n)\right), 2^{kN/p'}\chi_{(0,1)}(2^k\cdot -m)\right\rangle\right)_{n,m},$$

uniformly in $k \in \mathbb{Z}$.

(b) Using Shannon's sampling theorem (see §3) one can show that the boundedness of a linear operator on $L^p(\mathbb{R}^N)$ is equivalent to the boundedness on $\ell^p(\mathbb{Z}^N)$ uniformly in k of the operator associated to the matrix

$$\left(\langle T(2^{kN/p}\mathrm{sinc}\,(2^k\cdot -n)), 2^{kN/p'}\mathrm{sinc}\,(2^k\cdot -m)\rangle\right)_{n,m},$$

with $\mathrm{sinc}\, x = \prod_j \frac{\sin \pi x_j}{\pi x_j}$.

(c) In the context of Wavelet theory (see [M1], [M2]), the boundedness of a linear operator on $L^2(\mathbb{R})$ is equivalent to the boundedness on $\ell^2(\mathbb{Z}^2)$ of the operator

$$(a_{n,k})_{n,k} \to \left(\sum_{n,k}\left\langle 2^{kN/p}T(\varphi(2^k\cdot -n)), 2^{k'N/p'}\phi(2^{k'}\cdot -m)\right\rangle a_{n,k}\right)_{m,k'},$$

where φ and ϕ are wavelets. In [M2], they use this result to give a proof of the T1 theorem for singular operators.

(d) A result of de Leeuw and Jodeit (see [D] and [J]) shows that if $\operatorname{supp} m \subset (-1/2, 1/2)^N$ and $\hat{K} = m$, then m is a multiplier in $L^p(\mathbb{R}^N)$ if and only if the sequence $(K(n))_n$ gives a convolution kernel on ℓ^p.

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(e) Using the properties of the functions of exponential type, the previous result was extended in [AC] to show that if the function φ is such that

(1) $\hat{\varphi}$ is a multiplier on $L^p(\mathbb{R}^N)$,
(2) $\operatorname{supp}\hat{\varphi} \subset [-R, R]^N$, $R < 1$,
(3) for some $\varepsilon > 0$, $\chi_{(-\varepsilon,\varepsilon)^N}/\hat{\varphi} \in M_p(\mathbb{R}^N)$,

then a convolution kernel K is bounded on $L^p(\mathbb{R}^N)$ if and only if the sequence $((K_t * \varphi)(n))_n$ gives a convolution kernel on ℓ^p uniformly in $t > 0$.

Our aim in this paper is to show that in fact the three conditions we have just mentioned on φ are sufficient to discretize the boundedness of any linear operator and not only those of convolution type. Moreover, condition (3) can be replaced by an almost approximation of the identity condition (see Definition 3.1.1).

The paper is organized as follows: In Section 2, we study some necessary conditions so that the discretization result holds. Section 3 contains the main results of this paper and it is divided in three parts. Section 3.1 concerns the case when φ is an almost approximation of the identity, Section 3.2 is related to the condition (3) we have mentioned above (that we shall call the local invertibility property) and Section 3.3 concerns to integral operators. Section 4 presents some applications of our results and in Section 5 we study some connections of the local invertibility property with multiresolution analysis (MRA) on L^p (see [M1]).

We shall use $f \sim g$ to indicate the existence of two positive constants A and B such that $Af \leq g \leq Bf$. As usual, $\varphi_t(x) = t^{-N}\varphi(x/t)$ with N the dimension, M_p will be the class of Fourier multipliers on $L^p(\mathbb{R}^N)$ and constants such as C may change from one occurrence to the next.

Throughout this paper, we shall identify the torus $\mathbb{T}^N = (-1/2, 1/2)^N$.

2. Preliminaries

Let φ and ϕ be two functions in L^2.

Definition 2.1. We say that a pair of functions (φ, ϕ) discretize linear operators if the following condition holds: any linear operator T is bounded on $L^p(\mathbb{R}^N)$ if and only if the operators $\tilde{T}_{k,k'}$ associated to the matrices (whenever this makes sense)

$$\left(\langle 2^{kN/p} T(\varphi(2^k \cdot -n)), 2^{k'N/p'}\phi(2^{k'} \cdot -m)\rangle\right)_{n,m},$$

are bounded on ℓ^p uniformly in $k, k' \in \mathbb{Z}$.

Moreover,

$$\|T\|_{L^p(\mathbb{R}^N)\to L^p(\mathbb{R}^N)} \sim \sup_{k,k'} \|\tilde{T}_{k,k'}\|_{\ell^p\to\ell^p}.$$

For simplicity, we shall write $(\varphi, \phi) \in D_p$.

In the case that the boundedness of T can be characterized by the boundedness of $T_{k,k}$ uniformly in k, we shall write $(\varphi, \phi) \in DD_p$ (Diagonal Discretization result).

First of all, let us observe that using duality and the adjoint operator, $D_p = D_{p'}$ with p' the conjugate exponent of p. Then, using interpolation, we get $D_p \subset D_2$.

Let us now study some necessary conditions for the pair (φ, ϕ) to be in D_p.

If we take T to be the identity operator, then the sequence $\{\Psi(n) = (\varphi * \phi)(n)\}_n$ is a convolution kernel on ℓ^p. Similarly, if we take T to be the translation operator $Tf(x) = f(x+\theta)$, then $\{\Psi(n+\theta)\}_n$ is a convolution kernel on ℓ^p with norm uniform in θ. In this case, we write $\Psi \in U_p$. Observe that $U_p = U_{p'}$ and $U_p \subset L^p \cap L^{p'}$.

In fact, this U_p condition implies that $\hat{\Psi} \in M_p$, since, for every $f \in L^p$,

$$\begin{aligned} \|f * \Psi\|_p^p &= \int_{S^{n-1}} \sum_{n\in\mathbb{Z}^N} |(f*\Psi)(n+\theta)|^p \, d\theta \\ &= \int_{S^{n-1}} \sum_{n\in\mathbb{Z}^N} \left| \left(\int_{S^{n-1}} \sum_{m\in\mathbb{Z}^N} f(m+\beta)\Psi(n-m+\theta-\beta)\, d\beta \right) \right|^p d\theta \\ &\le C \int_{S^{n-1}} \sum_{m\in\mathbb{Z}^N} |f(m+\beta)|^p \, d\beta = C\|f\|_p^p. \end{aligned}$$

With an analogous argument one can easily see that the U_p condition also implies that Ψ satisfies the following upper Riesz condition, UR_p:

$$\left\| \sum_n a_n \Psi(\cdot - n) \right\|_p \le C\|a\|_p,$$

for every sequence $a \in \ell^p$. Now, since $UR_p \neq UR_{p'}$ U_p also implies $UR_{p'}$ and therefore UR_2. Thus, we have the following proposition.

PROPOSITION 2.2.

(a) $U_p \Rightarrow UR_p \cap UR_{p'} \Rightarrow UR_2$.
(b) $\varphi \in U_p$ *implies* $\hat{\varphi} \in M_p$.
(c) $(\varphi, \phi) \in D_p$ *implies* $\Psi = \varphi * \phi \in U_p$.

The converses of the above implications are not true in general. First of all, the fact that $\hat{\varphi} \in M_p$ does not imply, in general, that φ is well defined at every point and therefore the U_p condition may make no sense. However, if this is not the case, still the converse of (b) is not true. Take φ to be a function in L^1 such that, for every $n \in \mathbb{Z}$, $\varphi(n) = 1$. Then $\hat{\varphi} \in M_p$ but $\varphi \notin U_p$.

The $UR_p \cap UR_{p'}$ condition implies that the function has to be in $L^p \cap L^{p'}$ while the UR_2 condition does not.

However, if we assume that the support of $\hat{\varphi}$ is compact things change.

Let E_R be the set of slowly increasing C^∞ functions f with supp $\hat{f} \subset [-R, R]^N$. The elements of E_R are functions of exponential type R. We recall a well-known sampling theorem for such functions (see [B]):

If $g \in E_R$, then

$$g(x) = \sum_n g\Big(\frac{n}{2R}\Big) \operatorname{sinc}(2Rx - n), \tag{1}$$

and

$$\|g\|_p \sim R^{-N/p}\Big(\sum_n \Big|g\Big(\frac{n}{2R}\Big)\Big|^p\Big)^{1/p}. \tag{2}$$

We also need the following lemma (see [AC]).

LEMMA 2.3. *Let φ be in E_R such that $\hat{\varphi} \in M_p(\mathbb{R}^N)$. Then*

$$\Big\| \sum_{m\in\mathbf{Z}^N} a(m)\varphi(\cdot - m)\Big\|_p \leq C M_p(\hat{\varphi}) \max(1, R^{N(1-1/p)})\|a\|_p,$$

for all sequences $a = (a(m))_m$, where $C = C(p, N)$; that is, $\varphi \in UR_p$.

As a direct consequence of the previous results one can easily show the following proposition.

PROPOSITION 2.4. *If $\varphi \in E_R$, then the following conditions are equivalent:*

(*a*) $\hat{\varphi} \in M_p$
(*b*) $\varphi \in U_p$.
(*c*) $\varphi \in UR_p$.

Finally, as we shall see later, the fact that $\Psi = \varphi * \phi \in U_p$ does not imply that $(\varphi, \phi) \in D_p$ even in the case of exponential type functions (see Remark 3.5(b)).

3. Equivalence between boundedness of discrete and continuous versions

For the sake of completeness we shall start giving the proof of the well-known result (b) in the introduction.

THEOREM 3.1. *The boundedness of a linear operator T on $L^p(\mathbb{R}^N)$ is equivalent to the boundedness on ℓ^p of the operator associated to the matrix*

$$\big(2^{kN/p} < T(\operatorname{sinc}(2^k \cdot -n)), 2^{kN/p'} \operatorname{sinc}(2^k \cdot -m) > \big)_{n,m},$$

with norm uniformly bounded on k.

Proof. Let us assume that T is bounded on $L^p(\mathbb{R}^N)$ and let us consider two finite sequences $(a_n)_n$ and $(b_m)_m$. Then, by Lemma 2.3,

$$\begin{aligned}
&\sum_{n,m}\left\langle 2^{kN/p}T(\operatorname{sinc}(2^k\cdot-n)),2^{kN/p'}\operatorname{sinc}(2^k\cdot-m)\right\rangle b_m a_n\\
&\quad=\left\langle T\left(2^{kN/p}\sum_n a_n\operatorname{sinc}(2^k\cdot-n)\right),2^{kN/p'}\sum_m b_m\operatorname{sinc}(2^k\cdot-m)\right\rangle\\
&\quad\le\|T\|\left\|2^{kN/p}\sum_n a_n\operatorname{sinc}(2^k\cdot-n)\right\|_p\left\|2^{kN/p'}\sum_m b_m\operatorname{sinc}(2^k\cdot-m)\right\|_{p'}\\
&\quad\le\|T\|\,\|a\|_p\|b\|_{p'}.
\end{aligned}$$

Conversely, since by (1) and (2) the subspace

$$H_p=\left\{2^{kN/p}\sum a_n\operatorname{sinc}(2^k\cdot-n);\ k\in\mathbb{N},\ (a_n)_n\in\ell^p\right\},$$

is dense in $L^p(\mathbb{R})$, we only have to check that T is bounded on functions of H_p.

Now, let $f\in H_p$ and $g\in H_{p'}$; then $f=2^{kN/p}\sum_n a_n\operatorname{sinc}(2^k\cdot-n)$ and $g=2^{k'N/p'}\sum_m b_m\operatorname{sinc}(2^{k'}\cdot-m)$. We can assume without loss of generality that the sequences $(a_n)_n$ and $(b_m)_m$ are finite.

Now, if we write

$$V_k^p=\left\{2^{kN/p}\sum a_n\operatorname{sinc}(2^k\cdot-n);\ (a_n)_n\in\ell^p\right\},$$

we get $V_k^p\subset V_{k+1}^p$ and hence, if, say $k'\le k$, we can always write $g=2^{kN/p'}\sum_m c_m\operatorname{sinc}(2^k\cdot-m)$, with $c\in\ell^{p'}$.

Thus,

$$\begin{aligned}
\langle Tf,g\rangle&=\sum_{n,m}\left\langle T\left(2^{kN/p}\operatorname{sinc}(2^k\cdot-n)\right),2^{kN/p'}\operatorname{sinc}(2^k\cdot-m)\right\rangle c_m a_n\\
&\le C\|a\|_p\|c\|_{p'}\le C\left\|\sum_n a_n\operatorname{sinc}(\cdot-n)\right\|_p\left\|\sum_m c_m\operatorname{sinc}(\cdot-m)\right\|_{p'},
\end{aligned}$$

where we are using the fact that

$$\|a\|_p\sim\left\|\sum_n a_n\operatorname{sinc}(\cdot-n)\right\|_p. \qquad\square$$

We see from the proof that the conditions on the function sinc that we have used are the following: Let $q=p$ or p'; then

(i) H_q is dense in L^q.

(ii) For every $a = (a_n)_n \in \ell^q$, $\|a\|_q \sim \|\sum_n a_n \mathrm{sinc}\,(\cdot - n)\|_q$.

(iii) $V_k^q \subset V_{k'}^q$, with $k \leq k'$.

If $p = 2$, this is equivalent to saying that the function sinc is the scaling function of a MRA (see [M1]). The same argument proves the result (a) in the introduction.

Hence, the first question we want to consider is the following: Given φ, when is it true that the space

$$H_p^\varphi = \left\{ 2^{kN/p} \sum a_n \varphi(2^k \cdot - n);\ k \in \mathbb{N},\ (a_n)_n \in \ell^p \right\},$$

is dense in $L^p(\mathbb{R}^N)$?

After this problem is solved we shall see that (ii) and (iii) are too restrictive for our purpose and that the discretization result holds under weaker conditions.

In this paper, we shall restrict our attention to the case $\hat{\varphi} \in M_p(\mathbb{R}^N)$ such that $\varphi \in E_R$. Since, by Proposition 2.4, we know that then the UR_p holds, we will be interested in the opposite inequality, that we call the IR_p condition (Inferior Riesz condition).

3.1. *Almost approximations of the identity*

Definition 3.1.1. We say that φ is an almost approximation of the identity in L^p if there exist an invertible and bounded operator S in L^p and a sequence $(n_k)_k$ such that $\varphi_{2^{-n_k}} * f$ converges to Sf in L^p, for every $f \in L^p$.

For this case, we need the condition $\varphi \in E_{1/2}$.

THEOREM 3.1.2. *Let $\hat{\varphi} \in M_p(\mathbb{R}^N)$ be such that* $\operatorname{supp} \hat{\varphi} \subset (-1/2, 1/2)^N$. *Then, if φ is an almost approximation of the identity in L^p, H_p^φ is dense in $L^p(\mathbb{R}^N)$.*

Proof. Let us define

$$A^q f(x) = \sum_{n \in \mathbb{Z}^N} (\phi_{2^{-q}} * S^{-1} f)(2^{-q} n) \varphi(2^q x - n),$$

where $\phi = \mathrm{sinc}$ and $q \in \mathbb{N}$. We have $A^q f \in H_p^\varphi$.

Let us consider $H^q f = \sum_n (\phi_{2^{-q}} * f)(2^{-q} n) \phi(2^q x - n) = (\phi_{2^{-q}} * f)(x)$. We know that $H^q f$ converges to f in the L^p norm. Now, since $\varphi = \varphi * \phi$,

$$\begin{aligned} A^q f(x) &= \sum_{n \in \mathbb{Z}^N} (\phi_{2^{-q}} * S^{-1} f)(2^{-q} n) \int_{\mathbb{R}^N} \phi(2^q x - n - y) \varphi(y)\, dy \\ &= 2^{qN} \sum_n (\phi_{2^{-q}} * S^{-1} f)(2^{-q} n) \int_{\mathbb{R}^N} \phi(2^q (x - z) - n) \varphi(2^q z)\, dz \\ &= \int_{\mathbb{R}^N} H^q S^{-1} f(x - z) \varphi_{2^{-q}}(z)\, dz = (\varphi_{2^{-q}} * H^q S^{-1} f)(x). \end{aligned}$$

Therefore,

$$\begin{aligned}\|A^q f - f\|_p &\le \|A^q f - \varphi_{2^{-q}} * S^{-1} f\|_p + \|\varphi_{2^{-q}} * S^{-1} f - f\|_p \\ &= \|\varphi_{2^{-q}} * (H^q S^{-1} f - S^{-1} f)\|_p + \|\varphi_{2^{-q}} * S^{-1} f - f\|_p \\ &\le N_p(\hat{\varphi}) \|H^q S^{-1} f - S^{-1} f\|_p + \|\varphi_{2^{-q}} * S^{-1} f - f\|_p.\end{aligned}$$

Taking $q = n_k$, we see that the above expression converges to zero whenever k tends to infinity. □

For our purpose, we see that we do not need the IR_p condition to hold for every sequence $a \in \ell^p$ but only for those sequences that appear in the definition of A_q. That is, let us consider the set $\mathcal{A}_p$ of all sequences $(a_n)_n$ such that there exist $q \in (n_k)_k$ and $f \in L^p$ satisfying

$$a_n = 2^{-qN/p}(\phi_{2^{-q}} * S^{-1} f)(2^{-q} n), \qquad \forall n,$$

and such that $\|f\|_p \le 2\|A^q f\|_p$. Then, what we have shown above is that the set

$$\mathcal{A}_p^\varphi = \left\{ 2^{kN/p} \sum a_n \varphi(2^k \cdot - n);\ k \in \mathbb{N},\ (a_n)_n \in \mathcal{A}_p \right\},$$

is dense in L^p.

Moreover, if $a \in \mathcal{A}_p$, then

$$\begin{aligned}\|a\|_p &= \left\| 2^{-qN/p}(\phi_{2^{-q}} * S^{-1} f)(2^{-q} n) \right\|_p \\ &\sim \|\phi_{2^{-q}} * S^{-1} f\|_p \le C\|f\|_p \le C \left\| \sum_n a_n \varphi(\cdot - n) \right\|_p.\end{aligned}$$

Thus, following the same steps of Theorem 3.1, we can show the following result.

THEOREM 3.1.3. *Let $\varphi, \phi \in E_{1/2}$ such that $\hat{\varphi}$ and $\hat{\phi} \in M_p(\mathbb{R}^N)$. If φ and ϕ are almost approximations of the identity in L^p and $L^{p'}$ respectively, then $(\varphi, \phi) \in D_p$.*

We cannot prove that under the hypothesis of the previous theorem, $(\varphi, \phi) \in DD_p$, since, in general, if

$$V_k^\varphi = \left\{ 2^{kN/p} \sum a_n \varphi(2^k \cdot - n);\ (a_n)_n \in \ell^p \right\},$$

we do not have $V_k^\varphi \subset V_{k'}^\varphi$, for some $k' \ge k$. This will be solved in the following subsection.

3.2. *The local invertibility condition*

If we want to study (ii), we note that under the conditions we have assumed on φ, (ii) is equivalent to having $\hat{\varphi}$ bounded from below by a constant $C > 0$ (see [M1]).

Let us now consider a somehow weaker condition than this one.

Definition 3.2.1. We say that φ satisfies the local invertibility condition in L^p if for some $\varepsilon > 0$, $\chi_{(-\varepsilon,\varepsilon)^N}/\hat{\varphi} \in M_p(\mathbb{R}^N)$.

The following theorem shows that the local invertibility condition is enough to have condition (i).

THEOREM 3.2.2. *Let $\varphi \in E_R$ with $R < 1$, $\hat{\varphi} \in M_p(\mathbb{R}^N)$ and such that φ satisfies the local invertibility condition in L^p. Then the space H_p^φ is dense in $L^p(\mathbb{R}^N)$.*

Proof. Let $f \in L^2(\mathbb{R}^N) \cap E_{2^k}$ and let k_0 be such that $2^{-k_0} \leq \min(\varepsilon, 1 - R)$. Then, if $\hat{g}(\xi) = \hat{f}(2^{k+k_0}\xi)$, $g \in E_{2^{-k_0}}$ and therefore, if $h = \chi_{(-2^{-k_0},2^{-k_0})^N}$, we obtain

$$\hat{g}(\xi) = \hat{g}(\xi)\frac{h(\xi)}{\hat{\varphi}(\xi)}\hat{\varphi}(\xi) = P(\xi)\hat{\varphi}(\xi). \tag{3}$$

Now, since $\varphi \in E_R$ with $R \leq 1 - 2^{-k_0}$, we have $\hat{g}(\xi) = \big(\sum_k P(\xi + k)\big)\hat{\varphi}(\xi)$. Let us write $\sum_k P(\xi + k) = \sum_m a_m e^{2\pi i \xi m}$, where

$$a_m = \int_{\mathbb{R}^N} \hat{g}(\xi)\frac{h(\xi)}{\hat{\varphi}(\xi)} e^{2\pi i \xi m} d\xi = \left[\left(\frac{h}{\hat{\varphi}}\right)^{\vee} * g\right](m).$$

Since $\left(\frac{h}{\hat{\varphi}}\right)^{\vee} * g$ is of exponential type and $\left\| \left(\frac{h}{\hat{\varphi}}\right)^{\vee} * g\right\|_p \leq M_p\big(h/\hat{\varphi}\big)\|g\|_p$, we get $(a_m)_m \in \ell^p$ and

$$\|a\|_p \leq M_p\big(h/\hat{\varphi}\big)\|g\|_p = 2^{-(k+k_0)N/p'} M_p\big(h/\hat{\varphi}\big)\|f\|_p.$$

Moreover, from (3), we get $g = \sum_m a_m \varphi(\cdot - m)$ and hence,

$$f = 2^{k_0 N}2^{kN}\sum_m a_m \varphi(2^{k_0+k} \cdot - m), \tag{4}$$

with $(a_m)_m \in \ell^p$. Therefore, $f \in H_p^\varphi$. Since $L^2 \cap \big(\cup_k E_{2^k}\big)$ is dense in L^p, we are done. $\square$

Again, as in Theorem 3.1.2, we observe that we have shown more. In fact, if we write $b_m = 2^{(k+k_0)N/p'} a_m$ then every $f \in L^2 \cap E_{2^k}$ can be written as $f = 2^{k_0 N/p} 2^{kN/p} \sum_m b_m \varphi(2^{k_0+k} \cdot - m)$, with $\|b\|_p \leq M_p\big(h/\hat{\varphi}\big)\|f\|_p$.

Moreover, if $f = \sum_m a_m \varphi(2^k \cdot -m)$, then $f \in L^2 \cap E_{2^k}$ and therefore, we see from the proof of Theorem 3.2.2 that $V_k^{\varphi} \subset V_{k+k_0}^{\varphi}$.

This implies that, given k and k' we can find k'' such that both V_k and $V_{k'}$ are subsets of $V_{k''}$ and hence, we get the following result which is fundamental for the applications.

THEOREM 3.2.3. *Let $\hat{\varphi}$ and $\hat{\phi}$ be in $M_p(\mathbb{R}^N)$ and such that φ and ϕ are in E_R with $R < 1$. If φ and ϕ satisfy the local invertibility condition in L^p and $L^{p'}$ respectively, then $(\varphi, \phi) \in DD_p$.*

Remark 3.2. If we apply Theorem 3.2.2 to the function $\varphi = \phi_{(R)}$ where $\widehat{\phi_{(R)}} = \chi_{(-R,R)^N}$ with $R < 1$, we obtain, by (4), that if $f \in L^2 \cap E_{2^k}$,

$$f = \sum_n f\left(\frac{n}{2^{k+k_0}}\right)\phi_{(R)}(2^{k+k_0} \cdot -n),$$

with $2^{-k_0} \leq \min(R, 1-R)$. (Observe that the formula above is of the same type that (1).)

Now, let us assume that in Theorem 3.1.2, $\varphi \in E_R$ with $R < 1$. If we define

$$A^q f(x) = \sum_n (\phi_{2^{-q}} * S^{-1} f)(2^{-(q+k_0)}n)\varphi(2^{q+k_0}x - n),$$

and

$$H^q f = \sum_n (\phi_{2^{-q}} * f)(2^{-(q+k_0)}n)\phi_{(R)}(2^{q+k_0}x - n) = (\phi_{2^{-q}} * f)(x),$$

then H^q converges to f in L^p and, since, $\varphi * \phi_{(R)} = \varphi$, we get $A^q f = \varphi_{2^{-q}} * H^q S^{-1} f$. Therefore, Theorem 3.1.2 remains true under the hypothesis $\varphi \in E_R$ with $R < 1$.

As a consequence of this and Theorems 3.1.3 and 3.2.3, we obtain:

THEOREM 3.3. *Let $\hat{\varphi}$ and $\hat{\phi}$ be in $M_p(\mathbb{R}^N)$ and such that φ and ϕ are in E_R with $R < 1$. If φ satisfies the local invertibility condition in L^p and either the same holds for ϕ in $L^{p'}$ or ϕ is an almost approximation of the identity in $L^{p'}$, then $(\varphi, \phi) \in DD_p$.*

Remark 3.4. Now, let us see if there is some connection between the local invertibility property and the almost approximation of the identity condition. We shall work, for simplicity, with $p = 2$.

First, $\hat{\varphi} \in M_2$ implies that $(\varphi_{2^{-n}} * f)_n$ is a uniformly bounded family in L^2 and therefore, there exists a sequence $(n_k)_k$ such that $(\varphi_{2^{-n_k}} * f)_k$ converges in the weak topology to a function in L^2 that we call Sf. Obviously the operator S is bounded in L^2 and, in fact, it is a convolution operator, and hence, there exists m such that

$\widehat{Sf}(\xi) = m(\xi)\hat{f}(\xi)$. Moreover, by Plancherel's Theorem, there exists a subsequence $(n_k)_k$ such that

$$\lim_k \hat{\varphi}(2^{-n_k}\xi) = m(\xi), \qquad \text{a.e. } \xi. \tag{5}$$

Conversely, if (5) holds, then $\varphi_{2^{-n_k}} * f$ converges to Sf in L^2. That is, the almost approximation condition in L^2 is equivalent to (5), with $m \in L^\infty$ and $|m(\xi)| \geq C > 0$, a.e. ξ.

Now, if φ satisfies the local invertibility condition, then one can easily see that $\varphi_t * f$ converges weakly to an invertible operator, but the limit in (5) does not exist in general: Take $\hat{\varphi}(\xi) = \sin(1/\xi) + 3$. Therefore, the local invertibility condition in L^2 does not imply the almost approximation condition.

Conversely, let $(n_k)_k$ be defined by $n_1 = 1$ and $n_{k+1} = (k+1) + n_k$. Let us consider $I_j = (2^{-(j+1)}, 2^{-j}]$ and define

$$\hat{\varphi}(x) = \begin{cases} 1 & \text{if } x \in I_j,\ j \notin (n_k)_k \\ 0 & \text{elsewhere.} \end{cases}$$

Then φ does not satisfy the local invertibility property but, for every x, $\hat{\varphi}(x/2^{n_k})$ converges to 1 when k tends to infinity; that is, φ is an almost approximation of the identity.

Hence, both concepts, are independent.

Remark 3.5. (a) If the operator T is given by a convolution kernel K, then,

$$\left(2^{kN/p}\langle T(\varphi(2^k\cdot -n)), 2^{kN/p'}\phi(2^k\cdot -m)\rangle\right) = \left(K_{2^k} * \varphi * \phi\right)(n, m).$$

Thus:

(1) If φ satisfies the hypotheses in Theorem 3.2.3 and $\phi \in \mathcal{S}(\mathbb{R}^N)$ such that $\hat{\phi}(\xi) = 1$ in $[-R, R]^N$ then $K_{2^k} * \varphi * \phi = K_{2^k} * \varphi$ and, as a consequence, we get the results in [AC].

(2) If φ and ϕ satisfy the hypothesis in Theorem 3.1.3 with $\hat{\varphi} = 1$ in supp ϕ, then K is a convolution operator in L^p if and only if $(K_{2^k} * \phi)(n)$ is a convolution sequence on ℓ^p uniformly in k; that is, the condition of local invertibility can be changed by an almost approximation of the identity condition.

(b) As shown in [AC], the condition $R < 1$ in Theorem 3.2.3 is sharp in the sense that if $R = 1$ the result does not hold. Namely, if we take $\hat{\varphi}(\xi) = \chi_{[-1,1]}(\xi)$ then for $K(x) = \text{p.v.}\frac{1}{\pi x}$, $(K_{2^k} * f)(x) = H(f)(x)$ is the Hilbert transform of f while $K_{2^k} * \varphi(n) = 0$, for every n and therefore $(\varphi, \varphi) \notin D_p$. In this case, the space H_p^φ is not dense in L^p.

3.2. *Integral operators.* The discretization results for convolution operators studied in [AC] have been recently extended to the case of Hardy spaces H^p with $0 < p \leq 1$ (see [BC]). The extension of the results of this paper to Hardy spaces does not seem easy because we have used the duality L^p and $L^{p'}$ everywhere. However, if the operator T is an integral operator, we can give another proof (of a particular case of Theorem 3.3) that does not use duality.

Let

$$Tf(x) = \int_{\mathbb{R}^N} f(y)K(x,y)\,dy.$$

To avoid technicalities and to have a direct definition of the operator T, we shall assume that $K(x,\cdot) \in L^2$ with $\|K(x,\cdot)\|_2 \le C$ and $K(\cdot,y) \in L^2$ with $\|K(\cdot,y)\|_2 \le C$.

THEOREM 3.3.1. *Let $1 < p < \infty$ and $R \le 1/2$. Assume that φ and ϕ satisfy:*

(*a*) $\hat{\varphi} \in M_p$,
(*b*) φ *and* $\phi \in E_R$,
(*c*) φ *satisfies the local invertibility property,*
(*d*) $\phi \in L^1$ *and* $\int \phi = 1$.

Then T is bounded on $L^p(\mathbb{R}^N)$ if and only if the operators associated to the matrices

$$\left(\langle 2^{kN/p}T(\varphi(2^k\cdot - n)), 2^{kN/p'}\phi(2^k\cdot - m)\rangle\right)_{n,m},$$

(whenever this makes sense) are bounded on ℓ^p uniformly in $k \in \mathbb{Z}$.

Proof. The necessary condition is clear.

To prove that T is bounded, it is enough to show, by density and rescaling, that, for every $f \in \mathcal{S}(\mathbb{R}^N)$ satisfying $\operatorname{supp}\hat{f} \subset [-\delta,\delta]^N$ with δ small, we have $\|T_s f\|_p \le C\|f\|_p$ uniformly in $s > 0$, where T_s is the integral operator associated to the kernel $s^{-N}K(x/s, y/s)$.

To see this we shall prove that, for t small, the function $\tilde{\phi}_{t/R} * T_s f \in L^p$ with $\|\tilde{\phi}_{t/R} * T_s f\|_p \le C\|f\|_p$ and C independent of t and s, where, as usual, $\tilde{\phi}(x) = \phi(-x)$. Then, using (d) we can deduce that $\|T_s f\|_p \le C\|f\|_p$ as desired.

By condition (b),

$$\operatorname{supp}\,(\phi_{t/R} * T_s f)\hat{}\; \subset [-R^2/t, R^2/t]^N \subset [-R/(2t), R/(2t)]^N$$

and therefore, by (2),

$$\|\tilde{\phi}_{t/R} * T_s f\|_p \sim \left(\frac{t}{R}\right)^{N/p}\left(\sum_n \left|\left(\tilde{\phi}_{t/R} * T_s f\right)\left(\frac{tn}{R}\right)\right|^p\right)^{1/p}.$$

Now,

$$(\tilde{\phi}_{t/R} * T_s f)\left(\frac{tn}{R}\right) = \int_{\mathbb{R}^N} \tilde{\phi}_{t/R}\left(\frac{tn}{R} - x\right)\left(\int_{\mathbb{R}^N} K_s(x,y) f(y)\,dy\right)dx$$

$$\begin{aligned}
&= \int_{\mathbb{R}^N} f(y)\Big(\int_{\mathbb{R}^N} K_s(x,y)\tilde{\phi}_{t/R}\Big(\frac{tn}{R}-x\Big)\,dx\Big)dy\\
&= \int_{\mathbb{R}^N} f(y)\Big(K_s *_1 \tilde{\phi}_{t/R}\Big)\Big(\frac{tn}{R},y\Big)\,dy\\
&= \int_{\mathbb{R}^N} \hat{f}(\xi)\widehat{\Big(K_s *_1 \tilde{\phi}_{t/R}\Big)}^{2}\Big(\frac{tn}{R},\xi\Big)\,d\xi,
\end{aligned}$$

where $*_1$ denotes the convolution with respect to the first variable and $\widehat{\ }_2$ denotes the Fourier transform with respect to the second variable.

If we choose $t < (\varepsilon R)/\delta$, with $\varepsilon > 0$ and such that if $h = \chi_{(-\varepsilon,\varepsilon)^N}$, $h/\hat{\tilde{\varphi}} \in M_p$, we can write

$$\hat{f}(\xi) = \hat{f}(\xi)h\Big(\frac{t}{R}\xi\Big) = \hat{f}(\xi)\frac{h(\frac{t}{R}\xi)}{\hat{\tilde{\varphi}}_{t/R}(\xi)}\hat{\tilde{\varphi}}_{t/R}(\xi).$$

Therefore,

$$\begin{aligned}
(\tilde{\phi}_{t/R} * T_s f)\Big(\frac{tn}{R}\Big) &= \int_{[-R/(2t),R/(2t)]^N} \frac{h(\frac{t}{R}\xi)\hat{f}(\xi)}{\hat{\tilde{\varphi}}_{t/R}(\xi)}\widehat{\Big(K_s *_1 \tilde{\phi}_{t/R}\Big)}^{2}\Big(\frac{tn}{R},\xi\Big)\hat{\tilde{\varphi}}_{t/R}(\xi)\,d\xi\\
&= \int_{[-R/(2t),R/(2t)]^N} \frac{h(\frac{t}{R}\xi)\hat{f}(\xi)}{\hat{\tilde{\varphi}}_{t/R}(\xi)}\widehat{\Big(K_s * \tilde{\phi}_{t/R}\tilde{\varphi}_{t/R}\Big)}^{2}\Big(\frac{tn}{R},\xi\Big)\,d\xi\\
&= R^N t^{-N}\int_{[-1/2,1/2]^N} \frac{h(\xi)\hat{f}(\frac{R\xi}{t})}{\hat{\tilde{\varphi}}_{t/R}(\frac{R\xi}{t})}\widehat{\Big(K_s * J_{t/R}\Big)}^{2}\Big(\frac{tn}{R},\frac{R\xi}{t}\Big)\,d\xi\\
&= \sum_m \hat{P}_t(m)\Big(K_s * J_{t/R}\Big)\Big(\frac{tn}{R},\frac{tm}{R}\Big),
\end{aligned}$$

where

$$P_t(\xi) = \sum_{k\in\mathbf{Z}^N}\left(\frac{\hat{f}(\frac{R(\xi+k)}{t})h(\xi+k)}{\hat{\tilde{\varphi}}(\xi+k)}\right),$$

defines a 1-periodic function, whose Fourier coefficients are

$$\hat{P}_t(m) = \int_{\mathbb{R}^N}\frac{\hat{f}(\frac{R\xi}{t})h(\xi)}{\hat{\tilde{\varphi}}(\xi)}e^{-2\pi i m\xi}\,d\xi,$$

and thus

$$\Big(\sum_m |\hat{P}_t(m)|^p\Big)^{1/p} \le M_p\Big(\frac{h}{\hat{\tilde{\varphi}}}\Big)\Big(\sum_m\Big|\int_{\mathbb{R}^N}\hat{f}\Big(\frac{R\xi}{t}\Big)e^{im\xi}\,d\xi\Big|^p\Big)^{1/p}$$

$$= Ct^N \left(\sum_m \left| \int_{\mathbb{R}^N} \hat{f}(R\xi) e^{imt\xi}\, d\xi \right|^p \right)^{1/p}$$

$$= Ct^N \left(\sum_m |f_R(tm)|^p \right)^{1/p}.$$

On the other hand,

$$(K_s * J_{t/R}) \left(\frac{tn}{R}, \frac{tm}{R} \right)$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K_s(x, y) \tilde{\phi}_{t/R} \left(\frac{tn}{R} - x \right) \tilde{\varphi}_{t/R} \left(\frac{tm}{R} - y \right) dxdy$$

$$= s^N \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x, y) t^{-N} R^N \phi \left(\frac{Rsx}{t} - n \right) t^{-N} R^N \varphi \left(\frac{Rsy}{t} - m \right) dxdy,$$

and therefore, choosing t such that $Rs/t = 2^k$, we get

$$(K_s * J_{t/R}) \left(\frac{tn}{R}, \frac{tm}{R} \right) = t^{-N} R^N \langle T(2^{kN/p} \varphi(2^k \cdot - m)), 2^{kN/p'} \phi(2^k \cdot - n) \rangle,$$

and hence

$$\|\phi_{t/R} * T_s f\|_p \sim \left(\frac{t}{R} \right)^{N/p} \left(\sum_n \left| (\phi_{t/R} * T_s f) \left(\frac{tn}{R} \right) \right|^p \right)^{1/p}$$

$$= \left(\frac{t}{R} \right)^{N/p} \left(\sum_n \left| \sum_m \hat{P}_t(m) (K_s * J_{t/R}) \left(\frac{tn}{R}, \frac{tm}{R} \right) \right|^p \right)^{1/p}$$

$$\leq C \left(\frac{t}{R} \right)^{N/p} R^N t^{-N} \left(\sum_m |\hat{P}_t(m)|^p \right)^{1/p}$$

$$\leq C \left(\frac{t}{R} \right)^{N/p} R^N \left(\sum_m |f_R(tm)|^p \right)^{1/p} \sim \|f_R\|_p \sim \|f\|_p. \qquad \square$$

4. Applications

First of all since the theory generalizes the case of convolution operators, we get those studied in Examples I and II of [AC] as an application.

Example 1. *Singular Integrals.* We first consider the Hilbert transform on $\mathbb{R}$ whose multiplier is $m(x) =$ -i $\operatorname{sgn}(x)$. By choosing a smooth and even φ as above, we can explicitly compute $K_{2^k} * \varphi(m)$ and we obtain

$$\begin{aligned} K_{2^k} * \varphi(m) &= \int_{\mathbb{R}} -i \operatorname{sgn}(2^k\xi)\hat{\varphi}(\xi)e^{2\pi i m\xi}d\xi \\ &= 2\int_0^{\infty} \hat{\varphi}(\xi) \sin(2\pi m\xi)d\xi = \frac{1}{\pi m} + O\Big(\frac{1}{m^2}\Big). \end{aligned}$$

Thus, $(K_{2^k} * \varphi)(n) = 1/n + E_n$, where E_n is an operator bounded on all ℓ^p.

Therefore, the boundedness of H is equivalent to the boundedness of the discrete Hilbert transform. Similar results can be obtained in dimension N (see [AC]).

Example 2. *Convolution operators with compactly supported multipliers.* If $\hat{K}$ has compact support, then $K_{2^k} * \varphi$ reduces to K_{2^k} for k large, if $\hat{\varphi}$ is 1 in a neighborhood of 0. Thus, as a consequence of our results we get that for $1 < p < \infty$,

$$\|K * f\|_p \leq C\|f\|_p,$$

is equivalent to

$$\Big\|\sum_m K_{2^k}(m)a(n-m)\Big\|_p \leq C\|a\|_p,$$

uniformly in k (see [AC]).

Example 3. Let

$$H(x, y) = \sum_{n\in\mathbb{Z}^N} \varphi_1(x-n)\varphi_2(y-n).$$

We want to study when the linear operator associated to this kernel is bounded on L^p. As we show in Section 3.3, the operators $\tilde{T}_k$ are given by $(H_k * J)(n, m)$ where $J(x, y) = \varphi(x)\phi(y)$ and

$$H_k(x, y) = 2^{-kN}H(x/2^k, y/2^k).$$

Now, if we take φ and ϕ in $\mathcal{S}$ appropriately, we obtain

$$\begin{aligned} &(H_k * J)(n, m) \\ &\quad = 2^{kN}\sum_j \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi_1(x-j)\varphi_2(y-j)\varphi(n-2^kx)\phi(m-2^ky)\,dxdy \\ &\quad = 2^{-kN}\sum_j (\varphi_1 * \varphi_{2^{-k}})(2^{-k}n-j)(\varphi_2 * \phi_{2^{-k}})(2^{-k}m-j), \end{aligned}$$

that, in the particular case $k = 0$, is a convolution operator.

Example 4. Let us consider T to be the directional Hilbert transform; that is,

$$Tf(x) = \int_{\mathbb{R}} f(x-t)\,\frac{dt}{t},$$

where $x \in \mathbb{R}^N$, $(x-t) = (x_1 - t, \ldots, x_N - t)$.

From our results, with appropriate $\varphi = (\varphi_1, \ldots \varphi_N)$ and $\phi = (\phi_1, \ldots, \phi_N)$ the well-known boundedness on L^p of T is equivalent to the boundedness on ℓ^p of the convolution operators associated to the matrices

$$A(n) = \int_{\mathbb{R}} \prod_j (\varphi_j * \phi_j)(t - n_j)\,\frac{dt}{t}.$$

In particular, taking $\varphi_j \in S \cap E_{1/2}$ satisfying the hypotheses of Theorem 3.2.3 and $\phi_j = \text{sinc}$ we have

$$A(n) = \int_{\mathbb{R}} \prod_j \varphi_j(t - n_j)\,\frac{dt}{t}.$$

Now, if we define $A(x) = \int_{\mathbb{R}} \prod_j \varphi_j(t - x_j)\,\frac{dt}{t}$, we get $A \in E_{1/2}$ and therefore, the boundedness of the above matrix is equivalent to the fact that A is a convolution kernel on L^p (see [D]).

Hence, we have shown that the boundedness on L^p of T is equivalent to the existence of a function $\varphi \in E_{1/2}$ such that $\widehat{T\varphi} \in M_p$.

Example 5. Now let us consider the case where T is the Hilbert transform along a curve (see [NRW]),

$$Tf(x) = \int_{\mathbb{R}} \frac{f(x - \gamma(t))}{t}\,\frac{dt}{t},$$

where $x \in \mathbb{R}^N$ and $\gamma(t) = (\gamma_1(t), \ldots, \gamma_N(t))$ is a curve.

Then, with appropriate $\varphi = (\varphi_1, \ldots, \varphi_N)$ and $\phi = (\phi_1, \ldots, \phi_N)$, the boundedness on L^p of T is equivalent to the boundedness on ℓ^p of the convolution operators associated to the matrices

$$A_k(n) = \int_{\mathbb{R}} \prod_j (\varphi_j * \phi_j)(2^k \gamma_j(t) - n_j)\,\frac{dt}{t},$$

uniformly in k.

Again, taking $\varphi_j \in S$ satisfying the hypotheses of Theorem 3.2.3 and $\phi_j = \text{sinc}$ we have

$$A_k(n) = \int_{\mathbb{R}} \prod_j \varphi_j(2^k \gamma_j(t) - n_j)\,\frac{dt}{t}.$$

Example 6. Let b be a BMO function and let H be the Hilbert transform. It is known that the commutator

$$[H, b]\colon L^p(\mathbb{R}^N) \longrightarrow L^p(\mathbb{R}^N)$$

is bounded. This commutator operator can be defined as the linear operator T such that, for every $f \in L^p$ and every $g \in L^{p'}$, $\langle T(f), g\rangle = \langle b, fH(g) + gH(f)\rangle$. Hence, the boundedness of T is equivalent to the boundedness on ℓ^p of the operator associated to the matrix

$$\langle b, \varphi(\cdot - n)H(\phi(\cdot - m)) + \phi(\cdot - m)H(\varphi(\cdot - n))\rangle,$$

which is equivalent to having, for every $(a_n)_n \in \ell^p$ and $(b_m)_m \in \ell^{p'}$, the function

$$F(x) = \sum_n \sum_m a_n b_m \left(\varphi(x - n)H(\phi(x - m)) + \phi(x - m)H(\varphi(x - n))\right)$$

is the Hardy space H^1 with $\|F\|_{H^1} \le C\|a\|_p\|b\|_{p'}$.

Now, taking $\varphi = \phi = \operatorname{sinc} x$, the previous result follows by simple computations.

5. Multiresolution analysis on L^p

In this section, we shall study the connection between the local invertibility property and Multiresolution Analysis (MRA).

Let us recall that a MRA in L^2 (see [M1], [HW]) is a sequence of closed subspaces V_j $(j \in \mathbb{Z})$ such that

(i) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$,
(ii) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$,
(iii) $\cap V_j = 0$,
(iv) $\overline{\cup_j V_j} = L^2$, and
(v) there exist $\varphi \in V_0$ such that $\{\varphi(\cdot - n), n \in \mathbb{Z}\}$ is a Riesz basis for V_0.

The function φ is called the scaling function. Although this is the original definition of MRA, it turns out that condition (iii) is redundant (see [HW]). Let us now consider the following definition.

Definition 5.1. A Multiresolution Analysis (MRA) in L^p of order k is a sequence of closed subspaces of L^p, V_j $(j \in \mathbb{Z})$ such that

(i′) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$,
(ii′) $f(x) \in V_j$ if and only if $f(2^k x) \in V_{j+1}$,
(iii′) $\overline{\cup_j V_j} = L^p$, and
(iv′) there exists $\varphi \in V_0$ such that $\{\varphi(\cdot - n), n \in \mathbb{Z}\}$ is a Riesz basis for V_0.

The function φ will be called the scaling function of order k. Our aim is to give sufficient conditions on a function φ to be a scaling function of order k. We shall restrict ourselves to the case $\operatorname{supp}\hat{\varphi} \subset (-1, 1)^N$.

It is clear that if φ and ϕ are scaling functions of order k, then $(\varphi, \phi) \in DD_p$. It is well known (see [M1]) that a necessary and sufficient condition on a function φ to have (v) hold is

$$\sum_n |\hat{\varphi}(\xi + 2n\pi)|^2 \sim 1, \tag{6}$$

for almost every ξ. Hence, if the function $\hat{\varphi}$ is supported in $(-1/2, 1/2)^N$, this condition implies $\chi_{(-1/2,1/2)^N}/\hat{\varphi} \in M_2$.

Moreover, under this assumption on the support, condition (iv′) is equivalent to saying that the convolution operator $\varphi(n)\star$ is an invertible and bounded operator in ℓ^p and hence in $\ell^{p'}$ and the inverse operator is given by $(\chi_{(-1/2,1/2)^N}/\hat{\varphi})^\vee(n)\star$. Hence,

$$\chi_{(-1/2,1/2)^N}/\hat{\varphi} \in M_p. \tag{7}$$

The next result is implicit in the proof of Theorem 3.2.2 and the comment after it.

THEOREM 5.2. *Let $\varphi \in E_R$ such that $\hat{\varphi} \in M_p$ and satisfying (iv′). Then, if k is such that*

(*a*) $R \leq 1 - 2^{-k}$ *and*
(*b*) $\chi_{(-2^{-k},2^{-k})^N}/\hat{\varphi} \in M_p$,

then φ is a scaling function on L^p of order k.

Finally, if φ satisfies (7), $a = (a_n)$ is a finite sequence and $\phi(x) = \operatorname{sinc} x$,

$$\begin{aligned}
\|a\|_p &= \left\| \sum_n a_n \phi(\cdot - n) \right\|_p = \left\| \left(\sum_n a_n e^{2\pi i n \cdot} \chi_{(-1/2,1/2)^N}(\cdot) \right)^\vee \right\|_p \\
&= \left\| \left(\sum_n a_n e^{2\pi i n \cdot} \chi_{(-1/2,1/2)^N}(\cdot) \frac{\chi_{(-1/2,1/2)^N}(\cdot)}{\hat{\varphi}(\cdot)} \hat{\varphi}(\cdot) \right)^\vee \right\|_p \\
&\leq C M_p \left(\frac{\chi_{(-1/2,1/2)^N}}{\hat{\varphi}} \right) \left\| \left(\sum_n a_n e^{2\pi i n \cdot} \hat{\varphi} \right)^\vee \right\|_p \\
&= C M_p \left(\frac{\chi_{(-1/2,1/2)^N}}{\hat{\varphi}} \right) \left\| \sum_n a_n \varphi(\cdot - n) \right\|_p ,
\end{aligned}$$

and therefore $\varphi \in IR_p$. From this and Proposition 2.4 we obtain:

THEOREM 5.3. *Let $\varphi \in E_R$ $\hat{\varphi} \in M_p$ and $\chi_{(-1/2,1/2)^N}/\hat{\varphi} \in M_p$. Then φ is a scaling function of order $k \in \mathbb{N}$ where*

$$R \leq 1 - 2^{-k}.$$

Observe that $R < 1$ implicitly and if $k = 1$, then $R = 1/2$.

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Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, 08071 Barcelona, Spain
carro@cerber.mat.ub.es