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LINEAR ISOMETRIES BETWEEN CERTAIN SUBSPACES OF CONTINUOUS VECTOR-VALUED FUNCTIONS

JUAN J. FONT

Introduction

Throughout this note, X and Y will stand for locally compact Hausdorff spaces, and E and F for Banach spaces. Let $C_0(X, E)$ and $C_0(Y, F)$ be the Banach spaces of continuous E-valued and F-valued functions vanishing at infinity defined on X and Y respectively and endowed with the supremum norm $\|\cdot\|_\infty$. Let $\mathbf{K}$ denote the field of real or complex numbers. If $E = F = \mathbf{K}$, then we will write $C_0(X)$ and $C_0(Y)$ ($C(X)$ and $C(Y)$ if X, Y are compact).

The classical Banach-Stone theorem states that if there exists a linear isometry T of $C(X)$ onto $C(Y)$, then there is a homeomorphism ψ of Y onto X and a continuous map a: $Y \to \mathbf{K}$, $|a| \equiv 1$, such that T can be written as a weighted composition map; that is,

$$(Tf)(y) = a(y)f(\psi(y)) \text{ for all } y \in Y \text{ and all } f \in C(X).$$

An important generalization of the Banach-Stone theorem was given by W. Holsztyński in [9] by considering non-surjective isometries. Namely, he proved that, in this case, there is a closed subset Y_0 of Y where the isometry can still be represented as a weighted composition map. Recently, in [1], the authors have widely generalized this result by studying linear isometries between certain subspaces of $C_0(X)$ and $C_0(Y)$.

In the context of continuous vector-valued functions similar results are available. In [10], M. Jerison investigated the vector analogue of the Banach-Stone theorem: If X and Y are compact Hausdorff spaces and E is a strictly convex Banach space, then every linear isometry T of $C(X, E)$ onto $C(Y, E)$ can be written as a weighted composition map; namely, $(Tf)(y) = \omega(y)(f(\psi(y)))$, for all $f \in C(X, E)$ and all $y \in Y$, where ω is a continuous map from Y into the space of linear isometries from E onto E endowed with the strong operator topology. Furthermore, ψ is a homeomorphism of Y onto X. As in the scalar-valued case, Jerison's results have been extended in many directions (e.g., see [3] or [4]). Among them and in [6], M. Cambern obtained a formulation of Holsztyński's theorem for spaces of continuous vector-valued functions.

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In this paper we focus on such direction. In Section 1 we prove, assuming F strictly convex, that a linear isometry T of a certain linear subspace $\mathcal{A}[A]$ (see Definition 2 and Theorem 1) of $C_0(X, E)$ into $C_0(Y, F)$ can be written as a weighted composition map on a subspace Y_0 of Y for every function in $\mathcal{A}[A]$. Furthermore X is shown to be the continuous image of Y_0. In Section 2, after assuming that both E and F are strictly convex, we prove that X and Y are homeomorphic if T is a linear isometry of $\mathcal{A}[A]$ onto such a subspace $\mathcal{B}[B]$ of $C_0(Y, F)$. Let us recall that there are counterexamples (see [5] or [10]) which show that all the above results may not hold if the assumption of strict convexity is not observed.

Preliminaries

For a Banach space E, we will denote by $S_E := \{e \in E\colon \|e\| = 1\}$ its unit sphere and by $\|\cdot\|$ its norm. Recall that a Banach space E is said to be *strictly convex* if every element of S_E is an extreme point of the closed unit ball of E.

$X \cup \{\infty\}$ will stand for the Alexandroff compactification of the locally compact space X. For a function $f \in C_0(X, E)$, we will write $\operatorname{coz}(f)$ to denote the cozero set of f, this is, $\operatorname{coz}(f) := \{x \in X\colon f(x) \neq 0\}$. If V is a subset of X, we will write $\operatorname{cl}(V)$ to denote its closure in X.

Let A be a linear subspace of $C_0(X)$. A point $x_0 \in X$ is said to be a *strong boundary point* for A if for each neighborhood U of x_0 and $\epsilon > 0$, there is a function ξ in A such that $1 = \xi(x_0) = \|\xi\|_\infty$ and $|\xi(x)| < \epsilon$ for all $x \in X \setminus U$. We will write σA to denote the set of strong boundary points for A (i.e., *the strong boundary* for A).

A subset V of X is said to be a *boundary* for a linear subspace A (resp. $\mathcal{A}$) of $C_0(X)$ (resp. $C_0(X, E)$) if for every $\xi \in A$ (resp. $f \in \mathcal{A}$), there is $x \in V$ with $|\xi(x)| = \|\xi\|_\infty$ (resp. $\|f(x)\| = \|f\|_\infty$). ∂A will denote the *Shilov Boundary* for A, that is, the unique closed boundary for A. In [2], the authors show that the strong boundary for a point-separating closed subalgebra A of $C_0(X)$ is dense in ∂A.

Let us finally recall (e.g., see [8, p. 13]) that a linear subspace A of $C_0(X)$ is *regular* if for each closed subset W of X and each $x \in X \setminus W$, there is $\xi \in A$ such that $\xi(x) = 1$ and $\xi \equiv 0$ on W. Indeed, a regular closed subalgebra A of $C_0(X)$ can be proved to be normal (e.g., see [8, p. 3]); that is, for any pair, U and V, of disjoint compact subsets of X, there exists $\xi \in A$ such that $\xi \equiv 1$ on U and $\xi \equiv 0$ on V. It is also well known that the Shilov boundary of a regular closed subalgebra of $C_0(X)$ is X (e.g., see [8, p. 23]).

1. Into case

Definition 1. Let $\mathcal{A}$ be a linear subspace of $C_0(X, E)$ and let T be a linear isometry of $\mathcal{A}$ into $C_0(Y, F)$. If $e \in S_E$ and $x \in X$ such that there exists $f \in \mathcal{A}$ with

$f(x) = \|f\|_\infty \cdot e$, then we define

$I(x, e) := \{y \in Y\colon \|(Tf)(y)\| = \|f\|_\infty$ for all $f \in \mathcal{A}$ such that $f(x) = \|f\|_\infty \cdot e\}$.

LEMMA 1. *With the same hypothesis as in Definition 1, $I(x, e)$ is nonempty.*

Proof. It is clear that

$$I(x, e) \subset M_f = \left\{ y \in Y\colon \|(Tf)(y)\| \geq \frac{\|f\|_\infty}{2} \right\}$$

for any $f \in \mathcal{A}$ such that $f(x) = \|f\|_\infty \cdot e$. Since $I(x, e)$ is closed and M_f is compact (because $Tf \in C_0(Y, F)$), it suffices to check that if $f_1, \ldots, f_n$ satisfy $f_i(x) = \|f_i\|_\infty \cdot e$, then $\bigcap_{i=1}^n \{y \in Y\colon \|(Tf_i)(y)\| = \|f_i\|_\infty\} \neq \emptyset$.

Let us define $f_0 := \sum_{i=1}^n f_i$ and choose a point $y_0 \in Y$ where Tf_0 attains its norm (this point exists since $Tf_0 \in C_0(X, E)$). Hence

$$\begin{aligned} \sum_{i=1}^n \|(Tf_i)(y_0)\| \geq \|(Tf_0)(y_0)\| &= \|f_0\|_\infty \\ &\geq \|f_0(x)\| \\ &= \left\| \sum_{i=1}^n \|f_i\|_\infty \cdot e \right\| \\ &= \left\| e \cdot \sum_{i=1}^n \|f_i\|_\infty \right\| \\ &= \sum_{i=1}^n \|f_i\|_\infty . \end{aligned}$$

Consequently, since T is an isometry, we infer that

$$\sum_{i=1}^n \|(Tf_i)(y_0)\| = \sum_{i=1}^n \|f_i\|_\infty .$$

Hence $\|(Tf_i)(y_0)\| = \|f_i\|_\infty$ for all $i = 1, 2, \ldots, n$. That is, $y_0 \in \bigcap_{i=1}^n \{y \in Y : \|(Tf_i)(y)\| = \|f_i\|_\infty\}$. $\square$

Definition 2. Let A be a linear subspace of $C_0(X)$. We will denote by $\mathcal{A}[A]$ any linear subspace of $C_0(X, E)$ which contains the set $\{\xi \cdot e\colon \xi \in A, e \in S_E\}$.

Definition 3. Let A be a regular linear subspace of $C_0(X)$ and let T be a linear isometry of $\mathcal{A}[A]$ into $C_0(Y, F)$. For any $x \in \sigma A$, we define $I(x) := \bigcup_{e \in S_E} I(x, e)$.

In the remainder of this section we will assume that the Banach space F is strictly convex.

LEMMA 2. *Let A be a regular linear subspace of $C_0(X)$ and let T be a linear isometry of $\mathcal{A}[A]$ into $C_0(Y, F)$.*

(1) *Let $y \in I(x)$ for some $x \in \sigma A$. If we take $f \in \mathcal{A}[A]$ such that $f(x) = 0$, then $(Tf)(y) = 0$.*
(2) *$I(x_1) \cap I(x_2) = \emptyset$ for $x_1, x_2 \in \sigma A$.*

Proof. (1) Take $x_0 \in \sigma A$. From the definition of strong boundary point, we know that there is $\xi \in A$ with $1 = \xi(x_0) = \|\xi\|_\infty$. As a consequence, by Lemma 1, $I(x_0, e)$ is nonempty for every $e \in S_E$.

Fix $e \in S_E$ and $y_0 \in I(x_0, e)$. Let $f \in \mathcal{A}[A]$ such that f vanishes on some open neighbourhood U of x_0. We claim that $(Tf)(y_0) = 0$. To this end, we can assume, with no loss of generality, that $\|f\|_\infty < 1$. Let us choose ξ as above such that $\xi < \|\xi\|_\infty - \|f\|_\infty$ on $X \setminus U$. Let us define the functions

$$g := f + \xi \cdot e$$

and

$$h := \frac{1}{2}(g + \xi \cdot e).$$

It is obvious that $g(x_0) = h(x_0) = \xi(x_0) \cdot e$. Furthermore, $\|\xi \cdot e\|_\infty = \|g\|_\infty = \|h\|_\infty = \xi(x_0) = 1$. Hence, as $y_0 \in I(x_0, e)$, we have $\|T(\xi \cdot e)(y_0)\| = \|(Tg)(y_0)\| = \|(Th)(y_0)\| = \xi(x_0) = 1$. Since F is strictly convex, and $T(\xi \cdot e)(y_0)$, $(Tg)(y_0)$ and $(Th)(y_0)$ belong to S_E, we infer that $T(\xi \cdot e)(y_0)$ and $(Tg)(y_0)$ coincide (note that $(Th)(y_0)$ is on the segment which joins $T(\xi \cdot e)(y_0)$ and $(Tg)(y_0)$). As a consequence, $(Tf)(y_0) = 0$.

Let $\hat{T}\widehat{y_0}\colon \mathcal{A}[A] \longrightarrow F$ and $\hat{x_0}\colon \mathcal{A}[A] \longrightarrow E$ be the functionals defined by the requirement that $\hat{T}\widehat{y_0}(f) := (Tf)(y_0)$ and $\widehat{x_0}(f) := f(x_0)$, $f \in \mathcal{A}[A]$. It is straightforward to check that the functions in $\mathcal{A}[A]$ that vanish on a neighbourhood of x_0 are dense in $\ker(\widehat{x_0})$ since A is regular. Furthermore, $\ker(\widehat{x_0})$ is closed since the functional $\widehat{x_0}$ is continuous (e.g., see [7, p. 77]). Consequently, the above paragraph yields the inclusion $\ker(\widehat{x_0}) \subset \ker(\hat{T}\widehat{y_0})$; this is, if $f(x_0) = 0$, then $(Tf)(y_0) = 0$, as was to be proved.

(2) Suppose that there are $x_1, x_2 \in \sigma A$ and $y \in Y$ such that $y \in I(x_1) \cap I(x_2)$. Choose $\xi \in A$ such that $\xi(x_1) = 1$ and $\xi(x_2) = 0$. Since $(\xi \cdot e)(x_2) = 0$ for every $e \in E$, we have, by (1), that $T(\xi \cdot e)(y) = 0$ for all $e \in E$.

On the other hand, there is $e_1 \in S_E$ such that $y \in I(x_1, e_1)$ and, as x_1 is a strong boundary point for A, there is a function $0 \not\equiv f \in \mathcal{A}[A]$ such that $f(x_1) = \|f\|_\infty \cdot e_1$. By (1), we have $(Tf)(y) = T(\xi \cdot f(x_1))(y)$ since T is linear and $(f - \xi \cdot f(x_1))(x_1) = 0$. Besides, by the above paragraph, $(Tf)(y) = T(\xi \cdot f(x_1))(y) = 0$. But from the definition of the set $I(x_1, e_1)$, we know that $\|(Tf)(y)\| = \|f\|_\infty \neq 0$, which is a contradiction. $\square$

LEMMA 3. *Let A be a regular linear subspace of $C_0(X)$ and let T be a linear isometry of $\mathcal{A}[A]$ into $C_0(Y, F)$. Let $x \in \sigma A$ and $e \in S_E$. If $f(x) = e$ for $f \in \mathcal{A}[A]$, then $\|(Tf)(y)\| = \|e\|$ for all $y \in I(x, e)$.*

Proof. Since x is a strong boundary point, there is $\xi \in A$ with $1 = \xi(x) = \|\xi\|_\infty$. Define a function g in $\mathcal{A}[A]$ by $g := f - \xi \cdot e$. The clear fact that $g(x) = 0$ and Lemma 2(1) yield $(Tg)(y) = 0$. By the linearity of T, we have $(Tf)(y) = T(\xi \cdot e)(y)$. Finally, from the definition of $I(x, e)$, $\|(Tf)(y)\| = \|T(\xi \cdot e)(y)\| = \|\xi \cdot e\|_\infty = \|e\|$. $\square$

Definition 4. Let A be a regular closed subalgebra of $C_0(X)$ and let T be a linear isometry of $\mathcal{A}[A]$ into $C_0(Y, F)$. For $x \in X \setminus \sigma A$, we define $I(x) := \{y \in Y\colon\ (Tf)(y) = 0 \text{ for all } f \in \mathcal{A}[A] \text{ such that } f(x) = 0\}$.

LEMMA 4. *Let A be a regular closed subalgebra of $C_0(X)$ and let T be a linear isometry of $\mathcal{A}[A]$ into $C_0(Y, F)$. Then*

(1) *For any $x \in X$, $I(x)$ is nonempty.*
(2) *For any pair $x_1, x_2 \in X$, $I(x_1) \cap I(x_2) = \emptyset$.*

Proof. (1) For $x \in \sigma A$, the result can be found in Lemma 1. On the other hand, let us recall (see the Preliminaries) that σA is dense in $\partial A = X$. Hence, for $x_0 \in X \setminus \sigma A$, there is a net $\{x_\alpha\}$ in σA converging to x_0. Fix $e \in S_E$. For each α, there is $y_\alpha \in I(x_\alpha, e)$. Since $Y \cup \{\infty\}$ is compact, we can assume, by taking a subnet if necessary, that $\{y_\alpha\}$ converges to $y_0 \in Y \cup \{\infty\}$. Let us take a function $f \in \mathcal{A}[A]$ such that f vanishes on some neighbourhood V of x. Then there is α_0 such that $x_\alpha \in V$ for $\alpha > \alpha_0$. By Lemma 2(1), $(Tf)(y_\alpha) = 0$ for $\alpha > \alpha_0$ and, consequently, $(Tf)(y_0) = 0$. Now, arguments like those in the last paragraph of the proof of Lemma 2(1) show that $y_0 \in I(x_0)$.

Finally, let us check that $y_0 \neq \infty$. Since A is normal (see the Preliminaries), we can choose $\xi \in A$ such that $\xi \equiv 1$ on V. Let $g := \xi \cdot e$. Since $g(x_\alpha) = e$ for all $\alpha > \alpha_0$, Lemma 3 entails that $\|(Tg)(y_\alpha)\| = \|e\|$ for all $\alpha > \alpha_0$. That is, $\|(Tg)(y_0)\| = \|e\| \neq 0$, which shows that $y_0 \neq \infty$.

(2) If either x_1 or x_2 is a strong boundary point for A, then the result follows from the same arguments as in Lemma 2(2).

Assume that there exists $y \in I(x_1) \cap I(x_2)$ with $x_1, x_2 \in X \setminus \sigma A$. Let V_1 and V_2 be open neighbourhoods of x_1 and x_2 respectively with disjoint closures. By (1), there exist $e_1, e_2 \in S_E$, and two nets $\{x_\alpha\}$ and $\{x_\beta\}$ in σA converging to x_1 and x_2 respectively, such that both the nets $\{y_\alpha\} \subset I(x_\alpha, e_1)$ and $\{y_\beta\} \subset I(x_\beta, e_2)$ converge to y. Choose $\xi \in A$ such that $\xi \equiv 1$ on V_1 and vanishes on V_2. If we define $g := \xi \cdot e_1$, then, as in (1), it is apparent that $\|(Tg)(y)\| = \|e_1\| \neq 0$. But there is β_0 such that $(Tg)(y_\beta) = 0$ for $\beta > \beta_0$. Hence $(Tg)(y) = 0$. This contradiction shows that $I(x_1) \cap I(x_2) = \emptyset$. $\square$

Remark 1. Let us now introduce a linear map of E into F which will allow us to obtain a multiplicative representation of T. Note first that if $\xi \in A$ and $e \in E$, then $\xi \cdot e = (\xi \cdot \|e\|)\frac{e}{\|e\|} \in \mathcal{A}[A]$.

Definition. With the same hypothesis as in Lemma 2 (resp. Lemma 4), let $y \in I(x)$ for some $x \in \sigma A$ (resp. $x \in X$) and let $\xi \in A$ such that $\xi(x) = 1$. Then we define a linear map $\omega(y)$ of E into F as follows: $\omega(y)(e) := T(\xi \cdot e)(y)$ for all $e \in E$.

To see that $\omega(y)$ is well defined, suppose that there is another function $\xi' \in A$ such that $\xi'(x) = 1$. Hence, since $(\xi \cdot e - \xi' \cdot e)(x) = 0$ for all $e \in E$, Lemma 2(1) (resp. Definition 4) entails that $T(\xi \cdot e)(y) = T(\xi' \cdot e)(y)$ for all $e \in E$.

Definition 5. Let A be a regular linear subspace (resp. closed subalgebra) of $C_0(X)$. For a linear isometry T of $\mathcal{A}[A]$ into $C_0(Y, F)$, we define the set $Y_0 := \bigcup_{x \in \sigma A} I(x)$ (resp. $\mathcal{Y}_0 := \bigcup_{x \in X} I(x)$) and a mapping ψ of Y_0 (resp. $\mathcal{Y}_0$) onto σA (resp. X) by $\psi(y) := x$, where $y \in I(x)$.

THEOREM 1. *Let T be a linear isometry of $\mathcal{A}[A]$ into $C_0(Y, F)$.*

(1) *If A is a regular linear subspace of $C_0(X)$ such that $\sigma A \neq \emptyset$, then $\psi\colon Y_0 \longrightarrow \sigma A$ is a well defined surjective continuous mapping and $(Tf)(y) = \omega(y)(f(\psi(y)))$ for all $y \in Y_0$ and all $f \in \mathcal{A}[A]$.*
(2) *If A is a closed regular subalgebra of $C_0(X)$, then $\psi\colon \mathcal{Y}_0 \longrightarrow X$ is a well defined surjective continuous mapping and $(Tf)(y) = \omega(y)(f(\psi(y)))$ for all $y \in \mathcal{Y}_0$ and all $f \in \mathcal{A}[A]$. Furthermore, ω is a continuous mapping from $\mathcal{Y}_0$ into the space of bounded operators of E into F, when this latter space is given its strong operator topology.*

If, in addition, σA is a boundary for $\mathcal{A}[A]$, then Y_0 (resp. $\mathcal{Y}_0$) is a boundary for $T(\mathcal{A}[A])$.

Proof. (1) By Lemma 2(2), ψ is a well defined mapping. To check the continuity of ψ, let $\{y_\alpha\}$ be a net convergent to y in Y_0. Assume, contrary to what we claim, that $\{\psi(y_\alpha)\}$ does not converge to $\psi(y)$. By taking a subnet if necessary, we can consider that $\{\psi(y_\alpha)\}$ converges to an x in the compact space $X \cup \{\infty\}$. Let U and V be disjoint neighborhoods of x and $\psi(y)$ in $X \cup \{\infty\}$, respectively. There exist an

α_0 such that $\psi(y_\alpha) \in U$, for all $\alpha \geq \alpha_0$, and, since A is regular, a function $f \in \mathcal{A}[A]$ such that $\mathrm{coz}(f) \subset V$ and $\|(Tf)(y)\| \neq 0$. For $\alpha \geq \alpha_0$, $\psi(y_\alpha) \notin \mathrm{coz}(f)$. Hence, by Lemma 2(1), $(Tf)(y_\alpha) = 0$, for all $y_\alpha \geq \alpha_0$. Consequently $\{(Tf)(y_\alpha)\}$ does not converge to $(Tf)(y) \neq 0$, which contradicts the continuity of Tf.

Finally, to obtain the multiplicative representation of T, let $x \in \sigma A$ and $y \in I(x)$. Choose any function $\xi \in A$ such that $\xi(x) = 1$. For every $f \in \mathcal{A}[A]$, the function $f - \xi \cdot f(x)$ vanishes at x. Thus, by Lemma 2(1), we infer that $(Tf)(y) = T(\xi \cdot f(x))(y) = \omega(y)(f(x))$ for every $f \in \mathcal{A}[A]$.

(2) By Lemma 4(2), ψ is a well defined mapping. The results now follows from Lemma 4(1) and from the same arguments as in (1).

To prove the continuity of ω, let $\{y_\alpha\}$ be a net convergent to y in $\mathcal{Y}_0$. Fix $e \in E$ and define a function $f \in \mathcal{A}[A]$ by $f := \xi \cdot e$, where $\xi \equiv 1$ on a certain neighbourhood of $\psi(y)$. Since ψ is continuous, there is a α_0 such that, for all $\alpha \geq \alpha_0$, $\|\omega(y_\alpha)e - \omega(y)e\| = \|\omega(y_\alpha)f(\psi(y_\alpha)) - \omega(y)f(\psi(y))\| = \|(Tf)(y_\alpha) - (Tf)(y)\|$. Since $\{(Tf)(y_\alpha)\}$ converges to $(Tf)(y)$, the continuity of ω is thus verified.

Assume now that σA is a boundary for $\mathcal{A}[A]$. Take $g \in T(\mathcal{A}[A])$ and $f \in \mathcal{A}[A]$ such that $Tf = g$. We can find $x \in \sigma A$ and $e \in S_E$ with $f(x) = \|f\|_\infty \cdot e$. By Lemma 1, there exists $y \in I(x, e) \subseteq I(x) \subseteq Y_0$ (resp. $\mathcal{Y}_0$) such that $\|g(y)\| = \|(Tf)(y)\| = \|Tf\|_\infty = \|g\|_\infty$, that is, Y_0 (resp. $\mathcal{Y}_0$) is a boundary for $T(\mathcal{A}[A])$. □

Remark 2. Theorem 1 generalizes the main result of [6] by taking X compact and $\mathcal{A}[A] = C(X)$.

2. Surjective case

In this section we will assume that both E and F are strictly convex Banach spaces.

THEOREM 2. *Let T be a linear isometry of $\mathcal{A}[A]$ onto such a subspace $\mathcal{B}[B]$ of $C_0(Y, F)$, where A and B are regular closed subalgebras of $C_0(X)$ and $C_0(Y)$ respectively. Then ψ is a homeomorphism of Y onto X and $(Tf)(y) = \omega(y)(f(\psi(y)))$ for all $y \in Y$ and all $f \in \mathcal{A}[A]$. Furthermore, if $y \in I(x)$ for some $x \in \sigma A$, then $\omega(y)$ is a linear isometry of E into F.*

Proof. Fix $x \in X$ and let $y \in I(x)$. We first claim that $x \in I(y)$. Suppose that $x \notin I(y)$. Then, since $T^{-1}\colon \mathcal{B}[B] \longrightarrow \mathcal{A}[A]$ is a linear isometry, Lemma 4(1) entails that there exists $x' \in X$, $x' \neq x$, such that $x' \in I(y)$. Choose $f \in \mathcal{A}[A]$ such that $f(x) = 0$. By Lemma 2(1) and Definition 4, we infer that both $(Tf)(y) = 0$ and $T^{-1}(Tf)(x') = f(x') = 0$. This means that x and x' cannot be separated with functions of $\mathcal{A}[A]$, which contradicts the regularity of A.

Let us now suppose that $I(x)$ contains two elements, y and y'. By the above paragraph, $x \in I(y) \cap I(y')$. Since $T(\mathcal{A}[A])$ separates the points of Y, there is a function $f \in \mathcal{A}[A]$ such that $(Tf)(y) = 1$ and $(Tf)(y') = 0$. From Lemma 2(1) and Definition 4, we have $T^{-1}(Tf)(x) = f(x) = 0$ and, hence, $(Tf)(y) = 0$. This contradiction shows that $I(x)$ is a singleton.

As a straightforward consequence of the above two paragraphs and Theorem 1, we infer that $\mathcal{Y}_0 = Y$ and that $\psi\colon Y \longrightarrow X$ is a continuous bijection. Furthermore (see Definition 5) T^{-1} induces a continuous bijection of X onto Y which can be easily checked to be the inverse of ψ, which is to say that X and Y are homeomorphic.

Finally, take $y_0 \in Y$ such $y_0 = I(x_0)$ for some $x_0 \in \sigma A$. To see that $\omega(y_0)$ is a linear isometry of E into F, choose $e_0 \in S_E$ and $\xi \in A$ such that $1 = \xi(x_0) = \|\xi\|_\infty$. It suffices to check that $\|\omega(y_0)(e_0)\| = 1$. Let us first note that, by Lemma 1, it is apparent (see also the proof of Lemma 2(1)) that $I(x_0, e) \neq \emptyset$ for all $e \in S_E$. Hence, since $I(x_0)$ is a singleton, $y_0 = \bigcap_{e\in S_E} I(x_0, e)$. In particular, $y_0 \in I(x_0, e_0)$. Consequently,

$$\|\omega(y_0)(e_0)\| := \|T(\xi \cdot e_0)(y_0)\| = \|\xi \cdot e_0\|_\infty = 1. \qquad \square$$

Remark 3. Theorems 1 and 2 generalize the main result of [10] by taking X, Y compact, and $\mathcal{A}[A] = C(X)$ and $\mathcal{B}[B] = C(Y)$.

Definition 6. Let $\mathcal{A}$ be linear subspace of $C_0(X, E)$. We say that $x_0 \in X$ is a strong boundary point for $\mathcal{A}$ if for each neighborhood U of x_0, there is a function $f \in \mathcal{A}$ such that $\|f(x_0)\| = \|f\|_\infty$ and $\|f(x)\| < \|f\|_\infty$ for all $x \in X \setminus U$. We define the strong boundary for $\mathcal{A}$, $\sigma\mathcal{A}$, to be the set of all strong boundary points for $\mathcal{A}$.

It is a routine matter to verify that $\sigma A \subseteq \sigma\mathcal{A}[A]$ for any linear subspace A of $C_0(X)$.

THEOREM 3. *Let T be a linear isometry of $\mathcal{A}[A]$ onto such a subspace $\mathcal{B}[B]$ of $C_0(Y, F)$, where A and B are regular closed subalgebras of $C_0(X)$ and $C_0(Y)$ respectively. If we assume that $\sigma\mathcal{A}[A] = \sigma A$ and $\sigma\mathcal{B}[B] = \sigma B$, then the strong boundaries for $\mathcal{A}[A]$ and for $\mathcal{B}[B]$ are homeomorphic.*

Proof. Let $x_0 \in \sigma\mathcal{A}[A] = \sigma A$. By Theorem 2, $I(x_0)$ is a singleton. Let $y_0 = I(x_0)$.

Next we claim that $I(x_0) \subseteq \sigma\mathcal{B}[B]$. Fix $e_0 \in S_E$ and let V be a neighbourhood of y_0. Recall, as in the proof of Theorem 2, that $y_0 = \bigcap_{e\in S_E} I(x_0, e)$. If $y \notin V$, then, from the definition of $I(x_0)$, there is a function $f_y \in \mathcal{A}[A]$ such that

$$\left\| f_y(x_0) \right\| = \left\| f_y \right\|_\infty \cdot e_0$$

and

$$\left\| (Tf_y)(y) \right\| < \left\| f_y \right\|_\infty .$$

For each $y \in (Y \cup \{\infty\}) \setminus V$, we may take an open neighborhood V_y of y such that $\|(Tf_y)(y')\| < \|f_y\|_\infty$ for all $y' \in V_y$. Since $(Y \cup \{\infty\}) \setminus V$ is compact, there exist $\{y_1, \dots, y_n\} \subset (Y \cup \{\infty\}) \setminus V$ such that $(Y \cup \{\infty\}) \setminus V \subset \bigcup_{i=1}^n V_{y_i}$. Now, let us define the map

$$g := \sum_{i=1}^{n} f_{y_i}.$$

It is clear that

$$g(x_0) = \left(\sum_{i=1}^{n} \|f_{y_i}\|_\infty \right) \cdot e_0,$$

which implies that $\|g(x_0)\| = \|g\|_\infty$. Hence, since $y_0 = I(x_0)$, we infer that $\|(Tg)(y_0)\| = \|g\|_\infty$. Moreover, for every $y \in Y \setminus V$ we have

$$\|(Tg)(y)\| \leq \sum_{i=1}^{n} \left\|(Tf_{y_i})(y)\right\| < \sum_{i=1}^{n} \|f_{y_i}\|_\infty = \|g\|_\infty = \|Tg\|_\infty .$$

As a consequence, $y_0 \in \sigma\mathcal{B}[B]$.

In like manner we prove that, if $y_0 \in \sigma\mathcal{B}[B] = \sigma B$, then $I(y_0) \in \sigma\mathcal{A}[A]$. That is, ψ, which is a homeomorphism of Y onto X (see Theorem 2), sends $\sigma\mathcal{B}[B]$ onto $\sigma\mathcal{A}[A]$. □

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Departamento de Matemáticas, Universitat Jaume I, Campus Penyeta Roja, E-12071, Castellón, Spain
font@mat.uji.es