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SOBOLEV AND HÖLDER ESTIMATES FOR $\overline{\partial}$ ON BOUNDED CONVEX DOMAINS IN $\mathbb{C}^2$

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1. Introduction

The regularity of the Cauchy-Riemann operator $\overline{\partial}$ is a very important problem in both PDE's and Several Complex Variables. Numerous results have been proved by many mathematicians. Here we only list two recent results concerning $\overline{\partial}$ on convex domains.

In 1991, Polking [12] proved the following L^p estimates.

THEOREM 1. *Let $D = \{z \in \mathbb{C}^2 \mid \rho(z) < 0\}$ be a bounded convex domain with C^2 boundary ∂D. Then there exists an integral solution operator T for $\overline{\partial}$ on D such that*

$$\|Tf\|_{L^p(D)} \leq C(p)\|f\|_{L^p(D)}$$

for all $1 < p < +\infty$.

In 1992, Range [13] proved the following Hölder estimates.

THEOREM 2. *Let $D = \{z \in \mathbb{C}^2 \mid \rho(z) < 0\}$ be convex with C^∞ boundary. Then there exists an integral solution operator $T\colon C_{(0,1)}(\overline{D}) \longrightarrow C(D)$ for $\overline{\partial}$ such that*

$$\|Tf\|_{\Lambda_\alpha(D)} \leq C(\alpha)\|f\|_{\Lambda_\alpha(D)}$$

for all f with $\overline{\partial} f = 0$ and all $\alpha > 0$.

Without loss of generality, we will assume that f is a (0,1) form $f = f_1 d\overline{\xi}_1 + f_2 d\overline{\xi}_2$. A (0, 1) form is in $W^{\alpha,p}(D)$, $\Lambda^p_\alpha(D)$, if its coefficients are in $W^{\alpha,p}(D)$, $\Lambda^p_\alpha(D)$, respectively. For definitions and properties of the Sobolev and Hölder spaces $W^{\alpha,p}(D)$ and $\Lambda^p_\alpha(D)$, see [1], [17].

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In this paper, we prove the following results.

THEOREM 3. *Assume* $D = \{z \in \mathbb{C}^2 \mid \rho(z) < 0\}$ *is a bounded convex domain with smooth boundary* ∂D. *Then there exists an integral solution operator* T *for* $\overline{\partial}$ *on* D *such that if* $f \in W^{\alpha,p}(D)$, *then*

$$\|Tf\|_{W^{\alpha,p}(D)} \leq C(\alpha, p)\|f\|_{W^{\alpha,p}(D)}$$

for all $\alpha \geq 0,\ 1 < p < +\infty$.

THEOREM 4. *Assume* $D = \{z \in \mathbb{C}^2 \mid \rho(z) < 0\}$ *is defined as above. Then there exists an integral solution operator* T *for* $\overline{\partial}$ *on* D *such that if* $f \in \Lambda_\alpha^p(D)$, *then*

$$\|Tf\|_{\Lambda_\alpha^p(D)} \leq C(\alpha, p)\|f\|_{\Lambda_\alpha^p(D)}$$

for all $\alpha > 0$, *and* $1 \leq p \leq +\infty$.

Following the proof of Theorem 4, we can prove a similar result for the Cauchy tangential operator $\overline{\partial}_b$ on ∂D.

THEOREM 5. *Let* $D \subset \mathbb{C}^2$ *be a bounded convex domain with smooth boundary. Suppose a* $(0, 1)$ *form* f *satisfies the compatibility condition*

$$\int_{\partial D} f \wedge \varphi = 0$$

for any $(2, 0)$ *form* φ *which is* $\overline{\partial}$*-closed in* D *and continuous up to* ∂D. *Then there exists an integral solution operator* S *for* $\overline{\partial}_b$ *on* ∂D *such that if* $f \in \Lambda_\alpha^p(\partial D)$, *then*

$$\|Sf\|_{\Lambda_\alpha^p(\partial D)} \leq C(\alpha, p)\|f\|_{\Lambda_\alpha^p(\partial D)}$$

for all $\alpha > 0$, *and* $1 \leq p \leq +\infty$.

Remarks. (1) If $p > 2$, Fornaess-Sibony [8] gave many examples to show that there is no L^p estimate for $\overline{\partial}$ on general pseudoconvex domains.

(2) Chaumat-Chollet [5] proved Hölder Λ_α estimates for $\alpha > 1,\ \alpha \notin \mathbb{N}$, on convex domains in $\mathbb{C}^2$ with C^2 boundary.

(3) Sibony [7] provided a counter-example which shows that the Λ_α estimate is not true for general pseudoconvex domains.

(4) In Theorem 3, the $k = 0$ case is the L^p estimate which was proved in [12].

(5) In Theorem 4, the $\alpha > 0,\ p = +\infty$ case is the Λ_α estimate which was proved in [5] and [13].

(6) For some concepts, formulations, and results for $\overline{\partial}_b$, see [9], [16], [10], [3], [15].

This paper is presented as follows: In §1, we give two recent results about the regularity for $\overline{\partial}$ and state the main theorems. In §2, we prove the Sobolev estimates. In §3, examples will be given to show that there is no "gain" in the Sobolev estimates for the canonical solution. In §4, we prove the Hölder estimates for $\overline{\partial}$.

2. Proof of Theorem 3

Assume $D = \{z \in \mathbb{C}^2 \mid \rho(z) < 0\}$ is a bounded convex domain with smooth boundary ∂D and $|d\rho| = 1$ on ∂D. Let σ denote the Lebesgue measure, c denote a positive constant which may vary from line to line.

We choose a smooth defining function ρ for D, such that in a neighborhood U of ∂D,

$$\rho(z) = \begin{cases} -\mathrm{dist}(z, \partial D), & z \in U \cap \overline{D} \\ +\mathrm{dist}(z, \partial D), & z \in U \setminus D. \end{cases}$$

Define

$$\begin{aligned} \Phi_0(\xi, z) &= \Phi(\xi, z) - \rho(\xi), \\ \Phi(\xi, z) &= \frac{\partial \rho}{\partial \xi_1}(\xi_1 - z_1) + \frac{\partial \rho}{\partial \xi_2}(\xi_2 - z_2). \end{aligned}$$

By the convexity of D, it is well known (cf. [9], [13]) that

$$\mathrm{Re}\Phi(\xi, z) \geq c|\rho(z)|, \quad z \in \overline{D}, \quad \xi \in \partial D, \tag{1}$$

$$|\Phi_0(\xi, z)| \geq c(|\rho(\xi)| + |\rho(z)| + |\mathrm{Im}\Phi(\xi, z)|), \quad \xi, \ z \in \overline{D} \cap U. \tag{2}$$

The following lemma was proved in [5] and [12].

LEMMA 1. *Let* $(\xi_0, z_0) \in \partial D \times \partial D$ *such that* $\Phi(\xi_0, z_0) = 0$. *Then there exist neighborhoods* W *of* ξ_0 *and* V *of* z_0, *such that for each* $z \in V$, *there exists a* C^1 *local coordinate system* $\xi \longmapsto t^{(z)}(\xi) = (t_1, t_2, t_3, t_4)$ *on* W *with the following properties*:

$$\begin{cases} t_4 = -\rho(\xi) \\ t_3 = \mathrm{Im}\Phi(\xi, z) \\ \overline{t'} = t_1 - it_2 = p_2(z)(\overline{\xi}_1 - \overline{z}_1) - p_1(z)(\overline{\xi}_2 - \overline{z}_2) \end{cases} \tag{3}$$

$$|t^{(z)}(\xi) - t^{(z)}(\xi')| \sim |\xi - \xi'| \tag{4}$$

for all $\xi, \ \xi' \in W$, *with the constant in* (4) *independent of* $z \in V$.

Let T be Henkin's solution operator for $\overline{\partial}$. Then

$$Tf(z) = Hf(z) + Kf(z)$$

where Kf is given by integrating f against the Bochner-Martinelli kernel over D, and

$$Hf(z) = c \int_{\partial D} f(\xi) \wedge \frac{\frac{\partial \rho}{\partial \xi_1}(\overline{\xi}_2 - \overline{z}_2) - \frac{\partial \rho}{\partial \xi_2}(\overline{\xi}_1 - \overline{z}_1)}{|\xi - z|^2 \Phi(\xi, z)} \, d\xi_1 \wedge d\xi_2.$$

It is easy to prove that if $f \in W^{1,p}(D)$, $1 < p < \infty$, then

$$\|Kf\|_{W^{1,p}(D)} \le C_p \|f\|_{W^{1,p}(D)}.$$

We first want to prove the $W^{1,p}$ estimates for Tf which can be reduced to estimate Hf.

By Stokes' Theorem, we rewrite (cf. Polking [12])

$$Hf(z) = c \int_D f(\xi) \wedge \overline{\partial}_\xi \left(\frac{\chi(\xi) A_1(\xi, z)}{\tau(\xi, z)\Phi_0(\xi, z)} \right) d\xi_1 \wedge d\xi_2,$$

where $\tau(\xi, z) = |\xi - z|^2 + \rho(\xi)\rho(z)$, $A_1(\xi, z) = O(|\xi - z|)$, χ is a C^∞ function in $\mathbb{C}^2$ such that $\chi \equiv 1$ in $D_{\frac{\delta}{4}}$, supp$\chi \subset D_{\frac{\delta}{2}} \subset D_\delta \subset U$, and $D_\delta = \{z\colon \operatorname{dist}(z, \partial D) < \delta\}$ is a tube neighborhood of ∂D.
It is easy to show that

$$\tau(\xi, z) \ge c(|\xi - z|^2 + |\rho(\xi)|^2 + |\rho(z)|^2), \quad \xi,\ z \in \overline{D} \cap U. \tag{5}$$

In order to prove $\nabla Hf(z) \in L^p(D)$, $1 < p < \infty$, we need Schur's lemma (cf. [12]).

LEMMA 2. *Assume a kernel $k(\xi, z)$ is defined in $D \times D$ and an operator K is defined by $Kf(z) = \int_D k(\xi, z) f(\xi)\, d\sigma(\xi)$. Suppose for every $0 < \varepsilon < 1$, there exists C_ε such that*

$$\int_D |\rho(\xi)|^{-\varepsilon} |k(\xi, z)|\, d\sigma(\xi) \le C_\varepsilon |\rho(z)|^{-\varepsilon} \quad \textit{for all } z \in D,$$
$$\int_D |\rho(z)|^{-\varepsilon} |k(\xi, z)|\, d\sigma(z) \le C_\varepsilon |\rho(\xi)|^{-\varepsilon} \quad \textit{for all } \xi \in D.$$

Then for $1 < p < \infty$, there exists C_p such that $\|Kf\|_{L^p(D)} \le C_p \|f\|_{L^p(D)}$.

2.1. *The case $k = 1$.* A simple computation yields

$$\nabla Hf(z) = c \int_D f(\xi) \left(\frac{G_1(\xi, z)}{\tau^2(\xi, z)\Phi_0(\xi, z)} + \frac{G_2(\xi, z)}{\tau^2(\xi, z)\Phi_0^2(\xi, z)} + \frac{G_1(\xi, z)}{\tau(\xi, z)\Phi_0^3(\xi, z)} + \text{lower order singular terms} \right) d\sigma(\xi),$$

where $G_j(\xi, z) = O(|\xi - z|^j)$, $j = 1, 2, 3$ and the G_j's that appear in different places may not be the same. We need to estimate the three type terms:

$$\begin{aligned} I_1(z) &= \int_D f(\xi)\frac{G_1(\xi,z)}{\tau^2(\xi,z)\Phi_0(\xi,z)}\,d\sigma(\xi), \\ I_2(z) &= \int_D f(\xi)\frac{G_2(\xi,z)}{\tau^2(\xi,z)\Phi_0^2(\xi,z)}\,d\sigma(\xi), \\ I_3(z) &= \int_D f(\xi)\frac{G_1(\xi,z)}{\tau(\xi,z)\Phi_0^3(\xi,z)}\,d\sigma(\xi). \end{aligned}$$

LEMMA 3. *Estimating I_3 can be reduced to estimating an I_2 type integral.*

In order to estimate I_3, by the compactness of $\overline{D}$ and a partition of unity argument, it suffices to estimate the following integral

$$\int_{D\cap W} f(\xi)\frac{\chi_1(\xi)G_1(\xi,z)}{\tau(\xi,z)\Phi_0^3(\xi,z)}\,d\sigma(\xi)$$

Here, $\operatorname{supp}\chi_1 \subset W$, and $z \in V$, where W, V are neighborhoods chosen as in Lemma 1. Notice that the vector fields

$$T = \operatorname{Im}\left(\frac{\partial\rho}{\partial\xi_1}\frac{\partial}{\partial\overline{\xi}_1} + \frac{\partial\rho}{\partial\xi_2}\frac{\partial}{\partial\overline{\xi}_2}\right)$$

are tangential to the level sets of ∂D. Also, $Tt_3 = 1 + O(|\xi - z|)$. After making the coordinate change (3), we have

$$|T\Phi_0| = |T\mathrm{Re}\Phi_0| + 1 + O(|t|) \geq \frac{1}{2}.$$

Using the fact that $\frac{1}{\Phi_0^3} = -\frac{1}{2}(T\Phi_0)^{-1}T\left(\frac{1}{\Phi_0^2}\right)$ and integrating by parts with respect to the T direction, we have

$$\begin{aligned} &\int_{D\cap W} f(\xi)\frac{\chi_1(\xi)G_1(\xi,z)}{\tau(\xi,z)\Phi_0^3(\xi,z)}\,d\sigma(\xi) \\ &= \int_{|t|<c,\ t_4>0} Tf(\xi(t))\,\frac{\chi_1(\xi)G_1(\xi(t),z)}{\tau(\xi,z)\Phi_0^2(\xi,z)}\,d\sigma(t) \\ &\quad + \int_{|t|<c,\ t_4>0} f(\xi(t))\,\frac{G_0(\xi(t),z)}{\tau(\xi,z)\Phi_0^2(\xi,z)}\,d\sigma(t). \end{aligned}$$

As in Polking [12] , the first term is bounded by $\|Df\|_{L^p(D)} \leq C_p\|f\|_{W^{1,p}(D)}$. The second term can be reduced to an I_2 type integral, and the lemma follows.

The same proof shows that estimating I_2 can be reduced to estimating an I_1 type integral.

Let

$$k(\xi, z) = \frac{G_1(\xi, z)}{\tau^2(\xi, z)\Phi_0(\xi, z)}.$$

In order to estimate

$$\int_D |\rho(\xi)|^{-\varepsilon} |k(\xi, z)|\, d\sigma(\xi),$$

it suffices to estimate

$$I =: \int_{D\cap W} \chi_1 |\rho(\xi)|^{-\varepsilon} |k(\xi, z)|\, d\sigma(\xi)$$

with $\operatorname{supp}\chi_1 \subset W$.

By the estimates (1), (4), (5), and the coordinate change (3), we have

$$I \leq c \int_{|t|<c,\ t_j \geq 0} \frac{t_4^{-\varepsilon}}{(|t| + |\rho(z)|)^3 (t_3 + t_4 + |\rho(z)|)}\, d\sigma(t).$$

Let $t = |\rho(z)| s$. Then

$$\begin{aligned} I &\leq c|\rho(z)|^{-\varepsilon} \int_{s_j \geq 0} \frac{s_4^{-\varepsilon}}{(|s| + 1)^3 (s_3 + s_4 + 1)}\, d\sigma(s) \\ &= C_\varepsilon |\rho(z)|^{-\varepsilon}. \end{aligned}$$

By the compactness of $\overline{D}$ and the partition of unity, we see that

$$\int_D |\rho(\xi)|^{-\varepsilon} |k(\xi, z)|\, d\sigma(\xi) \leq C_\varepsilon |\rho(z)|^{-\varepsilon}, \quad z \in D.$$

By a symmetric argument, we have

$$\int_D |\rho(z)|^{-\varepsilon} |k(\xi, z)|\, d\sigma(z) \leq C_\varepsilon |\rho(\xi)|^{-\varepsilon}, \quad \xi \in D.$$

Therefore, in the $k = 1$ case, Theorem 3 follows by Schur's lemma.

2.2. *The case* $k \geq 2$. As in Range [13], we introduce the Seeley extension operator $E\colon C(\overline{D}) \longrightarrow C_0(D^\#)$, where $D^\# = D \cup U$. By Adams [1] (Theorem 4.28, p. 89),

$$\|Ef\|_{W^{k,p}(\mathbb{C}^2)} \leq C(k, p) \|f\|_{W^{k,p}(D)}, \qquad 1 \leq p < +\infty.$$

In the representation formula

$$f = \bar{\partial}_z T_1 f + T_2 \bar{\partial} f, \qquad f \in C^1_{(0,1)}(\overline{D}),$$

we let $g_1 = f,\ g_2 = \overline{\partial} f$, and $Eg_q = g_q$ on ∂D. Then

$$T_q g_q = \int_{\partial D\times I} Eg_q \wedge \Omega_{q-1}(\hat{W}) - \int_D g_q \wedge K_{q-1}, \qquad q = 1, 2.$$

Here, $T_1 = T$ is Henkin's solution operator.

Let $R = U \setminus (\mathbb{C}^2 \setminus D)$. For fixed $z \in D$, as in [14], we apply Stokes' Theorem on $R \times I$:

$$\begin{aligned} T_1 &= -\int_{R\times I} \overline{\partial}(Ef) \wedge \Omega_0(\hat{W}^r) + \int_R Ef \wedge \Omega_0(W^r) - \int_{D^\#} Ef \wedge K_0, \\ T_2\overline{\partial} f &= -\int_{R\times I} \overline{\partial}(E\overline{\partial} f) \wedge \Omega_1(\hat{W}^r) + \overline{\partial}_z \int_{R\times I} E\overline{\partial} f \wedge \Omega_0(\hat{W}^r) - \int_{D^\#} E\overline{\partial} f \wedge K_1. \end{aligned}$$

Notice that $\Omega_0(W^r)$ is holomorphic in z.

We rewrite

$$f = \overline{\partial}_z T_1^* f + T_2^* \overline{\partial} f,$$

where

$$T_q^* g_q = \int_{R\times I} (E\overline{\partial} g_q - \overline{\partial} E g_q) \wedge \Omega_{q-1}(\hat{W}) - \int_{D^\#} Eg_q \wedge K_{q-1}, \quad q = 1, 2.$$

Thus $T_1^* f$ is also a solution for $\overline{\partial} u = f$. Let $Q(f) = E\overline{\partial} f - \overline{\partial} E f$. Then

$$\begin{aligned} T_1^* f &= \int_R Q(f)(\xi) \wedge \frac{G_1(\xi, z)}{|\xi - z|^2 \Phi}\, d\xi_1 \wedge d\xi_2 - \int_{D^\#} Ef \wedge K_0 \\ &=: I_1(f) + I_2(f). \end{aligned}$$

Notice that

$$I_2(f) = -\int_{D^\#} Ef \wedge K_0 = -\int_{\mathbb{C}^2} Ef \wedge K_0$$

is a convolution integral. By Stein [18], it is in $W^{k,p}(D)$, $\forall\, k \in \mathbb{N}$, $1 < p < \infty$.

We will prove by induction that $I_1(f) \in W^{k,p}(D)$, $k \geq 2$.

If $k = 2$, we need to prove that $I_1(f) \in W^{2,p}(D)$. A computation gives that

$$\nabla_z^2 I_1(f) = \int_R Q(f)(\xi) \wedge \left(\frac{G_1(\xi, z)}{|\xi - z|^2 \Phi^3} + \frac{G_2(\xi, z)}{|\xi - z|^4 \Phi^2} + \frac{G_3(\xi, z)}{|\xi - z|^6 \Phi} \right) d\xi_1 \wedge d\xi_2.$$

Similarly, as before, we can show that $\nabla_z^2 I_1(f) \in L^p(D)$ with the folowing estimates

$$\|\nabla_z^2 I_1(f)\|_{L^p(D)} \leq C_p \|Q(f)(\xi)\|_{W^{1,p}(D)} \leq C_p \|f\|_{W^{2,p}(D)}.$$

So, $I_1(f) \in W^{2,p}(D)$.

Now suppose that

$$I_0(f) = \int_R f(\xi) \wedge \frac{G_1(\xi, z)}{|\xi - z|^2 \Phi} \, d\xi_1 \wedge d\xi_2$$

maps $W^{k,p}(D) \longrightarrow W^{k,p}(D)$ such that $\|I_0(f)\|_{W^{k,p}(D)} \leq C(k, p)\|f\|_{W^{k,p}(D)}$.

Let us consider $g \in W^{k+1,p}(D)$ such that $Q(g) \equiv 0$ on D. We want to prove that $\nabla I_1(g) \in W^{k,p}(D)$. By a computation as before,

$$\begin{aligned}
\nabla I_1(g) &= \int_R Q(g)(\xi) \wedge \left(\frac{G_1(\xi, z)}{|\xi - z|^2 \Phi^2} + \frac{G_2(\xi, z)}{|\xi - z|^4 \Phi} \right) d\xi_1 \wedge d\xi_2 \\
&= \int_R D(Q(g)(\xi)) \wedge \frac{G_1(\xi, z)}{|\xi - z|^2 \Phi} \, d\xi_1 \wedge d\xi_2 \\
&\quad + \int_R Q(g)(\xi) \wedge \frac{G_2(\xi, z)}{|\xi - z|^4 \Phi} \, d\xi_1 \wedge d\xi_2 \\
&=: J_1(g) + J_2(g).
\end{aligned}$$

By the inductive assumption, $J_1(g) \in W^{k,p}(D)$.

Since $Q(g) \equiv 0$ on D, we can rewrite

$$\begin{aligned}
J_2(g) &= \int_R (Q(g)(\xi) - Q(g)(z)) \wedge \frac{G_2(\xi, z)}{|\xi - z|^4 \Phi} \, d\xi_1 \wedge d\xi_2 \\
&= \int_R D(Q(g))(z + \theta(\xi, z)) \wedge \frac{G_1(\xi, z)}{|\xi - z|^2 \Phi} \, d\xi_1 \wedge d\xi_2,
\end{aligned}$$

for some $\theta(\xi, z) = O(|\xi - z|)$. $J_2(g)$ is again in $W^{k,p}(D)$. Therefore $\nabla I_1(g) \in W^{k,p}(D)$, i.e., $I_1(g) \in W^{k+1,p}(D)$, and

$$\|I_1(g)\|_{W^{k+1,p}(D)} \leq C(k, p)\|DQ(g)\|_{W^{k-1,p}(D)} \leq C(k, p)\|g\|_{W^{k+1,p}(D)}.$$

This implies that $I_1(f) \in W^{k,p}(D)$, and hence $T_1^* f \in W^{k,p}(D)$. In order to prove that $T_1 f \in W^{k,p}(D)$, notice that

$$T_1 f - T_1^* f = \int_R Ef \wedge \Omega_0(W^r),$$

where

$$\Omega_0(W^r) = c \, \frac{\partial\rho \wedge \bar{\partial}\partial\rho}{\Phi^2}.$$

By the compactness of $\overline{R}$ and the partition of unity, it suffices to prove that

$$M_0 f := \int_{R \cap W} Ef \wedge \chi(\xi)\Omega_0(W^r)$$

is in $W^{k,p}$, where $\operatorname{supp}\chi \subset W$.

By computations and integration by parts, we have

$$\begin{aligned}\nabla^k(M_0 f) &= \int_{R\cap W} Ef \wedge \chi_1(\xi)\,\frac{\partial\rho\wedge\overline{\partial}\partial\rho}{\Phi^{k+2}}\\ &= \int_{R\cap W} \sum_{j=1}^{k} D^j(Ef)\wedge\chi_2(\xi)\,\frac{\partial\rho\wedge\overline{\partial}\partial\rho}{\Phi^2},\end{aligned}$$

where χ_1, χ_2 have support in W. The last term is in L^p by the following lemma which can be proved by Schur's lemma again.

LEMMA 4. *If* $T_0 f := \int_{R\cap W} \frac{f}{\Phi^2}\,d\sigma(\xi)$, *then* T_0: $L^p \longrightarrow L^p$, *for all* $1 < p < +\infty$.

Therefore we conclude that $T_1 f \in W^{k,p}(D)$, and the proof of Theorem 3 is complete for the integer case.

2.3. *Non-integer case.* It is easy to prove that if $f \in W^{\alpha,p}(D)$, $0<\alpha<1$, $1<p<\infty$, then $\|Kf\|_{W^{\alpha,p}(D)} \le C_p\|f\|_{W^{\alpha,p}(D)}$. Therefore, we need to prove the $W^{\alpha,p}$ estimates for Tf which can be reduced to estimating Hf.

In order to prove $Hf(z) \in W^{\alpha,p}(D)$, the following lemma (cf. Bonami-Sibony [4]) is needed.

LEMMA 5. *Let* $1 \le p < \infty$, $\alpha > 0$. *If* $f \in C^1(D)$ *such that*

$$\int_D |\nabla f|^p\,\rho^{(1-\alpha)p}\,d\sigma < \infty, \tag{6}$$

Then $f \in W^{\alpha,p}(D)$.

Remark. Condition (6) is equivalent to $|\nabla f|\,\rho^{(1-\alpha)} \in L^p$.

As before,

$$\nabla Hf(z) = c\int_D f(\xi)\left(\frac{G_1(\xi,z)}{\tau^2(\xi,z)\Phi_0(\xi,z)} + \frac{G_2(\xi,z)}{\tau^2(\xi,z)\Phi_0^2(\xi,z)} + \frac{G_1(\xi,z)}{\tau(\xi,z)\Phi_0^3(\xi,z)}\right.$$
$$\left. + \text{ lower order singular terms}\right)d\sigma(\xi),$$

We need to estimate the three types of terms: $I_1(z)$, $I_2(z)$, $I_3(z)$. The $I_3(z)$ is the worst term.

By the compactness of $\overline{D}$ and a partition of unit, for $0 < \alpha < 1$, we have

$$
\begin{aligned}
|\rho(z)^{1-\alpha} I_3(z)|_{D\cap V} &\leq \int_{|t|<c, t_4\geq 0} \frac{|f(\xi)|\rho(z)^{1-\alpha}}{(|t|+\rho(z))(|t_3|+t_4+\rho(z))^3}\, d\sigma(t) \\
&\leq \int_{|t|<c, t_4\geq 0} \frac{|f(\xi)|}{(|t|+\rho(z))(|t_3|+t_4+\rho(z))^{2+\alpha}}\, d\sigma(t) \\
&\leq \int_{|t|<c, t_4\geq 0} |f(\xi)|\rho(\xi)^{-\alpha} \frac{t_4^\alpha}{(|t|+\rho(z))(|t_3|+t_4+\rho(z))^{2+\alpha}}\, d\sigma(t) \\
&\leq \int_{|t|<c, t_4\geq 0} |f(\xi)|\rho(\xi)^{-\alpha} \frac{d\sigma(t)}{(|t|+\rho(z))(|t_3|+t_4+\rho(z))^2}.
\end{aligned}
$$

By Polking [12], we know that

$$
T_0 g(z) = \int_{D\cap W} g(\xi) \frac{G_1(\xi, z)}{\tau(\xi, z)\Phi_0^3(\xi, z)}\, d\sigma(\xi)
$$

maps L^p to L^p for $1 < p < \infty$. Therefore our goal is to prove the following lemma.

LEMMA 6. *If* $0 < \alpha < \frac{1}{p}$, $f \in W^{\alpha,p}(D)$, *then* $\rho(\xi)^{-\alpha} f(\xi) \in L^p$, $1 < p < \infty$.

Remark. For the $p = 2$ case, one can find a proof in Lion-Magenes' book [11].

In order to prove the above lemma, Hardy's inequalities are required.

LEMMA 7 (HARDY'S INEQUALITY). *Assume* $1 < p \leq +\infty$ *and* q *is the conjugate exponent to* p. *Let*

$$
Tf(x) = \frac{1}{x}\int_0^x f(y)dy, \qquad Sg(x) = \int_x^\infty \frac{1}{y} g(y)\, dy.
$$

Then

$$
\begin{aligned}
\|Tf\|_{L^p} &\leq \tfrac{p}{p-1}\|f\|_{L^p}, \\
\|Sg\|_{L^q} &\leq \tfrac{p}{p-1}\|g\|_{L^q}.
\end{aligned}
$$

LEMMA 8 (MORE HARDY'S INEQUALITY). *Assume* $1 \leq p < +\infty$, $r > 0$, *and* h *is a non-negative measurable function on* $(0, \infty)$. *Then*

$$
\begin{aligned}
\int_0^\infty x^{-r-1}\left[\int_0^x h(y)\, dy\right]^p dx &\leq \left(\frac{p}{r}\right)^p \int_0^\infty x^{p-r-1} h(x)^p\, dx, \\
\int_0^\infty x^{r-1}\left[\int_x^\infty h(y)\, dy\right]^p dx &\leq \left(\frac{p}{r}\right)^p \int_0^\infty x^{p+r-1} h(x)^p\, dx.
\end{aligned}
$$

One can find proofs in Folland's book [6].

LEMMA 9. *Assume D is a bounded domain with smooth boundary. If $0 < \alpha < \frac{1}{p}$, then*

$$u \longrightarrow \rho^{-\alpha} u$$

is a continuous mapping of $W^{\alpha,p}(D) \longrightarrow L^p(D)$. The same is true for $D = \mathbb{R}^n_+$ with $\rho(x) = x_n$.

With the help of local maps, we need to verify that

$$\|x_n^{-\alpha}\varphi\|_{L^p(\mathbb{R}^n_+)} \leq c\|\varphi\|_{W^{\alpha,p}(\mathbb{R}^n_+)} \tag{7}$$

for $\varphi \in C_0^\infty(\overline{\mathbb{R}^n_+}) = r\left(C_0^\infty(\overline{\mathbb{R}^n_+})\right) = \{ f\colon f = u|_{\mathbb{R}^n_+},\ u \in C_0^\infty(\overline{\mathbb{R}^n}) \}$.

Let

$$\varphi(x) = \varphi(x', x), \qquad x > 0.$$

Then

$$\begin{aligned} \varphi(x) &= v(x) - w(x), \\ v(x) &= \frac{1}{x}\int_0^x (\varphi(x) - \varphi(\xi))\, d\xi, \\ w(x) &= \int_x^\infty \frac{1}{\xi} v(\xi)\, d\xi. \end{aligned}$$

In fact, $v(x) \to 0$, $w(x) \to 0$ as $x \to \infty$, since φ has a compact support in $\mathbb{R}^n_+$. Note that

$$\begin{aligned} v'(x) &= \varphi'(x) - \frac{1}{x^2}\int_0^x (\varphi(x) - \varphi(\xi))\, d\xi, \\ w'(x) &= -\frac{1}{x} v(x) = -\frac{1}{x^2}\int_0^x (\varphi(x) - \varphi(\xi))\, d\xi. \end{aligned}$$

Therefore, $\varphi'(x) = v'(x) - w'(x)$, and hence $\varphi(x) = v(x) - w(x)$. The inequality (7) follows from the inequalities

$$\|x_n^{-\alpha} v(x)\|_{L^p(\mathbb{R}^n_+)} \leq c\|\varphi\|_{W^{\alpha,p}(\mathbb{R}^n_+)}, \tag{8}$$

$$\|x_n^{-\alpha} w(x)\|_{L^p(\mathbb{R}^n_+)} \leq c\|\varphi\|_{W^{\alpha,p}(\mathbb{R}^n_+)}. \tag{9}$$

Notice that

$$\begin{aligned} \|v\|^p_{L^p_{x'}} &= \int_{\mathbb{R}^{n-1}} |v(x', x)|^p\, dx' \\ &= \frac{1}{x^p}\int_{\mathbb{R}^{n-1}} \left| \int_0^x (\varphi(x) - \varphi(\xi))\, d\xi \right|^p dx' \qquad \text{(by Hölder's inequality)} \end{aligned}$$

$$\leq \frac{1}{x^p}\int_{\mathbb{R}^{n-1}} x^{\frac{p}{q}} \int_0^x |\varphi(x)-\varphi(\xi)|^p \, d\xi dx'$$
$$= \frac{1}{x}\int_0^x \|\varphi(x)-\varphi(\xi)\|_{L^p_{x'}}^p \, d\xi;$$

we have

$$\begin{aligned}
\|x_n^{-\alpha} v(x)\|_{L^p(\mathbb{R}^n_+)}^p &= \int_0^\infty x^{-\alpha p}\|v\|_{L^p_{x'}}^p \, dx \\
&\leq \int_0^\infty x^{-\alpha p-1}\left(\int_0^x \|\varphi(x)-\varphi(\xi)\|_{L^p_{x'}}^p \, d\xi\right) dx \\
&= \int_0^\infty d\xi \int_\xi^\infty x^{-\alpha p-1}\|\varphi(x)-\varphi(\xi)\|_{L^p_{x'}}^p \, dx \\
&= \int_0^\infty d\xi \int_0^\infty (\xi+t)^{-\alpha p-1}\|\varphi(\xi+t)-\varphi(\xi)\|_{L^p_{x'}}^p \, dt \\
&\leq \int_0^\infty t^{-\alpha p-1} \int_0^\infty \|\varphi(\xi+t)-\varphi(\xi)\|_{L^p_{x'}}^p \, d\xi \, dt \\
&= \|\varphi\|_{W^{\alpha,p}(\mathbb{R}^n_+)}^p.
\end{aligned}$$

This proves inequality (8).

The inequality (9) follows from the following estimate.

Claim. $\|x_n^{-\alpha} w(x)\|_{L^p(\mathbb{R}^n_+)} \leq C_p \|x_n^{-\alpha} v(x)\|_{L^p(\mathbb{R}^n_+)}^p$.

In fact,

$$\begin{aligned}
\|x_n^{-\alpha} w(x)\|_{L^p(\mathbb{R}^n_+)} &= \int_0^\infty x^{-\alpha p}\left(\int_{\mathbb{R}^{n-1}} \left|\int_x^\infty \frac{1}{\xi} v(\xi)\, d\xi\right|^p dx'\right) dx \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty x^{-\alpha p}\left|\int_x^\infty \frac{1}{\xi} v(\xi)\, d\xi\right|^p dx\right) dx' \\
&\leq \int_{\mathbb{R}^{n-1}} \left[\int_0^\infty x^{-\alpha p}\left(\int_x^\infty \frac{1}{\xi}|v(\xi)|\, d\xi\right)^p dx\right] dx'.
\end{aligned}$$

Therefore, if $1-\alpha p > 0$, i.e. $\alpha < \frac{1}{p}$, then

$$\int_0^\infty x^{-\alpha p}\left(\int_x^\infty \frac{1}{\xi}|v(\xi)|\, d\xi\right)^p dx$$
$$= \int_0^\infty x^{(1-\alpha p)-1}\left(\int_x^\infty \frac{1}{\xi}|v(\xi)|\, d\xi\right)^p dx \quad \text{(by the second inequality in Lemma 8)}$$

$$\leq \left(\frac{p}{1-\alpha p}\right)^p \int_0^\infty x^{(1-\alpha)p} |\frac{1}{x} v(x)|^p \, dx$$
$$\leq \left(\frac{p}{1-\alpha p}\right)^p \int_0^\infty x^{-\alpha p} |v(x)|^p \, dx.$$

Hence

$$\|x_n^{-\alpha} w(x)\|_{L^p(\mathbb{R}^n_+)} \leq \left(\frac{p}{1-\alpha p}\right) \|x_n^{-\alpha} v(x)\|^p_{L^p(\mathbb{R}^n_+)}.$$

The proof of Lemma 9 is complete.

As a conseqence of Lemma 9 , we have proved Theorem 3 for $0 < \alpha < 1/p$.

By [1], if α is not an integer, the Sobolev space $W^{\alpha,p}$ is the Besov space $B^{\alpha,p}$. By an interpolation theorem for Besov spaces [2], Theorem 3 is true for all non-integer α. Therefore Theorem 3 is proved.

Remark. When α is not an integer, the Sobolev space $W^{\alpha,p}$ is different from the Besov space $B^{\alpha,p}$. This is why we prove Theorem 3 in integer and non-integer cases.

3. Examples

The following example shows that there is no "gain" in L^p estimates for the canonical solution of the $\overline{\partial}$-equation in a convex domain.

EXAMPLE 1. Let $0 < \alpha < +\infty$. We convexify the domain

$$\Omega = \left\{ z \in \mathbb{C}^2 \colon |z_1|^2 + e^{-\frac{1}{|z_2|^\alpha}} < 1 \right\}$$

to get a bounded convex domain D such that $D = \Omega$ if $|z_1| > \frac{1}{4}$ and D is strictly convex except on the circle $C = \{(e^{i\theta}, 0) \colon 0 \leq \theta < 2\pi\}$.

For any $2 \leq p < +\infty$, there is a $\overline{\partial}$-closed $f \in L^p(D)$, but $f \notin L^q(D)$ for $q > p$, such that the canonical solution to $\overline{\partial} u = f$ is in $L^p(D)$, but not in $L^q(D)$ for $q > p$.

For $1 \leq p < +\infty$, let

$$f(z) = \frac{\overline{\partial}(\chi(z_1)\overline{z}_2)}{(1-z_1)^{\frac{2}{p}} \left(\log \frac{1}{1-z_1}\right)^{\frac{\beta}{p}}}, \quad \beta > 2,$$

where $\log(1 - z_1)$ can be taken as the principal branch in D, $\chi \in C^\infty(\mathbb{C})$ such that $\chi \equiv 0$ in $\{|z-1| > \frac{1}{2}\}$, $\chi \equiv 1$ in $\{|z-1| < \frac{1}{4}\}$. Hence $\overline{\partial}\chi$ has a compact support in $\{\frac{1}{4} < |z-1| < \frac{1}{2}\}$.

Rewrite as

$$f(z) = \frac{(\partial\chi(z_1)/\partial\bar{z}_1)\,\bar{z}_2}{(1-z_1)^{\frac{2}{p}}\left(\log\frac{1}{1-z_1}\right)^{\frac{\beta}{p}}}\,d\bar{z}_1 + \frac{\chi(z_1)}{(1-z_1)^{\frac{2}{p}}\left(\log\frac{1}{1-z_1}\right)^{\frac{\beta}{p}}}\,d\bar{z}_2$$
$$= f_1 d\bar{z}_1 + f_2 d\bar{z}_2.$$

Clearly, f is $\bar{\partial}$-closed. The first term is uniformly bounded, but the second term is not.

Claim 1. $f \in L^p(D)$.

By the above observation, we only need to consider the second term in f. By the polar coordinate change $1 - z_1 = \rho e^{i\theta}$, we have

$$\int_D |f_2|^p d\sigma(z) \le c\int_{\substack{|z_1|<1\\ |z_1-1|<\frac{1}{2}}} \frac{d\sigma(z_1)}{|1-z_1|^2\left|\log\frac{1}{1-z_1}\right|^{\beta}}$$
$$\le c\int_0^{\frac{1}{2}} \frac{d\rho}{\rho\left(\log\frac{1}{\rho}\right)^{\beta}} < \infty.$$

Therefore, $f \in L^p(D)$.

Claim 2. $f \notin L^q(D)$ for $q > p$.
In fact,

$$\int_D |f_2|^q\, d\sigma(z) = \int_D \frac{\chi^q(z_1)\, d\sigma(z)}{|1-z_1|^{\frac{2q}{p}}\left|\log\frac{1}{1-z_1}\right|^{\frac{\beta q}{p}}}$$
$$\ge c\int_{|z_1|<1,|z_1-1|<\frac{1}{4},\ |\mathrm{Im}(1-z_1)|<\frac{1}{k}|\mathrm{Re}(1-z_1)|}$$
$$\times \frac{d\sigma(z_1)}{|1-z_1|^{\frac{2q}{p}}\left|\log\frac{1}{1-z_1}\right|^{\frac{\beta q}{p}}\left(\log\frac{1}{1-|z_1|^2}\right)^{\frac{2}{\alpha}}}$$

(Choose an integer $k \ge 2$. Then $1 - |z_1| \le |1 - z_1| \le 2(1 - |z_1|)$.)

$$\ge c\int_{|z_1|<1,|z_1-1|<\frac{1}{4},\ |\mathrm{Im}(1-z_1)|<\frac{1}{k}|\mathrm{Re}(1-z_1)|}$$
$$\times \frac{d\sigma(z_1)}{|1-z_1|^{\frac{2q}{p}}\left|\log\frac{1}{1-z_1}\right|^{\frac{\beta q}{p}}\left(\log\frac{1}{|1-z_1|}\right)^{\frac{2}{\alpha}}}$$
$$\ge c\int_0^{\frac{1}{4}} \frac{d\rho}{\rho^{\frac{2q}{p}-1-\varepsilon}}, \qquad \varepsilon > 0 \text{ very small.}$$

The last integral is divergent since $\frac{2q}{p} - 1 - \varepsilon > 1$ if ε is sufficiently small. Thus, $f \notin L^q(D)$ for $q > p$.

Notice that

$$v(z_1, z_2) = \frac{\chi(z_1)\bar{z}_2}{(1 - z_1)^{\frac{2}{p}} \left(\log \frac{1}{1-z_1}\right)^{\frac{\beta}{p}}}$$

is a solution for $\bar{\partial} u = f$.

The same proofs as before show that $v \in L^p(D)$, but $v \notin L^q(D)$ for $q > p$.

Claim 3. v is the canonical solution for $\bar{\partial} u = f$.

In fact, assume h is holomorphic and in $L^2(D)$. By the mean value theorem,

$$\begin{aligned} < h, v > &= \int_D h\bar{v}\, d\sigma(z_1, z_2) \\ &= \int_{|1-z_1|<\frac{1}{2}} \frac{\chi(z_1) d\sigma(z_1)}{(1 - \bar{z}_1)^{\frac{2}{p}} \left(\overline{\log \frac{1}{1-z_1}}\right)^{\frac{\beta}{p}}} \int_{|z_2|<(\log \frac{1}{1-|z_1|^2})^{-\frac{1}{\alpha}}} z_2 h(z_1, z_2)\, d\sigma(z_2) \\ &= 0. \end{aligned}$$

Therefore, v is the canonical solution for $\bar{\partial}$ in this domain D.

EXAMPLE 2. Take D as in Example 1. For any $2 \le p < +\infty$, there is $f \in W^{1,p}(D)$, $\bar{\partial} f = 0$, $f \notin W^{1,q}(D)$ for $q > p$, such that the canonical solution to $\bar{\partial} u = f$ is in $W^{1,p}(D)$, but not in $W^{1,q}(D)$ for $q > p$.

Let

$$f(z) = \frac{\bar{\partial}(\chi(z_1)\bar{z}_2)}{(1 - z_1)^{\frac{2}{p}-1} \left(\log \frac{1}{1-z_1}\right)^{\frac{\beta}{p}}}, \quad \beta > 2,$$

where χ is the same as in Example 1. It is easy to see that $f \in L^p(D)$.

Since $\frac{\partial f}{\partial z_1}$, $\frac{\partial f}{\partial \bar{z}_1}$, $\frac{\partial f}{\partial z_2}$, $\frac{\partial f}{\partial \bar{z}_2}$ are all in $L^p(D)$, then $f \in W^{1,p}(D)$. Also, $f \notin W^{1,q}(D)$, $q > p$.

By the mean value theorem again,

$$v(z) = \frac{\chi(z_1)\bar{z}_2}{(1 - z_1)^{\frac{2}{p}-1} \left(\log \frac{1}{1-z_1}\right)^{\frac{\beta}{p}}}$$

is the canonical solution to $\bar{\partial} u = f$.

The same proofs as in Example 1 show that $v \in W^{1,p}(D)$, but $v \notin W^{1,q}(D)$ for $q > p$. This implies that $v \notin W^{1+\delta,p}(D)$, since by the Sobolev imbedding theorem [1],

$$W^{1+\delta,p}(D) \hookrightarrow W^{1,r}(D), \quad r = \frac{4p}{4-\delta p} > p,$$

for small $\delta > 0$.

Therefore, on a convex domain, the canonical solution for $\overline{\partial}$ has no "gain" in the Sobolev estimates.

4. Hölder estimates for $\overline{\partial}$

In order to get Λ_α^p estimates, we need some classical lemmas. The first one is the Hardy-Littlewood lemma.

LEMMA 10. *Let* $0 < \alpha < 1$, $D_\delta = \{z \in D\colon \operatorname{dist}(z, \partial D) > \delta\}$. *If* $u \in C^1(D)$ *satisfies*

$$\|\operatorname{grad} u\|_{L^p(D_\delta)} \le M\delta^{-1+\alpha}, \quad 1 \le p \le +\infty$$

uniformly in δ, *then* $u \in \Lambda_\alpha^p(D)$ *and* $\|u\|_{\Lambda_\alpha^p(D)} \le cM$ *for some constant* $c > 0$.

For $p = +\infty$ case, one can find a proof in [9]. For general p, we can prove it similarly.

The second one is Minkowski's inequality for integrals (cf. [17]).

LEMMA 11. *For any* $1 \le p < +\infty$, *if* $f \in L^p(D_1 \times D_2)$, *then*

$$\left(\int_{D_2}\left|\int_{D_1} f(x,y)\,dx\right|^p dy\right)^{\frac{1}{p}} \le \int_{D_1}\left(\int_{D_2}|f(x,y)|^p\,dy\right)^{\frac{1}{p}} dx.$$

As in the computation in § 3, in order to estimate $\|\nabla Hf\|_{L^p(D_\delta)}$, we need to estimate the three types of terms: $I_1(z)$, $I_2(z)$, $I_3(z)$. The $I_3(z)$ is the worst term.

Here we give a proof for the $0 < \alpha < 1$ case. For the $\alpha \ge 1$ case, we can use the Seeley extension as in §3 to prove it. We omit the details.

By the compactness of $\overline{D}$ and the partition of unity, estimating $I_3(z)$ it can be reduced to estimating $\|J(z)\|_{L^p(V\cap D_\delta)}$, where

$$J(z) = \int_{D\cap W} f(\xi)\frac{\chi_1(\xi)G_1(\xi,z)}{\tau(\xi,z)\Phi_0^3(\xi,z)}\,d\sigma(\xi)$$

with V, W, χ_1 taken as before.

After making the coordinate change (3), we can write $\xi = g(t, z)$ and

$$\begin{aligned} J(z) &= \int_{|t|<c, t_4 \geq 0} f(g(t,z))k(t,z)\, d\sigma(t) \\ &= \int_{|t|<c, t_4 \geq 0} [f(g(t,z)) - f(g(t_1, t_2, 0, t_4, z))]k(t,z)\, d\sigma(t) \\ &\quad + \int_{|t|<c, t_4 \geq 0} f(g(t_1, t_2, 0, t_4, z))k(t,z)\, d\sigma(t) \\ &:= J_1(z) + J_2(z). \end{aligned}$$

By the usual Minkowski's inequality, we have

$$\begin{aligned} \|J(z)\|_{L^p(V\cap D_\delta)} &\leq \left(\int_{V\cap D_\delta} |J_1(z)|^p\right)^{\frac{1}{p}} + \left(\int_{V\cap D_\delta} |J_2(z)|^p\right)^{\frac{1}{p}}. \\ &:= \tilde{J}_1 + \tilde{J}_2. \end{aligned}$$

By the estimates (2), (4), (5) and Minkowski's Inequality for integrals, we get

$$\begin{aligned} \tilde{J}_1 &= c\int_{|t|<c, t_4 \geq 0} \left(\int_{V\cap D_\delta} |f(g(t,z)) - f(g(t_1, t_2, 0, t_4, z))|^p\, d\sigma(z)\right)^{\frac{1}{p}} \\ &\quad \times \frac{d\sigma(t)}{(|t|+\delta)(|t_3|+t_4+\delta)^3} \\ &\leq c\|f\|_{\Lambda^p_\alpha(D)} \int_{|t|<c, t_4 \geq 0} \frac{|t_3|^\alpha}{(|t|+\delta)(|t_3|+t_4+\delta)^3}\, d\sigma(t) \\ &\leq c\|f\|_{\Lambda^p_\alpha(D)}\, \delta^{-1+\alpha}. \end{aligned}$$

After integration by parts with respect to t_3, we can show that

$$\tilde{J}_2 \leq c\|f\|_{\Lambda^p_\alpha(D)}\, \delta^{-\beta}, \qquad \forall\, \beta > 0.$$

Direct computations show that

$$\begin{aligned} \|I_1(z)\|_{L^p(V\cap D_\delta)} &\leq c\|f\|_{\Lambda^p_\alpha(D)}\, \delta^{-\beta}, \\ \|I_2(z)\|_{L^p(V\cap D_\delta)} &\leq c\|f\|_{\Lambda^p_\alpha(D)}\, \delta^{-\beta}, \end{aligned}$$

which give $Hf(z) \in \Lambda^p_\alpha(D)$ by Lemma 10. It is easy to prove that $Kf(z) \in \Lambda^p_\alpha(D)$. Therefore $Tf(z) \in \Lambda^p_\alpha(D)$ and with estimates

$$\|Tf\|_{\Lambda^p_\alpha(D)} \leq c\|f\|_{\Lambda^p_\alpha(D)},$$

for all $\alpha > 0$, $1 \leq p \leq +\infty$. This proves Theorem 4.

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