ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 4, Winter 1998

DISCREPANCY NORMS ON THE SPACE M[0,1] OF RADON MEASURES

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Introduction

Let X be a Banach space with a symmetric basis (e_i), $i \in N$. We assume (see [L-T], p. 114) that the norm of X satisfies

$$\left\| \sum_{n=1}^{\infty} a_n \sigma_n e_{\pi(n)} \right\| = \left\| \sum_{n=1}^{\infty} a_n e_n \right\|$$

for every permutation π of the integers and every choice of signs $\sigma_n = \pm 1$. Let $M[0, 1]$ be the space of all Radon measures on the unit interval $[0, 1]$. For $\mu \in M[0, 1]$ we define $\|\mu\|_{M[X]}$ to be the quantity

$$\sup \left\{ \left\| \sum_{i=1}^{d} \mu(I_i) e_i \right\|, (I_i), i \leq d \text{ disjoint subintervals of } [0, 1],\ d \in N \right\}.$$

$M[X]$ is the space $M[0, 1]$ equipped with the norm $\|\cdot\|_{M[X]}$.

$L^1[0, 1]$ is the space of all Lebesgue integrable functions on $[0, 1]$ and m denotes the Lebesgue measure on $[0, 1]$. We can see $L^1[0, 1]$ as a closed subspace of $M[0, 1]$. A measure μ in $M[0, 1]$ is called diffuse if $\mu(\{x\}) = 0$ for each x in $[0, 1]$.

This paper consists of two sections. In Section 1 we study the structure of the $M[X]$ spaces.The papers [Wei] and [B] are cornerstones in our considerations. Theorem 4.2 in [Wei] about the M_0 space (related to Proposition 11 in [B]) can be extended to certain classes of $M[X]$ spaces.This is the content of Theorem 1.1 and Theorem 1.5. More precisely in Theorem 1.1 we show that if X has a symmetric basis and contains no copy of l^1 then every diffuse measure μ in $M[0, 1]$ is the limit in the $M[X]$-norm of a sequence (μ_n) of measures such that each μ_n is absolutely continuous with respect to the Lebesgue measure m on $[0, 1]$. Theorem 1.5 asserts the following.

Suppose X has a symmetric basis and contains no copy of l^1. Let (ξ_n, Σ_n) be an $L^1[0, 1]$ valued martingale and $\mu_x = w^* - \lim \xi_n(x)$ for almost all x $[m]$.

Received June 25, 1997.
1991 Mathematics Subject Classification. Primary 47B38, 47B07, 46B20, 46B22, 28A33, 60G57.

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The following are equivalent:

(i) The measures μ_x are diffuse for almost all x $[m]$.
(ii) The martingale (ξ_n, Σ_n) is Cauchy in the Bochner norm of $L^1_{M[X]}$.
(iii) The operator $T\colon L^1[0, 1] \to L^1[0, 1]$ associated to the martingale (ξ_n, Σ_n) becomes representable if we give to the target space of T the $M[X]$ norm.

In the proofs of Theorems 1.1 and 1.5, important role is played by Lemma 1.2. Lemma 1.2 asserts that if (e_i) is a subsymmetric basis for X and X contains no copy of l^1 then a convex combination of the form $\sum_{i=1}^d \lambda_i e_i$, λ_i scalars, can not be large in norm if all the coefficients λ_i remain small. In Lemma 1.3 we show that under certain conditions the convergence of a net (μ_j) of positive measures to a diffuse measure μ in the $M[X]$ norm is determined by the behaviour of (μ_j) on the dyadic intervals. Lemma 1.3 was suggested to us by the referee. It was pointed out to us by the referee that the proof of Lemma 1.3 was included in our earlier proof of Proposition 1.4 and that despite its simplicity Lemma 1.3 gives a proof of Proposition 1.4 and can also be used in the proof of Theorem 1.1. In the proof of Lemma 1.3 (b) we use Lemma 1.2. Proposition 1.4 is a result on the weak topology of $M[X]$. Corollary 1.6 asserts the following: Let X be a Banach space with a symmetric basis such that X contains no copy of l^1. Let (ξ_n, Σ_n) be a martingale associated to an operator $T\colon L^1[0, 1] \to L^1[0, 1]$. If the martingale (ξ_n, Σ_n) is Cauchy in the Pettis norm then (ξ_n, Σ_n) is Cauchy in the Bochner norm of $L^1_{M[X]}$. In the case $X = c_0$, Corollary 1.6 can be viewed as a rephrasing of Proposition 11 in [B].

In Section 2 we introduce and study the class of $M[X]$-continuous operators. Let X be a Banach space with a symmetric basis. Suppose Z is a Banach space. A bounded linear operator $T\colon L^1[0, 1] \to Z$ is called $M[X]$-continuous if it is continuous for the $M[X]$ norm on $L^1[0, 1]$ (Definition 2.1). It follows that if X has a symmetric basis and contains no copy of l^1 then every $M[X]$-continuous operator is nearly representable and therefore strongly regular (Proposition 2.2 and Corollary 2.3). The main result in this section (Theorem 2.9) is the following: Let Z, X be Banach spaces such that X has a symmetric basis and contains no copy of l^1. If every bounded operator $S\colon X \to Z$ is compact, then every $M[X]$-continuous operator $T\colon L^1[0, 1] \to Z$ is compact. In the proof of Theorem 2.9 we use Lemma 2.10 and Lemma 2.11. Lemma 2.10 is of independent interest. This result shows that the non-compactness of a general operator $T\colon L^1[0, 1] \to Z$ implies the existence of a bounded sequence (g_n), $n \in N$, in $L^1[0, 1]$ such that each function g_n is supported by a dyadic interval I_n so that the set $\{T(g_n), n \in N\}$ is not compact and the sequence (I_n) consists of pairwise disjoint intervals.

In Lemma 2.11 we show that if (e_i) is a symmetric basis for X then $\|\sum_{i=1}^r b_i u_i\| \le \|\sum_{i=1}^r b_i e_i\|$ for all scalars b_i, $i \le r$, where each u_i is a convex combination of suitable elements of the basis (e_i). Theorem 2.5 is a result on the behaviour of $M[l^p]$-continuous operators on orthonormal sequences.In Example 2.7 and in Proposition 2.8

we show that under certain conditions on X there are $M[X]$-continuous operators on $L^1[0, 1]$ that are not compact. Of related interest are Corollaries 2.4 and 2.6.

1. The structure of $M[X]$ spaces

In this section we study the normed spaces $M[X]$ which are defined in the Introduction. In most of our theorems X denotes a Banach space with a symmetric basis that contains no copy of l^1. In Lemma 1.2 the assumption on X is that X has a subsymmetric basis and contains no copy of l^1. We refer to [L-T], p. 114 for the definition of a subsymmetric basis.

We start with a few remarks.

(i) For $X = c_0$ and (e_i) the usual basis for c_0, the space $M[c_0]$ is the space M_0 considered in [B] and [Wei].

(ii) The formal identity maps $W_X\colon M[0, 1] \to M[X]$ and $V_X\colon M[X] \to M_0$ are continuous and

$$\|\mu\|_{M_0} \le \|\mu\|_{M[X]} \le \|\mu\|_{M[l^1]}, \qquad \mu \in M[0, 1].$$

(iii) The space $M[l^1]$ is the space $M[0, 1]$ with the usual norm.

(iv) The normed spaces $M[X]$ are not complete in general. If $M[X]$ is complete then $M[X]$ is isomorphic to $M[l^1]$.

(v) The completion of $L^1[0, 1]$ under the $M[c_0]$ norm is isomorphic to the space $C[0, 1]$ of continuous functions on $[0, 1]$. To see why this is true consider the operator

$$U\colon (L^1[0, 1], \|\cdot\|_{M[c_0]}) \to (C[0, 1], \|\cdot\|_\infty),$$

$$U(f)(x) = \int_0^x f(t)\, dm(t), \qquad x \in [0, 1], \qquad f \in L^1[0, 1].$$

Note that U is continuous and

$$\|U(f)\|_\infty \ge \frac{1}{2}\|f\|_{M[c_0]}, \qquad f \in L^1[0, 1] \quad \text{(see [Wei], p. 550)}.$$

The range of U is the set of absolutely continuous functions on $[0, 1]$ that vanish at zero.Hence the closure of the range of U is isomorphic to $C[0, 1]$.

(vi) The completion of $L^1[0, 1]$ under the $M[l^2]$ norm is the James function space JF studied in [L-S].

THEOREM 1.1. *Suppose X is a Banach space with a symmetric basis and that X contains no copy of l^1. The measure $\mu \in M[0, 1]$ belongs to the closure of $L^1[0, 1]$ under the $M[X]$ norm if and only if μ is a diffuse measure.*

We need the following result.

LEMMA 1.2. *Let X be a Banach space with a subsymmetric basis (e_i) so that X contains no copy of l^1. Given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that if $\lambda_i \geq 0$, $\sum_{i=1}^{d} \lambda_i = 1$ and $\sup \lambda_i < \delta(\varepsilon)$, $i \leq d$, then $\| \sum_{i=1}^{d} \lambda_i e_i \| < \varepsilon$.*

Proof of Lemma 1.2. Suppose there exists $\varepsilon > 0$ so that for all

$$n \in N \text{ there exists } \lambda_i^{(n)} \geq 0 \text{ such that } \sum_{i=1}^{k_n} \lambda_i^{(n)} = 1,\ \lambda_i^{(n)} < \frac{1}{n} \text{ for } 1 \leq i \leq k_n$$

and $\| \sum_{i=1}^{k_n} \lambda_i^{(n)} e_i \| \geq \varepsilon$. Let $x_n^* \in X^*$, $\|x_n^*\| = 1$, so that $x_n^*(\sum_{i=1}^{k_n} \lambda_i^{(n)} e_i) \geq \varepsilon$. Let $F_n = \{i \leq k_n : |x_n^*(e_i)| > \frac{\varepsilon}{2}\}$, $n \in N$.

Note that

$$\begin{aligned}\varepsilon \leq x_n^*\left(\sum_{i=1}^{k_n} \lambda_i^{(n)} e_i\right) &= \sum_{i=1}^{k_n} \lambda_i^{(n)} x_n^*(e_i) \\ &= \sum_{i \in F_n} \lambda_i^{(n)} x_n^*(e_i) + \sum_{i \notin F_n, i \leq k_n} \lambda_i^{(n)} x_n^*(e_i) \\ &\leq \sum_{i \in F_n} \lambda_i^{(n)} x_n^*(e_i) + \frac{\varepsilon}{2}.\end{aligned}$$

Therefore $\sum_{i \in F_n} \lambda_i^{(n)} \geq \frac{\varepsilon}{2}$. It follows that $\#F_n \cdot \frac{1}{n} \geq \frac{\varepsilon}{2}$ where $\#F_n$ is the cardinality of the set F_n. So $\#F_n \to \infty$ as $n \to \infty$. Every subsymmetric basis is unconditional and since X contains no copy of l^1 the biorthogonal functionals (e_i^*) associated to the basis (e_i) form an unconditional basis for X^* (see [D], p. 99).

Consider the functionals

$$\sum_{i \in F_n} \mu_i^{(n)} e_i^*, \text{ where } \mu_i^{(n)} = x_n^*(e_i), n \in N.$$

Note that $\mu_i^{(n)} > \frac{\varepsilon}{2}$ for $i \in F_n$. Also note that

$$\left\| \sum_{i \in F_n} \mu_i^{(n)} e_i^* \right\| \leq \left\| \sum_{i=1}^{\infty} \mu_i^{(n)} e_i^* \right\| = \|x_n^*\| \leq 1.$$

Suppose $F_n = \{i_1 < i_2 < \cdots < i_{l_n}\}$. Then $l_n = \#F_n$ and $l_n \to \infty$ as $n \to \infty$.

Now consider the functionals

$$y_n^* = \sum_{j=1}^{l_n} \mu_{i_j}^{(n)} e_j^*, \qquad n \in N.$$

By the subsymmetric property we have

$$\|y_n^*\| = \left\| \sum_{i \in F_n} \mu_i^{(n)} e_i^* \right\| \leq 1, \qquad n \in N.$$

Let y^* be a w^*-limit point of the infinite sequence (y_n^*). It follows that $y^*(e_i) > \varepsilon/2$ for $i \in N$. Now if J_1 is a finite subset of N we have

$$\left\| \sum_{i \in J_1} a_i e_i \right\| \geq \frac{\varepsilon}{K} \cdot \sum_{i \in J_1} |a_i|$$

where (a_i), $j \in J_1$ are scalars and K is the unconditionality constant of the basis (e_i), $i \in N$. Therefore X contains a copy of l^1. This is a contradiction. □

Let $I_{r,s} = [\frac{s-1}{2^r}, \frac{s}{2^r})$, $r = 0, 1, \ldots$; $s = 1, 2, \ldots, 2^r$ be the family of the dyadic intervals.

LEMMA 1.3. *Let (μ_j), $j \in J$ be a net of positive Radon measures with $\|\mu_j\| \leq 1$, $j \in J$. Suppose that μ is a positive diffuse measure and $\mu_j(I_{r,s}) \to \mu(I_{r,s})$ for every dyadic interval $I_{r,s}$, $r = 0, 1, 2, \ldots$; $s = 1, 2, \ldots 2^r$. Then*:

(*a*) *$\mu_j(I) \to \mu(I)$ for every interval I.*

(*b*) *$\mu_j \to \mu$ in the $M[X]$ norm if X has a symmetric basis and contains no copy of l^1.*

Proof of Lemma 1.3. (a) Let $\varepsilon > 0$. Since μ is a diffuse measure we can find $n \in N$ so that $\mu(I_{n,s}) < \varepsilon$, $s = 1, 2, \ldots, 2^n$. Find j_0 in the index set J so that

$$|\mu_j(I_{n,s}) - \mu(I_{n,s})| < \frac{\varepsilon}{2^n} \text{ for } s = 1, 2, \ldots, 2^n; \quad j > j_0.$$

Let I be any subinterval of [0, 1]. Assume that

$$I = I^{(1)} \cup I_{n,s} \cup I_{n,s+1} \cdots \cup I_{n,s+k} \cup I^{(2)}$$

where $I^{(1)}$, $I^{(2)}$ are intervals, $I^{(1)} \subseteq I_{n,s-1}$, $I^{(2)} \subseteq I_{n,s+k+1}$. Since μ_j is positive we have

$$\mu_j(I^{(1)}) \leq \mu_j(I_{n,s-1}) \leq \frac{\varepsilon}{2^n} + \varepsilon \quad \text{and} \quad \mu_j(I^{(2)}) \leq \mu_j(I_{n,s+k+1}) \leq \frac{\varepsilon}{2^n} + \varepsilon.$$

Hence for $j > j_0$ we have

$$\begin{aligned} |\mu_j(I) - \mu(I)| &\leq |\mu_j(I^{(1)}) - \mu(I^{(1)})| + |\mu_j(I^{(2)} - \mu(I^{(2)})| \\ &\quad + \sum_{\nu=0}^{k} |\mu_j(I_{n,s+\nu}) - \mu(I_{n,s+\nu})| \\ &\leq 2\left(\frac{\varepsilon}{2^n} + \varepsilon\right) + (k+1) \cdot \frac{\varepsilon}{2^n} \\ &\leq 4 \cdot \varepsilon. \end{aligned}$$

(b) Suppose X has a symmetric basis and contains no copy of l^1. Let $\varepsilon > 0$ and suppose $\delta(\varepsilon)$ satisfies the property in the statement of Lemma 1.2. Find $n \in N$ so that $\mu(I_{n,s}) < \delta(\varepsilon)$, $s = 1, 2, \ldots, 2^n$. Find j_0 in the index set J so that

$$|\mu_j(I_{n,s}) - \mu(I_{n,s})| < \frac{\delta(\varepsilon)}{4\,2^n} \text{ for } s = 1, 2, \ldots, 2^n; \ j > j_0.$$

For each $j > j_0$ there exists an interval I_j such that

$$|\mu_j(I_j) - \mu(I_j)| > \sup|\mu_j(I) - \mu(I)| - \frac{\delta(\varepsilon)}{8}$$

where the supremum is taken over all intervals $I \subseteq [0, 1]$. By the argument in the proof of (a) we have

$$|\mu_j(I_j) - \mu(I_j)| \leq \frac{\delta(\varepsilon)}{2}$$

and therefore

$$\sup\{|\mu_j(I) - \mu(I)|, \quad I \text{ subinterval of } [0, 1]\} \leq \frac{\delta(\varepsilon)}{2}$$

By Lemma 1.2. we get

$$\|\mu_j - \mu\|_{M[X]} \leq 2 \cdot \varepsilon \text{ if } j > j_0. \qquad \square$$

Proof of Theorem 1.1. Let $I_{r,s} = [\frac{s-1}{2^r}, \frac{s}{2^r})$, $r = 0, 1, \ldots$; $s = 1, 2, \ldots, 2^r$, be the family of dyadic intervals. For each measure $\nu \in M[0, 1]$ we consider the "table" $(\nu(I_{r,s}))$ $r = 0, 1, \ldots$; $s = 1, 2, \ldots, 2^r$, of ν. Let μ be a diffuse measure. We may assume that μ is positive otherwise we split μ in its positive and negative part. For each $n \in N$ we define a measure μ_n by determining its table in the following way:

If $r \leq n$ set $\mu_n(I_{r,s}) = \mu(I_{r,s})$, $s = 1, 2, \ldots, 2^r$.

If $r > n$ and $1 \leq s \leq 2^r$ consider the the unique dyadic interval of the form $I_{r-1,k}$ that contains $I_{r,s}$. Use induction to assign to $\mu_n(I_{r,s})$ the value $\frac{1}{2}\mu_n(I_{r-1,k})$. It follows that for each $n \in N$ the measure μ_n is absolutely continuous with respect to the Lebesgue measure m. Note that for every dyadic interval I the sequence $\mu_n(I)$ converges to $\mu(I)$ as $n \to \infty$. By Lemma 1.3, $\|\mu_n - \mu\|_{M[X]} \to 0$ as $n \to \infty$. $\square$

PROPOSITION 1.4. *Suppose X is a Banach space with a symmetric basis and that X contains no copy of l^1. Let K be the set of positive diffuse measures of norm less than one in $M[X]$. The weak and the norm topologies of $M[X]$ coincide on K.*

Proof. Assume that a net (μ_j), $j \in J$, from K converges weakly to $\mu \in K$. For any interval I the functional $\chi_I(\mu) = \mu(I)$ is continuous on $M[X]$ and $\|\chi_I\| = 1$. It follows that $\mu_j(I) \to \mu(I)$, $j \in J$, for each interval I. By Lemma 1.3, $\mu_j \to \mu$ in the $M[X]$ norm. $\square$

For terminology, notation and results on martingales and operators on L^1 we refer to [D-U], [B] and [Wei].

Let Z be a Banach space and $T\colon L^1[0,1] \to Z$ be a (bounded) operator. Let Σ_n, $n = 0, 1, 2, \dots$ be the finite algebra generated by the dyadic intervals $[\frac{k-1}{2^n}, \frac{k}{2^n}]$, $k = 1, 2, \dots, 2^n$. The Z-valued martingale (ξ_n, Σ_n) associated to T is defined by the formula

$$\xi_n(t) = \frac{1}{2^n}\sum_{k=1}^{2^n} T(h_{n,k})h_{n,k}(t),\ k = 1, 2, \dots, 2^n;\ n = 0, 1, 2, \dots;\ t \in [0,1]$$

where $h_{n,k}$ is the characteristic function of the interval $[\frac{k-1}{2^n}, \frac{k}{2^n}]$ normalized in $L^1[0,1]$. If $Z = L^1[0,1]$ then for each x in $[0,1]$, $(\xi_n(x))$, $n \in N$, is a sequence of functions in $L^1[0,1]$ and by Doob's theorem for $[m]$-almost all x in $[0,1]$ the weak*-limit of $(\xi_n(x))$, $n \in N$ exists as a measure μ_x in $M[0,1]$. It is known that the family (μ_x) is a random measure and the "kernel" $x \to \mu_x$ represents the dual operator T^* in the sense that

$$T^*(f)(x) = \int f\,d\mu_x,\ f \in L^\infty,\ [m] - \text{almost all } x \text{ in } [0,1]$$

(see [Wei], Prop. 2.8).

If Z is a Banach space and $f \in L^1_Z$, the Pettis norm of f is defined by

$$|||f||| = \sup_{x^* \in Z^*, \|x^*\| \le 1} \int |x^*(f)|\,dm$$

while the Bochner norm of f is the quantity $\int \|f(t)\|\,dm(t)$.

The next result follows from Theorem 4.2 in [Wei] and Lemma 1.2. (See also Cor. 4.4 in [Wei].) It can be considered as an extention of Theorem 1.1 to the case of a family (μ_x) of measures. In the special case that X is the space c_0 it becomes Theorem 4.2 (a), (e) of [Wei].

THEOREM 1.5. *Let X be a Banach space with symmetric basis such that X contains no copy of l^1. Let (ξ_n, Σ_n) be a martingale associated to an operator $T\colon L^1[0,1] \to L^1[0,1]$. Suppose $\mu_x = w^* - \lim \xi_n(x)$ for almost all x $[m]$. The following are equivalent*:

(*i*) *The measures μ_x are diffuse for almost all x $[m]$.*
(*ii*) *The martingale (ξ_n, Σ_n) is Cauchy in the Bochner norm of $L^1_{M[X]}$.*
(*iii*) *The operator $W_X \cdot T\colon L^1[0,1] \to M[X]$ is Bochner representable.*

(*Here W_X denotes the formal identity map $W_X\colon L^1[0,1] \to M[X]$.*)

Proof. The measures μ_x are diffuse for almost all x $[m]$ iff the operator T becomes Bochner representable if we give to the target space of T the weaker M_0 norm (Cor. 4.4,

[Wei]). This means that given $\varepsilon > 0$ there is a measurable set $A \subseteq [0, 1]$ so that $m([0, 1]\backslash A) < \varepsilon$ and the restriction T_A of T on $L^1(A)$ is a compact operator from $L^1(A)$ to M_0. Note that Lemma 1.2 implies that given $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ so that if $\|\mu\|_{M0} < \delta$ and $\|\mu\| = 1$ then $\|\mu\|_{M[X]} < \varepsilon$. So T_A is also compact from $L^1(A)$ to $M[X]$.Therefore T is Bochner representable in $M[X]$ and the martingale (ξ_n, Σ_n) is Cauchy in the Bochner norm of $L^1_{M[X]}$. □

COROLLARY 1.6. *Let X be a Banach space with a symmetric basis such that X contains no copy of l^1. Let (ξ_n, Σ_n) be a martingale associated to an operator $T\colon L^1[0, 1] \to L^1[0, 1]$. If the martingale (ξ_n, Σ_n) is Cauchy in the Pettis norm then (ξ_n, Σ_n) is Cauchy in the Bochner norm of $L^1_{M[X]}$.*

Proof. It is shown in Corollary 4.4 of [Wei] that if the martingale (ξ_n, Σ_n) is Cauchy in the Pettis norm then the measures $\mu_x = w^* - \lim \xi_n(x)$ are diffuse for $[m]$ almost all x. Now apply Theorem 1.5. □

Remark 1.7. For $X = c_0$, Corollary 1.6 becomes a restatement of Proposition 11 in [B]. In fact Corollary 1.6 can also be derived from Proposition 11 in [B] and Lemma 1.2.

2. $M[X]$-continuous operators

Recall [D-U] that an operator from $L^1[0, 1]$ to a Banach space Z is called *Dunford-Pettis* if it maps weakly compact subsets of $L^1[0, 1]$ into norm compact subsets of Z. It is known [B], [U] that an operator $T\colon L^1[0, 1] \to L^1[0, 1]$ is Dunford-Pettis iff the martingale (ξ_n, Σ_n) associated to T is Cauchy in the Pettis norm.

An operator $T\colon L^1[0, 1] \to Z$ is called *nearly representable* [K-P-R-U] if the composition $T \cdot D\colon L^1[0, 1] \to Z$ is Bochner representable for every Dunford-Pettis operator $D\colon L^1[0, 1] \to L^1[0, 1]$.

Definition 2.1. Suppose X has a symmetric basis. An operator $T\colon L^1[0, 1] \to Z$ is called $M[X]$-continuous if there is a constant C such that $\|T(f)\| \leq C \cdot \|f\|_{M[X]}$ for all f in $L^1[0, 1]$.

PROPOSITION 2.2. *Let Z, X be Banach spaces so that X has a symmetric basis and contains no copy of l^1. Every $M[X]$-continuous operator $T\colon L^1[0, 1] \to Z$ is nearly representable.*

Proof. By Theorem 1.5 (iii), the identity map $W_X\colon L^1[0, 1] \to M[X]$ is nearly representable. For every Dunford-Pettis $D\colon L^1[0, 1] \to L^1[0, 1]$ we have

$$\|T \cdot D(f)\| \leq C \cdot \|W_X D(f)\|_{M[X]}, \quad f \in L^1[0, 1].$$

The martingale associated to the operator $W_X D$ is Cauchy in the Bochner $L^1_{M[X]}$ norm and hence the martingale associated to the operator $T \cdot D$ is Cauchy in the Bochner L^1_Z norm. □

Recall [G-G-M-S, Theorem IV.10] that an operator $T\colon L^1[0,1] \to Z$ is strongly regular iff for each $A \subseteq [0,1]$, $m(A) > 0$ and $\varepsilon > 0$ there is a relatively weakly open set V of the set

$$F_A = \{f \in L^1[0,1] : f \geq 0, \int f = 1, \operatorname{supp}(f) \subseteq A\}$$

such that $\operatorname{diam}(T(V)) < \varepsilon$.

COROLLARY 2.3. *Suppose X has a symmetric basis and contains no copy of l^1. Every $M[X]$-continuous operator $T\colon L^1[0,1] \to Z$ is strongly regular.*

Proof. By Theorem 1 in [A-P], every nearly representable operator is strongly regular. □

COROLLARY 2.4. *Suppose X has a symmetric basis and contains no copy of l^1. Let Z be a separable Banach lattice that contains no copy of c_0. Then every $M[X]$-continuous operator $T\colon L^1[0,1] \to Z$ is Bochner representable.*

Proof. By [K-P-R-U] if Z is a separable Banach lattice that contains no copy of c_0, then every nearly representable operator $T\colon L^1[0,1] \to Z$ is Bochner representable. □

THEOREM 2.5. *Let Z be a Banach space, $p > 1$. If the operator $T\colon L^1[0,1] \to Z$ is $M[l^p]$-continuous and (u_n), $n = 1, 2, \ldots$ is an orthonormal sequence then there is some constant C such that*

$$\|T(u_n)\| \leq C \cdot \left(\frac{\log(n)}{\sqrt{n}}\right)^{1-\frac{1}{p}}$$

for infinitely many values of n.

Proof. Let $V\colon L^1[0,1] \to C[0,1]$ be the Volterra integral operator defined for f in $L^1[0,1]$ by

$$V(f)(x) = \int_0^x f(t)\,dm(t), \qquad x \in [0,1].$$

It is shown in [O], p. 95, that

$$\|V(u_n)\|_{C[0,1]} \leq K\frac{\log(n)}{\sqrt{n}} \quad \text{for infinitely many indices } n.$$

It is easy to see that for $X = l^p$, $p > 1$, a function $\delta(\varepsilon)$ as in the statement of Lemma 1.2 is $\delta(\varepsilon) = \varepsilon^{p/p-1}$. Note that

$$\|u_n\|_{M_0} \leq 2 \cdot K \frac{\log(n)}{\sqrt{n}} \quad \text{for infinitely many values of } n$$

and hence

$$\|u_n\|_{M[l^p]} \leq C \cdot \left(\frac{\log(n)}{\sqrt{n}}\right)^{1-\frac{1}{p}} \quad \text{for these indices.} \qquad \square$$

COROLLARY 2.6. *There are linear functionals on $L^1[0, 1]$ that are not $M[l^p]$-continuous for any $p > 1$.*

Proof. Consider the space $L^1[0, 2\pi]$ instead of $L^1[0, 1]$. There is a function g in $L^\infty[0, 2\pi]$ whose sequence of Fourier coefficients has order greater than that of the sequences $(\frac{\log(n)}{\sqrt{n}})^{1-\frac{1}{p}}$, $n \in N$, $p > 1$. The functional on $L^1[0, 2\pi]$ determined by such g can not be $M[l^p]$-continuous since the exponentials form an orthonormal system. $\square$

It is shown in [K-P-R-U] that if the Banach space Z contains no copy of c_0 then every M_0-continuous operator T: $L^1[0, 1] \to Z$ is a compact operator. The next example shows that the situation is not the same for general $M[X]$-continuous operators.

Example 2.7. There is an $M[l^2]$-continuous operator S: $L^1[0, 1] \to L^1[0, 1]$ which is not compact: Let (I_n) $n = 1, 2, 3, \ldots$ be a sequence of disjoint subintervals of $[0, 1]$. Consider the operator T: $L^1[0, 1] \to l^2$ defined by

$$T(f) = \sum_{n=1}^{\infty} \left(\int_{I_n} f\, dm\right) e_n, \quad f \in L^1[0, 1],$$

where (e_n) is the usual basis for l^2. The map that sents each e_n to the n^{th} Rademacher function r_n extends to an embedding U of l^2 into $L^1[0, 1]$. Now set $S = U \cdot T$ and notice that S maps the sequence

$$\left(\frac{1}{m(I_n)} \chi_{I_n}\right), \quad n = 1, 2, \ldots$$

into a non totally bounded set in $L^1[0, 1]$. It is also clear that S is an an $M[l^2]$-continuous operator. It is easy to see that using the identity map $l^p \to l^2$, $1 < p < 2$ one can construct $M[l^p]$-continuous operators from $L^1[0, 1]$ to $L^1[0, 1]$ that are not compact.

PROPOSITION 2.8. *Let X be an infinite dimensional reflexive space with a symmetric basis (e_i). Suppose that X is isomorphic to a subspace of the Banach space Z. Then there exists a non compact operator T: $L^1[0, 1] \to Z$ such that T is $M[X]$-continuous.*

Proof. Since X is reflexive the basis (e_i) is boundedly complete. Let (I_n), $n = 1, 2, 3, \ldots$, be a sequence of disjoint subintervals of $[0, 1]$. Note that for $k \in N$,

$$\left\| \sum_{n=1}^{k} \left(\int_{I_n} f \, dm \right) e_n \right\| \leq \|f\|_1$$

and therefore the series

$$\sum_{n=1}^{\infty} \left(\int_{I_n} f \, dm \right) e_n$$

converges. The operator T: $L^1[0, 1] \to X$ defined by

$$T(f) = \sum_{n=1}^{\infty} \left(\int_{I_n} f \, dm \right) e_n, \quad f \in L^1[0, 1]$$

is $M[X]$-continuous and non-compact. Let U: $X \to Z$ be an isomorphism. The composition $U \cdot T$ is the required operator. □

The next theorem is the main result in this section.

THEOREM 2.9. *Let X be a Banach space with a symmetric basis such that X contains no copy of l^1. Suppose Z is a Banach space and T: $L^1[0, 1] \to Z$ is a non-compact operator. If T is also $M[X]$-continuous then there exists a non-compact bounded operator S: $X \to Z$.*

For the proof of Theorem 2.9 we need two lemmas.

LEMMA 2.10. *Let Z be a Banach space and T: $L^1[0, 1] \to Z$ a non-compact operator. There is a bounded sequence (g_n), $n \in N$ in $L^1[0, 1]$ and a sequence (I_n), $n \in N$ of pairwise disjoint dyadic intervals such that* $\operatorname{supp}(g_n) \subseteq I_n$, $m(I_n) \to 0$ *as $n \to \infty$ and the set $\{T(g_n), n \in N\}$ is not compact in Z.*

Proof of Lemma 2.10. Since T is non-compact there is an $\varepsilon > 0$ such that for all finite $f_1, f_2, \ldots, f_l$ in B_{L^1} there is a g in B_{L^1} such that

$$d(T(g), \overline{\text{co}}(\{T(f_i), i \leq 1\})) > \varepsilon.$$

Here B_{L^1} denotes the unit ball $\{f \in L^1 : \|f\|_1 \leq 1\}$ of L^1 and d is the norm distance in Z. Now we make the following:

Claim. For $f_1, f_2, \dots, f_d \in B_{L^1}$ and for every $n \in N$ there is a $g \in B_{L^1}$ such that

$$d(T(g), \overline{\text{co}}(\{T(f_i),\ i \leq 1\})) > \frac{\varepsilon}{33} \text{ and } \operatorname{supp}(g) \subseteq I$$

where I is a dyadic interval of length $m(I) \leq \frac{1}{2^n}$.

To prove the claim, suppose there exist $f_1, f_2, \dots, f_d \in B_{L^1}$ and $n \in N$ such that for all h in B_{L^1} with $\operatorname{supp}(h) \subseteq I$, I a dyadic interval of length $m(I) \leq \frac{1}{2^n}$ we have

$$d(T(h), \overline{\text{co}}(\{T(f_i),\ i \leq d\})) \leq \frac{\varepsilon}{33}.$$

There is a g in B_{L^1} with

$$d(T(g), \overline{\text{co}}(\{T(f_i),\ i \leq d\})) > \varepsilon.$$

Let $I_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n}]$, $n = 0, 1, \dots k \leq 2^n$. If $\|g \mid I_{n,k}\|_1 \neq 0$ set

$$g_{n,k} = \frac{g \mid I_{n,k}}{\|g \mid I_{n,k}\|_1}.$$

Write g in the form $g = \sum_{k=1}^{2^n} \lambda_{n,k} g_{n,k}$ where $\lambda_{n,k} = \|g \mid I_{n,k}\|_1$, $k \leq 2^n$. Note that since $\operatorname{supp}(g_{n,k}) \subseteq I_{n,k}$ we have

$$d(T(g_{n,k}), \overline{\text{co}}(\{T(f_i),\ i \leq d\})) \leq \frac{\varepsilon}{33},\quad k \leq 2^n.$$

Let $K = \overline{\text{co}}(\{T(f_i),\ i \leq d\})$ and choose $w_{n,k} \in K$ so that

$$\|T(g_{n,k}) - w_{n,k}\| \leq \frac{\varepsilon}{33},\quad n = 0, 1, \dots,\ k \leq 2^n.$$

Note that for $n \in N$,

$$\left\| \sum_{k=1}^{2^n} \lambda_{n,k} T(g_{n,k}) - \sum_{k=1}^{2^n} \lambda_{n,k} w_{n,k} \right\| = \left\| \sum_{k=1}^{2^n} \lambda_{n,k} (T(g_{n,k}) - w_{n,k}) \right\| \leq \frac{\varepsilon}{33}.$$

This implies that

$$d(T(g), K)) \leq \frac{\varepsilon}{33},$$

a contradiction which proves the claim is true.

Now assume that (g_n), $n \in N$ is a sequence in B_{L^1} so that the set $\{T(g_n),\ n \in N\}$ is not compact, $\operatorname{supp}(g_n) \subseteq I_n$ where the I_n are dyadic intervals of length $\leq \frac{1}{2^n}$, $n \in N$. We may assume that either the sequence (I_n) is monotone (i.e., increasing, in the sense that $\max I_n \leq \min I_{n+1}$, or decreasing) (Case I) or that the sequence (I_n) is directed, in the sense that $I_m \subset I_n$ if $m > n$ (Case II). To see why this is true suppose $I_n = [a_n, b_n]$. By passing to subsequences we may assume that both (a_n) and (b_n)

are monotone. If (a_n) is increasing and (b_n) is decreasing we are in Case II. If both (a_n) and (b_n) are (strictly) increasing sequences we do the following: Start with an interval I_{n_1} of length less than $1/2$. Find $n_2 > n_1$ so that $b_{n_2} > b_{n_1}$. Now take a dyadic interval J from the sequence (I_n) so that the right endpoint of J is larger than b_{n_2} and the length of J is smaller than $b_{n_2} - b_{n_1}$. Note that I_{n_1} and J are disjoint. It is clear now that by induction we can contruct a subsequence (I_{n_k}) of (I_n) which consists of disjoint "increasing" dyadic intervals. So we are in Case I. Similar considerations show that we are in Case I if both (a_n) and (b_n) are decreasing sequences. If one of the monotone sequences (a_n), (b_n) is eventually constant it is easy to see we are in Case II. In Case I it is clear that Lemma 2.10 is true. We claim the same holds in Case II: In fact there is an increasing sequence $n_1 < n_2 < \cdots n_k < n_{k+1} < \cdots$ of integers so that

$$\|g_{n_k} \mid I_{n_{k+1}}\|_1 \to 0 \text{ as } k \to \infty$$

For each k write I_{n_k} in the form

$$I_{n_k} = I_{n_k}^1 \cup I_{n_{k+1}} \cup I_{n_k}^2$$

where $I_{n_k}^1$, $I_{n_k}^2$ are disjoint intervals which are also disjoint to $I_{n_{k+1}}$ and $\max I_{n_k}^1 < \min I_{n_k}^2$. By passing to a further subsequence we may assume that either the set $\{T(g_{n_k} \mid I_{n_k}^1)\ k \in N\}$ or the set $\{T(g_{n_k} \mid I_{n_k}^2), k \in N\}$ is not compact. □

LEMMA 2.11. *Suppose X is a Banach space with a symmetric basis (e_i), $i \in N$. For each $i \in N$ let $\{e_{ij}\colon j \le k_i\}$ be a set of elements of (e_i) such that $\{e_{mj}\colon j \le k_m\} \cap \{e_{ij}\colon j \le k_i\}$ is the empty set if $i \ne m$. Let $u_i = \sum_{j=1}^{k_i} a_{ij} e_{ij}$, where $\sum_{j=1}^{k_i} a_{ij} \le 1$, $a_{ij} \ge 0$. Then $\|\sum_{i=1}^r b_i u_i\| \le \|\sum_{i=1}^r b_i e_i\|$ for all scalars b_i, $i \le r$.*

Proof of Lemma 2.11. Let x^* in the unit ball B_{X^*} of the dual space X^* of X be such that

$$x^*\left(\sum_{i=1}^r b_i u_i\right) = \left\|\sum_{i=1}^r b_i u_i\right\|.$$

Then

$$\left\|\sum_{i=1}^r b_i u_i\right\| = \sum_{i=1}^r b_i x^*(u_i) = \sum_{i=1}^r b_i x^*\left(\sum_{i=1}^{k_i} a_{ij} e_{ij}\right) \le \sum_{i=1}^r b_i \sum_{i=1}^{k_i} a_{ij} x^*(\overline{e_i}),$$

where $\overline{e_i}$ is an element of the set $\{e_{ij}\colon j \le k_i\}$ such that

$$x^*(\overline{e_i}) = \max(x^*(e_{ij}),\ j \le k_i).$$

It follows that

$$\left\|\sum_{i=1}^{r} b_i u_i\right\| \leq x^*\left(\sum_{i=1}^{r} b_i \overline{e_i}\right) \leq \left\|\sum_{i=1}^{r} b_i \overline{e_i}\right\| \leq \left\|\sum_{i=1}^{r} b_i e_i\right\|.$$

In the last inequality we used the symmetry of the basis (e_i), $i \in N$. □

Proof of Theorem 2.9. Suppose T: $L^1[0, 1] \to Z$ is an $M[X]$-continuous operator which is not compact on $L^1[0, 1]$. By Lemma 2.10 there is a sequence (g_s) in B_{L^1} so that $\operatorname{supp}(g_s) \subseteq I_s$ where (I_s) is a sequence of disjoint dyadic intervals, $m(I_s) \to 0$ as $s \to \infty$, and the set $\{T(g_s),\ s \in N\}$ is not compact in Z. The proof of Lemma 2.10 shows that we can also suppose that

$$\|T(g_{2m-1}) - T(g_{2m}) - T(g_{2n-1}) - T(g_{2n})\| > \varepsilon \quad \text{if } m \neq n.$$

By splitting $g_s = g_s^+ - g_s^-$ into its positive and negative part and by passing to a further subsequence we may assume without loss of generality that each g_s is positive. For convenience we also assume that $\|g_s\|_1 = 1$ and that the sequence of the dyadic intervals (I_s) is increasing.

Now consider the sequence

$$f_1 = g_1 - g_2,\ f_2 = g_3 - g_4, \ldots,\ f_i = g_{2i-1} - g_{2i}, \ldots,\ i \in N.$$

Let

$$h = \sum_{i=1}^{m} a_i f_i, \qquad a_i \text{ real numbers, } 1 \leq i \leq m.$$

Suppose (e_i) is a 1-symmetric basis for X. We now make the following:

Claim. Given any interval I there is a subinterval J of I so that J is contained in one of the intervals (I_s) and

$$\left|\int_I h\,dm\right| \leq 4 \cdot \left|\int_J h\,dm\right|.$$

In fact if $I = [a, b]$ and $a \in I_{2k-1}$, $b \in I_{2l}$, $k < l$ write I in the form

$$I = (I \cap I_{2k-1}) \cup (I \cap I_{2k}) \cup M \cup (I \cap I_{2l-1}) \cup (I \cap I_{2l})$$

where M is an interval so that

$$\int_M h\,dm = 0.$$

Now choose J to be one of the 4 intervals

$$J_1 = (I \cap I_{2k-1}),\ J_2 = (I \cap I_{2k}),\ J_3 = (I \cap I_{2l-1}),\ J_4 = (I \cap I_{2l})$$

so that

$$\left|\int_J h\,dm\right| \geq \left|\int_{J_\tau} h\,dm\right|, \ \tau = 1, 2, 3, 4.$$

So the claim is true. Now consider a finite number of disjoint intervals $\Delta_1, \Delta_2, \dots \Delta_d$. The unconditionality of (e_i) and the claim imply that there exist intervals $H_1, H_2, \dots,$ H_d so that $H_q \subseteq \Delta_q$, $q = 1, 2, \dots d$, every H_q is contained in some interval from the sequence (I_n) and

$$\left\|\sum_{q=1}^{d}\left(\int_{\Delta_q} h\,dm\right)e_q\right\| \leq 4\left\|\sum_{q=1}^{d}\left(\int_{H_q} h\,dm\right)e_q\right\|.$$

Let $A_s = \{q \leq d\colon\ H_q \subseteq I_s\}$, $s \leq 2m$. Note that

$$\begin{aligned}&\left\|\sum_{q=1}^{d}\left(\int_{H_q} h\,dm\right)e_q\right\| \\ &= \left\|\sum_{q\in A_1}\left(\int_{H_q} h\,dm\right)e_q + \sum_{q\in A_2}\left(\int_{H_q} h\,dm\right)e_q + \cdots + \sum_{q\in A_{2m}}\left(\int_{H_q} h\,dm\right)e_q\right\|.\end{aligned}$$

By Lemma 2.11 the last quantity is less than

$$\begin{aligned}&\left\|\left(\int_{I_1} h\,dm\right)e_1 + \left(\int_{I_2} h\,dm\right)e_2 + \cdots + \left(\int_{I_{2m}} h\,dm\right)e_{2m}\right\| \\ &= \left\|a_1\left(\int_{I_1} g_1\,dm\right)e_1 + a_1\left(\int_{I_2} g_2\,dm\right)e_2\right. \\ &\quad + a_2\left(\int_{I_3} g_3\,dm\right)e_3 + a_2\left(\int_{I_4} g_4\,dm\right)e_4 \\ &\quad \left. + \cdots + a_m\left(\int_{I_{2m-1}} g_{2m-1}\,dm\right)e_{2m-1} + a_m\left(\int_{I_{2m}} g_{2m}\,dm\right)e_{2m}\right\| \\ &\leq \|a_1e_1 + a_1e_2 + a_2e_3 + a_2e_4 + \cdots + a_me_{2m-1} + a_me_{2m}\| \\ &\leq 2\|a_1e_1 + a_2e_2 + \cdots + a_me_m\|\end{aligned}$$

by the symmetry of the basis (e_i). The $M[X]$-continuity of the operator $T\colon L^1[0,1] \to Z$ shows that the map $e_i \to T(f_i)$, $i \in N$, extends to a bounded operator $S\colon X \to Z$ because

$$\left\|S\left(\sum a_ie_i\right)\right\|_Z = \left\|T\left(\sum a_if_i\right)\right\| \leq C \cdot \left\|\sum a_if_i\right\|_{M[X]} \leq 8 \cdot C \cdot \left\|\sum a_ie_i\right\|_X.$$

Since $\{T(f_i),\ i \in N\}$ is not compact in Z we get that the operator S is not compact. □

As a corollary of Theorem 2.9 we obtain a result from [K-P-R-U], Cor. 13.

COROLLARY 2.12. *If the Banach space Z contains no copy of c_0 then every M_0-continuous operator T: $L^1[0, 1] \to Z$ is compact.*

Proof. By a theorem of Pelczynski if S: $c_0 \to Z$ is non compact then S fixes a copy of c_0. □

Acknowledgments. I am grateful to Professor S. Argyros for fruitful discussions.I wish to thank the referee for valuable comments and suggestions.

REFERENCES

[A-P] S. Argyros and M. Petrakis, "A property of non-strongly regular operators," in *Geometry of Banach spaces*, Proceedings of the Conference held in Stroble, Austria, 1989, London Math. Society Lecture Notes Series 158, Cambridge University Press 1990, pp. 6–23.

[B] J. Bourgain, *Dunford-Pettis operators on L^1 and the Radon-Nikodym property*, Israel J. Math **37** (1980), 34–47.

[G-G-M-S] N. Ghoussoub, G. Godefroy, B. Maurey, W. Schachermayer, *Some topological and geometric structures in Banach spaces*, Mem. Amer. Math. Soc., Vol 70, Number 378, 1987.

[D] M. Day, *Normed linear spaces*, 3rd ed., Ergeb. Math. Grenzgeb., Band 21, Springer-Verlang, New York, 1973.

[D-U] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math Surveys, No. 15, Amer. Math. Soc., Providence, Rhode Island, 1977.

[K-P-R-U] R. Kaufman, M. Petrakis, L. Riddle and J. J. Uhl, Jr., *Nearly representable operators*, Trans. Amer. Math. Soc. **312** (1989), 315–333.

[L-S] J. Lindenstrauss and C. Stegall, *Examples of separable spaces which do not contain l^1 and whose duals are non separable*, Studia Math. **54** (1975), 81–105.

[L-T] J. Lindestrauss and L. Tzafriri, *Classical Banach spaces I, Sequence spaces*, Springer-Verlag, New York, 1977.

[O] A.M. Olevskiĭ, *Fourier series with respect to general orthonormal systems*, Springer-Verlag, New York, 1975.

[U] J. J. Uhl, Jr., *Pettis mean convergence of vector-valued asymptotic martingales*, Z. Wahrsch. Verw. Gebiete **37** (1977), 291–295.

[Wei] L. Weis, *On the representation of order continuous operators by random measures*, Trans. Amer. Math. Soc. **285** (1984), 535–563.

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