ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 4, Winter 1998

UNIFORM AND STRONG ERGODIC THEOREMS IN BANACH SPACES

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1. Introduction

In his study of the spectral theory of bounded linear operators on a Banach space, N. Dunford [3] gave some necessary and sufficient conditions for the convergence in various topologies of a sequence of operator functions to a projection and established a systematic theory of uniform and strong (i.e., mean) ergodic theorems in Banach spaces. But the equivalence of Cesàro, Hausdorff, and Abel summability of a sequence of operators had not yet been considered in the concrete. In connection with this problem, E. Hille [7] obtained, as applications of Abelian and Tauberian theorems to ergodic theorems, the uniform and strong ergodic theorems as stated below with a view to relating the (C, α) ergodic theorem for an operator T and the properties of the resolvent $R(\lambda; T)$. In particular, the fact that the uniform (or strong) convergence of $(\lambda - 1)R(\lambda; T)$ as $\lambda \to 1+0$ implies the (C, α) uniform (or strong) convergence for T has been established by supposing the power-boundedness of T. It appears, however, that the power-boundedness is not necessarily essential in the above implication. Our investigation is motivated by this very fact just mentioned, and we deal with ergodic theorems for operators which are not necessarily power-bounded.

THEOREM A (HILLE [7, THEOREM 6]). *A necessary condition for the existence of an operator E such that for some fixed $\alpha > 0$,*

(1) (uo) $\lim_{n\to\infty}[A_n^{(\alpha)}]^{-1} \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} T^k = E$

is that

(2) (uo) $\lim_{\lambda\to 1+0}(\lambda - 1)R(\lambda; T) = E$

and

(3) (uo) $\lim_{n\to\infty} T^n/n^\alpha = \theta$ *(the null operator).*

Conversely, if (3) *is replaced by the power-boundedness of T, then* (2) *implies* (1) *for every $\alpha > 0$. Here, $A_n^{(\alpha)}$, $n = 0, 1, 2, \ldots$, are the (C, α) coefficients of order α.*

Received July 7, 1996.
1991 Mathematics Subject Classification. Primary 47A35; Secondary 40E05.
Supported in part by the Inoue Enryo MRF at Toyo University.

THEOREM B (HILLE [7, THEOREM 7]). *A necessary condition for the existence of an operator E such that for some fixed $\alpha > 0$,*

(1) (so) $\lim_{n\to\infty}[A_n^{(\alpha)}]^{-1}\sum_{k=0}^{n} A_{n-k}^{(\alpha-1)}T^k = E$

is that

(2) (so) $\lim_{\lambda\to 1+0}(\lambda - 1)R(\lambda; T) = E$

and

(3) (so) $\lim_{n\to\infty} T^n/n^\alpha = \theta$.

Conversely, if (3) *is replaced by the power-boundedness of* T, *then* (2) *implies* (1) *for every* $\alpha > 0$.

In fact, we have a particular interest in the converse statements of the above theorems, when the operators in question are not necessarily power-bounded, because this case seems to have not been considered by Hille. More precisely, the question is whether the power-boundedness of the operators in question is indispensable to deduce (1) from (2). A partial negative answer to this question was first given by M. Lin [9] in the case $\alpha = 1$. The purpose of the present paper is to answer the question negatively for any real order $\alpha > 0$. The next section is devoted to the discussion concerning the relation between Cesàro and Abel summability of sequences of operators in the uniform operator topology. We shall establish a multiplication principle which reminds us of the so-called noncommuting ergodic theorems in the usual sense. This principle provides a new (one-parameter) method of treating the multiparameter ergodic theorems. The arguments used allow us to consider the case (corresponding to Theorem B) of the strong operator topology. In the last section we will deal with a similar question of relating Hausdorff and Abel summability. The proofs given here depend essentially on the operational calculus devised by Dunford in the spectral theory.

2. Cesàro and Abel summability

Throughout this paper, $(X, \|\cdot\|)$ will denote a complex Banach space and $B[X, X]$ will denote the Banach algebra of bounded linear operators on X to itself. For a real $\alpha > 0$ and each integer $n \geq 0$ let $A_n^{(\alpha)}$ be the (C, α) coefficient of order α which is defined by the generating function $(1-\mu)^{-(\alpha+1)} = \sum_{n=0}^{\infty} A_n^{(\alpha)}\mu^n (0 < \mu < 1)$. In particular,

$$A_0^{(\alpha)} = A_0^{(\alpha-1)} = A_n^{(0)} = 1$$

and

$$A_n^{(\alpha)} = \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} = \binom{\alpha+n}{n} = p_{n,\alpha} \cdot n^{\alpha}/\Gamma(\alpha+1) \quad \left(\lim_{n\to\infty} p_{n,\alpha} = 1\right).$$

In what follows we take as the basis of this consideration the general formula

$$C_n^{(\alpha)}[T] = \frac{1}{A_n^{(\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} T^k \quad (T \in B[X, X]).$$

The main result is stated as follows.

THEOREM 1. *Fix a real $\alpha > 0$ and let $T \in B[X, X]$ satisfy the condition $\|T^n/n^{\omega}\| \to \theta$ when $n \to \infty$, where $\omega = \min(1, \alpha)$. Then there exists an operator $E \in B[X, X]$ such that $\|C_n^{(\alpha)}[T] - E\| \to 0$ as $n \to \infty$ if and only if $\|(\lambda - 1)R(\lambda; T) - E\| \to 0$ as $\lambda \to 1 + 0$.*

In order to prove Theorem 1 we need the following lemma which is of interest in itself.

LEMMA 1. If $\|T^n/n^{\omega}\| \to 0$ as $n \to \infty$ then $\|(I - T)C_n^{(\alpha)}[T]\| \to 0$ as $n \to \infty$.

Proof. For every positive integer n, no matter how large, $(I - T)C_n^{(\alpha)}[T]$ can be rewritten as

$$(I - T)C_n^{(\alpha)}[T] = \frac{1}{A_n^{(\alpha)}} \left\{ A_n^{(\alpha-1)} I - T^{n+1} + \sum_{k=1}^{n} (A_{n-k}^{(\alpha-1)} - A_{n-k+1}^{(\alpha-1)}) T^k \right\}.$$

Since the sequences $\{p_{n,\alpha-1}/p_{n,\alpha}\}$ and $\{1/p_{n,\alpha}\}$ are both bounded, we get

$$\begin{aligned} \|(I - T)C_n^{(\alpha)}[T]\| &\le \frac{1}{A_n^{(\alpha)}} \left\{ A_n^{(\alpha-1)} + \|T^{n+1}\| + \sum_{k=1}^{n} |A_{n-k}^{(\alpha-1)} - A_{n-k+1}^{(\alpha-1)}| \|T^k\| \right\} \\ &\le \frac{1}{A_n^{(\alpha)}} \left\{ A_n^{(\alpha-1)} + \|T^{n+1}\| + \max(1, A_n^{(\alpha-1)}) \max_{0\le k\le n} \|T^k\| \right\} \\ &\le \frac{M}{n^{\omega}} \left\{ 1 + \|T^n\| + \max_{0\le k\le n} \|T^k\| \right\} \end{aligned}$$

for some constant $M > 0$. However, it is easily seen that $\|T^n/n^{\omega}\| \to 0$ as $n \to \infty$ implies $\max_{0\le k\le n} \|T^k\|/n^{\omega} \to 0$ as $n \to \infty$. Hence $\|(I - T)C_n^{(\alpha)}[T]\| \to 0$ as $n \to \infty$ and the lemma follows. □

Proof of Theorem 1. Suppose that there exists an operator $E \in B[X, X]$ such that

$$\|C_n^{(\alpha)}[T] - E\| \to 0 \quad \text{as} \quad n \to \infty.$$

We claim that

$$\|(\lambda - 1)R(\lambda; T) - E\| \to 0 \quad \text{as} \quad \lambda \to 1 + 0.$$

Though this follows from Hille's theorem (Theorem A), we sketch its proof in the present situation. In view of Lemma 1 we may say that E is a projection so that, taking into account that the functions $C_n^{(\alpha)}(\cdot)$ with complex variables are polynomials, we can apply Dunford's uniform ergodic theorem [3, Theorem 3.16] to assert that $X = N(I - T) \oplus R(I - T)$, $EX = N(I - T)$, $R(I - T)$ is closed, where $N(I - T)$ and $R(I - T)$ denote the null space and range of I–T respectively. Note further that $R(I - T)$ is invariant under T and let S denote the restriction of T to $R(I - T)$. Then, using the uniform ergodic theorem, we see that $I - S$ is invertible on $R(I - T)$. So, all that is required is to show that

$$\lim_{\lambda \to 1+0} \|(\lambda - 1)R(\lambda; S)\| = 0.$$

Now, for sufficiently small $\epsilon > 0$, by assumption there exists a positive integer $N_1 = N_1(\epsilon)$ say, such that $\|S^n/n^\omega\| < \epsilon$ for all $n > N_1$. For the number N_1 so obtained,

$$\sum_{n=1}^{\infty} \frac{\|S^n\|}{\lambda^n} \le \sum_{n=1}^{N_1} \frac{\|S^n\|}{\lambda^n} + \frac{\epsilon}{\lambda} \sum_{n=1}^{\infty} \frac{n^\omega}{\lambda^{n-1}} \le \sum_{n=1}^{N_1} \frac{\|S^n\|}{\lambda^n} + \frac{\epsilon\lambda}{(\lambda - 1)^2}.$$

Thus, using the equality $(I - S)(I - S)^{-1} = I$ which holds on $R(I - T)$, we obtain

$$\begin{aligned} \|(\lambda - 1)R(\lambda; S)\| &\le (\lambda - 1)\|(I - S)^{-1}\| \left\| \frac{1}{\lambda} I + \sum_{k=1}^{\infty} \left(\frac{1}{\lambda^{k+1}} - \frac{1}{\lambda^k} \right) S^k \right\| \\ &\le (\lambda - 1)\|(I - S)^{-1}\| \left\{ \frac{1}{\lambda} + \frac{\lambda - 1}{\lambda} \sum_{k=1}^{\infty} \frac{\|S^k\|}{\lambda^k} \right\} \\ &\le \|(I - S)^{-1}\| \left\{ \frac{\lambda - 1}{\lambda} + \frac{(\lambda - 1)^2}{\lambda} \sum_{k=1}^{N_1} \frac{\|S^k\|}{\lambda^k} + \epsilon \right\}; \end{aligned}$$

whence the required one on first letting $\lambda \to 1+0$ and then $\epsilon \to 0$. Therefore we have proved that $\|(\lambda - 1)R(\lambda; T) - E\| \to 0$ as $\lambda \to 1 + 0$. Next we suppose conversely that $\|(\lambda - 1)R(\lambda; T) - E\| \to 0$ as $\lambda \to 1 + 0$. Let $\epsilon > 0$ be fixed sufficiently small and choose a positive integer $N_2 = N_2(\epsilon)$ say, such that $\|T^n/n^\omega\| < \epsilon$ for all $n > N_2$. As before, we get

$$\|(\lambda - 1)R(\lambda; T)(I - T)\| \le \frac{\lambda - 1}{\lambda} + \frac{(\lambda - 1)^2}{\lambda} \sum_{k=1}^{N_2} \frac{\|T^k\|}{\lambda^k} + \epsilon,$$

so that $E = TE = ET$ and $E = (\text{uo})\lim_{\lambda \to 1+0} (\lambda - 1)R(\lambda; T)E = E^2$; that is to say, E is a projection in $B[X, X]$ and $EX = N(I - T)$. Now, the series $\sum_{n=0}^{\infty} T^n/\lambda^{n+1}$ being well defined in the uniform operator topology since $\|T^n/n^\omega\| \to 0$ when $n \to \infty$, we have that

$$(\lambda - 1)\sum_{n=0}^{\infty} \frac{I - T^n}{\lambda^{n+1}} = (\lambda - 1)(I - T)\sum_{k=0}^{\infty}\left(\sum_{n=k+1}^{\infty} \frac{1}{\lambda^{n+1}}\right)T^k = (I - T)\sum_{k=0}^{\infty}\frac{T^k}{\lambda^{k+1}}$$

is also well defined in the uniform operator topology. Thus, if for any $x \in X$ we write $\bar{x} = x - Ex$ then clearly $Ex \in N(I - T)$ and $\bar{x} \in \overline{R(I - T)}$, because

$$\bar{x} = (\mathrm{s})\lim_{\lambda \to 1+0}(\lambda - 1)\sum_{n=0}^{\infty}\frac{(I - T^n)x}{\lambda^{n+1}} = (\mathrm{s})\lim_{\lambda \to 1+0}(I - T)\sum_{n=0}^{\infty}\frac{T^n x}{\lambda^{n+1}}.$$

Moreover, we claim that $N(I - T) \cap \overline{R(I - T)} = \{0\}$. To verify this, first we remark that there exists a constant $K > 0$ such that $\sup_{1<\lambda\le 2}\|(\lambda - 1)R(\lambda; T)\| \le K$ by the principle of uniform boundedness (e.g., see [4, page 66]). If y is of the form $y = (I - T)x + z$, $x, z \in X$, $\|z\| < \epsilon$, then

$$\begin{aligned}\left\|(\lambda - 1)\sum_{n=0}^{\infty}\frac{T^n y}{\lambda^{n+1}}\right\| &= \left\|(\lambda - 1)\left\{\sum_{n=0}^{\infty}\frac{T^n(I - T)x}{\lambda^{n+1}} + \sum_{n=0}^{\infty}\frac{T^n z}{\lambda^{n+1}}\right\}\right\| \\ &\le \left\{\frac{\lambda - 1}{\lambda} + \frac{(\lambda - 1)^2}{\lambda}\sum_{k=1}^{N_2}\frac{\|T^k\|}{\lambda^k} + \epsilon\right\}\|x\| + \epsilon K.\end{aligned}$$

This means that for any $y \in \overline{R(I - T)}$,

$$\|(\lambda - 1)R(\lambda; T)y\| \to 0 \quad \text{as} \quad \lambda \to 1 + 0.$$

Accordingly, if $x \in N(I - T) \cap \overline{R(I - T)}$ then $x = Ex = 0$ as asserted. Now, $\overline{R(I - T)}$ is manifestly a T-invariant subspace of X and we let S be the restriction of T to $\overline{R(I - T)}$. Then one gets

$$\lim_{\lambda \to 1+0}\|(\lambda - 1)R(\lambda; S)y\| = 0 \quad \text{for all} \quad y \in \overline{(R(I - T)}$$

which follows from what has been observed above and hence

$$\|(\lambda - 1)R(\lambda; S)\| \to 0 \quad \text{as} \quad \lambda \to 1 + 0.$$

From this we infer that for a fixed $\lambda > 0$ close enough to 1, $I - (\lambda - 1)R(\lambda; S)$ is invertible on $\overline{R(I - T)}$. Hence, so is the operator $I - S$ and $R(I - T)$ must be closed because we have $I - (\lambda - 1)R(\lambda; S) = (\lambda - 1)^{-1}(I - S)R(\lambda; S)$. We have therefore proved that

$$X = N(I - T) \oplus R(I - T)$$

(which also means that the representation $x = Ex + \bar{x}$ is unique). In view of Lemma 1 and the fact that $C_n^{(\alpha)}(1) = 1$ for all n, this yields $\|C_n^{(\alpha)}[T] - E\| \to 0$ as $n \to \infty$

by Dunford's uniform ergodic theorem, since all the functions $C_n^{(\alpha)}(\cdot)$ of complex variables are analytic in a fixed neighborhood of the spectrum $\sigma(T)$ of T. The proof of Theorem 1 has hereby been completed.

Next we make mention of the corresponding question in the strong operator topology. If (so) $\lim_{n\to\infty} T^n/n^\omega = \theta$, where $\omega = \min(1, \alpha)$ for some fixed $\alpha > 0$ then an easy observation gives

$$\text{(s)} \lim_{n\to\infty} \left\{ \max_{0\le k\le n} \|T^k x\| \right\} / n^\omega = 0 \text{ for all } x \in X.$$

In addition, it can be easily checked that as in Lemma 1, the analytic functions $C_n^{(\alpha)}(\cdot)$ of complex variables satisfy the condition (so) $\lim_{n\to\infty}(I - T)C_n^{(\alpha)}[T] = \theta$.

Taking this into consideration, the corresponding theorem in the strong operator topology can be obtained from a minor modification of the arguments used in the proof of Theorem 1 by applying Dunford's mean ergodic theorem [3, Theorem 3.19]. Here we state only the result without proof.

THEOREM 2. *Fix a real $\alpha > 0$ and let $T \in B[X, X]$ satisfy the condition $\|T^n x/n^\omega\| \to 0$ for all $x \in X$ when $n \to \infty$, where $\omega = \min(1, \alpha)$. Suppose that $\sup_n \|C_n^{(\alpha)}[T]x\| < \infty$ for all $x \in \overline{R(I - T)}$. Then there exists an operator $E \in B[X, X]$ such that $\|C_n^{(\alpha)}[T]x - Ex\| \to 0$ for all $x \in X$ when $n \to \infty$ if and only if $\|(\lambda - 1)R(\lambda; T)x - Ex\| \to 0$ for all $x \in X$ when $\lambda \to 1 + 0$.*

COROLLARY 1. *Let $\alpha > 0$ be fixed and let $T \in B[X, X]$ satisfy the condition $\|T^n/n^\omega\| \to 0$ as $n \to \infty$ where $\omega = \min(1, \alpha)$. Suppose $\sup_n \|\sum_{k=0}^n A_{n-k}^{(\alpha-1)} T^k x\| < \infty$ for every $x \in \overline{R(I - T)}$. Then there exists an operator $E \in B[X, X]$ such that $C_n^{(\alpha)}[T]$ converges to E in the uniform operator topology when $n \to \infty$.*

COROLLARY 2. *Let $\alpha > 0$ be fixed and let $T \in B[X, X]$ satisfy the condition $\|T^n/n^\omega\| \to 0$ as $n \to \infty$ where $\omega = \min(1, \alpha)$. Suppose that there is an integer $k > 0$ such that T^k is quasi-compact. Then there exists a compact projection $E \in B[X, X]$ such that $\|C_n^{(\alpha)}[T] - E\| \to 0$ as $n \to \infty$ if and only if $\sup_n \|C_n^{(\alpha)}[T]\| < \infty$.*

Proof. The necessity of the condition $\sup_n \|C_n^{(\alpha)}[T]\| < \infty$ follows from the principle of uniform boundedness. Conversely, if $C_n^{(\alpha)}[T]$ is uniformly (norm-) bounded then

$$C_n^{(\alpha)}[T] \in B[X, X],\ C_n^{(\alpha)}[T]x \in \overline{\text{co}}\ \text{Orbit}(x) \quad \text{for all } n \ge 0 \text{ and every } x \in X.$$

In accordance with Lemma 1, $\{C_n^{(\alpha)}[T]\}$ becomes a system of almost uniformly invariant integrals for the cyclic semigroup $\{T^n\colon n \ge 0\}$. Hence the uniform convergence of $C_n^{(\alpha)}[T]$ to a compact projection follows from Eberlein's uniform ergodic theorem [5, Theorem 6.1]. □

Remark 1. If $0 < \alpha \leq 1$ and $T \in B[X, X]$ is quasi-compact, then there exists a compact projection $E \in B[X, X]$ such that $\|C_n^{(\alpha)}[T] - E\| \to 0$ as $n \to \infty$ if and only if T^n/n^α converges to θ in the weak operator topology when $n \to \infty$. This follows from Hille's theorem (Theorem A) and Theorem 3.1 of [13]. Incidentally, the equation

$$(\lambda - 1)R(\lambda; T) = \left(\frac{\lambda - 1}{\lambda}\right)^{\alpha+1} \sum_{n=0}^{\infty} \left(\frac{1}{\lambda}\right)^n A_n^{(\alpha)}\, C_n^{(\alpha)}\, [T]$$

shows that $\|(\lambda - 1)R(\lambda; T) - E\| \to 0$ as $\lambda \to 1+0$ whenever $\|C_n^{(\alpha)}[T] - E\| \to 0$ as $n \to \infty$.

We shall now prove the following theorem which may be regarded as an operator-theoretical generalization of Cesàro's multiplication formula for sequences.

THEOREM 3. *Let $\alpha_i > 0$ and $\omega_i = \min(1, \alpha_i), i = 1, 2, \ldots, N$. Let $T_i \in B[X, X], i = 1, 2, \ldots, N$, be uniformly Abel ergodic and satisfy the conditions $\lim_{n\to\infty} \|T_i^n/n^{\omega_i}\| = 0, i = 1, 2, \ldots, N$. Put*

$$M_n^{(\alpha_i)}[T_i] = \sum_{k=0}^{n} A_{n-k}^{(\alpha_i-1)}\, T_i^k, i = 1, 2, \ldots, N,$$

$$\Omega_n^{(1)}[T_1] = M_n^{(\alpha_1)}[T_1],$$

$$\Omega_n^{(m)}[T_1, \ldots, T_m] = \sum_{p+q=n} M_p^{(\alpha_m)}[T_m]\Omega_q^{(m-1)}[T_1, \ldots, T_{m-1}], \quad m = 2, 3, \ldots, N.$$

Then there exist projections $E_i \in B[X, X], i = 1, 2, \ldots, N$, such that

$$(\mathrm{uo}) \lim_{n\to\infty} \frac{\Omega_n^{(N)}[T_1, \ldots, T_N]}{n^{\alpha_1+\cdots+\alpha_N+N-1}} = \frac{E_N \cdots E_2 E_1}{\Gamma(\alpha_1 + \cdots + \alpha_N + N)}.$$

Proof. In view of Theorem 1 there are projections $E_i \in B[X, X], i = 1, 2, \ldots, N$, with

$$(\mathrm{uo}) \lim_{n\to\infty} C_n^{(\alpha_i)}[T_i] = E_i \text{ for each } i,$$

so that

$$(\mathrm{uo}) \lim_{n\to\infty} M_n^{(\alpha_i)}[T_i]/n^{\alpha_i} = E_i/\Gamma(\alpha_i + 1) \quad \text{for each } i.$$

This also shows that the theorem holds for the case $N = 1$. Suppose that the theorem has been established for $N - 1$ operators $T_1, \ldots, T_{N-1}$. So, letting

$$V_n^{(N-1)}[T_1, \ldots, T_{N-1}] = \Omega_n^{(N-1)}[T_1, \ldots, T_{N-1}]/A_n^{(\alpha_1+\cdots+\alpha_{N-1}+N-2)},$$

one gets

$$(\text{uo}) \lim_{n\to\infty} V_n^{(N-1)}[T_1, \ldots, T_{N-1}] = E_{N-1} \cdots E_2 E_1$$

by assumption. Cesàro's multiplication formula enables us to assert that

$$(\text{uo}) \lim_{n\to\infty} \Omega_n^{(N)}[T_1, \ldots, T_N]/n^{\alpha_1+\cdots+\alpha_N+N-1} = E_N \cdots E_2 E_1/\Gamma(\alpha_1 + \cdots + \alpha_N + N)$$

by the induction hypothesis. In fact, we have

$$\frac{n^{\alpha_1+\cdots+\alpha_N+N-1}}{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}} \left\{ \frac{\Omega_n^{(N)}[T_1, \ldots, T_N]}{n^{\alpha_1+\cdots+\alpha_N+N-1}} - \frac{E_N \cdots E_2 E_1}{\Gamma(\alpha_1 + \cdots + \alpha_N + N)} \right\}$$

$$= \frac{\sum_{p+q=n} A_p^{(\alpha_N)} A_q^{(\alpha_1+\cdots+\alpha_{N-1}+N-2)} \left(C_p^{(\alpha_N)}[T_N]\, V_q^{(N-1)}[T_1, \ldots, T_{N-1}] - E_N \cdots E_2 E_1\right)}{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}}$$
$$+ \frac{n^{\alpha_1+\cdots+\alpha_N+N-1}}{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}} \left\{ \frac{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)} E_N \cdots E_2 E_1}{n^{\alpha_1+\cdots+\alpha_N+N-1}} - \frac{E_N \cdots E_2 E_1}{\Gamma(\alpha_1 + \cdots + \alpha_N + N)} \right\}$$

and

$$\frac{\sum_{p+q=n} A_p^{(\alpha_N)} A_q^{(\alpha_1+\cdots+\alpha_{N-1}+N-2)} \left(C_p^{(\alpha_N)}[T_N]\, V_q^{(N-1)}[T_1, \ldots, T_{N-1}] - E_N \cdots E_2 E_1\right)}{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}}$$

$$= \frac{\sum_{q=0}^{n} A_{n-q}^{(\alpha_N)} A_q^{(\alpha_1+\cdots+\alpha_{N-1}+N-2)} C_{n-q}^{(\alpha_N)}[T_N] \left(V_q^{(N-1)}[T_1, \ldots, T_{N-1}] - E_{N-1} \cdots E_2 E_1\right)}{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}}$$
$$+ \frac{\sum_{p=0}^{n} A_p^{(\alpha_N)} A_{n-p}^{(\alpha_1+\cdots+\alpha_{N-1}+N-2)} \left(C_p^{(\alpha_N)}[T_N] - E_N\right) E_{N-1} \cdots E_2 E_1}{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}}.$$

Let $\epsilon > 0$ be arbitrarily small and choose a number $n_0 = n_0(\epsilon) > 0$ such that

$$\|V_n^{(N-1)}[T_1, \ldots, T_{N-1}] - E_{N-1} \cdots E_2 E_1\| < \epsilon$$

and

$$\|C_n^{(\alpha_N)}[T_N] - E_N\| < \epsilon \quad \text{for all} \quad n > n_0.$$

Note further that there exists a constant $K_0 > 1$ such that $\|E_i\| \le \sup_n \|C_n^{(\alpha_i)}[T_i]\| <$

K_0, $i = 1, 2, \ldots, N$. Then it follows that

$$\begin{aligned}
&\frac{n^{\alpha_1+\cdots+\alpha_N+N-1}}{K_0^{N-1}\, A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}} \left\| \frac{\Omega_n^{(N)}\,[T_1, \ldots, T_N]}{n^{\alpha_1+\cdots+\alpha_N+N-1}} - \frac{E_N \cdots E_2 E_1}{\Gamma(\alpha_1+\cdots+\alpha_N+N)} \right\| \\
&\leq \frac{A_n^{(\alpha_N+1)}}{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}} \\
&\quad \times \sum_{q=0}^{n_0} A_q^{(\alpha_1+\cdots+\alpha_{N-1}+N-2)} \left\| V_q^{(N-1)}\,[T_1, \ldots, T_{N-1}] - E_{N-1} \cdots E_2 E_1 \right\| \\
&\quad + \frac{A_n^{(\alpha_1+\cdots+\alpha_{N-1}+N-1)}}{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}} \sum_{p=0}^{n_0} A_p^{(\alpha_N)} \left\| C_p^{(\alpha_N)}[T_N] - E_N \right\| + 2\epsilon \\
&\quad + \frac{K_0 n^{\alpha_1+\cdots+\alpha_N+N-1}}{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}} \left| \frac{A_n^{(\alpha_1+\cdots+\alpha_N+N-1)}}{n^{\alpha_1+\cdots+\alpha_N+N-1}} - \frac{1}{\Gamma(\alpha_1+\cdots+\alpha_N+N)} \right|
\end{aligned}$$

for all $n > n_0$. Therefore

$$\overline{\lim_{n\to\infty}} \left\| \frac{\Omega_n^{(N)}\,[T_1, \ldots, T_N]}{n^{\alpha_1+\cdots+\alpha_N+N-1}} - \frac{E_N \cdots E_2 E_1}{\Gamma(\alpha_1+\cdots+\alpha_N+N)} \right\| \leq \frac{2\,K_0^{N-1}\,\epsilon}{\Gamma(\alpha_1+\cdots+\alpha_N+N)}$$

as required. The proof of Theorem 3 has hereby been completed. □

THEOREM 4. *With the hypotheses of Theorem* 3, *let* $T_i \in B[X, X]$, $i = 1, 2, \ldots, N$, *be strongly Abel ergodic and satisfy the conditions* $\sup_n \|C_n^{(\alpha_i)}[T_i]\| < \infty$ *and* $\lim_{n\to\infty} \|T_i^n x/n^{\omega_i}\| = 0$, $i = 1, 2, \ldots, N$, *for all* $x \in X$. *Then there exist projections* $E_i \in B[X, X]$, $i = 1, 2, \ldots, N$, *such that for all* $x \in X$,

$$\text{(s)} \lim_{n\to\infty} \Omega_n^{(N)}\,[T_1, \ldots, T_N]x/n^{\alpha_1+\cdots+\alpha_N+N-1} = E_N \cdots E_2 E_1 x/\Gamma(\alpha_1+\cdots+\alpha_N+N).$$

The proof follows exactly the same line as the proof of Theorem 3, and therefore it is omitted.

COROLLARY 3. *Let* $S, T \in B[X, X]$ *satisfy the conditions* $\|S^n\| = o(n)$ *and* $\|T^n\| = o(n)$. *Let* $\lambda = 1$ *be a pole of* $R(\lambda; S)$ *and* $R(\lambda; T)$ *of order one. Then with* $E(1; T)$ *and* $E(1; S)$ *the projections corresponding to the pole* $\lambda = 1$ *it follows that*

(i) (uo) $\lim_{n\to\infty} \sum_{p+q=n} \{(p+1) \sum_{k=0}^{q} T^k\}/n^3 = E(1; T)/\Gamma(4)$ *and*
(ii) (uo) $\lim_{n\to\infty} \sum_{p+q=n} \{\sum_{i=0}^{p} S^i \sum_{j=0}^{q} T^j\}/n^3 = E(1; S)E(1; T)/\Gamma(4)$.

This corollary holds also in real Banach spaces, when the assumption that 1 is a simple pole is replaced by uniform ergodicity, with $E(1; S)$ and $E(1; T)$ the $(C, 1)$ ergodic projections. In the complex case, these are equivalent to the statement of Corollary 3.

COROLLARY 4. *Let $S, T \in B[X, X]$ be quasi-compact and $0 < \alpha \leq 1$, $0 < \beta \leq 1$. Suppose that $S^n/n^\alpha \to \theta$, $T^n/n^\beta \to \theta$ in the weak operator topology when $n \to \infty$. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real or complex numbers with $\lim_{n\to\infty} a_n/n^\alpha = a$ ($|a| < \infty$). Then with the same projections $E(1; S)$ and $E(1; T)$ as in Corollary 3 it follows that*

(i) (uo) $\lim_{n\to\infty}[n^{\alpha+\beta+1}]^{-1} \sum_{p+q=n} \left\{a_p \sum_{k=0}^{q} A_{q-k}^{(\beta-1)} T^k\right\}$
$= a[\Gamma(\alpha+\beta+2)]^{-1}\, \Gamma(\alpha+1)E(1; T)$,

(ii) (uo) $\lim_{n\to\infty}[n^{\alpha+\beta+1}]^{-1} \sum_{p+q=n} \left\{\sum_{i=0}^{p} A_{p-i}^{(\alpha-1)} S^i \sum_{j=0}^{q} A_{q-j}^{(\beta-1)} T^j\right\}$
$= [\Gamma(\alpha+\beta+2)]^{-1}\, E(1; S)E(1; T)$.

It is known that there exists a non power-bounded operator on X which is strongly (C, α) ergodic for some $\alpha > 0$. Following Hille [7], to illustrate this, we take X to be the space $C_0[0, 1]$ of functions $f(x)$ continuous for $0 \leq x \leq 1$ which vanish at 0, with $\|f\| = \max|f(x)|$. Let $\beta > 0$ be fixed and define

$$Q_\beta f = (I - J_\beta)f, \quad (J_\beta f)(x) = \int_0^x [\Gamma(\beta)]^{-1}(x-u)^{\beta-1} f(u)du, 0 \leq x \leq 1$$

for $f \in X$. Obviously Q_β and J_β are bounded linear operators on X. Also, it is seen that $\|J_\beta\| \leq [\Gamma(\beta+1)]^{-1}$, $J_{\beta_1}J_{\beta_2} = J_{\beta_1+\beta_2}$ ($\beta_1, \beta_2 > 0$), and $Q_\beta J_\beta = J_\beta Q_\beta$. Moreover, the iterate Q_β^n for each $n \geq 1$ has the form

$$(Q_\beta^n f)(x) = f(x) - \int_0^x P_n(x-u, \beta) f(u)du$$

where

$$P_n(x-u, \beta) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} [\Gamma(k\beta)]^{-1}(x-u)^{k\beta-1}.$$

Then Hille's theorem [7, Theorem 11] shows that (i) $\|Q_1^n\| = O(n^{1/4})$, $\lim_{n\to\infty}\|Q_1^n\| = \infty$, and (ii) Q_1 is strongly (C, α) ergodic for $\alpha > 1/2$. It is worth while to note that another example of a strongly $(C, 1)$ ergodic operator which is not power-bounded is given by Derriennic-Lin [2] and Emilion [6] for a positive operator on a reflexive Banach lattice. Now, using the operators Q_β and J_β, we define $T_\beta = \Gamma(\beta+1)Q_\beta J_\beta$ for $\beta \geq 3/2$. Clearly

$$T_\beta^n = \{\Gamma(\beta+1\}^n Q_\beta^n J_\beta^n \quad \text{for all } n \geq 1.$$

So, making use of Hille's estimate

$$\int_0^1 |P_n(w, \beta)|dw \leq C(\beta) \begin{cases} (n-1)^{1/2}\log(n-1), & \beta = \frac{3}{2} \\ (n-1)^{\beta-1}, & \beta > \frac{3}{2} \end{cases}$$

for sufficiently large n, we have $\|T_\beta^n\| = o(n^{1/2+\gamma})$ for $\beta = 3/2$, $\gamma > 0$ and $\|T_\beta^n\| = O(n^{\beta-1})$ for $\beta > 3/2$. Furthermore, T_β is compact since the Volterra integral operator is known to be compact. Hence from Theorem 3.1 of [13] it follows that if $3/2 \le \beta < 2$ then the operator T_β is uniformly (C, α) ergodic for $\alpha > \beta - 1$. The question is whether T_β is a non power-bounded operator for some $3/2 \le \beta < 2$, but it is still open for the time being.

3. Hausdorff and Abel summability

When $T \in B[X, X]$ is given, we denote by $\Phi(T)$ the class of all functions of complex variables which are analytic in some open set containing the spectrum $\sigma(T)$. The open set need not be connected and may depend on $f \in \Phi(T)$. If $f \in \Phi(T)$ is analytic in an open set D containing $\sigma(T)$ and the boundary ∂D of D consists of a finite number of rectifiable Jordan curves, oriented in the usual sense, then the operator $f(T)$ is defined by

$$f(T) = (2\pi i)^{-1} \int_{\partial D} f(\lambda) R(\lambda; T) d\lambda$$

since $R(\lambda; T)$ is analytic in the resolvent $\rho(T)$ of T. The operator $f(T)$ so defined depends only on the function f but not on the domain D. Recall that for $|\lambda| > \gamma(T)$ (the spectral radius of T) the series $\sum_{n=0}^{\infty} T_n/\lambda^{n+1}$ converges in the uniform operator topology. Then

$$\begin{aligned} f(T) &= \frac{1}{2\pi i} \int_{\partial D} f(\lambda) \left(\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}} \right) d\lambda \\ &= \sum_{n=0}^{\infty} T^n \left(\frac{1}{2\pi i} \int_{\partial D} \frac{f(\lambda)}{n+1} d\lambda \right) \\ &= \sum_{n=0}^{\infty} c_n T^n, \end{aligned}$$

say, and in particular it follows that $I - f(T) = (I - T) \sum_{n=0}^{\infty} (1 - \sum_{p=0}^{n} c_p) T^n$.

LEMMA 2. Let $H_n = H_n(\cdot)$, $n = 0, 1, 2, \ldots$, be the functions of complex variables defined by

$$H_n(\lambda) = \int_0^1 \{\lambda t + (1 - t)\}^n dt.$$

Then each function H_n belongs to the class $\Phi(T)$ and

$$H_n[T] = \int_0^1 \{(1 - t) I + t T\}^n dt.$$

Proof. Clearly $H_n \in \Phi(T)$ for all n, because each H_n is a polynomial function. Let C be the circle $|\lambda| = \rho$ with $\gamma(T) < \rho \leq \gamma(T) + \epsilon$ for an $\epsilon > 0$ sufficiently small. Then using Cauchy's integral formula we have

$$\begin{aligned} H_n[T] &= \frac{1}{2\pi i}\int_C \left(\int_0^1 \{\lambda t + (1-t)\}^n\, dt\right) \sum_{p=0}^{\infty} \frac{T^p}{\lambda^{p+1}}\, d\lambda \\ &= \int_0^1 \left[\sum_{k=0}^{n}\binom{n}{k} t^k (1-t)^{n-k} \left(\frac{1}{2\pi i}\sum_{p=0}^{\infty} T^p \int_C \frac{\lambda^k}{\lambda^{k+1}}\, d\lambda\right)\right] dt \\ &= \int_0^1 \sum_{k=0}^{n}\binom{n}{k} t^k\, (1-t)^{n-k}\, T^k\, dt = \int_0^1 \{(1-t)I + \lambda T\}^n\, dt \end{aligned}$$

as desired.

By the way we note that the functions $C_n^{(\alpha)} = C_n^{(\alpha)}(\cdot)$ belong to the class $\Phi(T)$ and that

$$\begin{aligned} C_n^{(\alpha)}[T] &= \frac{1}{2\pi i}\int_C \frac{1}{A_n^{(\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)}\, \lambda^k \sum_{p=0}^{\infty} \frac{T^p}{\lambda^{p+1}}\, d\lambda \\ &= \frac{1}{A_n^{(\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} \left(\frac{1}{2\pi i}\sum_{p=0}^{\infty} T^p \int_C \frac{\lambda^k}{\lambda^{p+1}}\, d\lambda\right) \\ &= \frac{1}{A_n^{(\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)}\, T^k. \end{aligned}$$

The sequence of functions H_n appearing in Lemma 2 is known to constitute a strongly regular Hausdorff method H_g with the generating function $g(t) = t$, $0 \leq t \leq 1$. We call the operator averages $H_n[T]$ defined by Hausdorff method H_g in the sense of Lemma 2 the Hausdorff means of the sequence of powers T^k, or simply the Hausdorff means for T. So far, no way to relate the Cesàro (C, α) limit and the Hausdorff limit has been known in ergodic theory. In connection with this matter, the next aspect we wish to consider is the new question of relations between Hausdorff and Abel summability in the uniform and strong operator topology. It will be seen later that the Abel limit plays an important role in relating the (C, α) limit and the Hausdorff limit. □

THEOREM 5. *Let $T \in B[X, X]$ satisfy the condition $\|T^n/\sqrt{n}\| \to \theta$ when $n \to \infty$. Then there exists a projection $E \in B[X, X]$ such that $\|H_n[T] - E\| \to 0$ as $n \to \infty$ if and only if $\|(\lambda - 1)R(\lambda; T) - E\| \to 0$ as $\lambda \to 1 + 0$.*

Proof. First of all we prove that $(I - T)H_n[T] \to \theta$ in the uniform operator topology when $n \to \infty$. It follows from Lemma 2 that

$$\begin{aligned}(I-T)H_n[T] &= \frac{I - T^{n+1}}{n+1} \\ &+ \int_0^1 \sum_{k=1}^{n} \left\{ \binom{n}{k} t^k (1-t)^{n-k} - \binom{n}{k-1} t^{k-1}(1-t)^{n-k+1} \right\} T^k \, dt.\end{aligned}$$

Note that for a fixed t with $0 < t < 1$, $\max_{0 \le k \le n} \binom{n}{k} t^k (1-t)^{n-k}$ is attained for a $k = k_0 (= k_0(t))$ such that $|k_0/n - t| < 1/n$. For $0 < t < 1$, put

$$A(n) = \{t\colon\ 10/n \le t \le 1 - 10/n\}$$

and

$$B(n,t) = \{k\colon\ |k/n - t| < t/10,\ |k/n - t| < (1-t)/10\}.$$

We make use of Lorentz's estimate [11, page 543]

$$\binom{n}{k} t^k (1-t)^{n-k} < \frac{1}{\sqrt{2t(1-t)n}} \exp\left[-\frac{n}{4t(1-t)} \left(\frac{k}{n} - t \right)^2 \right], \qquad k \in B(n,t).$$

Then for sufficiently large n, $t \in A(n)$ implies $k_0 \in B(n,t)$, so that there exists an integer $N > 0$ such that for all $n > N$,

$$\max_{0 \le k \le n} \binom{n}{k} t^k (1-t)^{n-k} \le \binom{n}{k_0} t^{k_0} (1-t)^{n-k_0} < \frac{1}{\sqrt{2t(1-t)n}}, \qquad t \in A(n).$$

Therefore, for all $n > N$,

$$\begin{aligned}&\left\{ \int_0^{10/n} + \int_{1-10/n}^{1} + \int_{10/n}^{1-10/n} \right\} \max_{0 \le k \le n} \binom{n}{k} t^k (1-t)^{n-k} \, dt \\ &\le \frac{20}{n} + \frac{1}{\sqrt{2n}} \int_{10/n}^{1-10/n} \frac{1}{\sqrt{t(1-t)}} \, dt \\ &\le \frac{\pi + 20}{\sqrt{n}}\end{aligned}$$

and we thus have

$$\begin{aligned}\|(I-T)H_n[T]\| &\le \frac{1 + \|T^{n+1}\|}{n+1} + 2 \max_{0 \le k \le n} \|T^k\| \int_0^1 \max_{0 \le k \le n} \binom{n}{k} t^k (1-t)^{n-k} \, dt \\ &\le \frac{1}{\sqrt{n}} \{1 + \|T\| \|T^n\| + (2\pi + 40) \max_{0 \le k \le n} \|T^k\|\}\end{aligned}$$

which approaches zero as $n \to \infty$ since $\{\max_{0 \le k \le n} \|T^k\|\}/\sqrt{n} \to 0$ when $n \to \infty$. Now suppose $\|H_n[T] - E\| \to 0$ as $n \to \infty$. Since $H_n(1) = 1$ for all n, Dunford's uniform ergodic theorem applies and we see that $X = N(I - T) \oplus R(I - T)$, $EX = N(I - T)$, and $R(I - T)$ is closed. If S denotes the restriction of T to $R(I - T)$ then $\lim_{n\to\infty} \|H_n[S]\| = 0$ holds on $R(I - T)$, so that $I - H_n[S]$ is invertible on $R(I - T)$ for sufficiently large n. On the other hand, one can easily find appropriate analytic functions $G_n(\cdot) \in \Phi(T)$ such that $I - H_n[T] = (I - T)G_n[T]$ for each n. Hence $I - S$ is also invertible on $R(I - T)$. Therefore by the same calculation as in the first half of the proof of Theorem 1 we obtain $\|(\lambda - 1)R(\lambda; S)\| \to 0$ when $\lambda \to 1 + 0$, which implies that $\|(\lambda - 1)R(\lambda; T) - E\| \to 0$ when $\lambda \to 1 + 0$. The converse follows from the general result, and the theorem is proved. □

The above theorems can be generalized to the case of more general operator functions $f_n(T)$ for $f_n \in \Phi(T)$. We state only the results without the proofs.

THEOREM 6. *Let $T \in B[X, X]$ satisfy $\lim_{n\to\infty} \|T^n/n^\alpha\| = 0$ for some $0 < \alpha \le 1$. Suppose that the functions $f_n \in \Phi(T)$ satisfy $\lim_{n\to\infty} f_n(1) = 1$ and $\lim_{n\to\infty} \|(I - T)f_n(T)\| = 0$. Then the following statements are equivalent*:

(i) (uo) $\lim_{n\to\infty} f_n(T) = E$, $E^2 = E$, $EX = N(I - T)$.
(ii) (uo) $\lim_{\lambda\to 1+0} (\lambda - 1)R(\lambda; T) = E$, $E^2 = E$, $EX = N(I - T)$.
(iii) *$R(I - T)$ is closed.*

THEOREM 7. *Let $T \in B[X, X]$ satisfy $\lim_{n\to\infty} \|T^n x/n^\alpha\| = 0$ for all $x \in X$ and some $0 < \alpha \le 1$. Let $f_n \in \Phi(T)$ satisfy $\lim_{n\to\infty} f_n(1) = 1$ and $\lim_{n\to\infty} \|(I - T)f_n(T)x\| = 0$ for all $x \in X$. Suppose that $\sup_n \|f_n(T)x\| < \infty$ and $\sup_{1<\lambda\le 2} \|(\lambda - 1)R(\lambda; T)x\| < \infty$ for all $x \in X$. Then the following statements are equivalent*:

(i) (so) $\lim_{n\to\infty} f_n(T) = E$, $E^2 = E$, $EX = N(I - T)$.
(ii) (so) $\lim_{\lambda\to 1+0} (\lambda - 1)R(\lambda; T) = E$, $E^2 = E$, $EX = N(I - T)$.
(iii) $X = N(I - T) \oplus \overline{R(I - T)}$.

THEOREM 8. *Let $T \in B[X, X]$ be quasi-compact and satisfy the condition $T^n/\sqrt{n} \to \theta$ in the weak operator topology when $n \to \infty$. Then $H_n[T]$ converges (as $n \to \infty$) to a compact projection $E \in B[X, X]$ in the uniform operator topology.*

Proof. By virtue of Lemma VIII, 8.1 of [4] (cf. [13, Lemma 2.2]), the spectrum $\sigma(T)$ is a subset of the unit disk $\{z\colon |z| \le 1\}$ and any pole λ of $R(\mu; T)$ with $|\lambda| = 1$ has order one. Moreover, by Theorem VIII, 8.3 of [4] (cf. [13, Lemma 2.3]), there exist at most a finite number of points $\lambda_1, \dots, \lambda_p$ of unit modulus in $\sigma(T)$. Each point λ_k is a simple pole and the corresponding projection $E(\lambda_k; T)$ has a finite dimensional range. Let

$$\sigma = \sigma(T) \cap \{z\colon |z| < 1\} \quad \text{and} \quad \sigma' = \sigma(T) - \sigma.$$

Clearly $\sigma' = \{\lambda_1, \dots, \lambda_p\}$. Since σ is compact, one can choose a number δ with $0 < \delta < 1$ such that $\sup_{\lambda\in\sigma} |\lambda| < \delta$. It follows that

$$H_n[T]E(\sigma'; T) = \sum_{\lambda_i\in\sigma'} H_n(\lambda_i)E(\lambda_i; T)$$

and

$$H_n[T]E(\sigma; T) = (H_n[T])_\sigma E(\sigma; T) = H_n[T_\sigma]E(\sigma; T)$$

(see [4, Theorems VII, 3.20 and VII, 3.22]). Each function $H_n(\cdot)$ belongs to $\Phi(T_\sigma)$, and so

$$H_n[T_\sigma] = (2\pi i)^{-1} \int_{\partial U} H_n(\lambda)R(\lambda; T_\sigma)\, d\lambda,$$

where U is some neighborhood of $\sigma = \sigma(T_\sigma)$ with $U \subset \{z\colon |z| < \delta\}$ and its boundary ∂U is rectifiable. If n is sufficiently large then by using Lorentz's estimate, for λ with $|\lambda| \le \delta$ we have

$$\begin{aligned}
|H_n(\lambda)| &\le \left\{\int_0^{10/n} + \int_{1-10/n}^{1} + \int_{10/n}^{1-10/n}\right\} \{\delta t + (1-t)\}^n\, dt \\
&\le \sum_{k=0}^{n} \delta^k \left\{\frac{20}{n} + \int_{10/n}^{1-10/n} \max_{0\le k\le n} \binom{n}{k} t^k(1-t)^{n-k}\, dt\right\} \\
&\le \frac{1}{1-\delta}\left\{\frac{20}{n} + \frac{1}{\sqrt{2n}}\int_{10/n}^{1-10/n} \frac{1}{\sqrt{t(1-t)}}\, dt\right\} \\
&\le \frac{\pi + 20}{(1-\delta)\sqrt{n}}
\end{aligned}$$

and for λ with $|\lambda| = 1$, $\lambda \ne 1$,

$$\begin{aligned}
|(\lambda - 1)H_n(\lambda)| &\le 2\left\{\int_0^1 t^n dt + \int_0^1 \max_{0\le k\le n} \binom{n}{k} t^k(1-t)^{n-k}\, dt\right\} \\
&\le 2\left\{\frac{1}{n+1} + \frac{20}{n} + \frac{1}{\sqrt{2n}}\int_{10/n}^{1-10/n} \frac{1}{\sqrt{t(1-t)}}\, dt\right\} \\
&\le \frac{1}{\sqrt{n}}(2\pi + 42).
\end{aligned}$$

Therefore, since $R(\lambda; T_\sigma)$ is uniformly bounded on ∂U in the sense of the norm $\|\ \ \|$,

$$\begin{aligned}
\|H_n[T_\sigma]\| &\le \frac{1}{2\pi}\int_{\partial U} |H_n(\lambda)|\|R(\lambda; T_\sigma)\||d\lambda| \\
&\le \frac{\pi + 20}{2\pi(1-\delta)\sqrt{n}}\int_{\partial U} \|R(\lambda; T_\sigma)\||d\lambda| = O\left(\frac{1}{\sqrt{n}}\right),
\end{aligned}$$

$$\|H_n[T]E(\sigma'; T) - E(1; T)\| \le \frac{2\pi + 42}{\sqrt{n}} \sum_{\substack{\lambda_i \in \sigma' \\ \lambda_i \ne 1}} \frac{1}{|\lambda_i - 1|} \|E(\lambda_i; T)\| = O\left(\frac{1}{\sqrt{n}}\right).$$

All in all we get

$$\begin{aligned} \|H_n[T] - E(1; T)\| &\le \|H_n[T_\sigma]E(\sigma; T)\| + \|H_n[T]E(\sigma'; T) - E(1; T)\| \\ &= O\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

which means that $\lim_{n\to\infty} \|H_n[T] - E(1; T)\| = 0$. Finally, it remains to show that $E = E(1; T)$ is compact. Since T is assumed to be quasi-compact, there exist some integer $m > 0$ and some compact operator $P \in B[X, X]$ such that $\|T^m - P\| < 1$. Put $Q = T^m - P$. Then $(I - Q)^{-1}$ exists and

$$P(I - Q)^{-1} + (I - T^m)(I - Q)^{-1} = I.$$

Hence, $EP(I - Q)^{-1} = E$ which implies that E is compact. This completes the proof of the theorem. □

From what we have already observed we can derive the following equivalence of Cesàro, Hausdorff, and Abel summability as mentioned in the introduction (cf. [1], [8]).

COROLLARY 5. *Let $T \in B[X, X]$ satisfy the condition $\lim_{n\to\infty} \|T^n/n^\alpha\| = 0$ for some $0 < \alpha \le 1/2$. Then the following statements are equivalent*:

(i) (uo) $\lim_{n\to\infty} C_n^{(\alpha)}[T] = E$, $E^2 = E$, $EX = N(I - T)$.
(ii) (uo) $\lim_{n\to\infty} H_n[T] = E$, $E^2 = E$, $EX = N(I - T)$.
(iii) (uo) $\lim_{\lambda\to 1+0}(\lambda - 1)R(\lambda; T) = E$, $E^2 = E$, $EX = N(I - T)$.
(iv) $X = N(I - T) \oplus R(I - T)$, *$R(I - T)$ is closed.*
(v) *$R((I - T)^2)$ is closed.*
(vi) *$R(I - T)$ is closed.*

COROLLARY 6. *Let $T \in B[X, X]$ satisfy the condition $\lim_{n\to\infty} \|T^n x/n^\alpha\| = 0$ for all $x \in X$ and for some $0 < \alpha \le 1/2$. Suppose that $\sup_n \|C_n^{(\alpha)}[T]x\| < \infty$ and $\sup_n \|H_n[T]x\| < \infty$ for all $x \in \overline{R(I - T)}$. Then the following statements are equivalent*:

(i) (so) $\lim_{n\to\infty} C_n^{(\alpha)}[T] = E$, $E^2 = E$, $EX = N(I - T)$.
(ii) (so) $\lim_{n\to\infty} H_n[T] = E$, $E^2 = E$, $EX = N(I - T)$.
(iii) (so) $\lim_{\lambda\to 1+0}(\lambda - 1)R(\lambda; T) = E$, $E^2 = E$, $EX = N(I - T)$.
(iv) $X = N(I - T) \oplus \overline{R(I - T)}$.

THEOREM 9. *Let $T_i \in B[X, X]$, $i = 1, 2, \ldots, N$, be uniformly Abel ergodic and satisfy the conditions* $\lim_{n\to\infty} \|T_i^n/\sqrt{n}\| = 0$, $i = 1, 2, \ldots, N$. *Write* $G_n^{(1)}[T_1] = H_n[T_1]$ *and*

$$G_n^{(m)}[T_1, \ldots, T_m] = \sum_{p+q=n} H_p[T_m] G_q^{(m-1)}[T_1, \ldots, T_{m-1}] \quad m = 2, 3, \ldots, N.$$

Then there exist projections $E_i \in B[X, X]$, $i = 1, 2, \ldots, N$, such that

$$\text{(uo)} \lim_{n\to\infty} \frac{G_n^{(n)}[T_1, \ldots, T_N]}{n^{N-1}} = \frac{E_N \cdots E_2 E_1}{\Gamma(N)}.$$

Proof. By virtue of Theorem 5, there exist projections $E_i \in B[X, X]$, $i = 1, 2, \ldots, N$, such that

$$\text{(uo)} \lim_{n\to\infty} H_n[T_i] = E_i, i = 1, 2, \ldots, N$$

which shows, of course, that the theorem holds for the case $N = 1$. Suppose that in the case $N \geq 2$, the theorem has been established for $N - 1$ operators $T_1, \ldots, T_{N-1}$. So, writing

$$W_n^{(N-1)}[T_1, \ldots, T_{N-1}] = G_n^{(N-1)}[T_1, \ldots, T_{N-1}]/A_n^{(N-2)}$$

we see by the induction hypothesis that

$$\text{(uo)} \lim_{n\to\infty} W_n^{(N-1)}[T_1, \ldots, T_{N-1}] = E_{N-1} \cdots E_2 E_1.$$

Now let $\epsilon > 0$ be arbitrarily small and choose a number $n_0 = n_0(\epsilon) > 0$ such that

$$\|H_n[T_N] - E_N\| < \epsilon,$$

$$\|W_n^{(N-1)}[T_1, \ldots, T_{N-1}] - E_{N-1} \cdots E_2 E_1\| < \epsilon,$$

and

$$|A_n^{(N-1)}/n^{(N-1)} - 1/\Gamma(N)| < \epsilon$$

for all $n > n_0$. Moreover, by the principle of uniform boundedness, $\|E_i\| \leq \sup_{n\geq 0} \|H_n[T_i]\| < K_0$, $i = 1, 2, \ldots, N$ for some constant $K_0 > 1$. Therefore we have

$$\begin{aligned}
&\left\| \frac{G_n^{(N)}[T_1, \dots, T_N]}{n^{N-1}} - \frac{E_N \cdots E_2 E_1}{\Gamma(N)} \right\| \\
&\quad \le \frac{A_n^{(N-1)} K_0}{n^{N-1}} \\
&\quad \times \frac{\sum_{q=0}^{n_0} A_{n-q}^{(0)} A_q^{(N-2)} \| W_q^{(N-1)}[T_1, \dots, T_{N-1}] - E_{N-1} \cdots E_2 E_1 \|}{A_n^{(N-1)}} \\
&\quad + \frac{A_n^{(N-1)} K_0^{N-1}}{n^{N-1}} \cdot \frac{\sum_{q=0}^{n_0} A_p^{(0)} A_{n-p}^{(N-2)} \| H_p[T_N] - E_N \|}{A_n^{(N-1)}} \\
&\quad + \left(\frac{A_n^{(N-1)} K_0}{n^{N-1}} + \frac{A_n^{(N-1)} K_0^{N-1}}{n^{N-1}} + K_0^N \right) \cdot \epsilon,
\end{aligned}$$

so that

$$\varlimsup_{n \to \infty} \| [n^{N-1}]^{-1} G_n^{(N)}[T_1, \dots, T_N] - [\Gamma(N)]^{-1} E_N \cdots E_2 E_1 \| < 3 K_0^N \epsilon,$$

which proves the theorem. $\square$

Remark 2. Let $T_\beta = \Gamma(\beta + 1) Q_\beta J_\beta$ with the operators J_β and Q_β defined in §2. First we consider the case $1 < \beta < 3/2$. Hille's estimate

$$\int_0^1 |P_n(w, \beta)|\, dw \le C(\beta)(n-1)^{\beta/2 - 1/4}$$

for sufficiently large n gives $\lim_{n\to\infty} \|T_\beta^n\|/\sqrt{n} = 0$. Recall that T_β is compact. Then, in view of Theorem 8, T_β turns out to be uniformly Hausdorff ergodic and it is also uniformly (C, α) ergodic if $\alpha \ge 1/2$. Next we consider the case $-1 < \beta < 3/2$. For $|\lambda - 1| > \|J_\beta\|$ we have

$$R(\lambda; Q_\beta) = [(\lambda - 1)I + J_\beta]^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda - 1)^{n+1}} J_\beta^n,$$

which converges in the uniform operator topology. It follows from Hille's theorem that $\lim_{n\to\infty} \|Q_\beta^n\|/n = 0$, but it is obviously false that the point 1 is at most a simple pole of $R(\lambda; Q_\beta)$. Hence by Mbekhta and Zemánek's theorem [10, Théorème 1] the operator Q_β fails to be uniformly $(C, 1)$ ergodic. This also implies that Q_1 is not uniformly (C, α) ergodic when $1/2 < \alpha < 1$. This fact seems to have been unnoticed by Hille. In [12], Wacker proved that if the point 1 is a pole of order less than or equal to an integer $p \geq 1$ of $R(\lambda; T)(T \in B[X, X])$ and $\lim_{n\to\infty} \|T^n\|/n^p = 0$, then $(1/n^p)\sum_{k=0}^{n-1} T^k$ converges in $B[X, X]$. The converse implication does not hold in general. For example, we have $\lim_{n\to\infty} \|Q_\beta^n\|/n^p = 0$ and (uo) $\lim_{n\to\infty}(1/n^p)\sum_{k=0}^{n-1} Q_\beta^k = \theta$ when $p \geq 2$. But the point 1 fails to be a pole of $R(\lambda; Q_\beta)$ of order less than or equal to p. Incidentally, if $p \geq 2$ then Q_β is easily shown to satisfy Burlando's condition $E(k, p)$ for some positive integer k. Hence by Burlando's theorem [1, Theorem 3.4] we see that $\delta(I - Q_\beta) = \infty$ where $\delta(T) = \inf\{n \in N\colon R(T^n) = R(T^{n+1})\}$.

Acknowedgments I would like to express my thanks to the referee for many helpful comments which improved the presentation of this paper.

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