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DENSE SUBSETS OF BANACH $*$-ALGEBRAS

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ABSTRACT. Some subsets of a Banach *-algebra A are shown to be dense. In the special case of the algebra of $L(H)$ of all bounded linear operators on a Hilbert space H, the set of all T in $L(H)$ for which T^n is quasi-normal for no positive integers n is dense in $L(H)$.

1. Introduction

We study dense subsets of Banach $*$-algebras in order to obtain results which are new and relevant even in the case of the well-studied $L(H)$, the algebra of all bounded linear operators on a Hilbert space H. As in [5, p. 69], $T \in L(H)$ is called quasi-normal if T permutes with T^*T. See also [4, Chapt. II]. This notion was first introduced and studied (under a different name) by A. Brown [2].

Now let A be a Banach $*$-algebra. It is natural to say that $x \in A$ is *quasi-normal* if x permutes with x^*x. Our results, when applied to $L(H)$, show that the set $\mathfrak{S}$ of all $T \in L(H)$ for which T^n is quasi-normal for no positive integer n is dense. Let W be any $*$-subalgebra of $L(H)$, closed or not, which is not commutative and contains the identity operator E. Then the set of scalar multiples of E lies in the closure of $\mathfrak{S} \cap W$.

It is not difficult to exhibit $T \in \mathfrak{S}$. In the case of the algebra of all two-by-two matrices any matrix with a zero row (column) where the entries of the other row (column) are all non-zero is in $\mathfrak{S}$. More involved examples involving shifts can be readily devised.

For Banach $*$-algebras we provide a more general pattern in which the above result lies. We restrict our discussion to the case where A is not commutative and has no nilpotent ideal $\neq (0)$. Say $a \in A$ is *anti-central* if the set of $x \in A$ for which $[a^m, x^n] \neq 0$ and $[a^m, x^{n*}] \neq 0$ for all positive integers m and n is dense. (Here $[x, y] = xy - yx$ as usual.) The set $\mathfrak{Q}$ of anti-central elements of A is dense. Moreover, $\mathfrak{S} \subset \mathfrak{Q}$ as well as some dense subsets of $\mathfrak{Q}$ such as the set of $x \in A$ where $[x, (x^*x)^n] \neq 0$ for all positive n and the set of $x \in A$ where $[x^m, x^{*n}] \neq 0$ for all positive m and n.

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2. On quasi-normality

Throughout, A will be a complex Banach $*$-algebra with involution $x \to x^*$. We denote the center of A by Z. We set $\rho(x) = \lim \|x^n\|^{1/n}$. In [3, p. 420] the involution in A is said to be regular if $\rho(h) = 0$ and h self-adjoint imply that $h = 0$. It is readily verified that A is semi-simple if A has a regular involution. Also A has such an involution if A has a faithful $*$-representation as bounded linear operators on a Hilbert space.

Here and below we use the following fact. Let $p(t) = \sum_{k=0}^{n} a_k t^k$ be a polynomial in the real variable t with coefficients in A. Let M be a closed linear subspace of A. If $p(t) \in M$ for an infinite subset of the reals then each $a_k \in M$.

THEOREM 2.1. *Suppose that A has a regular involution. Then either A is commutative or the set of $x \in A$ for which $[x^n, x^{n*}x^n] \in Z$ for no positive integer n is dense in A.*

Proof. Suppose that the set of $x \in A$ in question is not dense. Then there is a non-void open set G where, to each $x \in G$, there corresponds a positive integer $n = n(x)$ with $[x^n, x^{n*}x^n] \in Z$.

For each positive integer m let

$$W_m = \{x \in A : [x^m, x^{m*}x^m]\} \notin Z .$$

As A is semi-simple the involution is continuous [1, p. 191]. Thus each W_m is open. If every W_m were dense then, by the Baire category theorem, the intersection of all the sets W_m would also be dense, contrary to the existence of G. Hence there is a positive integer n with W_n not dense. Let Ω be a non-void open set in its complement.

Pick $a \in \Omega$. For any $y \in A$ we have $a + ty \in \Omega$ for infinitely many real values of t. For these values of t,

$$[(a+ty)^n, \quad (a^* + ty^*)^n(a+ty)^n] \in Z .$$

The coefficient of the highest power of t in this polynomial is $[y^n, y^{*n}y^n]$. Thus $[y^n, y^{*n}y^n] \in Z$ for all $y \in A$.

Now let $y = h + itk$ where h and k are self-adjoint and t is real. Set $B = \sum_{j=0}^{n-1} h^j k h^{n-1-j}$. Then $y^n = h^n + iBt + \cdots$ and $y^{*n} = h^n - iBt + \cdots$, where we have omitted the terms involving higher powers of t. A direct calculation shows that $[h, B] = [h^n, k]$.

Let $w = (y^n + y^{n*})/2$, then $[w, y^{*n}y^n] \in Z$ for all y. Now $w = h^n +$ terms involving t to powers two and higher. Then we have

$$[h^n + \cdots, \quad (h^n + itB + \cdots)\,(h^n - itB + \cdots)] \in Z$$

for all h, k self-adjoint. (Here again we omitted terms in powers of t greater than one.) This gives

$$[h^n + \cdots, h^{2n} + it[B, h^n] + \cdots] \in Z$$

for all h, k self-adjoint. The coefficient of t in the polynomial here is $[h^n, i[B, h^n]]$. Therefore $[h^n, [h^n, B]] \in Z$ and consequently

$$[h, [h^n, [h^n, B]]] = 0$$

for all h, k self-adjoint. Recall that $[a^p, [a^q, b]] = [a^q, [a^p, b]]$ for all a, b. Therefore

$$[h^n, [h^n, [h, B]]] = 0$$

and so

$$[h^n, [h^n, [h^n, k]]] = 0$$

for all h, k self-adjoint.

We employ the Kleinecke-Shirokov theorem [1, p. 91] which asserts that if $[a, [a, b]] = 0$ then $\rho([a, b]) = 0$. This gives $\rho([h^n, [h^n, k]]) = 0$. Now $[h^n, k]$ is skew and $[h^n, [h^n, k]]$ is self-adjoint. By hypotheses we have $[h^n, [h^n, k]] = 0$. Again using the Kleinecke-Shirokov theorem we have $\rho([h^n, k]) = 0$ so our hypothesis on $\rho(x)$ shows that $[h^n, k] = 0$ for all h, k self-adjoint. Consequently $[h^n, x] = 0$ for all h self-adjoint and all $x \in A$. Thus $h^n \in Z$ for all h self-adjoint. We then use [9, Lemma 3.1] to see that $x^n \in Z$ for all $x \in A$. By standard ring theory [7, Theorem 3.22] we see that A is commutative.

Theorem 2.1 is applicable to all group algebras of locally compact groups as well as to C^*-algebras.

COROLLARY 2.2. *Suppose that A has a regular involution. Then the set of* $x \in A$ *for which* $[x^n, x^{n*}x^n] \in Z$ *for no positive integer n is dense if and only if the set of* $x \in A$ *for which* $[x^n, x^{n*}x^n] = 0$ *for no n is dense.*

Proof. The proof of Theorem 2.1 carries through if everywhere we replace Z by (0).

In the following theorem we drop the requirement of completeness. Let B be a normed $*$-algebra with an identity e and let $\mathfrak{S}$ be the set of $x \in B$ such that x^n is quasi-normal for no positive integer n.

THEOREM 2.3. *If B is not commutative then the set of scalar multiples of e lies in the closure of* $\mathfrak{S}$.

Proof. Since $\lambda x \in \mathfrak{S}$ whenever $x \in \mathfrak{S}$ for any scalar $\lambda \neq 0$ it is enough to show that e is in the closure of $\mathfrak{S}$ whenever B is not commutative.

Suppose otherwise; then there is a neighborhood $\mathfrak{N}$ of e disjoint with $\mathfrak{S}$. Let $x \in B$. There is an interval $[0, c]$, $c > 0$ so that, for each t, $0 \leq t \leq c$, $e + tx \in \mathfrak{N}$. To each such t there corresponds a positive integer $n(t)$ where

$$[(e + tx)^{n(t)},\ (e + tx^*)^{n(t)}\ (e + tx)^{n(t)}] = 0.$$

For each positive integer m let W_m be the set of $t \in [0, c]$ where $n(t) = m$. At least one W_m, say W_r, must be infinite. Hence

$$[(e + tx)^r,\ (e + tx^*)^r\ (e + tx)^r] = 0$$

for infinitely many values of t. We omit powers of t at least two in the expansions of $(e + tx)^r$ and $(e + tx^*)^r$ to have

$$[(e + rtx + \cdots,\ (e + rtx^* + \cdots)(e + rtx + \cdots)] = 0$$

so that

$$[rtx + \cdots,\ rt(x + x^*) + \cdots] = 0.$$

Therefore $[x, x + x^*] = 0 = [x, x^*]$ for all $x \in B$. Let $x = u + iv$ where u and v are self-adjoint. We see that $[u, v] = 0$ for these u, v and so B is commutative.

3. Anti-central elements

Henceforth we assume that A has a continuous involution $x \to x^*$. We use M to represent a closed linear subspace of A where $M = M^*$. Our final conclusions involve $M = (0)$ and $M = Z$. We adopt the following notation of Herstein [7, p. 5]. We set $T(M) = \{x \in A : [x, A] \subset M\}$. Of course $T(M) = Z$ if $M = (0)$. $T(Z)$ is more interesting algebraically.

Consider A as a Jordan algebra A^J under the Jordan multiplication $a \cdot b = ab + ba$. By the standard definition of the center of a non-associative algebra [8, p. 18], inasmuch as $a \cdot b = b \cdot a$, the center Z^J of A^J is the set of all $z \in A^J$ where

$$(z, x, y) = (x, z, y) = (x, y, z) = 0$$

for all $x, y \in A^J$. Here (a, b, c) is the associator of a, b and c:

$$(a, b, c) = (a \cdot b) \cdot c - a \cdot (b \cdot c).$$

A straight-forward calculation shows that

$$(a, b, c) = [b, [a \cdot c]].$$

Then Z^J is the set of all $z \in A$ such that, for all x and y,

$$[x, [z, y]] = [z, [x, y]] = [y, [x, z]] = 0.$$

For $z \in Z^J$, $[z, x] \in Z$ for all x or $z \in T(Z)$.

Hence $Z^J \subset T(Z)$. Conversely suppose $z \in T(Z)$ so that $[[z, x], y] = 0$ for all $x, y \in A$. The Jacobi identity gives $[[x, y], z] + [[y, z]x] + [[z, x], y] = 0$ for all x, y, z so that $z \in Z^J$. Therefore $T(Z) = Z^J$.

THEOREM 3.1. *If A has no non-zero nilpotent ideals then $Z = Z^J$.*

Proof. Let $a \in Z^J$. As noted above, $[a, x] \in Z$ for all $x \in A$. Hence a permutes with $[a, x]$ for all $x \in A$. By a result of Herstein [7, p. 5] we see that $a \in Z$.

We say that $a \in A$ is *anti-central modulo M* if the set of $x \in A$ for which $[a^m, x^n] \notin M$ and $[a^m, x^{n*}] \notin M$ for all positive integers m and n is dense in A. We use $X(M)$ to denote the set of anti-central elements modulo M.

THEOREM 3.2. *A is the union of two disjoint sets, $X(M)$ and the set of $x \in A$ for which some power of x lies in $T(M)$.*

Proof. Suppose that $a \notin X(M)$. We use the general strategy as in the proof of Theorem 2.1 by applying the Baire Category Theorem to the open sets

$$W_{m,n,r,s} = \{x \in A : [a^m, x^n] \notin M \text{ and } [a^r, x^{s*}] \notin M\}.$$

so that at least one of them, say $W_{m,n,r,s}$, is not dense. Then, for each $y \in A$, either (1) $[a^m, y^n] \in M$ or (2) $[a^r, y^{s*}] \in M$. Hence A is the union of two closed sets where, respectively, (1) and (2) hold. At least one of these must contain a non-void open set. From this we see that either $[a^m, x^n] \in M$ for all $x \in A$ or $[a^r, x^s] \in M \in A$ for all $x \in A$. By [9, Lemma 2.1] there is an integer p so that $a^p \in T(M)$.

Let $S(M)$ denote the set of $x \in A$ for which x^n is quasi-normal modulo M for no positive integer n.

3.3 COROLLARY. $S(M) \subset X(M)$.

Proof. Let $a \in S(M)$ so that $[a^n, a^{n*}a^n] \in M$ for no positive integer n. Then there is no integer p so that $[a^p, x] \in M$ for all $x \in A$. Theorem 3.2 then shows that $a \in X(M)$.

3.4 COROLLARY. *If A has no non-zero nilpotent ideals then either A is commutative or $S(Z)$ is dense.*

Proof. Suppose $S(Z)$ is not dense. Then, by Theorem 3.2, there is a non-void open subset G of A where, to each $x \in G$ there corresponds a positive integer $n = n(x)$ so that $[x^n, y] \in Z$ for all $y \in A$. By [9, Lemma 2.2] there is a fixed integer n so that $x^r \in T(Z)$ for all $x \in A$. But $T(Z) = Z$ by Theorem 3.1. Hence A is commutative [7, Theorem 3.2.2].

4. On some dense subsets

4.1 THEOREM. *Either there exists a positive integer r so that $x^r \in T(M)$ for all $x \in A$ or the set of $x \in A$ such that $[x, (x^*x)^n] \in M$ for no positive integer n is dense in A.*

Proof. Suppose the set in question is not dense. We apply the Baire Category Theorem to the sets $H_n = \{x \in A : [x, (x^*x)^n] \notin M\}$ to see, reasoning as above, that for some positive integer r, $[y, (y^*y)^r] \in M$ for all $y \in A$.

In dealing with $[y, (y^*y)^r] \in M$ we set $y = u + itv$ where u and v are self-adjoint and t is real. Then $y^*y = u^2 + t^2v^2 + i[u, v]t$. For convenience we set $u^2 + t^2v^2 = w$ and $z = i[u, v]$ so that $(y^*y)^r = (w + tz)^r$. Let Q_k be the sum of the terms in the expansion of $(w + tz)^r$ for which the sum of the exponents of the z^j factors is k. Then $(y^*y)^r = \sum_{k=0}^{r} Q_k t^k$.

As $[u + itv, (y^*y)^r] \in M$ also $[u - itv, (y^*y)^r] \in M$ for all u, v self-adjoint and t real; thus $[v, (y^*y)^r] \in M$ for all v, y in question. We have

$$[v, Q_0 + tQ_1 + \cdots + t^r Q_r] \in M$$

for all v self-adjoint and t real. Notice that $Q_0 = w^r = (u^2 + t^2v)^r$. Letting $t \to 0$ we see that $[v, u^{2r}] \in M$ for all u, v self-adjoint. Thus $[h^{2r}, y] \in M$ for all h self-adjoint and $y \in A$; it follows from [9, Lemma 3.1] that $[x^{2r}, y] \in M$ for all $x, y \in A$.

Say $a \in A$ is *anti-normal modulo M* if for all positive integers m and n we have $[a^m, a^{*n}] \notin M$.

4.2 LEMMA. *The set W of $x \in A$, x anti-normal modulo M, is either dense or empty.*

Proof. Suppose that the set in question is not dense. By applying the Baire Category Theorem to the sets $Q_{r,s} = \{x \in A : [x^r, x^{*s}] \notin M\}$ we can, by reasoning as above, deduce that there are positive integers m and n so that $[y^m, y^{*n}] \in M$ for all $y \in M$. This shows that if W is not dense then W is void.

Next we consider the case where $M = (0)$.

4.3 THEOREM. *Suppose A has no non-zero nilpotent ideals. Then either A is commutative or the set W of its anti-normal elements is dense.*

Proof. Suppose W is not dense. As in the proof of Lemma 4.2 there are positive integers m and n so that $[y^m, y^{*n}] = 0$ for all $y \in A$. It follows that $[y^r, y^{*r}] = 0$ for $r = mn$ and all $y \in A$; that A is commutative follows from [9, Theorem 3.6].

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