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ON THE ALAOGLU–BIRKHOFF EQUIVALENCE OF POSETS

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ABSTRACT. We show that under the Proper Forcing Axiom, the Alaoglu-Birkhoff equivalence on separative posets of the first uncountable size roughly coincides with regular embeddability. We also investigate the behavior of the equivalence under the Continuum Hypothesis.

Introduction

Attempts to classify small partially ordered sets according to certain criteria are by now a standard part of set theory [T1], [T2]. In practice this usually means counting classes of certain equivalences of posets. The following definitions tell at least part of the story:

(1) $P \sim Q$ if there is a poset R such that both P, Q are (isomorphic to) its dense subsets.
(2) $P > Q$ if there is a function $f\colon P \to Q$ such that for every $q \in Q$ there is $p \in P$ such that every $p' \leq_P p$ has $f(p') \leq_Q q$. Such a function is called Moore-Smith convergent [MS] or a Tukey map [Tu]. $>$ is a preorder on posets and it naturally generates an equivalence via the definition $P \equiv Q$ if $P > Q$ and $Q > P$.
(3) $P \succ Q$ if there is a function $f\colon P \to Q$ such that preimage of every open dense subset of Q under f contains an open dense subset of P. Such a function is called Alaoglu–Birkhoff convergent [AB] and the preorder $\succ$ generates the Alaoglu–Birkhoff equivalence $\approx$ through $P \approx Q$ if $P \succ Q$ and $Q \succ P$.
(4) On separative posets P, Q define $Q \lessdot P$ if $RO(Q)$ can be completely embedded into $RO(P)$. This is a basic forcing-theoretic notion; let $P \bowtie Q$ if $Q \lessdot P$ and $P \lessdot Q$.

Classical results [D] say that each of these equivalences has at most countably many equivalence classes of countable posets. Since the Proper Forcing Axiom tends

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to generalize certain properties of ω to ω_1 it is interesting to ask whether there can be a sensible classification of equivalence classes of posets of size $\aleph_1$ under PFA. Indeed, for the directed partial orders the first three equivalences defined above coincide and it was proved in [T1] that under PFA there are only finitely many equivalence classes of directed posets of size $\aleph_1$. For the general partially ordered sets the situation is more complex, since in ZFC there are the maximum possible $2^{\aleph_1}$ many equivalence classes of posets of size $\aleph_1$ in both $\sim$ and $\bowtie$. However, in the context of PFA there are only countably many $\equiv$-equivalence classes of posets of size $\aleph_1$. The study of $\approx$ along these lines was suggested in [T2].

In this paper we show that there are $2^{\aleph_1}$ many $\approx$-classes of posets of size $\aleph_1$ under PFA. This is nevertheless proved in a way which provides a good understanding of the relations $\prec$, $\approx$:

THEOREM 1. *Suppose PFA holds and P, Q are arbitrary separative posets of size $\aleph_1$. Then $Q \prec P$ if and only if there is an open subposet $Q' \subset Q$ with $Q' \lessdot P$.*

Thus in the context of PFA the relations $\prec$, $\approx$ can be reduced to the logically simpler $\lessdot$, $\bowtie$. The nonclassification result for $\bowtie$ then easily gives $2^{\aleph_1}$ many $\approx$-nonequivalent posets of size $\aleph_1$ under PFA.

Our notation follows the set-theoretic standard as set forth in [J]. C_{ω_1} is the forcing for adding ω_1 many Cohen reals with finite support product, $RO(P)$ for a separative poset P is the completion of P and for a Boolean algebra B the set of nonzero elements of B is denoted by B^+.

1. Simple properties of the Alaoglu-Birkhoff preorder

In this section we make several basic observations about the nature of the relation $\prec$.

CLAIM 2. *Suppose $P \sim P'$, $Q \sim Q'$ and $P \succ Q$. Then $P' \succ Q'$.*

Proof. Without loss of generality the four posets can be assumed pairwise disjoint. Since P, P' and Q, Q' are codense, there are partial orders $\leq^*$ on $P \cup P'$ and $\leq^{**}$ on $Q \cup Q'$ so that

(1) $\leq^*$ restricted to P or P' is exactly equal to $\leq_P$ or $\leq_{P'}$ respectively, and the same on the Q side
(2) both P, P' form dense parts of the poset $\langle P \cup P', \leq^* \rangle$ and the same holds on the Q-side.

Now fix an Alaoglu-Birkhoff convergent function $f\colon P \to Q$. We must produce a convergent function from P' to Q'. For each $p' \in P'$ choose a condition $q' \in Q'$ such that there are $p \leq^* p'$ in P and $q \geq^{**} q'$ in Q with $f(p) = q$. Then the function $f'\colon P' \to Q'$, $f'\colon p' \mapsto q'$ is Alaoglu-Birkhoff convergent.

To see that, suppose $D' \subset Q'$ is an open dense set. The set $D \subset Q$ given by $D = \{q \in Q\colon \exists q' \in D'\ q \leq^{**} q'\}$ is open dense as well and by the convergence of the function f there must be an open dense set $E \subset P$ with $f''E \subset D$. Then the set $E' = \{p' \in P'\colon \exists p \in E\ p' \leq^{*} p\} \subset P'$ is open dense and its image under f' is included in D' as required. □

CLAIM 3. *Suppose $P \succ Q'$ and Q' is an open subset of a poset Q. Then $P \succ Q$.*

Proof. If $f\colon P \to Q'$ is Alaoglu-Birkhoff convergent then so is $f\colon P \to Q$. For whenever $D \subset Q$ is open dense then $D \cap Q' \subset Q'$ is open dense and so there is an open dense set $E \subset P$ such that $f''E \subset D \cap Q' \subset D$. □

CLAIM 4. *Suppose Q, P are separative posets and $Q \lessdot P$. Then $Q \prec P$.*

Proof. Fix a complete embedding $\pi\colon RO(Q) \to RO(P)$ and the corresponding projection $pr\colon RO(P) \to RO(Q)$ given by $pr(b) = 1 - \Sigma\{c \in RO(Q)\colon \pi(c) \wedge b = 0\}$. The following are well known [J] and easy to verify:

(1) pr preserves order.
(2) $pr(1_{RO(P)}) = 1_{RO(Q)}$ and pr maps nonzero elements of $RO(P)$ to nonzero elements of $RO(Q)$.
(3) Whenever $a \leq pr(b)$ in $RO(Q)$ then $\pi(a) \wedge b \neq 0$ in $RO(P)$.
(4) $pr(a \wedge b) \leq pr(a) \wedge pr(b)$ in $RO(Q)$, for all $a, b \in RO(P)$.
(5) $pr(\pi(a)) = a$ for all $a \in RO(Q)$.

Since $Q \sim RO(Q)^{+}$ and $P \sim RO(P)^{+}$ it is enough to prove that pr is an Alaoglu-Birkhoff convergent function from $RO(P)^{+}$ into $RO(Q)^{+}$. Fix an open dense subset $D \subset RO(Q)+$ and let $E = \{b \in RO(P)\colon pr(b) \in D\}$. We shall complete the proof by showing that $E \subset RO(P)^{+}$ is open dense.

And indeed, E is open since pr preserves order. For the density, note that if $b \in RO(P)^{+}$ then any element of $RO(P)^{+}$ of the form $b \wedge \pi(a)$ belongs to E where $a \in D$, $a \leq pr(b)$. This follows from the fact that $b \wedge \pi(a) \neq 0$ by (3) and $pr(b \wedge \pi(a)) \leq pr(\pi(a)) \leq a \in D$ by (4) and (5). □

Note that $Q \leq P$ is a $\Sigma_1(P, Q)$ fact, a statement about existence of Boolean algebras B_P, B_Q codense with P, Q respectively and a projection function $pr\colon B_P \to B_Q$ satisfying the propeties (1)–(6) above. Thus $Q \lessdot P$ is upwards absolute between models of set theory. This is in sharp contrast to $Q \prec P$ which is a $\Sigma_2(P, Q)$ statement and generally not upwards absolute.

We shall now associate with every poset P ideals $\mathfrak{J}_P(\kappa)$, κ a cardinal, such that $Q \prec P$ is equivalent to $\mathfrak{J}_Q(\kappa) \subset \mathfrak{J}_P(\kappa)$ for $\kappa = |Q|^{|P|}$.

Definition 5. Let P be a poset and X a set of subsets of $\bigcup X$. We say that X is a *character* of P if for every sequence D_i: $i \in \bigcup X$ of open dense subsets of P there is some $x \in X$ with $\bigcap_{i \in x} D_i \neq 0$.

If the poset P is separative, then saying that X is a character of P is to say that there is a condition in P forcing "every ordinal valued total function from $\bigcup X$ has a ground model subfunction with domain in X".

Definition 6. $\mathfrak{I}_P(\kappa)$ is the set of those $X \subset \text{Power}(\kappa)$ which are not characters of P.

It is not hard to see that $\mathfrak{I}_P(\kappa)$ is closed under subsets and unions. For the latter, note that if X, Y are not characters of P as witnessed by sequences $\langle D_\alpha: \alpha \in \kappa\rangle$, $\langle E_\alpha: \alpha \in \kappa\rangle$ of open dense subsets of P respectively, then $X \cup Y$ is not a character of P either—consider the sequence $\langle D_\alpha \cap E_\alpha: \alpha \in \kappa\rangle$.

LEMMA 7. *Let P, Q be posets and $\kappa = |Q|^{|P|}$. Then $Q \prec P$ if and only if $\mathfrak{I}_Q(\kappa) \subset \mathfrak{I}_P(\kappa)$.*

Proof. On one hand, if $P \succ Q$—as witnessed by a function f: $P \to Q$—then every character X of P is also a character of Q: If D_α: $\alpha \in \kappa$ are open dense subsets of Q, fix E_α: $\alpha \in \kappa$, open dense subsets of P such that $f''E_\alpha \subset D_\alpha$. Since X is a character of P, there is a set $x \in X$ such that $\bigcap_{\alpha \in x} E_\alpha$ is nonempty, containing some condition $p \in P$. But then $\bigcap_{\alpha \in x} D_\alpha$ is nonempty as well, containing $f(p)$. Thus X is a character of Q and $\mathfrak{I}_Q(\kappa) \subset \mathfrak{I}_P(\kappa)$.

On the other hand suppose $P \not\succ Q$. Let $A = \{\langle f, Z\rangle$: f: $P \to Q$ is a function and $Z \subset P$ is a somewhere dense set such that $f''Z \subset Q$ is nowhere dense$\}$ and define X by $x \in X$ if $x \subset A$ and $\bigcup\{f''Z$: $\langle f, Z\rangle \in x\} \subset Q$ is dense. Since $|A| \leq |Q|^{|P|}$ it is enough to show that X is a character of P and not a character of Q.

On the Q side, for each $\langle f, Z\rangle \in A$ choose an open dense set $D_{\langle f,Z\rangle} \subset Q$ disjoint from the nowhere dense set $f''Z$. Suppose that for some $x \in X$ the intersection $\bigcap\{D_{\langle f,Z\rangle}, \langle f, Z\rangle \in x\}$ is nonempty, containing some condition $q \in Q$; then, since $x \in X$, there must be some $\langle g, Y\rangle \in X$ and $p \in Y$ such that $g(p) \leq q$. So $g(p)$ together with q belongs to all the sets $D_{\langle f,Z\rangle}$, $\langle f, Z\rangle \in x$, in particular, to $D_{\langle g,Y\rangle}$, which is a contradiction to the choice of $D_{\langle g,Y\rangle}$. Thus the collection $\{D_{\langle f,Z\rangle}$: $\langle f, Z\rangle \in A\}$ shows that X is not a character of Q.

To prove that X is a character of P, suppose by way of contradiction it is not, as witnessed by a family $\{E_{\langle f,Z\rangle}$: $\langle f, Z\rangle \in A\}$. Then for each $p \in P$ there is a condition $q \in Q$ such that no element of the set $\bigcup\{f''Z$: $p \in E_{\langle f,Z\rangle}\} \subset Q$ is below or equal to q—otherwise the set $x = \{\langle f, Z\rangle$: $p \in E_{\langle f,Z\rangle}\}$ would be in X, the intersection $\bigcap\{E_{\langle f,Z\rangle}$: $\langle f, Z\rangle \in x\}$ is nonempty containing p and $\{E_{\langle f,Z\rangle}$: $\langle f, Z\rangle \in A\}$ would not be a counterexample to P having a character X. Now the function g: $P \to Q$, g: $p \mapsto q$ is not a witness to $P \succ Q$ and so there must be a somewhere dense set

$Y \subset P$ such that $g''Y \subset Q$ is nowhere dense. Look at the pair $\langle g, Y\rangle \in A$ and choose some condition $p \in Y \cap E_{\langle g, Y\rangle}$. By the definition of the function g, there should be no element of the set $\bigcup\{f''Z\colon\ p \in E_{\langle f, Z\rangle}\} \subset Q$ below or equal to $g(p)$; on the other hand, $g(p)$ belongs to that set. A contradiction. □

While the right-to-left direction of the previous lemma has a purely existential proof, in practice the comparison of characters of posets gives a strong hint about the possible candidates for Alaoglu-Birkhoff convergent functions. Interesting characters include $\{\omega\}$—in separative posets this character means somewhere $\aleph_0$-distributivity—$[\omega_1]^{\aleph_0}$—for example the random algebra has this character—and $\{S \subset \omega_1\colon\ S \text{ stationary}\}$—Sacks forcing has this character under the Proper Forcing Axiom.

2. Under $\diamondsuit$

If one assumes a strong construction principle like $\diamondsuit$, many posets of size $\aleph_1$ will be $\prec$-comparable. The following lemma says that, in particular, $Q \prec C_{\omega_1}$ for every poset Q of size $\aleph_1$ under $\diamondsuit$. Note that C_{ω_1} does not have the character $[\omega_1]^{\aleph_0}$ since the C_{ω_1}-generic function from ω_1 to 2 has no infinite subfunction from the ground model.

LEMMA 8. ($\diamondsuit$) *Suppose P, Q are posets of size $\aleph_1$ and P does not have character $[\omega_1]^{\aleph_0}$. Then $Q \prec P$.*

Proof. Fix an enumeration $\{q_\alpha\colon\ \alpha \in \omega_1\}$ of Q and a $\diamondsuit$ sequence $\{D_\alpha\colon\ \alpha \in \omega_1\}$ guessing subsets of Q. Let $S \subset \omega_1$ be the stationary set $\{\alpha \in \omega_1\colon\ D_\alpha \subset \{q_\beta\colon\ \beta \in \alpha\}$ is open dense$\}$. Fix an enumeration $\{p_\alpha\colon\ \alpha \in S\}$ of P and a sequence $E_\alpha\colon\ \alpha \in S$ of open dense subsets of P such that every infinite subcollection has an empty intersection.

Note that we may assume that $\{p_\beta\colon\ \beta \in \alpha \cap S\} \cap E_\alpha = 0$ for all $\alpha \in S$. For if this failed on a stationary set of $\alpha \in S$ then by a Fodor-style argument it would be possible to find even a stationary subcollection of $\{E_\alpha\colon\ \alpha \in S\}$ with a nonempty intersection. And if this failed on only a nonstationary set of $\alpha \in S$ it would be easy to remove these and rearrange the rest so that we get $\{p_\beta\colon\ \beta \in \alpha \cap S\} \cap E_\alpha = 0$ for all $\alpha \in S$ as required.

Now for each $p \in P$, say $p = p_\alpha$ for some $\alpha \in S$, there is $q \in Q$ such that for all $\beta \in \alpha + 1$ with $p \in E_\beta$ the condition q has an element of D_β above it. To see this, enumerate the finite set $\{\beta \in \alpha + 1\colon\ p \in E_\beta\}$ in an increasing order as $\{\beta_0, \dots, \beta_n\}$ and by induction on $i \in n + 1$ build conditions $q_i \in D_{\beta_i}$ so that $q_0 \geq q_1 \geq \cdots \geq q_n$ in Q. This is certainly possible as the sets $D_{\beta_i} \subset \{q_\beta\colon\ \beta \in \beta_i\}$ are dense, and $q = q_n$ is as desired.

The function $f\colon\ P \to Q$, $f\colon\ p \mapsto q$ is a witness to $P \succ Q$. For fix an open dense subset $D \subset Q$ and an ordinal $\alpha \in S$ such that $D \cap \{q_\beta\colon\ \beta \in \alpha\} = D_\alpha$. We claim that $f''E_\alpha \subset D$, and this will complete the proof of $P \succ Q$. And indeed, if $p = p_\gamma \in E_\alpha$ then $\alpha \leq \gamma$ and by the construction of f the condition $f(p_\gamma)$ has an element of $D_\alpha \subset D$ above it. Since D is open, $f(p_\gamma) \in D$ as desired. □

Already under the Continuum Hypothesis it is not hard to produce $2^{\aleph_1}$ many $\approx$-nonequivalent separative posets of size $\aleph_1$. Let $S \subset \omega_1$ and define $P_S = \{f\colon \mathrm{dom}(f)$ is a countable subset of ω_1 all of whose accumulation points belong to S and $\mathrm{rng}(f) = 2\}$ ordered by $f \geq g$ if $f \subset g$ and the set $\sup(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \setminus \mathrm{dom}(f)$ is finite. It is not hard to see that P_S adds reals and its generic extension is determined by the function from ω_1^V to 2 that is the union of all functions in the generic filter.

CLAIM 9. *Suppose $S, T \subset \omega_1$. Then P_S has character $X_T = \{A \subset \omega_1\colon o.t.A = \omega, \sup(A) \in T\}$ if and only if $S \cap T$ is a stationary subset of ω_1.*

Proof. First, suppose that $S \cap T$ is stationary and $D_\alpha\colon \alpha \in \omega_1$ is a sequence of open dense subsets of P_S. We must find a set $A \subset \omega_1$ with $o.t.A = \omega$, $\sup(A) \in T$ and $\bigcap_{\alpha \in A} D_\alpha \neq 0$. To that end, move into any generic extension $V[G]$ with the same reals and a closed unbounded set $C \subset S \cap T$, $C \in V[G]$ (see [B]). There, build a decreasing sequence $p_\alpha\colon \alpha \in \omega_1$ of conditions in P_S and a sequence $f_\alpha\colon \alpha \in \omega_1$ of finite functions from ω_1 into 2 so that

(1) $\sup(\mathrm{dom}(p_\alpha)) \in C$,
(2) $p_{\alpha+1} \restriction \sup(\mathrm{dom}(p_\alpha)) = p_\alpha$,
(3) for α limit $p_\alpha = \bigcup_{\beta \in \alpha} p_\beta$,
(4) $p_{\alpha+1} \cup f_\alpha \in D_\alpha$.

This is easily done; the only difficulty is at successor stages where we first find a condition $q \leq p_\alpha$ in D_α with $\sup(\mathrm{dom}(q)) \in C$ and then let $f_\alpha = q \restriction \sup(\mathrm{dom}(p_\alpha)) \cap \mathrm{dom}(q) \setminus \mathrm{dom}(p_\alpha)$ and $p_{\alpha+1} = q \setminus f_\alpha$.

By a Fodor-style argument it is now possible to find a stationary set $U \subset \omega_1$ and a finite function f such that every $\alpha \in U$ has $f = f_\alpha \restriction \alpha$. Fix a set $A \subset U$ of ordertype ω whose limit is in C and such that writing $A = \{\alpha_0, \alpha_1, \dots\}$ in increasing order we have $\mathrm{dom}(f_{\alpha_n}) \subset \alpha_{n+1}$. Let $\beta = \sup(A) \in C \subset S \cap T$ and $p = p_\beta \cup \bigcup_{n \in \omega} f_{\alpha_n}$.

On the other hand, assume $S \cap T$ nonstationary. We must find a sequence $D_\alpha\colon \alpha \in \omega_1$ such that for every set $A \subset \omega_1$ of ordertype ω and with supremum in T the intersection $\bigcap_{\alpha \in A} D_\alpha$ is empty. Fix a club $C \subset \omega_1$ disjoint from $S \cap T$, enumerate each infinite maximal interval I of ω_1 disjoint from C by $\alpha_n^I\colon n \in \omega$ and fix an inclusion-decreasing sequence $E_n\colon n \in \omega$ of open dense subsets of P_S whose intersection is empty. Then for $\alpha \in \omega_1$ define the sets $D_\alpha = \{p \in P_S\colon p$ decides the α-th bit of the P_S-generic function$\} \cap E_n$ if $\alpha = \alpha_n^I$ for some (unique) infinite maximal interval $I \subset \omega_1$ disjoint from C and $D_\alpha = \{p \in P_S\colon p$ decides the α-th bit of the P_S-generic function$\}$ otherwise.

Now suppose $A = \{\alpha_0 \in \alpha_1 \in \ldots\}$ is a set of countable ordinals with limit in T. There are two cases.

(1) If $\sup(A) \in C$ then $\sup(A) \notin S$ and no $p \in P_S$ decides the values of the P_S-generic function on all the α_n, $n \in \omega$, since such p would have to include A in its domain and would therefore have an accumulation point $\sup(A)$ outside of S, contrary to the definition of P_S. Consequently $\bigcap_{\alpha\in A} D_\alpha = 0$.
(2) If $\sup(A) \notin C$ then there must be an infinite maximal interval $I \subset \omega_1$ disjoint from C such that for almost all $n \in \omega$, $\alpha_n \in I$. Then $\bigcap_{\alpha\in A} D_\alpha \subset \bigcap_n E_n = 0$.

In any case, $\bigcap_{\alpha\in A} D_\alpha = 0$ and the sets $D_\alpha, \alpha \in \omega_1$, witness the fact that X_T is not a character of P_S. □

Now there is a collection $\{S_i\colon i \in 2^{\aleph_1}\}$ of $2^{\aleph_1}$ many stationary subsets of ω_1 which are pairwise not equal modulo the nonstationary ideal. Then $P_{S_i}, i \in 2^{\aleph_1}$, are mutually $\approx$-nonequivalent posets: if $i \neq j \in I$ then there will be a stationary subset T of one of S_i, S_j which is disjoint from the other one, and consequently one of the posets P_{S_i}, P_{S_j} has the character $X_T = \{A \subset \omega_1\colon o.t.A = \omega, \sup(A) \in T\}$ and the other does not. This shows that $P_{S_i} \not\approx P_{S_j}$ via Lemma 7. And of course under the Continuum Hypothesis the posets P_S have size $\aleph_1$.

The separative σ-centered posets $X(A)$ defined in Section 4 of [T2] can also be proved to be non-$\approx$-equivalent. It is interesting to compare the above example with the $2^{\aleph_1}$ many non-$\bowtie$-equivalent posets of size $\aleph_1$ produced in ZFC by the first author in [T2].

Thus we proved that under $\diamondsuit$ the quasiorder $\prec$ on posets of size $\aleph_1$ has a top (namely, C_{ω_1}) and a complicated structure. The last claim of this section shows that Souslin trees constitute a bottom, if we look only at the posets with no countable locally dense subsets.

CLAIM 10. *Suppose P is a poset of size $\aleph_1$ with no somewhere dense countable subsets and T is a Suslin tree. Then $P \succ T$.*

Proof. Any injection $f\colon P \to T$ is Alaoglu-Birkhoff convergent. To see this, fix an open dense set $D \subset T$; as T is Souslin, $|T \setminus D| \leq \aleph_0$, as f is one-to-one, $f^{-1}(T \setminus D)$ is at most countable, and since there are no small somewhere dense sets in P, there must be an open dense set $E \subset P$ disjoint from $f^{-1}(T \setminus D)$. Then $f''E \subset D$ as desired in the definition of Alaoglu-Birkhoff convergence. □

3. Under the Proper Forcing Axiom

Our goal in this section is to prove Theorem 1 from the introduction. Fix posets P and Q and a function $f\colon P \to Q$. The question of interest is whether $f\colon P \to Q$

is Birkhoff-Alaoglu convergent absolutely, that is, in any model containing f, P and Q, or whether it is perhaps possible to disturb the convergence of the function by a reasonably regular forcing, that is, if one can force a somewhere dense subset $X \subset P$ such that $f''X \subset Q$ is nowhere dense.

Assume for now that the posets P, Q are separative and Q has a finite character, meaning for every $q \in Q$ there are only finitely many elements of Q above it. In such a situation we split into two cases depending on whether or not the following formula holds:

$$\begin{aligned} &\text{there is } p \in P \text{ and an open dense set } D \subset Q \\ &\quad \text{such that for every finite set } d \subset D \text{ and every } p' \leq_P p \\ &\qquad\qquad \text{there is } p'' \leq_P p' \\ &\qquad\qquad\quad \text{such that for every } q \in d \ f(p'') \not\leq_Q q \text{ holds.} \end{aligned} \tag{$*$}$$

LEMMA 11. *Suppose* $(*)$ *holds. Then there is a c.c.c. forcing* R, $R \Vdash$ *there is a somewhere dense set* $X \subset \check{P}$ *such that* $\check{f}''X \subset \check{Q}$ *is nowhere dense, so* $\check{f}$ *ceases to be convergent.*

Proof. Fix $p \in P$ and $D \subset Q$ as in $(*)$ and for notational simplicity suppose that p is the largest element of the poset P. Let $R = \{\langle c, d\rangle \colon c \subset P, d \subset D$ are finite sets and $\forall p \in c\ \forall q \in d\ f(p) \not\leq_Q q\}$ ordered by coordinatewise reverse inclusion. So R is a straightforward attempt to force a dense (below p) set $X \subset P$ and an open dense set $Y \subset Q$ such that $f''X \cap Y = 0$: if $G \subset R$ is a generic filter set $X = \bigcup\{c\colon \langle c, 0\rangle \in G\}$ and $Y = \bigcup\{d\colon \langle 0, d\rangle \in G\}$. Standard density arguments using $(*)$ show that $X \subset P$ will indeed be dense and $Y \subset Q$ will be open dense with $f''X \cap Y = 0$. So the lemma follows once we show that the forcing R satisfies the countable chain condition.

Here the finite character of Q is used. To prove one of the strong forms of c.c.c. let $\{\langle c_\alpha, d_\alpha\rangle\colon \alpha \in \omega_1\}$ be a collection of conditions in R; a subcollection of size $\aleph_1$ consisting of pairwise compatible conditions will be found.

For $\alpha \in \omega_1$ define $e_\alpha = \{q \in D\colon \exists p \in c_\alpha\ f(p) \leq_Q q\} \subset D$. Note that $e_\alpha \cap d_\alpha = 0$, the sets c_α, d_α and (by the finite character of Q) e_α are finite and conditions $\langle c_\alpha, d_\alpha\rangle$, $\langle c_\beta, d_\beta\rangle$ for $\alpha \neq \beta$ are compatible just in case $e_\alpha \cap d_\beta = e_\beta \cap d_\alpha = 0$—then their lower bound is $\langle c_\alpha \cup c_\beta, d_\alpha \cup d_\beta\rangle \in R$.

By standard Δ-system arguments a subset $I \subset \omega_1$ of full cardinality can be found such that the sets $\{e_\alpha\colon \alpha \in I\}$, $\{d_\alpha\colon \alpha \in I\}$, $\{e_\alpha \cup d_\alpha\colon \alpha \in I\}$ form Δ-systems with respective roots e, d, b.

CLAIM 12. $e = b \cap e_\alpha$ *and* $d = b \cap d_\alpha$ *for every* $\alpha \in I$.

Proof. Note that both e and d are subsets of b since they are both subsets of every

$e_\alpha \cup b_\alpha$: $\alpha \in I$. On the other hand if $q \in b$ then there are two mutually exclusive cases.

(1) If q belongs to e_α for infinitely many $\alpha \in I$ then $q \in e$, so actually every $\alpha \in I$ has $q \in e_\alpha$ and $q \notin d_\alpha$.
(2) If q belongs to d_α for infinitely many $\alpha \in I$ then $q \in d$, and $q \notin e_\alpha$ and $q \in d_\alpha$ holds for every $\alpha \in I$.

Thus b is a disjoint union of e and d and the equalities in the claim immediately follow. □

CLAIM 13. *$e_\alpha \cap d_\beta = 0$ for every $\alpha, \beta \in I$.*

Proof. We have $e_\alpha \cap d_\beta = b \cap e_\alpha \cap d_\beta = e \cap d_\beta \subset e_\beta \cap d_\beta = 0$, where the first inclusion follows from the definition of the root b, the second equality is a consequence of the previous claim, the next inclusion follows from $e \subset e_\beta$ end the last equality comes from the definition of R. □

The last claim shows that the conditions $\langle c_\alpha, d_\alpha \rangle$: $\alpha \in I$ are pairwise compatible—they even form a centered system. The Lemma has been proven. □

On the other hand, suppose $(*)$ fails. In such a case it is not hard to see that f is a witness for $P \prec Q$ in any universe containing P, Q and f and the results of the previous lemma cannot be used. Rather, we shall find a P-name for a Q-generic filter.

LEMMA 14. *Suppose $(*)$ fails. Then there is an open subset $Q' \subset Q$ such that $Q' \lessdot P$.*

Proof. Recall that the failure of $(*)$ means that for every $p \in P$ and an open dense set $D \subset Q$ one can find a strengthening $p' \leq_P p$ and a finite set $d \subset D$ such that for every $p'' \leq_P p'$ there is $q \in d$ with $f(p'') \leq_Q q$. Now suppose $G \subset P$ is a generic filter. For each open dense set $D \subset Q$ in the ground model a condition $p(D) \in G$ and a finite set $d(D)$ can be found such that for every $p \leq_P p(D)$ there is $q \in d(D)$ with $f(p) \leq_Q q$. We shall use the collection $\{d(D)$: D a ground model open dense subset of $Q\}$ to construct a V-generic filter $H \subset Q$ in $V[G]$, completing the task.

Note that for each finite set I of ground model open dense subsets of Q there is a function h_I: $I \to Q$ such that $h_I(D) \in d(D) \subset D$ and the range of h_I is centered. To see that, just choose a condition $p \in G$ below all the $p(D)$: $D \in I$ and let $h_I(D)$ be some condition in $d(D)$ above $f(p)$. This is well defined by $p \leq p(D)$ and the choice of $p(D)$ and $d(D)$, and certainly $f(p)$ is a lower bound of the range of h_I. Now by the compactness principle applied in $V[G]$ there is a function h defined on all ground model open dense subsets of Q such that $h(D) \in d(D)$ and the range of h is centered as a subset of Q. Since $d(D) \subset D$, the upwards closure H of the range of h is a centered set meeting all open dense ground model subsets of Q, therefore $H \in V[G]$ is a V-generic filter on the poset Q. □

Theorem 1 now immediately follows. Suppose the Proper Forcing Axiom holds and P, Q are separative posets of size $\aleph_1$. On one hand, if there is an open subset $Q' \subset Q$ with $Q' \lessdot P$ then $P \succ Q$ by virtue of Claims 3 and 4. On the other hand, suppose $P \succ Q$. From PFA it follows that Q has a dense subset of finite character [T2] and since both $\prec$ and $\lessdot$ are preserved under the transfer to dense subsets we may assume that in fact Q itself is of finite character. Pick an Alaoglu-Birkhoff convergent function $f\colon P \to Q$. There are two cases according to whether $(*)$ above holds or not.

(1) If $(*)$ holds then by Lemma 11 there is a c.c.c. forcing R violating the convergence of f—it introduces a somewhere dense set $X \subset P$ whose image under f is nowhere dense in Q. A routine application of Martin's Axiom, a consequence of PFA, now gives such a set $X \subset P$ already in our universe. Thus f is not Alaoglu-Birkhoff convergent, a contradiction.
(2) If $(*)$ fails then by Lemma 14 there is an open set $Q' \subset Q$ such that $Q' \lessdot P$.

Since the first case leads to a contradiction, this completes the proof of Theorem 1.

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