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INJECTIVITY AS A TRANSVERSALITY PHENOMENON IN GEOMETRIES OF NEGATIVE CURVATURE

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ABSTRACT. The global asymptotic stability conjecture in dynamical systems was solved recently and independently by Feller, Glutsiuk and Gutierrez. Crucial to the approach of Gutierrez is the following theorem of his: A local diffeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ for which the eigenvalues of $Df(x)$ miss $(0, \infty)$ must be injective. The present paper gives a partial generalization of this theorem to local diffeomorphisms between Hadamard surfaces, the spectral condition being replaced by transversality conditions among certain foliations associated to horocycles. The proofs use arguments from global analysis.

Introduction

The question often comes up of determining if the local diffeomorphisms in a particular class are injective. Recent results of this type are the counterexample to the strong real Jacobian conjecture in $\mathbb{R}^2$ by Pinchuk [8], the solution of the global asymptotic stability conjecture by Gutierrez [6], Fessler [4], Glutsiuk [5], and the global injectivity theorem with nearly spectral hypothesis of Smyth-Xavier [9].

In this paper we approach the problem of injectivity of two dimensional local diffeomorphisms from the point of view of geometries of negative curvature. Theorem 1 below gives a set of sufficient conditions for injectivity of a local diffeomorphism $f\colon M_1 \to M_2$ between non-compact simply connected surfaces which is based entirely on certain simple transversality conditions to be satisfied by the horocycle foliations associated to metrics of variable non-positive curvature on M_1 and M_2, and the pull-backs under f of such foliations. Theorem 1 provides a geometric setting for some of the results of [6] and [9]; in particular, it gives a partial extension of [6] to the case of Hadamard manifolds.

Our arguments represent a refinement, on the analytic side, of the method introduced in [9] to show injectivity of local diffeomorphisms of $\mathbb{R}^n$ that satisfy certain algebraic and nearly algebraic conditions. In the present paper however our conditions are entirely geometric and we work in the context of complete simply-connected Riemannian surfaces of non-positive curvature (Hadamard surfaces). The underlying

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principle is that the Busemann functions of a Hadamard manifold play a role similar to the one played by linear functionals in euclidean spaces. In particular there are enough of them to separate points. The question of injectivity of a map f then reduces to the problem of selecting a point at infinity whose Busemann function distinguishes between $f(a)$ and $f(b)$ when $a \neq b$. This is done using degree theory.

We now give a broad outline of the proof of Theorem 1 which, as we mentioned before, is based on the technique introduced in [9]. In order to show that a map f is injective we first construct an auxiliary map, to be denoted $\psi_{1,1}$, whose surjectivity implies the injectivity of f. The map $\psi_{1,1}$ is naturally embedded in a family $\{\psi_{\alpha,s}\colon \alpha \geq 1,\ 0 \leq s \leq 1\}$ obtained by taking integral curves of a suitable 3-parameters family of vector fields. For $\alpha > 1$, the maps $\psi_{\alpha,s}$ satisfy a coercivity condition and topological degree theory is then used to show that they are surjective. A careful geometric analysis allows us to prove that $\psi_{\alpha,1}$ is surjective when $\alpha = 1$ also.

Before we state our results we must introduce a few concepts. We begin by recalling that if v is a point in the unit circle T a *v-horocycle* (relative to the Poincaré metric on the open unit disc D) is a euclidean circle in D that is tangent to T at v (the point v itself being excluded). Horocycles will be oriented in the counterclockwise sense. The foliation of D by v-horocycles will be denoted by H_v. The region between two distinct v-horocycles will be called a *v-horoannulus*.

The notion of horocycle can be defined purely in terms of the hyperbolic geometry of D. Actually, this and other classical concepts of hyperbolic geometry are best understood in the more general context of *Hadamard manifolds*. By definition these are complete simply connected Riemannian manifolds M of non-positive sectional curvature. A Hadamard manifold with curvature bounded from above and below by negative constants has many structural and analytic properties similar to the Poincaré disc. This analogy has been pursued vigorously in recent years, revealing many fruitful connections with analysis, topology and ergodic theory (see [3] for a recent survey). In this paper we will be concerned with the two dimensional case only.

We shall briefly review the basics on Hadamard manifolds. A full account of the discussion below can be found in [1] or [2]. A Hadamard surface M is diffeomorphic to $\mathbb{R}^2$ and can be compactified in a natural way by the introduction of $M(\infty)$, the *circle at infinity*, or *ideal boundary*. An element of $M(\infty)$ is an equivalence class of geodesics that remain at a finite distance from each other as time goes to $+\infty$. The geodesics of the Poincaré metric are arcs of circles perpendicular to the unit circle and in this case $M(\infty)$ can be naturally identified with T. Given a point at infinity v and a point p in M, there is exactly one geodesic γ through p in the class of v and we write $\gamma = \gamma_v$. Furthermore, if the curvature of M is bounded away from zero there is exactly one geodesic joining any two distinct points of $M(\infty)$. The ideal boundary can be topologized in such a way that for every point p in M the natural identification between the unit circle in the tangent plane at p and $M(\infty)$ is a homeomorphism. Given $v \in M_\infty$, a *v-horocycle* through $p \in M$ is the boundary of the region formed by taking the union over all $t > 0$ of the geodesic balls centered at $\gamma_v(t)$ and having radius t, where $\gamma_v(0) = p$. This definition coincides with the previous one in the case of the Poincaré disc. As before, one can consider the horocycle foliations H_v. They are orthogonal to the foliation of all geodesics in the class v. Again, a *v-horoannulus* is a region between two v-horocycles. For instance, in the flat plane a v-horoannulus is a strip bounded by lines perpendicular to v.

An important alternative way to define a v-horocycle is as a level set of a *Busemann function* B_v. Fix a point $o \in M$ and let γ be the geodesic emanating from o that belongs to the equivalence class v. Let, for $p \in M$, $B_v(p) = \lim_{t\to\infty}(d(p, \gamma(t)) - t)$, where d stands for the Riemannian distance. The Busemann function B_v enjoys some important properties: it is a function of class C^2 [7], its gradient has length one and $-\nabla B_v(p)$ determines a geodesic in the class v. If M is an oriented Hadamard surface we can orient the v-horocycle $H_v(p)$ through p by declaring a unit tangent vector w to H_v to be positive if the oriented orthonormal basis $\{\nabla B_v(p), w\}$ is positive.

In order to state our results we need a concept weaker than transversality:

Definition. Two smooth oriented planar foliations (line fields) are said to be loosely transversal if, for every point p in their common domain of definition, either their leaves are transversal at p or they are tangent there but have *opposite* orientations at p.

THEOREM 1. *let f: $(M_1, g_1) \to (M_2, g_2)$ be an orientation-preserving local diffeomorphism between oriented Hadamard surfaces. Then f is injective, provided there is a continuous map h: $M_1(\infty) \to M_2(\infty)$ such that:*

(i) *For every $v \in M_1(\infty)$, the horocycle foliation H_v and the pull-back foliation $f^*H_{h(v)}$ are loosely transversal.*
(ii) *For every $v \in M_1(\infty)$, the set where H_v and $f^*H_{h(v)}$ are tangent intersects every v-horoannulus in a compact set.*

The above theorem represents a partial generalization to Hadamard surfaces of the Gutierrez theorem [6] (see below). In order to see this, in Theorem 1 take $(M_1, g_1) = (M_2, g_2) =$ the flat plane and $h =$ the identity. It is easy to see that condition (i) is equivalent to the spectral condition $[0, \infty) \cap \text{Spec }(Df(x))^t = \emptyset$ for all x in the plane, where A^t denotes the transpose of A. In order to see this, let $\{v, v^\perp\}$ be a positive orthonormal basis and suppose, by way of contradiction, that for f as in the theorem we have $Df(x)^t v = \lambda v$, where $\lambda > 0$. It follows that $Df(x)v^\perp = \mu v^\perp$ for some real number μ. The map f preserves orientation so that the basis $\{Df(x)v, Df(x)v^\perp\} = \{Df(x)v, \mu v^\perp\}$ is positive. Since $\langle Df(x)v, v\rangle > 0$ this implies that $\mu > 0$, which is a contradiction to loose transversality.

Condition (ii) means that the set of points in $\mathbb{R}^2$ that has v as an eigenvector of $(Df(x))^t$ intersects any strip in the plane with sides perpendicular to v in a compact set. In particular [9, Theorem 3] f is injective if the spectra of the linearizations of f miss the non-negative real axis everywhere and miss the real line in a neighborhood of infinity. In fact, the central result in the recent important paper of Gutierrez [6] states that the spectral condition $[0, \infty) \cap \operatorname{Spec} Df(x) = \emptyset$ *alone* suffices to prove that f is injective. An important consequence of the theorem of Gutierrez (see also [4] and [5]) is the affirmative solution of the global asymptotic stability conjecture in the theory of dynamical systems: if f is a smooth vector field in $\mathbb{R}^2$ with a singularity at p and if the linearizations of f are everywhere stable then the stable manifold of p is the entire plane.

In view of the developments described in the paragraph above, it is natural to ask if condition (ii) of Theorem 1 is superfluous. If this can be shown to be the case one would have a full generalization of the theorem of Gutierrez to Hadamard surfaces.

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1. Proof of the theorem

Let $Df(x)\colon T_xM_1 \to T_{f(x)}M_2$ be the differential of f at x and denote by $Df(x)^t\colon T_{f(x)}M_2 \to T_xM_1$ the transpose of $Df(x)$, defined in the usual way.

In what follows, B_w will denote the Busemann function corresponding to $w \in M_1(\infty)$ or $M_2(\infty)$, for a fixed choice of base points in M_1 and M_2. Given $\alpha \geq 1$, $s \in [0, 1]$ and $v \in M_1(\infty)$, we define a vector field $G_{v,\alpha,s}$ on M_1 by

$$G_{v,\alpha,s}(x) = \nabla B_v(x) - \frac{s}{\alpha}\frac{Df(x)^t \nabla B_{h(v)}(f(x))}{|Df(x)^t \nabla B_{h(v)}(f(x))|}. \tag{1}$$

Since Busemann functions are C^2, the vector field $G_{v,\alpha,s}$ is C^1 and therefore its local trajectories are uniquely determined. In fact, since $|G_{v,\alpha,s}| \leq 2$ and M_1 is complete, the trajectories are defined for all times. We also observe that $G_{v,\alpha,s}$ varies continuously with the parameters v, α and s. In particular, the associated flow also varies continuously with these parameters.

Now, suppose that $f(a) = f(b)$ with $a \neq b$ and denote by $\phi_{v,\alpha,s}$ the integral curve of $G_{v,\alpha,s}$ passing through a at time $t = 0$. Hence $\phi_{v,\alpha,s}$ solves the initial-value problem $\dot{x} = G_{v,\alpha,s}(x)$, $x(0) = a$.

Letting $\alpha > 1$, $x = \phi_{v,\alpha,s}(t)$ and using $|\nabla B_v| = 1$, we have

$$\frac{d}{dt}B_v(x) = \langle \nabla B_v(x), \nabla B_v(x) - \frac{s}{\alpha}\frac{Df(x)^t \nabla B_{h(v)}(f(x))}{|Df(x)^t \nabla B_{h(v)}(f(x)|}\rangle \geq 1 - \frac{1}{\alpha}.$$

Integration between 0 and t gives

$$B_v(\phi_{v,\alpha,s}(t)) - B_v(a) \geq \frac{(\alpha - 1)}{\alpha}t, \tag{2}$$

valid for $t \geq 0, \alpha > 1,\ 0 \leq s \leq 1,\ v \in M_1(\infty)$. For $\alpha > 1,\ 0 \leq s \leq 1$, we define a (continuous) map $\psi_{\alpha,s}\colon M_1 \to M_1$ by

$$\psi_{\alpha,s}(w) = \begin{cases} \phi_{v(a,w),\alpha,s}(d(w, a)), & w \neq a \\ a, & w = a. \end{cases}$$

Here d is the Riemannian distance and for $w \neq a$, $v(a, w)$ stands for the point at infinity corresponding to the unique oriented geodesic joining a to w. From the definition of $\psi_{\alpha,s}$ and (2) we have

$$B_{v(a,w)}(\psi_{\alpha,s}(w)) - B_{v(a,w)}(a) \geq \frac{(\alpha-1)}{\alpha} d(w, a). \tag{3}$$

For $\alpha > 1$ we consider the homotopy K_α: $[0, 1] \times M_1 \to M_1$, given by $K_\alpha(s, w) = \psi_{\alpha,s}(w)$. If $s_n \in [0, 1]$ and $w_n \in M_1$ are sequences and $\lim_{n\to\infty} d(w_n, a) = \infty$, then it follows from (3) that

$$\lim_{n\to\infty} d(K_\alpha(s_n, w_n), a) = \infty.$$

Hence the (continuous) homotopy K_α is *proper*. In particular, the continuous maps $K_\alpha(0, .)$ and $K_\alpha(1, .)$ are also proper maps of M_1. It is then possible to define their topological degree. Then the invariance of the degree under proper homotopies gives

$$deg(\psi_{\alpha,1}) = deg(\psi_{\alpha,0}). \tag{4}$$

Clearly $\psi_{\alpha,0}$ is the geodesic symmetry about the point a. The latter is a diffeomorphism of M_1 and by (4), $\deg(\psi_{\alpha,1}) = \deg(\psi_{\alpha,0})$ is non-zero. In particular, $\psi_{\alpha,1}$ is surjective. Therefore for every $\alpha > 1$ there exist $\tau_\alpha > 0$, $v_\alpha \in M_\infty$ such that

$$\phi_{v_\alpha,\alpha,1}(0) = a, \qquad \phi_{v_\alpha,\alpha,1}(\tau_\alpha) = b. \tag{5}$$

At this point in the proof we analyze two alternatives.

Alternative 1. $\liminf_{\alpha\to 1} \tau_\alpha < \infty$.

Let α_n be a sequence tending to 1 such that $\tau_{\alpha_n} \to c < \infty$ and $v_{\alpha_n} \to v$ for some c and v. From the continuous dependence of the flow on parameters and from (5) we have

$$\phi_{v,1,1}(0) = a, \qquad \phi_{v,1,1}(c) = b. \tag{6}$$

Letting $x = \phi_{v,1,1}(t)$ we have

$$\frac{d}{dt} B_{h(v)}(f(x)) = \langle \nabla B_{h(v)}(f(x)), Df(x)\dot{x} \rangle,$$

which is equal to

$$\langle Df(x)^t \nabla B_{h(v)}(f(x)), \nabla B_v(x) - \frac{Df(x)^t \nabla B_{h(v)}(f(x))}{|Df(x)^t \nabla B_{h(v)}(f(x))|} \rangle \leq 0,$$

since $|\nabla B_v| = 1$. The inequality will be strict provided

$$\nabla B_v(x) \neq \frac{Df(x)^t \nabla B_{h(v)}(f(x))}{|Df(x)^t \nabla B_{h(v)}(f(x))|}. \tag{7}$$

If (7) is violated then

$$(8) \qquad Df(x)^t \nabla B_{h(v)}(f(x)) = \beta(x)\nabla B_v(x),$$

where $\beta(x)$ is positive. We want to show that (8) leads to a contradiction. Let w (resp. w') be a positively oriented tangent vector to the v-horocycle at x (resp., the $h(v)$-horocycle at $f(x)$), so that the bases $\{\nabla B_v(x), w\}$ and $\{\nabla B_{h(v)}(f(x)), w'\}$ are positive.

From $\langle \nabla B_{h(v)}(f(x)), Df(x)w\rangle = 0$ we have $Df(x)w = \mu w'$ for some nonzero μ. In particular, $f^* H_{h(v)}$ is tangent to H_v at x. For $0 \le t \le 1$ we let $X_t = tDf(x)\nabla B_v(x) + (1-t)\nabla B_{h(v)}(f(x))$. From $\langle X_t, \nabla B_{h(v)}(f(x))\rangle = 1 - t + t\beta(x) \ge \min\{1, \beta(x)\} > 0$ we see that $\{X_t, w'\}$ is a basis for $0 \le t \le 1$ and that the orientations of $\{X_0, w'\}$ and $\{X_1, w'\}$ are the same. It follows that $\{X_1, w'\} = \{Df(x)\nabla B_v(x), \frac{1}{\mu}Df(x)w\}$ is a positive basis. Since f preserves orientation, this forces $\mu > 0$. But this contradicts hypothesis (i) on loose transversality. Hence (7) is satisfied .

Integrating $\frac{d}{dt}B_{h(v)}(f(x)) < 0$ between 0 and c ($c > 0$ in view of of (6) since $a \neq b$) and using (7), we have $B_{h(v)}(f(b)) < B_{h(v)}(f(a))$ which is a contradiction to $f(a) = f(b)$. We have therefore finished the proof of the theorem under the assumption that Alternative 1 prevails. It remains to analyze the other case.

Alternative 2. $\operatorname{Lim}_{\alpha\to 1} \tau_\alpha = \infty$.

We are going to show that this case cannot happen either. Let α_n be a sequence tending to 1 with $\tau_n = \tau_{\alpha_n} \to \infty$ and $v_n = v_{\alpha_n} \to v_0$. We claim that

$$(9) \qquad \cup_{n=1}^{\infty} \phi_{v_n,\alpha_n,1}([0, \tau_n])$$

is an unbounded subset of M_1. Suppose, by way of contradiction, that the union above is contained in a fixed compact set K.

Letting $x = \phi_{v_n,\alpha_n,1}(t)$, we have

$$\frac{d}{dt}B_{v_n}(x) = \langle \nabla B_{v_n}(x), \nabla B_{v_n}(x) - \frac{1}{\alpha_n}\frac{Df(x)^t\nabla B_{h(v_n)}(f(x))}{|Df(x)^t\nabla B_{h(v_n)}(f(x)|}\rangle.$$

It follows from $\alpha_n \ge 1$, $|\nabla B_{v_n}| = 1$, Schwarz's inequality and loose transversality (see the discussion of (8)) that as long as the trajectory $\phi_{v_n,\alpha_n,1}(t)$ remains in K there is a positive number δ, depending on K but not on n, such that

$$\frac{d}{dt}B_{v_n}(\phi_{v_n,\alpha_n,1}) \ge \delta.$$

Integrating this inequality between 0 and τ_n and using (6) we have $B_{v_n}(b) - B_{v_n}(a) \ge \delta\tau_n$, a contradiction to $\tau_n \to \infty$ and the fact that $|\nabla B_v| = 1$. This establishes that the set in (9) is unbounded.

A *truncation* of a closed v-horoannulus L_v is a closed subset T_v bounded by the horocycles in the boundary of L_v and by two distinct v-geodesics. Let $\delta = \max_{v \in M_1(\infty)} |B_v(a) - B_v(b)| = d(a,b)$. For all $v \in M_1(\infty)$, consider the v-horoannulus $L_{v,\delta} = B_v^{-1}([B_v(a) - 2\delta, B_v(a) + 2\delta])$. In particular, a, $b \in L_{v,\delta}$ for all $v \in M_1(\infty)$. Let v_0 be as before. By hypothesis ii) in the theorem, it is possible to find a truncation $T_{v_0,\delta}$ of $L_{v_0,\delta}$ that contains both a and b, and such that

$$(10) \qquad L_{v_0,\delta} \cap \{\text{points where } H_{v_0} \text{ and } f^* H_{h(v_0)} \text{ are tangent}\} \subset T'_{v_0,\delta}.$$

Here $T'_{v_0,\delta}$ is obtained from $T_{v_0,\delta}$ by deleting the v_0-geodesics in its boundary. As $n \to \infty$, $L_{v_n,\delta} \to L_{v_0,\delta}$ uniformly over compact sets. Clearly, it is also possible to find truncations $T_{v_n,\delta}$ of $L_{v_n,\delta}$ that contain a and b and converge to $T_{v_0,\delta}$.

Let now K be the union over all $v \in M_1(\infty)$ of all the truncations $T_{v_n,\delta}$ of $L_{v_n,\delta}$ given above. Since K is a bounded set, it follows from (9) that the set $(M_1 \setminus K) \cap \phi_{v_n,\alpha_n,1}([0, \tau_n])$ is non-empty for all large enough n. In particular,

$$(11) \qquad (M_1 \setminus T_{v_n,\delta}) \cap \phi_{v_n,\alpha_n,1}([0, \tau_n]) \neq \emptyset, n \geq N.$$

Since $\frac{d}{dt} B_{v_n}(\phi_{v_n,\alpha_n,1})(t) > 0$, the trajectory $\phi_{v_n,\alpha_n,1}([0, \tau_n])$ is transversal to the v_n-horocycle foliation and by (5) we have

$$(12) \qquad \phi_{v_n,\alpha_n,1}([0, \tau_n]) \subset L_{v_n,\delta}.$$

Since a and b belong to all truncations, it follows from (11) that the trajectory $\phi_{v_n,\alpha_n,1}([0, \tau_n])$, $n \geq N$, must exit and then re-enter $T_{v_n,\delta}$. By (12), the exit and entrance points cannot be on the v_n-horocycle boundaries of the truncation. By passing to subsequences we may assume that the exit and entrance points lie in the same geodesic component γ_n of the boundary of $T_{v_n,\delta}$. In particular, the vector field $G_{v_n,\alpha_n,1}$ becomes tangent to γ_n somewhere. In the limit we conclude that the vector field $G_{v_0,1,1}$ becomes tangent to a v_0-geodesic at a point x_0 in the v_0-geodesic that is part of the boundary of $L_{v_0,\delta} \setminus T_{v_0,\delta}$. Hence there exists l such that

$$(13) \qquad G_{v_0,1,1}(x_0) = l \nabla B_{v_0}(x_0), \qquad x_0 \in (L_{v_0,\delta} \setminus T'_{v_0,\delta}).$$

It follows from (13) and (1) that there is $\lambda \in \mathbb{R}$ such that

$$Df(x_0)^t \nabla B_{h(v_0)}(f(x_0)) = \lambda \nabla B_{v_0}(x_0), \qquad x_0 \in (L_{v_0,\delta} \setminus T'_{v_0,\delta}).$$

As explained in the discussion immediately following (8), this implies that the foliations H_v and $f^* H_{h(v)}$ are tangent at the point $x_0 \in (L_{v_0,\delta} \setminus t'_{v_0,\delta})$, a contradiction to (10) and ultimately to the fact that we assumed $f(a) = f(b)$ with $a \neq b$. Thus Alternative 2 also leads to a contradiction and the proof of the theorem is complete.

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