ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 3, 1999

ALMOST EVERYWHERE CONVERGENCE AND BOUNDEDNESS OF CESÀRO-α ERGODIC AVERAGES

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Dedicated to Professor Alexandra Bellow

ABSTRACT. We study the almost everywhere convergence of the ergodic Cesàro-α averages $R_{n,\alpha} f = \frac{1}{A_n^\alpha} \sum_{i=0}^{n} A_{n-i}^{\alpha-1} T^i f$ and the boundedness of the ergodic maximal operator $M_\alpha f = \sup_{n \in \mathbb{N}} |R_{n,\alpha} f|$, associated with a positive linear operator T with positive inverse on some $L^p(\mu)$, $1 < p < \infty, 0 < \alpha \leq 1$.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let T be a positive linear operator on some $L^p(\mu)$, $1 \leq p < \infty$ (positive means that if $f \geq 0$ a.e. then $Tf \geq 0$ a.e.). For every $f \in L^p(\mu)$ we consider the averages

$$R_n f = \frac{1}{n+1} \sum_{i=0}^{n} T^i f, \quad n \in \mathbb{N},$$

and the maximal operator

$$Mf = \sup_{n \in \mathbb{N}} |R_n f|.$$

Akcoglu [1] proved that if T is a positive linear contraction on $L^p(\mu)$, $1 < p < \infty$, then

$$\int_X |Mf|^p \, d\mu \leq \left(\frac{p}{p-1} \right)^p \int_X |f|^p \, d\mu$$

and $R_n f$ converges almost everywhere and in $L^p(\mu)$ for all $f \in L^p(\mu)$. Actually, one does not need to have a contraction to obtain the boundedness of M and the a.e convergence of the averages $R_n f$. In fact, it was shown in [16] that if $T\colon L^p(\mu) \to L^p(\mu)$, $1 < p < \infty$, is a positive linear operator with positive inverse then the ergodic dominated estimate

$$\int_X |Mf|^p \, d\mu \leq C \int_X |f|^p \, d\mu$$

Received November 9, 1998.
1991 Mathematics Subject Classification. Primary 47A35, 28D05; Secondary 42B25.
This research has been partially supported by a D.G.I.C.Y.T. grant (PB94-1496), a D.G.E.S. grant (PB97-1097) and Junta de Andalucía.

holds for all $f \in L^p(\mu)$ if, and only if, the operator is Cesàro bounded, i.e.,

$$\sup_{n\in\mathbb{N}} \int_X |R_n f|^p \, d\mu \leq C \int_X |f|^p \, d\mu$$

for all $f \in L^p(\mu)$ and, in that case, the averages $R_n f$ converge a.e. and in $L^p(\mu)$ for every $f \in L^p(\mu)$. (A. Brunel [4] proved that this equivalence holds assuming only that T is a positive linear operator on $L^p(\mu)$, $1 < p < \infty$.) It is worth mentioning that by Theorem 4.2 in [8], a positive operator is Cesàro bounded in $L^p(\mu)$ if, and only if, the averages $R_n f$ converge in $L^p(\mu)$ for all $f \in L^p(\mu)$.

The averages R_n are the Cesàro-1 averages of the sequence $\{T^n f\}$. In this paper we are interested in studying the a.e. convergence of the Cesàro-α averages with $0 < \alpha \leq 1$, which are stronger processes of convergence [23]. The Cesàro-α averages and the Cesàro-α maximal operator associated with T are defined by

$$R_{n,\alpha} f = \frac{1}{A_n^\alpha} \sum_{i=0}^{n} A_{n-i}^{\alpha-1} T^i f$$

and

$$M_\alpha f = \sup_{n\in\mathbb{N}} |R_{n,\alpha} f|,$$

where $A_n^\alpha = \frac{(\alpha+1)\cdots(\alpha+n)}{n!}$ and $A_0^\alpha = 1$. Note that $R_{n,1} = R_n$ and $M_1 = M$. In his thesis [11], R. Irmisch proved the following theorem which generalizes Akcoglu's theorem to Cesàro-α averages.

THEOREM A [11]. *Let α and p be such that $0 < \alpha \leq 1$ and $\alpha p > 1$. Let $T\colon L^p(\mu) \to L^p(\mu)$ be a positive linear contraction. Then there exists $C > 0$ such that*

$$\int_X |M_\alpha f|^p \, d\mu \leq C \int_X |f|^p \, d\mu$$

and $R_{n,\alpha} f$ converges a.e. and in $L^p(\mu)$ for all $f \in L^p(\mu)$.

In the limit case $\alpha p = 1$, the result does not hold even if T is induced by an ergodic measure-preserving transformation [5]. In this limit case, Broise, Deniel and Derriennic [3] have obtained that a restricted weak type inequality holds for operators defined by composition with a measure-preserving transformation. More precisely, they obtained the following theorem.

THEOREM B [3]. *Let $(X, \mathcal{M}, \mu)$ be a probability measure space and assume that $\tau\colon X \to X$ is a measure-preserving transformation. Let $Tf = f \circ \tau$. Then the maximal operator M_α maps the Lorentz space $L_{1/\alpha,1}(\mu)$ into $L_{1/\alpha,\infty}(\mu)$. Furthermore, the sequence $R_{n,\alpha} f$ converges a.e. for all $f \in L_{1/\alpha,1}(\mu)$. (See [10] for the definition of the $L_{p,q}(\mu)$ spaces.)*

Other results related to the ones stated above can be found in [6].

Our goal is to study the behaviour of the maximal operator M_α and the a.e. convergence of $R_{n,\alpha}f$, $0 < \alpha \leq 1$, assuming that T is a positive linear operator on $L^p(\mu)$ with positive inverse, i.e., for the same class of operators considered in [16]. In Theorem 3.1 we give a sufficient condition for the boundedness of M_α and the a.e. convergence of the averages $R_{n,\alpha}f$. As a corollary we obtain the dominated estimate and the a.e. convergence of the averages $R_{n,\alpha}f$ in the following cases:

(1) T is a positive power bounded linear operator with positive inverse and $p\alpha > 1$ (Corollary 3.3).
(2) T is a positive linear operator with positive inverse such that the operator $\widetilde{T}$, defined by $\widetilde{T}f = (Tf^\alpha)^{1/\alpha}$ for nonnegative functions, is Cesàro bounded in $L^{p\alpha}(\mu)$, $p\alpha > 1$ (Corollary 3.4).
(3) $T = f \circ \tau$ where $\tau: X \to X$ is an invertible nonsingular transformation such that T is Cesàro bounded in $L^{p\alpha}(\mu)$, $p\alpha > 1$ (Corollary 3.5).

These results leave open the question of the equivalence between the ergodic dominated estimate

$$\int_X |M_\alpha f|^p \, d\mu \leq C \int_X |f|^p \, d\mu$$

and the uniform boundedness of the Cesàro-α averages

$$\sup_{n\in\mathbb{N}} \int_X |R_{n,\alpha} f|^p \, d\mu \leq C \int_X |f|^p \, d\mu.$$

Unlike the case $\alpha = 1$, this equivalence does not hold for $0 < \alpha < 1$ even in the good case $\alpha p > 1$. In §3 we show an example for which the Cesàro-α averages are uniformly bounded but the ergodic dominated estimate does not hold for M_α.

Taking into account this example, the following question arises: are there any kind of Cesàro-α averages, let us say $\{R'_{n,\alpha}\}$, such that the boundedness of M_α is equivalent to the uniform boundedness of $\{R'_{n,\alpha}\}$? In §4 we answer this question in the affirmative for operators T of the form $Tf = g(f \circ \tau)$, where g is a positive function and τ is an ergodic invertible transformation, working in $L^p(\omega \, d\mu)$ where $\omega > 0$ and μ is preserved by τ (see Theorem 4.6, where we prove also that the sufficient condition in Theorem 4.1 is equivalent to the boundedness of M_α). The averages that we introduce in §4 can be viewed as generalizations of the Cesàro-Hardy averages defined for functions f on the integers by $H_n f(k) = \frac{1}{A^\alpha_{n-k}} \sum_{i=k}^n A^{\alpha-1}_{n-i} f(i)$ if $k \leq n$ and $H_n f(k) = 0$ if $k > n$.

The statements and the proofs of the theorems need some notation and several results that we establish in §2.

Throughout the paper the letter C means a positive constant not necessarily the same at each occurrence. If $1 < p < \infty$ then p' is the number such that $1/p + 1/p' = 1$. Finally, if A and B are measurable sets, $A = B$ means that A equals B up to a set of measure zero.

2. Some previous results

We are going to need some results about the maximal operator m_α^+ associated with the Cesàro-α averages of functions on the set of the integer numbers.

Definition 2.1. Let $0 < \alpha \leq 1$. If a is a real-valued function on $\mathbb{Z}$, we define the Cesàro-α maximal function $m_\alpha^+ a$ by

$$m_\alpha^+ a(i) = \sup_{n\geq 0} \frac{1}{A_n^\alpha} \left| \sum_{j=0}^{n} A_{n-j}^{\alpha-1} a(i+j) \right|, \quad i \in \mathbb{Z}. \tag{2.1}$$

LEMMA 2.2 [21]. *Let ω be a positive function on $\mathbb{Z}$, $0 < \alpha \leq 1$ and $1 < p < \infty$. The following statements are equivalent*:

(*i*) *There exists a positive constant C such that*

$$\sum_{i=-\infty}^{\infty} \left[m_\alpha^+ a(i)\right]^p \omega(i) \leq C \sum_{i=-\infty}^{\infty} |a(i)|^p \omega(i), \tag{2.2}$$

for any function a on $\mathbb{Z}$.

(*ii*) *ω satisfies the condition $A_{p;\alpha}^+(\mathbb{Z})$, or $\omega \in A_{p;\alpha}^+(\mathbb{Z})$, i.e., there exists a positive constant C such that*

$$\left(\sum_{i=r}^{s} \omega(i)\right)^{1/p} \left(\sum_{i=s}^{k} \omega^{1-p'}(i) \left(A_{k-i}^{\alpha-1}\right)^{p'}\right)^{1/p'} \leq C A_{k-r}^\alpha, \tag{2.3}$$

for all $r, s, k \in \mathbb{Z}$ with $r \leq s \leq k$.

Lemma 2.2 is a particular case of Theorem 2.16 in [21]. Alternatively, just look at the proof in [19] and write it in the setting of the integers. Observe that if $w(i) = 1$ for all i then (2.3) holds if, and only if, $p\alpha > 1$.

The following result states a relationship between the classes $A_{p;\alpha}^+(\mathbb{Z})$ and the classical ones $A_p^+(\mathbb{Z}) = A_{p;1}^+(\mathbb{Z})$; it also gives the analogue in our setting of the implication $\omega \in A_p^+(\mathbb{Z}) \Rightarrow \omega \in A_{p-\varepsilon}^+(\mathbb{Z})$ (see [20], [22], [16] and [14]).

LEMMA 2.3. *Let ω be a positive function on $\mathbb{Z}$. Let $0 < \alpha \leq 1$ and $p > 1$.*

(1) *If $\omega \in A_{p;\alpha}^+(\mathbb{Z})$ with a constant C, then there exists $\varepsilon > 0$, which depends only on C, such that $\omega \in A_{p-\varepsilon,\alpha}^+(\mathbb{Z})$. Furthermore, ω is also in $A_p^+(\mathbb{Z})$ with the same constant C.*

(2) *If $\alpha p > 1$ and $\omega \in A_{\alpha p}^+(\mathbb{Z})$, then ω is also in $A_{p;\alpha}^+(\mathbb{Z})$.*

We shall sketch the proof of this lemma (alternatively, one can look at the corresponding proof in [19] and write it in the setting of the integer numbers).

Theorem 2.16 in [21] (see the proof of (ii)$\Rightarrow$ (iii)) and the fact that, for $-1 < \alpha \leq 0$, the coefficients A_n^α are decreasing as a function of n [23], give us (1).

To prove (2) we simultaneously use the fact that if $\omega \in A_{\alpha p}^+(\mathbb{Z})$ with $\alpha p > 1$, then $\omega \in A_r^+(\mathbb{Z})$ for some r with $1 < r < \alpha p$, together with Hölder's inequality with exponents γ and γ', where $\gamma = \frac{1-r'}{1-p'}$.

Analogous results hold for the operator

$$m_\alpha^- a(i) = \sup_{n \geq 0} \frac{1}{A_n^\alpha} \left| \sum_{j=-n}^{0} A_{n+j}^{\alpha-1} a(i+j) \right|, \quad i \in \mathbb{Z},$$

with the $A_{p;\alpha}^+(\mathbb{Z})$ changed by the $A_{p;\alpha}^-(\mathbb{Z})$ classes: a positive function ω defined on $\mathbb{Z}$ satisfies the condition $A_{p;\alpha}^-(\mathbb{Z})$ if there exists a positive constant C such that

$$\left(\sum_{i=s}^{k} \omega(i) \right)^{1/p} \left(\sum_{i=r}^{s} \omega^{1-p'}(i) \left(A_{i-r}^{\alpha-1} \right)^{p'} \right)^{1/p'} \leq C A_{k-r}^\alpha,$$

for all $r, s, k \in \mathbb{Z}$ with $r \leq s \leq k$

The following result characterizes the power functions of the form $\omega_\gamma = (1+|i|)^\gamma$ belonging to the $A_{p;\alpha}^{+(-)}(\mathbb{Z})$ classes. This lemma will be important in the next section.

LEMMA 2.4. *Let $0 < \alpha \leq 1$ and $\alpha p > 1$. Then $\omega_\gamma \in A_{p;\alpha}^{+(-)}(\mathbb{Z})$ if and only if $-1 < \gamma < \alpha p - 1$.*

Proof of Lemma 2.4. We shall give the proof only for the $A_{p;\alpha}^+(\mathbb{Z})$ classes. Assume that $-1 < \gamma < \alpha p - 1$. As in the classical case of Muckenhoupt weights (see [9]) it can be proved that $\omega_\gamma \in A_{\alpha p}^+(\mathbb{Z})$. Then, by Lemma 2.3 (2), we have $\omega_\gamma \in A_{p;\alpha}^+(\mathbb{Z})$. For the converse we shall need the following lemma.

LEMMA 2.5. (1) *If $\omega \in A_{p;\alpha}^+(\mathbb{Z})$ then, for every natural number N, we have $\sum_{i \leq -2N} \frac{\omega(i)}{|i|^{\alpha p}} < \infty$.*

(2) *If $\omega \in A_{p;\alpha}^-(\mathbb{Z})$ then, for every natural number N, we have $\sum_{i \geq 2N} \frac{\omega(i)}{i^{\alpha p}} < \infty$.*

This lemma is nothing but the translation to our setting of Lemma 4 in [17]. Therefore, we omit the proof.

Once we have Lemma 2.5 it is not difficult to prove the converse of Lemma 2.4. Assume that $\omega_\gamma \in A_{p;\alpha}^+(\mathbb{Z})$. On one hand, by Lemma 2.5 (1), we have $\sum_{i \leq -2N} (1+|i|)^\gamma |i|^{-\alpha p} < \infty$. Therefore $\alpha p - \gamma > 1$ or, equivalently, $\gamma < \alpha p - 1$. On the other hand, by Lemma 2.3, we have $\omega_\gamma \in A_p^+(\mathbb{Z})$ which is equivalent to saying that $\omega_\gamma^{1-p'} \in A_{p'}^-(\mathbb{Z})$. Now applying Lemma 2.5 (2) as above, but with $\alpha = 1$, we obtain $p' - \gamma(1-p') > 1$ or, equivalently, $\gamma > -1$, which finishes the proof of Lemma 2.4.

3. The Cesàro-α ergodic averages for positive linear transformations with positive inverse

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. Let T be an invertible positive linear operator on $L^p(\mu) = L^p(X, \mathcal{F}, \mu)$, with $1 < p < \infty$, and suppose that T^{-1} is also positive. Then, as is well known [12], T and T^{-1} are Lamperti operators, i.e., they separate supports, and they have the following properties:

(a) For each integer i, there exists a positive function g_i such that

$$T^i f = g_i S^i f \quad \text{and } g_{i+j} = g_i S^i g_j,$$

where S is a positive multiplicative invertible linear map acting on measurable functions.

(b) For each integer i, there exists a positive function J_i such that

$$J_{i+j} = J_i S^i J_j \quad \text{and } \int_X J_i S^i f \, d\mu = \int_X f \, d\mu.$$

(c) If $h_i = g_i^{-p} J_i$ then

$$\int_X |T^i f|^p h_i \, d\mu = \int_X |f|^p \, d\mu. \tag{3.1}$$

THEOREM 3.1. *Let $(X, \mathcal{F}, \mu)$, p, T and h_i be as above. Let $0 < \alpha \leq 1$ and suppose that, for almost all x, the function h_x defined on $\mathbb{Z}$ by $i \to h_i(x)$ satisfies $A^+_{p;\alpha}(\mathbb{Z})$ with a constant independent of x. Then we have*:

(i) *The maximal operator M_α is bounded in $L^p(\mu)$.*
(ii) *For every $f \in L^p(\mu)$ the averages $R_{n,\alpha} f$ converge almost everywhere and in $L^p(\mu)$.*

Proof. We start with the proof of (i). It suffices to work with nonnegative functions belonging to $L^p(\mu)$. Given $L > 0$, $L \in \mathbb{N}$, let $M_{\alpha,L}$ denote the truncated maximal operator defined by

$$M_{\alpha,L} f = \sup_{0 \leq n \leq L} R_{n,\alpha} f.$$

For such a function f there exist pairwise disjoint measurable subsets of X, E_0, E_1, $\ldots$, E_L, such that

$$M_{\alpha,L} f = \sum_{j=0}^{L} \chi_{E_j} R_{j,\alpha} f.$$

For every $i \in \mathbb{Z}$, we have

$$T^i(M_{\alpha,L} f) = \sum_{j=0}^{L} T^i(\chi_{E_j} R_{j,\alpha} f). \tag{3.2}$$

Since T^i is positive and separates supports, there exist pairwise disjoint measurable sets, $E_{i,0}, E_{i,1}, \ldots, E_{i,L}$, such that

$$T^i(\chi_{E_j} R_{j,\alpha} f) \leq \chi_{E_{i,j}} T^i(R_{j,\alpha} f) \leq \chi_{E_{i,j}} M_{\alpha,L}(T^i f), \quad j = 0, 1, \ldots, L. \tag{3.3}$$

Adding up in (3.3) and using (3.2), we obtain

$$T^i(M_{\alpha,L} f) \leq M_{\alpha,L}(T^i f), \quad i \in \mathbb{Z}. \tag{3.4}$$

On the other hand, given $N \in \mathbb{N}$, and $x \in X$, the definitions of $M_{\alpha,L}$ and m_α^+ imply

$$M_{\alpha,L}(T^i f)(x) = m_\alpha^+(G_x \chi_{[0,N+L]})(i), \quad i = 0, 1, \ldots, N, \tag{3.5}$$

where $G_x \chi_{[0,N+L]}$ is the function defined in $\mathbb{Z}$ by

$$\begin{cases} G_x \chi_{[0,N+L]}(j) = T^j f(x), & \text{if } j \in \mathbb{Z} \cap [0, N+L], \\ G_x \chi_{[0,N+L]}(j) = 0, & \text{if } j \notin \mathbb{Z} \cap [0, N+L]. \end{cases}$$

Then, for fixed $L > 0$ and $N \geq 0$, using property (c) of the operator T, (3.4) and (3.5) we see that

$$\begin{aligned} \int_X \left(M_{\alpha,L} f(x)\right)^p \, d\mu(x) &= \frac{1}{N+1} \int_X \sum_{i=0}^{N} \left[T^i(M_{\alpha,L} f)(x)\right]^p h_i(x)\, d\mu(x) \qquad (3.6) \\ &\leq \frac{1}{N+1} \int_X \sum_{i=0}^{N} \left[m_\alpha^+(G_x \chi_{[0,N+L]})(i)\right]^p h_i(x)\, d\mu(x). \end{aligned}$$

Taking into account that $1 < p < \infty$ and that, for almost every $x \in X$, $h_x \in A_{p;\alpha}^+(\mathbb{Z})$ with a constant independent of x, Lemma 2.2 implies that there exists a positive constant C such that

$$\sum_{i=0}^{N} \left[m_\alpha^+(G_x \chi_{[0,N+L]})(i)\right]^p h_x(i) \leq C \sum_{i=-\infty}^{\infty} \left[G_x \chi_{[0,N+L]}(i)\right]^p h_x(i), \tag{3.7}$$

for almost every $x \in X$. Now, using (3.6), (3.7) and property (c) of T, we obtain

$$\begin{aligned} \int_X (M_{\alpha,L} f)^p \, d\mu &\leq \frac{C}{N+1} \sum_{i=0}^{N+L} \int_X [T^i f(x)]^p h_i(x)\, d\mu(x) \\ &= C \frac{N+L+1}{N+1} \int_X [f(x)]^p \, d\mu(x). \qquad (3.8) \end{aligned}$$

Letting N tend to ∞ in (3.8) gives

$$\int_X (M_{\alpha,L} f)^p \, d\mu \leq C \int_X [f(x)]^p \, d\mu(x).$$

Then let L tend to ∞ to complete the proof of (i).

Now we turn to the proof of (ii). It is clear that it suffices to prove the a.e. convergence. Using Lemma 2.3, we see that $h_x \in A_p^+(\mathbb{Z})$ for almost every $x \in X$ with a constant independent of x. Then Theorem 2.1 of [16] implies that this is equivalent to the uniform boundedness of the Cesàro-1 averages in $L^p(\mu)$, that is,

$$\sup\{\|R_{n,1}\|, n \geq 0\} < \infty.$$

Furthermore, Theorem 2.1 of [16] (see also [8]) shows that for every $f \in L^p(\mu)$ the averages $R_{n,1}f$ converge in the L^p-norm. Therefore, the set of all functions of the form $h + f - Tf$ with h invariant and f simple is dense in $L^p(\mu)$ (see [7, Corollary VIII.5.2]). The almost everywhere convergence of the Cesàro-α averages is clear for the invariant functions. We shall prove it also for the functions $f - Tf$ with f simple. Then, keeping (i) in mind, by the Banach Principle, the Cesàro-α averages $R_{n,\alpha}f$ converge a.e. for every $f \in L^p(\mu)$. Hence, it only remains to prove the almost everywhere convergence of the Cesàro-α averages for all functions of the form $f - Tf$ with f simple and, clearly, this will follow from the next result.

PROPOSITION 3.2. *Let A be a measurable subset of X with $\mu(A) < \infty$ and let $f = \chi_A$. Then*

$$\lim_{n\to\infty} R_{n,\alpha}(f - Tf) = 0 \quad \text{a.e.} \tag{3.9}$$

Proof of Proposition 3.2. We write $R_{n,\alpha}(f - Tf)$ as the sum of three terms:

$$\begin{aligned}
R_{n,\alpha}(f - Tf)(x) &= \sum_{i=0}^{n} \frac{A_{n-i}^{\alpha-1}}{A_n^\alpha}(T^i f(x) - T^{i+1} f(x)) \\
&= \sum_{i=0}^{n} \frac{A_{n-i}^{\alpha-1}}{A_n^\alpha} T^i f(x) - \sum_{i=1}^{n+1} \frac{A_{n+1-i}^{\alpha-1}}{A_n^\alpha} T^i f(x) \\
&= \frac{A_n^{\alpha-1}}{A_n^\alpha} f(x) - \frac{T^{n+1} f(x)}{A_n^\alpha} + \sum_{i=1}^{n}\left(\frac{A_{n-i}^{\alpha-1} - A_{n+1-i}^{\alpha-1}}{A_n^\alpha}\right) T^i f(x) \\
&= \frac{\alpha}{\alpha + n} f(x) - \frac{T^{n+1} f(x)}{A_n^\alpha} + \frac{1-\alpha}{A_n^\alpha}\sum_{i=0}^{n} \frac{A_{n-i}^{\alpha-1}}{n+1-i} T^i f(x) \\
&= A_n(x) - B_n(x) + C_n(x).
\end{aligned}$$

Clearly $\lim_{n\to\infty} A_n(x) = 0$ a.e. Now, using Lemma 2.3, we see that there exists $\varepsilon > 0$ such that $h_x \in A_{p-\varepsilon;\alpha}^+(\mathbb{Z})$ for almost every $x \in X$ with a constant independent of x. Then part (i) of Theorem 3.1, which has already been proved, implies that the maximal operator M_{α,T_ε} associated with the transformation T_ε, defined in $L^{p-\varepsilon}(\mu)$ by

$$T_\varepsilon \varphi = g_1^r S\varphi, \quad r = \frac{p}{p-\varepsilon},$$

is bounded in $L^{p-\varepsilon}(\mu)$ because the definition of T_ε and property (c) of the operator T imply that for every $\varphi \in L^{p-\varepsilon}(\mu)$ we have

$$\int_X \left|T_\varepsilon^i \varphi(x)\right|^{p-\varepsilon} h_i(x)\, d\mu(x) = \int_X \left|T^i\left(\varphi^{(p-\varepsilon)/p}\right)\right|^p h_i(x)\, d\mu(x) = \int_X |\varphi(x)|^{p-\varepsilon}\, d\mu(x).$$

Then, keeping in mind that S is multiplicative and f is a characteristic function, we have

$$\begin{aligned} B_n(x) &= \frac{T^{n+1} f(x)}{A_n^\alpha} \\ &= \left(\frac{\alpha+1+n}{n+1}\right) \frac{T^{n+1} f(x)}{A_{n+1}^\alpha} \\ &= \frac{\alpha+1+n}{(n+1)A_{n+1}^\alpha} g_{n+1}(x) S^{n+1} f(x) \\ &= \frac{\alpha+1+n}{(n+1)(A_{n+1}^\alpha)^{\varepsilon/p}} \left[\frac{1}{A_{n+1}^\alpha} g_{n+1}^r(x) S^{n+1} f(x)\right]^{1/r} \\ &\le \frac{\alpha+1+n}{(n+1)(A_{n+1}^\alpha)^{\varepsilon/p}} \left[M_{\alpha,T_\varepsilon} f(x)\right]^{1/r}. \end{aligned}$$

Since we have already seen that M_{α,T_ε} is bounded in $L^{p-\varepsilon}(\mu)$, we have $M_{\alpha,T_\varepsilon} f(x) < \infty$ for almost every $x \in X$. Then, using the fact that

$$\lim_{n\to\infty} \frac{\alpha+1+n}{(n+1)(A_{n+1}^\alpha)^{\varepsilon/p}} = 0,$$

we obtain $\lim_{n\to\infty} B_n(x) = 0$ a.e. Finally,

$$\begin{aligned} C_n(x) &= \frac{1-\alpha}{A_n^\alpha} \sum_{i=0}^n \frac{A_{n-i}^{\alpha-1}}{n+1-i} T^i f(x) \\ &= \frac{1-\alpha}{A_n^\alpha} \sum_{i=0}^n \frac{A_{n-i}^{\alpha-1}}{n+1-i} \left[g_i^r(x) S^i f(x)\right]^{1/r} \\ &\le \left[\frac{1}{A_n^\alpha} \sum_{i=0}^n A_{n-i}^{\alpha-1} g_i^r(x) S^i f(x)\right]^{1/r} \\ &\quad \times \left[\frac{1}{A_n^\alpha} \sum_{i=0}^n A_{n-i}^{\alpha-1} \left(\frac{1-\alpha}{n+1-i}\right)^{r'}\right]^{1/r'}, \end{aligned} \tag{3.10}$$

using the Hölder inequality. The first factor of the last term of (3.10) is dominated by $(M_{\alpha,T_\varepsilon} f(x))^{1/r}$. Consequently, in order to show that $C_n(x) \to 0$ almost everywhere it suffices to prove that

$$\lim_{n\to\infty} \frac{1}{A_n^\alpha} \sum_{i=0}^n A_{n-i}^{\alpha-1} \left(\frac{1-\alpha}{n+1-i}\right)^{r'} = 0. \tag{3.11}$$

Note that

$$A_n^\alpha \uparrow \infty \quad \text{as } n \to \infty \tag{3.12}$$

and that

$$\begin{aligned}\frac{\sum_{i=0}^{n+1} A_{n+1-i}^{\alpha-1}\left(\frac{1-\alpha}{n+2-i}\right)^{r'} - \sum_{i=0}^{n} A_{n-i}^{\alpha-1}\left(\frac{1-\alpha}{n+1-i}\right)^{r'}}{A_{n+1}^\alpha - A_n^\alpha} &= \frac{A_{n+1}^{\alpha-1}\left(\frac{1-\alpha}{n+2}\right)^{r'}}{A_{n+1}^\alpha - A_n^\alpha} \\ &= \frac{A_{n+1}^{\alpha-1}\left(\frac{1-\alpha}{n+2}\right)^{r'}}{A_n^{\alpha-1}\left(\frac{\alpha+1}{n+1}\right)} \\ &= \left(\frac{1-\alpha}{n+2}\right)^{r'} \to 0 \\ &\quad \text{as } n \to \infty.\end{aligned} \tag{3.13}$$

Then (3.11) follows from (3.12) and (3.13) using Stolz's criterium. This finishes the proof of Proposition 3.2.

COROLLARY 3.3. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let T be a positive linear operator on $L^p(\mu)$. Suppose that T is invertible and that T^{-1} is also positive. If $0 < \alpha \leq 1$, $\alpha p > 1$ and T is power bounded, i.e.,*

$$\sup_{n \geq 0} \|T^n\| = C < \infty, \tag{3.14}$$

then $h_x \in A_{p;\alpha}^+(\mathbb{Z})$ for almost every $x \in X$ with a constant independent of x and, hence, (i) *and* (ii) *in* Theorem 3.1 *hold.*

Proof. We shall start by proving that, for almost every x, h_x is a quasi-increasing function on $\mathbb{Z}$ with a constant independent of x, that is, there exists a positive constant C such that

$$h_j(x) \leq C h_i(x), \quad \text{for all } i, j \in \mathbb{Z} \text{ with } j \leq i \text{ and a.e. } x \in X. \tag{3.15}$$

Let A be a measurable subset of X with $\mu(A) < \infty$ and let $i \in \mathbb{Z}$. Since T^i is invertible, there exists $f \in L^p(\mu)$ such that $T^i f = \chi_A$. This fact and property (c) of the operator T imply

$$\int_A h_j \, d\mu = \int_X |T^i f|^p h_j \, d\mu = \int_X |T^{i-j} f|^p \, d\mu, \quad j \in \mathbb{Z}. \tag{3.16}$$

Now, (3.14) implies that

$$\int_X |T^{i-j} f|^p \, d\mu \leq C \int_X |f|^p \, d\mu \quad \text{for all } i, j \in \mathbb{Z} \text{ with } j \leq i. \tag{3.17}$$

Using property (c) of the operator T again, we deduce that

$$\int_X |f|^p \, d\mu = \int_X |T^i f|^p h_i \, d\mu = \int_A h_i \, d\mu. \tag{3.18}$$

Then (3.15) follows putting together (3.16), (3.17) and (3.18).

Finally, we shall see that (3.15) implies that $h_x \in A^+_{p;\alpha}(\mathbb{Z})$ for almost every $x \in X$ with a constant independent of x. Let r, k and m be integers with $r \le k \le m$. We may assume without loss of generality that $r < m$. Then, for almost every x, we have

$$\begin{aligned}
&\left(\sum_{i=r}^{k} h_x(i)\right)^{1/p} \left(\sum_{i=k}^{m} h_x(i)^{1-p'} \left(A^{\alpha-1}_{m-i}\right)^{p'}\right)^{1/p'} \\
&\quad \le C h_x(k)^{1/p} \left(\sum_{i=r}^{k} 1\right)^{1/p} \left(h_x(k)^{1-p'}\right)^{1/p'} \left(\sum_{i=k}^{m} \left(A^{\alpha-1}_{m-i}\right)^{p'}\right)^{1/p'} \\
&\quad \le C(m-r+1)^{1/p} \left(\sum_{i=k}^{m} \left(A^{\alpha-1}_{m-i}\right)^{p'}\right)^{1/p'} \\
&\quad \le C(m-r+1)^{1/p} \left(\sum_{i=0}^{m-r} \left(A^{\alpha-1}_{m-r-i}\right)^{p'}\right)^{1/p'}
\end{aligned} \tag{3.19}$$

Since (see [23])

$$A^\alpha_n = \frac{n^\alpha}{\Gamma(\alpha+1)}[1 + O(\frac{1}{n})], \quad \alpha \ne -1, -2, \dots,$$

and $\alpha p > 1$ or, equivalently, $0 \le (1-\alpha)p' < 1$, the last term is dominated by

$$C(m-r+1)^{1/p} \left(\int_0^{m-r} t^{(\alpha-1)p'} \, dt\right)^{1/p'} \le C A^\alpha_{m-r},$$

where the constant C does not depend on x, and therefore $h_x \in A^+_{p;\alpha}(\mathbb{Z})$ for almost every $x \in X$ with a constant independent of x. Then by Theorem 3.1 we are done.

COROLLARY 3.4. *Let T be as in* Theorem 3.1 *and let $\widetilde{T}$ be the operator defined by $\widetilde{T} f = g_1^{1/\alpha} S f$ ($f \in L^{\alpha p}(\mu)$), $\alpha p > 1$, $0 < \alpha \le 1$.(Note that $\widetilde{T} f = (T f^\alpha)^{1/\alpha}$ for nonnegative measurable functions.) If $\widetilde{T}$ is Cesàro-bounded in $L^{\alpha p}(\mu)$, i.e., if*

$$\sup_n \int_X \left| \frac{f + \widetilde{T} f + \dots + \widetilde{T}^n f}{n+1} \right|^{\alpha p} d\mu \le C \int_X |f|^{\alpha p} \, d\mu$$

for all $f \in L^{\alpha p}(\mu)$, then (i) *and* (ii) *of* Theorem 3.1 *hold.*

Proof. By Theorem 2.1 in [16], for almost every x, h_x satisfies $A^+_{\alpha p}(\mathbb{Z})$ with a constant independent of x. By Lemma 2.3, this implies that, for almost every x, h_x satisfies $A^+_{p;\alpha}(\mathbb{Z})$ with a constant independent of x and, therefore, (i) and (ii) in Theorem 3.1 hold.

As a consequence of Corollary 3.4, we immediately obtain the following result.

COROLLARY 3.5. *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $\tau\colon X \to X$ be an invertible nonsingular transformation. Let $Tf = f \circ \tau$, $0 < \alpha \le 1$ and $p\alpha > 1$. If T is Cesàro bounded in $L^{p\alpha}(\mu)$ then* (i) *and* (ii) *of* Theorem 3.1 *hold.*

Remark 3.6. We observe that for $\alpha = 1$, Corollary 3.4 tells us that the uniform boundedness in $L^p(\mu)$, $p > 1$, of the Cesàro-1 averages associated with T is equivalent to the boundedness in $L^p(\mu)$ of the ergodic maximal operator. This result, which is a part of Theorem 2.1 in [16], could induce us to also think that in the case $0 < \alpha < 1$ the uniform boundedness of the Cesàro-α averages and the boundedness of the ergodic maximal operator associated with them are equivalent. The following example shows that, at least in the case $\max\{1/p, 1/p'\} < \alpha < 1$, this equivalence does not hold.

Example 3.7. Let $X = \mathbb{Z}$, the set of the integers, and let μ be the counting measure on $\mathbb{Z}$. Let T be the operator defined for every real-valued function a on $\mathbb{Z}$ by $Ta = a \circ \tau$ where τ is the invertible measure-preserving transformation on $\mathbb{Z}$ defined by $i \mapsto i + 1$. Note that given $0 < \alpha \le 1$, the operator M_α associated with T coincides with the operator m^+_α defined in the previous section. We know that this operator is bounded in $L^p(\mu)$ if, and only if, $p > 1/\alpha$. On the other hand, if $p \ge 1$ and a is a real-valued function defined in $\mathbb{Z}$, we have

$$\left[\sum_{i=-\infty}^{\infty} \left(R_{n,\alpha}|a|(i)\right)^p\right]^{1/p} = \left[\sum_{i=-\infty}^{\infty} \left(\sum_{j=0}^{n} \frac{A_{n-j}^{\alpha-1}}{A_n^\alpha} |a(i+j)|\right)^p\right]^{1/p}$$
$$\le \sum_{j=0}^{n} \frac{A_{n-j}^{\alpha-1}}{A_n^\alpha} \left(\sum_{i=-\infty}^{\infty} |a(i+j)|^p\right)^{1/p} = \|a\|_p,$$

i.e., the averages $R_{n,\alpha}$ are uniformly bounded in $L^p(\mu)$, $p \ge 1$. Therefore, at least in the case $1 \le p \le \frac{1}{\alpha}$, the uniform boundedness of the Cesàro-α averages does not imply the boundedness of the ergodic maximal operator M_α. However, if we want to show the differences between the cases $0 < \alpha < 1$ and $\alpha = 1$ we have to see that even in the "good" case $\alpha p > 1$ the uniform boundedness of the Cesàro-α averages and the boundedness of the maximal operator M_α are not equivalent.

We shall work in the case $1/\alpha < p < 1/(1-\alpha)$, that is, $\alpha p > 1$ and $\alpha p' > 1$, and we shall see that there exist positive measurable functions ω defined on $\mathbb{Z}$ for which the uniform boundedness of the averages in $L^p(\omega\, d\mu)$ does not imply the boundedness of the maximal operator in $L^p(\omega\, d\mu)$. In order to find such a function ω, we shall make use of the adjoints of the averages $R_{n,\alpha}$ in $L^p(d\mu)$ which will be denoted by $R^-_{n,\alpha}$ and are defined for every function a on $\mathbb{Z}$ by

$$R^-_{n,\alpha}a(i) = \frac{1}{A_n^\alpha}\sum_{j=-n}^{0} A_{n+j}^{\alpha-1}a(i+j), \quad i \in \mathbb{Z}.$$

The maximal operator associated with these operators is nothing but the operator m^-_α introduced in §2.

It follows from the results in §2 that our problem will be solved if we find a positive measurable function ω such that

$$\omega^{1-p'} \in A^-_{p';\alpha}(\mathbb{Z}) \quad \text{and} \quad \omega \notin A^+_{p;\alpha}(\mathbb{Z}).$$

In fact, for such a function ω, the analogue of Lemma 2.2 implies the boundedness of the operator m^-_α in $L^{p'}(\omega^{1-p'}\, d\mu)$ and, hence, the uniform boundedness of the averages $R^-_{n,\alpha}$ in the same space. Then, by duality, we obtain the uniform boundedness of the averages $R_{n,\alpha}$ in $L^p(\omega\, d\mu)$. On the other hand, since $\omega \notin A^+_{p;\alpha}(\mathbb{Z})$, the maximal operator $M_\alpha = m^+_\alpha$ is not bounded in $L^p(\omega\, d\mu)$. To finish the example it only remains to show a function ω such that $\omega^{1-p'} \in A^-_{p';\alpha}(\mathbb{Z})$ and $\omega \notin A^+_{p;\alpha}(\mathbb{Z})$. If we take $w_\gamma(i) = (1+|i|)^\gamma$, it follows from Lemma 2.4 that ω_γ satisfies the desired properties if $\alpha p - 1 < \gamma < p - 1$. Note that the example does not include the case $\alpha = 1$ since then $\alpha p - 1 = p - 1$.

4. The Cesàro-α ergodic maximal operator associated with ergodic transformations

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space which is nonatomic if $\mu(X) < \infty$ and let $\tau\colon X \to X$ be an invertible ergodic measure-preserving transformation. We shall work in this section with the Lamperti operator associated with τ and a positive measurable function g, i.e., the operator T defined for all measurable functions f by

$$Tf(x) = g(x)f(\tau x).$$

Of course, this operator is a particular case of the one treated in §3. For that reason, in what follows we shall use the notations introduced in §3.

For the operator T introduced above, with arbitrary positive g, we shall study the characterization of the boundedness of the Cesàro-α ergodic maximal operator in $L^p(w\, d\mu)$, where w is a positive measurable function (see the final remark). It follows from the results of the previous section that if, for almost every x, the functions $h_x(i) = g_i^{-p}(x)w(\tau^i x)/w(x)$ satisfy $A^+_{p;\alpha}(\mathbb{Z})$ with a constant independent of x then M_α is bounded in $L^p(w\, d\mu)$. The goal of this section is to show that the converse holds and that, actually, the boundedness of M_α is equivalent to the uniform boundedness of a countable family of some kind of Cesàro-α averages. In order to fix these averages and to state the theorem we need to introduce some notations, definitions and a lemma.

Definition 4.1. If B is a measurable subset and $x \in \bigcup_{j=0}^{\infty} \tau^{-j} B$ we define

$$n_B(x) = \inf\{k \geq 0\colon\ \tau^k x \in B\}$$

and

$$L_B(x) = \begin{cases} \sup\{j \geq 1\colon\ \tau^{-1}x, \dots, \tau^{-j}x \notin B\}, & \text{if } \{j \geq 1\colon\ \tau^{-1}x, \dots, \tau^{-j}x \notin B\} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $L_B(x)$ can take the value $+\infty$.

Definition 4.2. If B is a measurable subset we define the average $R_{B,\alpha} f$ as

$$R_{B,\alpha} f(x) = \begin{cases} (A^\alpha_{n_B(x)})^{-1} \sum_{i=0}^{n_B(x)} A^{\alpha-1}_{n_B(x)-i} T^i f(x), & \text{if } x \in \bigcup_{j=0}^{\infty} \tau^{-j} B, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$\sup_{B \in \mathcal{F}} R_{B,\alpha} f(x) \leq M_\alpha f(x). \tag{4.1}$$

In fact, it can be proved that the equality holds. Moreover, we can obtain the equality if we take the supremum over a countable family of measurable subsets. In order to determine this countable family we need a definition and a lemma (see [13] and [2]).

Definition 4.3 [2]. Let k be a natural number. The measurable set $B \subset X$ is said to be the base of an (ergodic) rectangle of length $k+1$ if $\tau^i B \cap \tau^j B = \emptyset$ whenever $i \neq j$, $0 \leq i, j \leq k$. In such a case the set $R = \bigcup_{i=0}^{k} \tau^i B$ will be called an (ergodic) rectangle with base B and length $k+1$.

LEMMA 4.4 [13]. *For every nonnegative integer k there exists a countable family of bases of ergodic rectangles of length* $k+1$, $\{B_n^{(k)}\colon\ n \in \mathbb{N}\}$, *such that* $X = \bigcup_n B_n^{(k)}$. *We shall denote by* $\mathcal{B}$ *the family* $\{\tau^k(B_n^{(k)})\colon\ k, n \in \mathbb{N}\}$.

Our first result in this section shows that the countable family $\mathcal{B}$ is enough to obtain the equality in (4.1).

PROPOSITION 4.5. *With the above notations and assumptions we have*

$$\sup_{B \in \mathcal{B}} R_{B,\alpha} f(x) = M_\alpha f(x) \quad \textit{for almost every } x \in X.$$

The uniform boundedness of the averages $\{R_{B,\alpha}\colon B \in \mathcal{B}\}$ is equivalent to the boundedness of the Cesàro-α ergodic maximal operator. This fact is part of the following theorem which is the main result in this section. It shows also that for operators induced by invertible ergodic measure-preserving transformations the converse of part (i) in Theorem 3.1 holds.

THEOREM 4.6. *Let $(X, \mathcal{F}, \mu)$, τ, g, ω and $\mathcal{B}$ be as above. Let $0 < \alpha \leq 1$ and $p > 1$. The following statements are equivalent:*

(1) *There exists $C > 0$ such that*

$$\int_X |M_\alpha f|^p w\, d\mu \leq C \int_X |f|^p w\, d\mu$$

for all $f \in L^p(w\, d\mu)$.

(2) *There exists $C > 0$ such that*

$$\sup_{B\in\mathcal{B}} \int_X |R_{B,\alpha} f|^p w\, d\mu \leq C \int_X |f|^p w\, d\mu$$

for all $f \in L^p(w\, d\mu)$.

(3) *For almost every $x \in X$ the function $h_x(i) = g_i^{-p}(x) w(\tau^i x)/w(x)$ satisfies $A^+_{p;\alpha}(\mathbb{Z})$ with a constant independent of x.*

Proof of Proposition 4.5. Let $X' = \bigcap_k \bigcup_n B_n^{(k)}$. Then $\mu(X \setminus X') = 0$. Therefore, it suffices to prove that the equality holds for every $x \in X'$. Assume that $x \in X'$ and $m \in \mathbb{N}$. Then there exists $B_n^{(m)}$ such that $x \in B_n^{(m)}$. If $B = \tau^m(B_n^{(m)})$ then $B \in \mathcal{B}$ and $n_B(x) = m$. Therefore $R_{m,\alpha} f(x) = R_{B,\alpha} f(x) \leq \sup_{B\in\mathcal{B}} R_{B,\alpha} f(x)$ which proves the proposition.

Proof of Theorem 4.6. It is obvious that (1) implies (2), and (3) $\Rightarrow$ (1) follows from Theorem 3.1. Therefore we only have to prove that (2) $\Rightarrow$ (3). In order to prove this implication we follow ideas of Rubio de Francia (see [9] for instance). For that reason we need to compute the adjoint of $R_{B,\alpha}$.

LEMMA 4.7. *Under the assumptions of* Theorem 4.6, *if B is a measurable subset and $R_{B,\alpha}$ is bounded in $L^p(w\, d\mu)$ then the adjoint of $R_{B,\alpha}$ is the operator $R^*_{B,\alpha}\colon L^{p'}(w\, d\mu) \to L^{p'}(w\, d\mu)$ defined by*

$$R^*_{B,\alpha} h(x) = w^{-1}(x) A^{\alpha-1}_{n_B(x)} \left(\sum_{j=0}^{L_B(x)} \frac{1}{A^\alpha_{j+n_B(x)}} (g_j h w)(\tau^{-j} x) \right) \chi_{\bigcup_{j=0}^\infty \tau^{-j} B}.$$

Proof of Lemma 4.7. Let $B_k = \{x \in \bigcup_{j=0}^{\infty}\tau^{-j}B\colon n_B(x) = k\}$. By the definition of $R_{B,\alpha}$ and since τ preserves the measure μ we obtain

$$\int_X (R_{B,\alpha}f)\, hw\, d\mu = \int_{\bigcup_{i=0}^{\infty}\tau^{-i}B} f(x)\sum_{j=0}^{\infty}(g_j hw)(\tau^{-j}x)\sum_{k=j}^{\infty}\frac{A_{k-j}^{\alpha-1}}{A_k^{\alpha}}\chi_{B_k}(\tau^{-j}x)\, d\mu(x).$$

For fixed x and j, the sum

$$\sum_{k=j}^{\infty}\frac{A_{k-j}^{\alpha-1}}{A_k^{\alpha}}\chi_{B_k}(\tau^{-j}x)$$

is not zero if there exists $k \geq j$ such that $\tau^{-j}x \in B_k$. In this case, it is clear that there exists only one value of k with that property and $k - j = n_B(x)$. Notice also that, for each x, the j's satisfying $\tau^{-j}x \in B_{j+n_B(x)}$ are exactly the j's such that $j \leq L_B(x)$ (see Definition 4.1). Therefore

$$\int_X R_{B,\alpha}f(x)h(x)w(x)\, d\mu(x) = \int_{\bigcup_{i=0}^{\infty}\tau^{-i}B} f(x)\sum_{j=0}^{L_B(x)}(g_j hw)(\tau^{-j}x)\frac{A_{n_B(x)}^{\alpha-1}}{A_{j+n_B(x)}^{\alpha}}\, d\mu(x),$$

and the lemma follows.

Proof of (2) $\Rightarrow$ (3). Assume that (2) holds, i.e., the family $\{R_{B,\alpha}\colon B \in \mathcal{B}\}$ is uniformly bounded in $L^p(w\, d\mu)$. By duality, the family of the adjoint operators $\{R_{B,\alpha}^*\colon B \in \mathcal{B}\}$ is uniformly bounded in $L^{p'}(w\, d\mu)$. Therefore, there exists a constant $C > 0$ such that for all $B \in \mathcal{B}$,

$$\begin{aligned}\|R_{B,\alpha}f\|_{p,w\,d\mu} &= \left(\int_X |R_{B,\alpha}f|^p w\, d\mu\right)^{1/p}\\ &\leq C\|f\|_{p,w\,d\mu} \quad \text{for every } f \in L^p(w\, d\mu)\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\|R_{B,\alpha}^*f\|_{p',w\,d\mu} &= \left(\int_X |R_{B,\alpha}^*f|^{p'} w\, d\mu\right)^{1/p'}\\ &\leq C\|f\|_{p',w\,d\mu} \quad \text{for every } f \in L^{p'}(w\, d\mu).\end{aligned} \tag{4.3}$$

For every $B \in \mathcal{B}$, let us consider the sublinear operators P_B and Q_B defined, on every measurable function f, by

$$P_B f = \left(R_{B,\alpha}|f|^{p'}\right)^{1/p'} \quad \text{and} \quad Q_B f = \left(R_{B,\alpha}^*|f|^p\right)^{1/p}.$$

It is clear from (4.2) and (4.3) that the family of sublinear operators $\{S_B = P_B + Q_B\colon B \in \mathcal{B}\}$ is uniformly bounded in $L^{pp'}(w\, d\mu)$. Now let f be any positive function in $L^{pp'}(w\,d\mu)$ and define

$$h_B = \sum_{i=0}^{\infty} \frac{S_B^{(i)} f}{(2C)^i}$$

where $S_B^{(i)}$ is the i-th power of S_B and C is a constant such that $\|S_B\| \le C$ for all $B \in \mathcal{B}$. Then $h_B \in L^{pp'}(w\,d\mu)$, h_B is positive ($h_B \ge f > 0$), $\|h_B\|_{pp',w\,d\mu} \le 2\|f\|_{pp',w\,d\mu}$ and

$$S_B h_B \le 2C h_B \quad \text{a.e.}$$

Since P_B and Q_B are positive sublinear operators, it follows that

$$P_B h_B \le 2C h_B \text{ a.e. and } Q_B h_B \le 2C h_B \text{ a.e.},$$

which is the same as

$$R_{B,\alpha} h_B^{p'} \le (2C h_B)^{p'} \text{ a.e.} \tag{4.4}$$

and

$$R_{B,\alpha}^* h_B^p \le (2C h_B)^p \text{ a.e.} \tag{4.5}$$

Let $u_B = h_B^p w$ and $v_B = h_B^{p'}$. By these definitions $w = u_B v_B^{1-p}$. Therefore

$$w(x)h_x(i) = g_i^{-p}(x) u_B(\tau^i x) v_B^{1-p}(\tau^i x) = g_i^{-1}(x) u_B(\tau^i x)(g_i(x) v_B(\tau^i x))^{1-p}. \tag{4.6}$$

Once we have the functions $w(x)h_x(i)$ factorized in this fashion, we are going to prove that (3) holds, i.e., there exists a positive constant C such that

$$\left(\sum_{i=0}^{k} \omega(x) h_x(i+r)\right)^{1/p} \left(\sum_{i=k}^{n} (\omega(x) h_x(i+r))^{1-p'} \left(A_{n-i}^{\alpha-1}\right)^{p'}\right)^{1/p'} \le C A_n^\alpha$$

for almost every $x \in X$ and all r, k and n in $\mathbb{Z}$ with $0 \le k \le n$.

Let r, k and n be in $\mathbb{Z}$ with $0 \le k \le n$. Notice that if $n = 0$, the above inequality holds for every $x \in X$ with any constant $C \ge 1$ and, hence, it suffices to prove it in the case $n > 0$. Observe, also, that if we define X' as in the proof of Proposition 4.5, i.e., $X' = \bigcap_k \bigcup_n B_n^{(k)}$, then for almost every $x \in X'$ and all $r \in \mathbb{Z}$ we have $\tau^r x \in X'$ and, therefore, for fixed r, we have that, for almost every $x \in X'$, there exists $B \in \mathcal{B}$ such that $n_B(\tau^r x) = n$. Applying (4.5) to $\tau^{i+r} x$ with $k \le i \le n$, we obtain

$$A_{n_B(\tau^{i+r}x)}^{\alpha-1} \sum_{j=0}^{L_B(\tau^{i+r}x)} \frac{1}{A_{j+n_B(\tau^{i+r}x)}^\alpha} (g_j u_B)(\tau^{i+r-j} x) \le (2C)^p u_B(\tau^{i+r} x) \quad \text{a.e.}$$

Now observe that $n_B(\tau^{i+r}x) = n - i$ and $L_B(\tau^{i+r}x) \ge i$ since $n > 0$. Therefore, we have

$$A_{n-i}^{\alpha-1} \sum_{j=0}^{i} \frac{1}{A_{j+n-i}^\alpha} (g_j u_B)(\tau^{i+r-j} x) \le (2C)^p u_B(\tau^{i+r} x) \quad \text{a.e.},$$

or, changing the variable,

$$A_{n-i}^{\alpha-1}\sum_{l=0}^{i}\frac{1}{A_{n-l}^{\alpha}}(g_{i-l}u_B)(\tau^{l+r}x)\leq(2C)^p u_B(\tau^{i+r}x)\quad\text{a.e.}$$

Since $0\leq k\leq i$, we obtain

$$A_{n-i}^{\alpha-1}\sum_{l=0}^{k}\frac{1}{A_{n-l}^{\alpha}}(g_{i-l}u_B)(\tau^{l+r}x)\leq(2C)^p u_B(\tau^{i+r}x)\quad\text{a.e.}$$

Multiplying by $g_{-(i+r)}(\tau^{i+r}x)=g_{i+r}^{-1}(x)$ and taking into account that

$$\begin{aligned} g_{-(i+r)}(\tau^{i+r}x)g_{i-l}(\tau^{l+r}x) &= g_{-(i+r)}(\tau^{i+r}x)g_{(i+r)-(l+r)}(\tau^{l+r}x)\\ &= g_{-(l+r)}(\tau^{l+r}x)=g_{l+r}^{-1}(x), \end{aligned}$$

we get, for all $i\in\mathbb{Z}$ with $k\leq i\leq n$ and for almost every x, the inequality

$$A_{n-i}^{\alpha-1}\sum_{l=0}^{k}\frac{1}{A_{n-l}^{\alpha}}g_{l+r}^{-1}(x)u_B(\tau^{l+r}x)\leq(2C)^p g_{i+r}^{-1}(x)u_B(\tau^{i+r}x). \tag{4.7}$$

Now let $i\in\mathbb{N}$ with $0\leq i\leq k$. Applying (4.4) to $\tau^{r+i}x$ and keeping in mind that $n_B(\tau^{r+i}x)=n-i$ we obtain

$$\frac{1}{A_{n-i}^{\alpha}}\sum_{j=0}^{n-i}A_{n-i-j}^{\alpha-1}g_j(\tau^{r+i}x)v_B(\tau^{r+i+j}x)\leq(2C)^{p'}v_B(\tau^{r+i}x)\text{ a.e.}$$

Changing the variable, we have

$$\frac{1}{A_{n-i}^{\alpha}}\sum_{l=i}^{n}A_{n-l}^{\alpha-1}g_{l-i}(\tau^{r+i}x)v_B(\tau^{r+l}x)\leq(2C)^{p'}v_B(\tau^{r+i}x)\text{ a.e.}$$

Since $i\leq k$ we have

$$\frac{1}{A_{n-i}^{\alpha}}\sum_{l=k}^{n}A_{n-l}^{\alpha-1}g_{l-i}(\tau^{r+i}x)v_B(\tau^{r+l}x)\leq(2C)^{p'}v_B(\tau^{r+i}x)\text{ a.e.}$$

Multiplying by $g_{r+i}(x)$ we obtain, for all $i\in\mathbb{Z}$ with $0\leq i\leq k$ and almost every x, the inequality

$$\frac{1}{A_{n-i}^{\alpha}}\sum_{l=k}^{n}A_{n-l}^{\alpha-1}g_{r+l}(x)v_B(\tau^{r+l}x)\leq(2C)^{p'}g_{r+i}(x)v_B(\tau^{r+i}x). \tag{4.8}$$

By (4.6) and (4.8),

$$\left(\sum_{i=0}^{k} \omega(x) h_x(i+r)\right)^{1/p} \tag{4.9}$$

$$= \left(\sum_{i=0}^{k} g_{i+r}^{-1}(x) u_B(\tau^{i+r}x)(g_{i+r}(x) v_B(\tau^{i+r}x))^{1-p}\right)^{1/p}$$

$$\leq 2C \left(\sum_{i=0}^{k} g_{i+r}^{-1}(x) u_B(\tau^{i+r}x)(A_{n-i}^{\alpha})^{p-1}\right)^{1/p} \left(\sum_{l=k}^{n} A_{n-l}^{\alpha-1} g_{l+r}(x) v_B(\tau^{l+r}x)\right)^{-1/p'}$$

for almost every x. On the other hand, by (4.6) and (4.7),

$$\left(\sum_{i=k}^{n} (\omega(x) h_x(i+r))^{1-p'} (A_{n-i}^{\alpha-1})^{p'}\right)^{1/p'} \tag{4.10}$$

$$= \left(\sum_{i=k}^{n} (g_{i+r}^{-1}(x) u_B(\tau^{i+r}x))^{1-p'} (A_{n-i}^{\alpha-1})^{p'-1} A_{n-i}^{\alpha-1} g_{i+r}(x) v_B(\tau^{i+r}x)\right)^{1/p'}$$

$$\leq 2C \left(\sum_{i=k}^{n} A_{n-i}^{\alpha-1} g_{i+r}(x) v_B(\tau^{i+r}x)\right)^{1/p'} \left(\sum_{l=0}^{k} \frac{1}{A_{n-l}^{\alpha}} g_{l+r}^{-1}(x) u_B(\tau^{l+r}x)\right)^{-1/p}.$$

Multiplying (4.9) and (4.10) we get

$$\left(\sum_{i=0}^{k} \omega(x) h_x(i+r)\right)^{1/p} \left(\sum_{i=k}^{n} (\omega(x) h_x(i+r))^{1-p'} (A_{n-i}^{\alpha-1})^{p'}\right)^{1/p'}$$

$$\leq (2C)^2 \left(\sum_{i=0}^{k} g_{i+r}^{-1}(x) u_B(\tau^{i+r}x)(A_{n-i}^{\alpha})^{p-1}\right)^{1/p} \left(\sum_{l=0}^{k} \frac{1}{A_{n-l}^{\alpha}} g_{l+r}^{-1}(x) u_B(\tau^{l+r}x)\right)^{-1/p}$$

for almost every x, and, finally, since the coefficients A_{n-i}^{α} increase [23], we obtain

$$\left(\sum_{i=0}^{k} \omega(x) h_x(i+r)\right)^{1/p} \left(\sum_{i=k}^{n} (\omega(x) h_x(i+r))^{1-p'} (A_{n-i}^{\alpha-1})^{p'}\right)^{1/p'} \leq (2C)^2 A_n^{\alpha} \quad \text{a.e.},$$

which proves (3).

Remark 4.8. Assume, for a while, that $g = 1$ and τ is an invertible measurable transformation which is nonsingular with respect to a finite measure ν. If $0 < \alpha < 1$ and the Cesàro-α averages $R_{n,\alpha} f$ converge a.e. for every $f \in L^p(d\nu)$, then the same happens for the Cesàro-1 averages. Then it is known (see [15]) that the measure ν is equivalent to a finite measure μ which is preserved by τ. That is the reason why we have worked with the measures $\omega\, d\mu$.

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