ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 3, 1999

GENERAL AND WEIGHTED AVERAGES OF ADMISSIBLE SUPERADDITIVE PROCESSES

DOĞAN ÇÖMEZ

ABSTRACT. It is shown that if $\{\mu_n\}$ is a sequence of measures good a.e. or resp. in the p-mean for additive processes, then it is good a.e. or resp. in the p-mean, for the class of strongly bounded admissible superadditive processes. Using the method developed, it is shown also that weighted averages of strongly bounded admissible superadditive processes converge a.e. or in the p-mean for weights that are good a.e. or in the p-mean for additive processes.

1. Introduction

Using some techniques of harmonic analysis, Bellow, Jones and Rosenblatt [BeJR$_2$] and Rosenblatt [R] studied the behaviour of weighted and general averages and obtained various conditions on (probability) sequences $\{\mu_n\}$ that ensure the a.e. and norm convergence. In particular, moving averages sequences satisfying the cone condition are good a.e. (and in the p-mean, $1 \le p < \infty$) [BeJR$_1$]. It is proved in [F] and [ÇF] that such sequences are also good a.e. and in the 1-mean for bounded superadditive processes, respectively. Recently, in [Ç] it was shown that moving averages sequences are good in the p-mean for admissible superadditive processes (see the definitions below). In the spirit of these results, it is natural to ask the following questions:

If $\{\mu_n\}$ is a sequence of probabilities which is good a.e. or in the 2-mean (hence good in the p-mean), is it also good a.e. or in the mean for superadditive processes? If not, for what type of superadditive processes (if any) is the answer affirmative?

The question of which sequences $\{\mu_n\}$ are good in the p-mean (a.e.) for superadditive processes is a delicate problem: there are some simple sequences $\{\mu_n\}$ which are good in the 2-mean (a.e.) for additive processes but not so for superadditive processes (see the example below as well as the ones in [ÇF]). For some class of superadditive processes, however, one can obtain an affirmative answer to this question. In this article, we show that if $\{\mu_n\}$ is good in the p-mean (a.e.), then it is good in the p-mean (a.e.) for some classes of bounded superadditive processes.

Let (X, Σ, m) be a probability space and $T\colon X \to X$ an invertible measure-preserving transformation (MPT). Given a probability measure μ on $\mathbb{Z}$, and $f\colon X \to \mathbb{C}$, define $\mu f(x) = \mu(T)f(x) = \sum_{k=-\infty}^{\infty} \mu(k)f(T^k x)$ for all $x \in X$. If $f \in L_p$,

Received August 4, 1998.
1991 Mathematics Subject Classification. Primary 47A35; Secondary 28D99.
This work was supported in part by ND-EPSCoR through a grant from the National Science Foundation.

then $\mu f \in L_p$, with $\|\mu f\|_p \le \|f\|_p$, for all $1 \le p \le \infty$. The Fourier transform of μ is defined by $\hat{\mu}(z) = \sum_{k=-\infty}^{\infty} \mu(k)\bar{z}^k$ for $z \in \Gamma$, the unit circle in the complex plane $\mathbb{C}$. If $\mathbf{n} = \{n_k\}$ is a sequence of positive integers, then the averages $\frac{1}{N}\sum_{i=0}^{N-1} f(T^{n_i}x)$ along $\mathbf{n}$ can also be defined as $\mu_N^{\mathbf{n}} f(x)$ using the sequence of probabilities $\mu_N^{\mathbf{n}} = \frac{1}{N}\sum_{k=0}^{N-1} \delta_{n_k}$, where δ_j is the unit (point) mass at $j \in \mathbb{Z}$. Similarly, if $\mathbf{a} = \{a_i\}$ is a sequence of weights, the weighted averages $\frac{1}{N}\sum_{i=0}^{N-1} a_i T^i f(x)$ are defined by the sequence of (signed) measures $\mu_N^{\mathbf{a}} f(x)$, where $\mu_N^{\mathbf{a}} = \frac{1}{N}\sum_{k=0}^{N-1} a_i \delta_i$. In this paper, the averages $\mu_n f$ will be referred to as *general averages*, whereas $\mu_N^{\mathbf{a}} f$ will be referred to as *weighted averages*. We will write $T^i f$ instead of $f \circ T^i$. For technical reasons, each $\{\mu_n\}$ is assumed to be uniformly dissipative , i.e., $\lim_{n\to\infty}(\sup_i |\mu_n(i)|) = 0$.

A sequence $\{\mu_n\}$ is called *good in the p-mean for T* if $\lim_n \mu_n(T)f(x)$ exists in L_p-norm for all $f \in L_p$, and is called *good a.e. in L_p for T* if $\lim_n \mu_n(T)f(x)$ exists a.e. for all $f \in L_p$. If $\{\mu_n\}$ is good a.e. (good in the p-mean) in L_p for all MPTs, it is called *good a.e. in L_p (good in the p-mean)*. We will say that $\{\mu_n\}$ *admits a maximal inequality* if $m\{x\colon \sup_n |\mu_n f(x)| > \alpha\} \le \frac{C_1}{\alpha}\|f\|_1$, for $f \in L_1$, $\alpha > 0$, when $p = 1$, or if $\|\sup_n |\mu_n f(x)|\|_p \le C_p\|f\|_p$ for $f \in L_p$ when $1 < p < \infty$, where C_1, C_p are constants (which may not be the same at each appearance below).

A family $F = \{F_n\}_{n\ge 1} \subset L_p$ is called a *T-superadditive process* if $F_{n+m} \ge F_n + T^n F_m$ for all $n, m \ge 0$. If the equality holds, it is called *T-additive*, and if the reverse inequality holds, it is called *T-subadditive*. T-additive processes are necessarily of the form $F_n = \sum_{i=0}^{n-1} T^i F_1$. A T-superadditive process $F \subset L_1$ is called *bounded* if $\sup_{n\ge 1} \frac{1}{n}\|F_n\|_1 < \infty$. If F is a T-superadditive process, then $F_n \ge \sum_{i=0}^{n-1} T^i F_1$ for all $n \ge 1$, hence $F_n' = F_n - \sum_{i=0}^{n-1} T^i F_1$ is a positive superadditive process (and necessarily increasing). It also follows that, if a result is valid for additive processes, then the same holds for F if and only if it holds for F'.

In order to define the general averages of a T-superadditive process $F = \{F_n\}$ along a sequence $\{\mu_n\}$, it will be convenient to view F as a sequence of functions $\{f_k\}_{k\ge 0} \subset L_p$ with partial sums $F_n = \sum_{i=0}^{n-1} f_i$ satisfying $T^m F_n \le F_{m+n} - F_m$, $m, n \ge 1$. Thus, if F is positive, $f_i \ge 0$, for all $i \ge 0$. Furthermore, F is called *strongly bounded* if $\sup_n \|f_n\|_p < \infty$. If $\{\mu_n\}$ is a sequence of measures on $\mathbb{Z}$, and F is a T-superadditive process, we will define $\mu_n F = \sum_{k\in\mathbb{Z}} \mu_n(k) f_k$ (for $k < 0$, let $f_k \equiv 0$.) If F is a T-superadditive process, a sequence of measures $\{\mu_n\}$ is called *good in the p-mean (a.e.) for F* if $\lim_n \mu_n F$ exists in L_p-norm (a.e.)

2. Convergence of general averages

In [DK], Derriennic and Krengel constructed examples of superadditive processes in L_2 satisfying $\sup_n \frac{1}{n}\|F_n\|_2 < \infty$ such that $\mu_n^{\mathbf{n}} F$ does not converge in the norm, where $\mathbf{n}$ is the full sequence of positive integers. Hence, one needs some stronger hypotheses to obtain the norm convergence of superadditive processes when $p > 1$. Indeed, as the following example shows, the problem of obtaining subsequential results for superadditive processes seems more intricate than expected (see also [ÇF]).

Example. Let $f_n = (-1)^n$, $n \geq 0$. Clearly $F = \{F_n\}$ is a bounded subadditive process (on a one point space). If **n** is the sequence of odd (or even) integers or the the sequence of squares or primes, then clearly $\mu_n^{\mathbf{n}} F$ converges. Now, we will define, inductively, a sequence $\{n_k\}$ such that $\lim_N \frac{1}{N} \sum_{k=0}^{N-1} f_{n_k}$ fails to exist. To see this, let $n_0 = 0,\ n_1 = 1,\ n_2 = 2,\ n_3 = 4,\ n_4 = 5,\ n_5 = 7$, and

if $0 \leq j < 3^i 2$, let $n_{3^i 2 + j}$ be the next $3^i 2$ even numbers after $n_{3^i 2 - 1}$,

if $0 \leq j < 3^i 2$, let $n_{3^i 4 + j}$ be the next $3^i 2$ odd numbers after $n_{3^i 4 - 1}$.

Then $\frac{1}{N} \sum_{k=0}^{N-1} f_{n_k} = 0$ if $N = 3^i 2$, and is $\frac{1}{2}$ if $N = 3^i 4$. Hence, $\liminf_N \frac{1}{N} \sum_{k=0}^{N-1} f_{n_k} = 0$, whereas $\limsup_N \frac{1}{N} \sum_{k=0}^{N-1} f_{n_k} = \frac{1}{2}$.

These examples suggest that if a sequence $\{\mu_n\}$ is good in the p-mean (a.e.), we cannot expect it be good in the p-mean (a.e.) for superadditive processes. Below, we will show that this pathology does not exist for some classes of superadditive processes.

Definition. A sequence $\{f_n\}_{n \geq 0} \subset L_p$ of functions is said to be a *Chacon T-admissible family*, (or simply *T-admissible*) if $Tf_i \leq f_{i+1}$ for $i \geq 0$.

Clearly, if $\{f_n\}$ is a T-admissible family, then the sequence $\{F_n\}_{n \geq 1}$, where $F_n = \sum_{i=0}^{n-1} f_i$, is a T-superadditive process, called a *T-admissible process*. Observe that the process F in the example above is *not* admissible.

Remarks. 1. When $p = 1$, strong boundedness follows from the boundedness and admissibility [ÇF]. However, when $p > 1$, the condition $\sup_n \frac{1}{n} \| F_n \|_p < \infty$ and admissibility need not imply that $\sup_n \| f_n \|_p < \infty$. One condition that implies strong boundedness is $\sup_n \frac{1}{n} \| F_n \|_p^p < \infty$, which, on the other hand, is too strong to include any nonconstant positive superadditive processes.

2. If a superadditive processes is strongly bounded, then $\sup_{n \geq 1} \frac{1}{n} \| F_n \|_p < \infty$. However, as remarked above, when $p > 1$, the converse implication is not valid, even if F is admissible. As it is written, in Proposition 2.7 of [Ç] such a claim was made, hence the proof of that statement is incomplete. Nevertheless, there, the proofs of all the norm convergence results for $p > 1$ follow without any change if the processes are assumed to be strongly bounded. (Or they can be obtained as a corollary of Theorem 2.1 below.)

By the remarks above, strong boundedness and boundedness are the same for the admissible processes in L_1. That is why, in the following, when $p = 1$, if we make the assumption of strong boundedness for an admissible process, it will not be a further restriction than boundedness.

THEOREM 2.1. *Let $F \subset L_p$ be a strongly bounded T-admissible process, $1 \leq p < \infty$. If $\{\mu_n\}$ is a sequence of uniformly dissipative probabilities good in the 2-mean for T-additive processes, then $\mu_n F$ converges in the p-mean.*

Proof. We will employ the idea in [ÇF]. By assumption $\{\mu_n\}$ is good in the p-mean for additive processes, hence we can assume (if necessary by passing to F') that $f_i \geq 0$ for each $i \geq 1$. Let $P_i = f_i - Tf_{i-1}$, where $P_0 = f_0$. Hence $P_i \geq 0$ for $i \geq 1$. Then by Clarkson's inequalities (when $p > 1$) and admissibility, $\int P_i^p \leq C_p(\|f_i\|_p^p - \|f_{i-1}\|_p^p) < \infty$, where C_p is a constant depending on p only [Ç]. Fix $k \in \mathbb{Z}^+$, and define

$$g_i^k(x) = \begin{cases} f_k(T^{i-k}x) & \text{for } i > k \\ f_i(x) & \text{for } 0 \leq i \leq k \end{cases}$$

Then, it follows that

$$f_i(x) - g_i^k(x) = \begin{cases} 0 & \text{if } 0 \leq i \leq k \\ \sum_{j=1}^m P_{k+j}(T^{m-j}x) & \text{for } i > k, \text{where } m = i - k. \end{cases}$$

Now, define

$$D_n(x) = \sum_{i=0}^{\infty} \mu_n(i)(f_i(w) - g_i^k)(x).$$

Then $D_n(x) \leq \sum_{i=0}^{\infty} \mu_n(i) \sum_{r=k+1}^{i} P_r(T^{i-r}x)$, where the terms on the void sets are zero. If

$$b_{k,s}(w) = \sum_{r=k+1}^{s} P_r(T^r w) \quad \text{and} \quad b_k(w) = \lim_{s\to\infty} b_{k,s}(w),$$

then $b_{k,s} \geq 0$, $b_k \geq 0$. Using the Lebesgue monotone convergence theorem and strong boundedness, we obtain

$$\int_X b_k^p \, dm = \lim_{s\to\infty} \int_X b_{k,s}^p \, dm \leq \sum_{r=k+1}^{\infty} \int P_r^p \leq C_p \lim_{j\to\infty} \|f_j\|_p^p < \infty.$$

Because $b_{k,s} \uparrow b_k$ and $b_k \in L_p$ we conclude that $T^j b_{k,s} \uparrow T^j b_k$ in L_p, for all j, since T is strongly continuous. Therefore,

$$D_n(x) \leq \sum_{i=0}^{\infty} \mu_n(i) T^i b_k.$$

Consequently, $\|D_n\|_p^p \leq \|b_k\|_p^p \leq \sum_{r=k+1}^{\infty} \int P_r^p \downarrow 0$, as $k \to \infty$.

By assumption, $G_k := L_p - \lim_{n\to\infty} \sum_{i=0}^{\infty} \mu_n(i) g_i^k$ exists. Since, for all $n \geq 1$, $g_n^k \leq g_n^{k+1}$, we also have $G^k \leq G^{k+1}$. Therefore, $\{G^k\}$ is a monotone increasing bounded sequence of functions in L_p, and consequently, $G = \lim_{k\to\infty} G^k$ exists in L_p. Now, given $\epsilon > 0$, find a positive integer K such that for $k \geq K$, $\|b_k\|_p^p < \epsilon/3$, $\|\sum_{i=0}^{\infty} g_i^k - G^k\|_p^p < \epsilon/3$, and $\|G - G^k\|_p^p < \epsilon/3$. Then,

$$\left\|\sum_{i=0}^{\infty} \mu_n(i) f_i - G\right\|_p^p \leq \left\|\sum_{i=0}^{\infty} \mu_n(i)(f_i - g_i^k)\right\|_p^p + \left\|\sum_{i=0}^{\infty} \mu_n(i) g_i^k - G^k\right\|_p^p + \left\|G - G^k\right\|_p^p < \epsilon,$$

proving the assertion. ■

Remarks. 1. If $\mu = \frac{1}{2}(\delta_0 + \delta_1)$, then the sequence of measures $\{\mu^n\}$ is good in the p-mean, $1 \leq p < \infty$, where μ^n denotes the n times convolution of μ by itself. Hence it is good in the p-mean for strongly bounded admissible processes.

2. Sequences of probabilities having asymptotically trivial transforms, i.e., $\{\mu_n\}$ with $\mu_n(\gamma) \to 0$ for all $\gamma \in \Gamma$, $\gamma \neq 1$, are good in the p-mean [BeJR2], hence they are good in the p-mean for admissible processes as well. By the same token, if μ is strictly aperiodic, then $\{\mu^n\}$ is good in the p-mean for additive processes, hence good in the p-mean for strongly bounded admissible processes. Another interesting example is the sequence $\mu_n^{\mathbf{n}} = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{[\sqrt{k}]}$ studied in [JW]. (Using Lemma 2.3 in [JW], and applying the idea employed in Example 2.4 there, it is straightforward to show that $\{\mu_n^{\mathbf{n}}\}$ has asymptotically trivial transforms.)

The main result in [ÇF] states that if a sequence of strictly increasing positive integers $\mathbf{n}$ is good in the p-mean for superadditive processes relative to MPTs, then it is good in the p-mean for T-superadditive processes, where T is a positive L_p-contraction when $1 < p < \infty$, or a Dunford-Schwartz operator on L_1. The same result also holds if the processes involved are admissible. Theorem 2.1, combined with this observation gives:

COROLLARY 2.2. *If* $\mathbf{n}$ *is good in the 2-mean, then it is good in the p-mean for strongly bounded admissible T-superadditive processes, where T is a positive L_p-contraction , $1 < p < \infty$, or a Dunford-Schwartz operator on L_1.*

In [ÇF] it has also been proved that moving averages sequences satisfying the *cone condition* are good a.e. for admissible processes relative to MPTs. This result has been extended in [Ç] to superadditive processes relative to positive L_p-contactions, $1 < p < \infty$,. It turns out that same conclusions can be drawn for the general averages of admissible processes:

THEOREM 2.3. *Let $F \subset L_p$ be a strongly bounded T-admissible process, $1 \leq p < \infty$. If $\{\mu_n\}$ is a uniformly dissipative sequence of probabilities that admits a* *maximal inequality and is good a.e. for T-additive processes, then $\{\mu_n\}$ is good a.e. for F.*

Proof. Since $\{\mu_n\}$ is good a.e. for T, we can assume that F is positive. We will use the same setup as in Theorem 2.1 and consider the cases $p = 1$ and $1 < p < \infty$ separately. First assume $p = 1$, and let

$$f^*(x) = \limsup_n \mu_n F(x) \quad \text{and} \quad f_*(x) = \liminf_n \mu_n F(x).$$

Given $\alpha > 0$, define $E = \{x\colon\ f^*(x) - f_*(x) > \alpha\}$. In order to prove a.e. convergence of $\mu_n F$ it is enough to show that $m(E) = 0$. To do this, let $H_k(x) = \lim_n \sum_{i=0}^{\infty} \mu_n(i) f_k(T^{i-k}x)$, which exists a.e. by assumption. Furthermore, the uniform dissipativity of $\{\mu_n\}$ implies that $H_k(x) = \lim_n \sum_{i=0}^{\infty} \mu_n(i) g_i^k(x)$ a.e. Now, since $D_n(x) \le \sum_{i=0}^{\infty} \mu_n(i) T^i b_k$,

$$E \subset \left\{x\colon\ \sup_n D_n(x) > \frac{\alpha}{2}\right\} \subset \left\{x\colon\ \sup_n \sum_{i=0}^{\infty} \mu_n(i) T^i b_k(x) > \frac{\alpha}{2}\right\}.$$

By hypothesis, the additive process $\{\sum_{i=0}^{n-1} T^i b_k\}$ admits a maximal inequality along $\{\mu_n\}$. Therefore,

$$m(E) \le m\left(\left\{x\colon\ \sup_n \sum_{i=0}^{\infty} \mu_n(i) T^i b_k(x) > \frac{\alpha}{2}\right\}\right) \le \frac{2C_1}{\alpha}\int b_k\, dm \le \frac{2C_1}{\alpha}\sum_{r=k+1}^{\infty}\int P_r\, dm.$$

By letting $k \to \infty$, we obtain $m(E) = 0$.

Next, assume $1 < p < \infty$. In this case, for a fixed integer $k \ge 1$, consider the "additive" process $G = \{\sum_{i=0}^{n-1} g_i^k\}$. Then

$$\begin{aligned}
0 &\le \int (f^* - f_*)\, dm \le 2\int \limsup_n |\mu_n F - \mu_n G|\, dm \\
&\le 2\int \sup_n |\mu_n F - \mu_n G|\, dm \\
&\le 2\int \sup_n |\mu_n F - \mu_n G|^p\, dm \le 2\int \sup_n |D_n|^p\, dm \\
&\le 2\int \sup_n \left|\sum_{i=0}^{\infty} \mu_n(i) T^i b_k\right|^p dm \le C_p\, \|b_k\|_p^p,
\end{aligned}$$

where the last inequality follows from the hypothesis that $\{\mu_n\}$ admits a maximal inequality. Since $\|b_k\|_p \downarrow 0$, $\lim_n \mu_n F$ exists a.e. ■

3. Convergence of weighted averages

In this section we will obtain another consequence of the machinery employed in the previous section, namely, a.e. and norm convergence of the *weighted* averages of admissible processes.

For $1 \leq p < \infty$, the class W_p of complex sequences is defined as

$$W_p = \left\{ \mathbf{a} = (a_i) \colon \limsup_n \frac{1}{n} \sum_{k=0}^{n-1} |a_k|^p < \infty \right\},$$

which contains unbounded sequences as well. W_∞ is the class of all bounded sequences. On W_p we have a seminorm, called *p-seminorm,* defined by $\|\mathbf{a}\|_{W_p}^p = \limsup_n \frac{1}{n} \sum_{k=0}^{n-1} |a_k|^p$. (We will also write $\|\mathbf{a}\|_{W_\infty} = \|\mathbf{a}\|_\infty$.) Clearly, $W_q \subset W_p$ if $1 \leq p < q \leq \infty$. A sequence $\mathbf{a}$ of complex numbers is said to *have a mean* if $\lim_n \frac{1}{n} \sum_{i=0}^{n-1} a_i$ exists.

If F is a T-superadditive process and $\mathbf{a} = \{a_k\}$ is a sequence of weights, then we define the *weighted* averages of F along the sequence of weights $\mathbf{a}$ by $\mu_n^{\mathbf{a}} F = \frac{1}{n} \sum_{k=0}^{n-1} a_k f_k$. $\mathbf{a} = \{a_k\}$ is called *good in the p-mean (a.e.) for F* if $\lim \mu_n^{\mathbf{a}} F$ exists in L_p-norm (a.e.)

THEOREM 3.1. *Let $F \subset L_p$ be a strongly bounded T-admissible process, $1 \leq p < \infty$. Assume that $\mathbf{a} \in W_\infty$ when $p = 1$, or $\mathbf{a} \in W_q$ when $1 < p < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathbf{a}$ is good a.e. for additive processes, then it is good a.e. for F.*

Proof. Again, we will assume that F is positive and use the same setup as in the proof of Theorem 2.1. Let $f^* = \limsup_n \mu_n^{\mathbf{a}} F$ and $f_* = \liminf_n \mu_n^{\mathbf{a}} F$. When $p = 1$,

$$0 \leq f^* - f_* \leq 2 \limsup_n \frac{1}{n} \left| \sum_{i=0}^{n-1} a_i (f_i - g_i^k) \right| \leq 2 \|\mathbf{a}\|_\infty \sup_n \frac{1}{n} \left| \sum_{i=0}^{n-1} T^i b_k \right|.$$

If $E = \{x \colon (f^* - f_*)(x) > \alpha\}$, $\alpha > 0$, then

$$m(E) \leq m\left(\left\{ x \colon \sup_n \frac{1}{n} \left| \sum_{i=0}^{n-1} T^i b_k \right| > \frac{\alpha}{2 \|\mathbf{a}\|_\infty} \right\} \right) \leq \frac{2 \|\mathbf{a}\|_\infty C_1}{\alpha} \|b_k\|_1.$$

Since $\|b_k\|_1 \downarrow 0$ as $k \to \infty$, we have $m(E) = 0$.

When $1 < p < \infty$, observe first that the sequence $\{v_i\} \subset L_p$, where $v_i = T^{-i} f_i$, is increasing. Hence $v_i \uparrow v$ for some $v \in L_p$, and $f_i \leq T^i v$ for all $i \geq 1$. Define sequences $\mathbf{w}$ by $w_i(x) = f_i(x)$, and $\mathbf{w}^k$ by $w_i^k(x) = g_i^k(x)$, $k \geq 1$. Then, by the maximal inequality (2.11) in [BO] for the measure preserving case, for any $\alpha > 0$,

$$\alpha m \left\{ x \colon \sup_n \frac{1}{n} \sum_{i=0}^{n-1} |f_i|^p > \alpha \right\} \leq \alpha m \left\{ x \colon \sup_n \frac{1}{n} \sum_{i=0}^{n-1} \left| T^i v \right|^p > \alpha \right\} \leq C_p \|v\|_p^p,$$

which shows that $\mathbf{w} \in W_p$ a.e. Similarly, $\mathbf{w}^k \in W_p$ a.e., $k \geq 1$. On the other hand,

$$\begin{aligned}\int \|\mathbf{w}^k - \mathbf{w}\|_{W_p}\, dm &= \int \limsup_n \frac{1}{n}\sum_{i=0}^{n-1} |w_i^k - w_i|^p\, dm \\ &\leq \int \sup_n \frac{1}{n}\sum_{i=0}^{n-1} |f_i(x) - g_i^k(x)|^p\, dm \\ &\leq \int \sup_n \frac{1}{n}\sum_{i=0}^{n-1} |T^i b_k|^p\, dm \to 0 \text{ as } k \to \infty.\end{aligned}$$

Thus, $\|\mathbf{w}^k - \mathbf{w}\|_{W_p} \to 0$ a.e. as $k \to \infty$. By assumption, $\mathbf{w}^k\mathbf{a}$ has a mean a.e., hence, Lemma 2.2 in [JO] implies that the sequence $\mathbf{aw}$ has a mean a.e., i.e., $\mu_n^{\mathbf{a}} F$ converges a.e. ■

THEOREM 3.2. *Let $F \subset L_p$ be a strongly bounded T-admissible process $1 \leq p < \infty$. Assume that $\mathbf{a} \in W_\infty$ when $p = 1$, or $\mathbf{a} \in W_q$ when $1 < p < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathbf{a}$ is good in the mean for additive processes, then it is good in the p-mean for F.*

Proof. A simple calculation shows that

$$\left\| \frac{1}{n}\sum_{i=0}^{n-1} a_i(f_i - g_i^k) \right\|_1 \leq \|\mathbf{a}\|_\infty \|b_k\|_1 \text{ if } p = 1,$$

$$\left\| \frac{1}{n}\sum_{i=0}^{n-1} a_i(f_i - g_i^k) \right\|_p \leq \|\mathbf{a}\|_{W_q} \|b_k\|_p \text{ if } 1 < p < \infty.$$

Therefore, for $1 \leq p < \infty$,

$$\begin{aligned}\left\|\mu_m^{\mathbf{a}} F - \mu_n^{\mathbf{a}} F\right\|_p &\leq \left\| \frac{1}{m}\sum_{i=0}^{m-1} a_i(f_i - g_i^k) \right\|_p + \left\|\mu_m^{\mathbf{a}} G^k - \mu_n^{\mathbf{a}} G^k\right\|_p \\ &\quad + \left\| \frac{1}{n}\sum_{i=0}^{n-1} a_i(f_i - g_i^k) \right\|_p \\ &\leq 2\, \|\mathbf{a}\|_{W_q} \| b_k\|_p + \left\|\mu_m^{\mathbf{a}} G^k - \mu_n^{\mathbf{a}} G^k\right\|_p,\end{aligned}$$

where G^k is the process $\{\sum_{i=0}^{n-1} g_i^k\}$. Now, it follows that $\{\mu_n^{\mathbf{a}} F\}_n$ is Cauchy in the p-mean, since $\|b_k\|_p \to 0$, $1 \leq p < \infty$ as $k \to \infty$, and $\{\mu_n^{\mathbf{a}} G^k\}_n$ is Cauchy in the p-mean, for every $k \geq 1$. ■

Remark. Many sequences studied in [BeL], [CLO] and [JO], in particular the class B_p of *p-Besicovitch* sequences, satisfy the hypotheses of Theorems 3.1 and 3.2. Furthermore, in [CLO] it has been established that the sequences $\mathbf{a} \in W_p$ having Fourier coefficients, $1 < p < \infty$, and 1-Besicovitch sequences are good in the p-mean for contractions T on L_p induced by MPTs.

The tools utilized in the proofs above can (almost verbatim) be repeated for positive invertible L_p-isometries, $1 \leq p < \infty$. Hence Theorem 2.3 and Theorem 3.2 are valid if F is assumed to be a T-admissible process, where T is positive invertible L_p-isometry. For Theorem 3.1, however, while the proof of the case $p = 1$ is the same, in the case $p > 1$ one needs the maximal inequality (2.11) of [BO] for isometries. Since this is the only modification needed for the proof of the statement analogous to Theorem 3.1 for positive invertible L_p-isometries, we will state the theorem only:

THEOREM 3.3. *Let T be a positive invertible L_p-isometry, and let $F \subset L_p$ be a strongly bounded T-admissible process, $1 \leq p < \infty$. Assume that $\mathbf{a} \in W_\infty$ when $p = 1$, or $\mathbf{a} \in W_s$, where $s > q$ and $\frac{1}{p} + \frac{1}{q} = 1$, when $1 < p < \infty$. If $\mathbf{a}$ is good a.e. for T-additive processes, then it is good a.e. for F.*

By Bourgain's return times theorem [BoFKO], the weights associated with return time sequences are bounded sequences of weights which are good a.e for T-additive processes, where T is a MPT. Thus Theorem 3.1 immediately implies:

COROLLARY 3.4. *Let $F \subset L_1$ be a bounded T-admissible process, and $\mathbf{a}$ be a return time sequence. Then it is good a.e. for F.*

Lastly, we observe an interesting feature of admissible processes: let $\mathbf{a} = \{a_i\}$ be a *dynamically generated* sequence of weights, that is, $a_i = \phi(\tau^i \omega)$ for some $\omega \in \Omega$, and $\phi \in L_\infty(\Omega)$ of a dynamical system $(\Omega, \Sigma', \nu, \tau)$. If $\phi \in L_\infty^+(\Omega)$ and F is a positive bounded T-admissible process,

$$S_{m+n}^{\mathbf{a}} F := \sum_{i=0}^{m+n-1} a_i f_i \geq \sum_{i=0}^{m-1} a_i f_i + T^m \sum_{i=m}^{m+n-1} a_i f_{i-m} = S_m^{\mathbf{a}} F + U^m S_n^{\mathbf{a}} F,$$

where $U: X \times \Omega \to X \times \Omega$ is the $m \times \nu$-measure preserving transformation $T \times \tau$. Therefore, the "process" $\{S_n^{\mathbf{a}} F\}_n$ is a U-superadditive process. By assumption, this process is bounded. Hence, by the results in [AS], for a.e. $\omega \in \Omega$, $\lim_n \mu_n^{\mathbf{a}} F = \lim_n \frac{1}{n} S_n^{\mathbf{a}} F$ exists m-a.e. and in the 1-mean. It should be remarked here, however, that the null set in Ω that is involved in this argument *depends on T and F.*

Acknowledgement. The author would like to thank the referee for various comments and corrections on the original version of the paper and suggesting the idea that improved the proof of Theorem 3.1.

REFERENCES

[AS] M. A. Akcoglu and L. Sucheston, *A ratio ergodic theorem for superadditive processes*, Z. Warsch. Verw. Gebiete **44** (1978), 269–278.

[BO] J. R. Baxter and J. Olsen, *Weighted and subsequential ergodic theorems*, Canad. J. Math. **35**, (1983), 145–166.

[BeJR$_1$] A. Bellow, R. Jones and J. Rosenblatt, *Convergence for moving averages*, Ergodic Theory Dynamical Systems **10** (1990), 43–62.

[BeJR$_2$] ———, *Almost everywhere convergence of weighted averages*, Math. Ann. **293** (1992), 399–426.

[BeL] A. Bellow and V. Losert, *The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences*, Trans. Amer. Math. Soc. **288** (1985), 307–345.

[BoFKO] J. Bourgain, H. Fürstenberg, Y. Katznelson and D. Ornstein, *Return times in dynamical systems*, appendix to J.Bourgain, *Pointwise ergodic theorems for arithmetic sets*, Publ. Math. IHES **69** (1989), 5–45.

[Ç] D.Çömez, *Convergence of moving averages of multiparameter superadditive processes*, New York J. Math. **3A** (1998), 135–148.

[ÇF] D.Çömez and S. Ferrando, *On the sequences that are good in the mean for positive L_p-contractions*, $1 \leq p < \infty$, Illinois J. Math. **42** (1998), 617–629.

[CLO] D.Çömez, M. Lin and J. Olsen, *Weighted ergodic theorems for mean ergodic L_1-contractions*, Trans. Amer. Math. Soc. **350** (1998), 101–117.

[DK] Y. Derriennic and U. Krengel, *Subadditive mean ergodic theorems*, Ergodic Theory Dynamical Systems **1** (1981), 33–48.

[F] S. Ferrando, *Moving ergodic theorems for superadditive processes*, Canad. J. Math. **47** (1995), 728–743.

[JO] R. Jones and J. Olsen, *Multiparameter weighted ergodic theorems*, Canad. J. Math. **46** (1994), 343–356.

[JW] R. Jones and M. Wierdl, *Convergence and divergence of ergodic averages*, Ergodic Theory Dynamical Systems **14** (1994), 515–535.

[R] J. M.Rosenblatt, *Norm convergence in ergodic theory and the behavior of Fourier transforms*, Canad. J. Math. **46** (1994), 184–199.

Doğan Çömez, Department of Mathematics, North Dakota State University, Fargo, ND 58105
comez@plains.nodak.edu