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METRICS OF CONSTANT CURVATURE 1 WITH THREE CONICAL SINGULARITIES ON THE 2-SPHERE

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ABSTRACT. A necessary and sufficient condition for the existence and the uniqueness of a conformal metric on a 2-sphere of constant curvature 1 and with three conical singularities of prescribed order is given.

Introduction

Let $\mathrm{Met}_1(\Sigma)$ be the set of positive semi-definite conformal metrics of constant curvature 1 with conical singularities on a compact Riemann surface Σ. Suppose that $d\sigma^2 \in \mathrm{Met}_1(\Sigma)$ has conical singularities at points $p_j \in \Sigma$ $(j = 1, \dots, n)$ with order β_j (> -1); that is, the metric admits a tangent cone of angle $2\pi(\beta_j + 1) > 0$ at each p_j. We call a formal sum

$$D = \beta_1 p_1 + \cdots + \beta_n p_n$$

the *divisor* of $d\sigma^2$. By the Gauss-Bonnet formula, the total curvature

$$\chi(\Sigma, D) := \frac{1}{2\pi} \int_{\Sigma} dA_{d\sigma^2} \tag{0.1}$$

of the metric $d\sigma^2$ satisfies

$$\chi(\Sigma, D) = \chi(\Sigma) + \sum_{j=1}^{n} \beta_j > 0. \tag{0.2}$$

We define a constant $\delta(\Sigma, D)$ as

$$\delta(\Sigma, D) := \chi(\Sigma, D) - 2\mathrm{Min}_{j=1,\dots,n}\{1, \beta_j + 1\}. \tag{0.3}$$

The divisor D is called *subcritical*, *critical*, or *supercritical* when $\delta(\Sigma, D)$ is negative, zero, or positive, respectively. Troyanov [T2] showed that for a divisor D satisfying $\chi(\Sigma, D) > 0$, there exists a pseudometric in $\mathrm{Met}_1(\Sigma)$ with the desired conical singularities whenever it is subcritical. Moreover, when the genus of Σ is zero and

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$-1 < \beta_j < 0$, the uniqueness of such a metric is shown in Luo and Tian [LT], and this unique metric is realized as a spherical polytope. On the other hand, when the genus is zero and the divisor D is supercritical, several obstructions are known [LT], [T2], [CL]: For example, there is no such metric with only one conical singularity. Troyanov [T2] gave a classification of metrics of constant curvature 1 with at most two conical singularities on the 2-sphere.

In this paper, we shall give a necessary and sufficient condition for the existence and uniqueness of a metric with three conical singularities of given order on the 2-sphere. Conformal metrics of constant curvature 1 with conical singularities on a closed Riemann surface Σ correspond bijectively to branched CMC-1 (constant mean curvature 1) surfaces in the hyperbolic 3-space with given hyperbolic Gauss map defined on Σ excluding finite points (see [UY2] and also [RUY1]). Several techniques as in [UY2], [UY3] and also [RUY1] will play important roles. In Section 1, we recall some basic properties of null meromorphic curves in $\mathrm{PSL}(2, \mathbf{C})$. The irreducible metrics are classified in Section 2 using the study of CMC-1 surfaces developed in [RUY1] (Theorem 2.4). Moreover, applying the same method, we classify all genus zero irreducible CMC-1 surfaces with three embedded regular ends (Theorem 2.6). In Section 3, we give a method for explicit construction of all reducible metrics with three singularities (Theorems 3.3 and 3.5).

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1. Preliminaries

In this section, we recall fundamental properties of null meromorphic curves in $\mathrm{PSL}(2, \mathbf{C}) := \mathrm{SL}(2, \mathbf{C})/\{\pm \mathrm{id}\}$.

Definition 1.1. Let $F \colon \Sigma \to \mathrm{PSL}(2, \mathbf{C})$ be a meromorphic map defined on Riemann surface Σ. Then F is called *null* if

$$\det(F^{-1} \cdot F_z) = 0$$

holds on Σ, where z is a complex coordinate. (The condition does not depend on the choice of coordinates.)

Let $F \colon \Sigma \to \mathrm{PSL}(2, \mathbf{C})$ be a null meromorphic map. We define a matrix α by

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} := F^{-1} \cdot dF,$$

and set

$$g := \alpha_{11}/\alpha_{21}, \qquad \omega := \alpha_{21}. \tag{1.1}$$

Then the pair (g, ω) of a meromorphic function g and a meromorphic 1-form ω on Σ satisfies the equality

$$F^{-1} \cdot dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega. \tag{1.2}$$

Conversely, let g be a meromorphic function and ω a meromorphic 1-form on Σ. Then the ordinary differential equation (1.2) is integrable and the solution F is a null map into PSL(2, **C**), however F may not be single-valued on Σ. Moreover, F may have essential singularities. We call the pair (g, ω) the *Weierstrass data* of F.

Definition 1.2. Let

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

be a null meromorphic map of Σ into PSL(2, **C**). We call

$$G := \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}$$

the *hyperbolic Gauss map* of F. Furthermore, we call g in (1.1) the *secondary Gauss map* and $Q = \omega dg$ the *Hopf differential* of F.

Let F: $\Sigma \to$ PSL(2, **C**) be a null meromorphic map. Then for $a, b \in$ PSL(2, **C**), $\hat{F} = a \cdot F \cdot b^{-1}$ is also a null meromorphic map. The associated two Gauss maps $\hat{G}$, $\hat{g}$, and the Hopf differential $\hat{Q}$ of $\hat{F}$ are given by

$$\hat{G} = a \star G, \quad \hat{g} = b \star g, \quad \text{and} \quad \hat{Q} = Q. \tag{1.3}$$

Here, for a matrix $a = (a_{ij}) \in$ PSL(2, **C**) and a meromorphic function g, we denote by

$$a \star g = \frac{a_{11} g + a_{12}}{a_{21} g + a_{22}} \tag{1.4}$$

the Möbius transformation of g by a.

Let $(U; z)$ be a complex coordinate of Σ. Now we consider the Schwarzian derivatives $S(G)$ and $S(g)$ of G and g respectively, where

$$S(G) = \left[\left(\frac{G''}{G'} \right)' - \frac{1}{2} \left(\frac{G''}{G'} \right)^2 \right] dz^2 \qquad \left(' = \frac{d}{dz} \right). \tag{1.5}$$

The description of the Schwarzian derivative depends on the choice of complex coordinates. However, any difference of two Schwarzian derivatives, as a holomorphic 2-differential, does not depend on the choice of complex coordinate. The following identity can be checked:

$$S(G) = S(g) - 2Q. \tag{1.6}$$

We remark that the Schwarzian derivative is invariant under Möbius transformations:

$$S(G) = S(a \star G) \qquad (a \in \mathrm{PSL}(2, \mathbf{C}))\,. \tag{1.7}$$

Conversely, the following fact is known:

FACT 1.3 [Sm], [UY2]. *Let G and g ($S(G) \not\equiv S(g)$) be non-constant meromorphic functions on a Riemann surface Σ. Then there exists a unique null meromorphic map $F : \Sigma \to \mathrm{PSL}(2, \mathbf{C})$ with the hyperbolic Gauss map G and the Hopf differential*

$$Q = -\frac{1}{2}\left(S(G) - S(g)\right),$$

such that $(g, Q/dg)$ is the Weierstrass data of F.

Moreover, the following fact plays an important role in the latter discussions:

FACT 1.4 [UY3]. *Let $F\colon \Sigma \to \mathrm{PSL}(2, \mathbf{C})$ be a null meromorphic map with hyperbolic Gauss map G and secondary Gauss map g. Then the inverse map F^{-1} is a null meromorphic map with hyperbolic Gauss map g and secondary Gauss map G. In particular, F satisfies the ordinary differential equation*

$$-F \cdot d(F^{-1}) = dF \cdot F^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG}.$$

Finally, we point out the following elementary fact from linear algebra:

FACT 1.5 [RUY1]. *A matrix $a \in \mathrm{SL}(2, \mathbf{C})$ satisfies $a \cdot \bar{a} = \mathrm{id}$ if and only if a is of the form*

$$a = \begin{pmatrix} p & i\gamma_1 \\ i\gamma_2 & \bar{p} \end{pmatrix} \qquad (\gamma_1, \gamma_2 \in \mathbf{R},\ p\bar{p} + \gamma_1\gamma_2 = 1).$$

Moreover, a can be diagonalized by a real matrix in $\mathrm{SL}(2, \mathbf{R})$ whenever it is semi-simple.

2. Irreducible metrics with three singularities

Recall that $\mathrm{Met}_1(\Sigma)$ denotes the set of (non-vanishing) conformal pseudometrics of constant curvature 1 on Σ with finitely many conical singularities. Let $d\sigma^2 \in \mathrm{Met}_1(\Sigma)$ with divisor

$$D = \beta_1 p_1 + \cdots + \beta_n p_n \qquad (p_j \in \Sigma,\ \beta_j > -1).$$

Then there exists a meromorphic function g defined on the universal cover $\tilde{\Sigma}_{p_1,\dots,p_n}$ of $\Sigma_{p_1,\dots,p_n} := \Sigma \setminus \{p_1,\dots,p_n\}$ such that the metric is the pull-back of the canonical metric $d\sigma_0^2 := 4\,dz\,d\bar{z}/(1+|z|^2)^2$ on $S^2 = \mathbf{C} \cup \{\infty\}$, that is, we have an expression

$$d\sigma^2 = g^* d\sigma_0^2 = \frac{4\,dg\,d\bar{g}}{(1+|g|^2)^2}. \tag{2.1}$$

Such a function g is unique up to the change

$$g \mapsto a \star g, \qquad \text{where } a \in \mathrm{PSU}(2) := \mathrm{SU}(2)/\{\pm \mathrm{id}\}. \tag{2.2}$$

For a metric $d\sigma^2 \in \mathrm{Met}_1(\Sigma)$, we define the *Schwarzian derivative* as

$$\tilde{S}(d\sigma^2) := S(g) \tag{2.3}$$

where g is the function satisfying (2.1). This definition is independent of choice of g because of (2.2).

We denote by $\pi\colon \tilde{\Sigma}_{p_1,\dots,p_n} \to \Sigma_{p_1,\dots,p_n}$ the covering projection. Fix a base point $\tilde{z}_0$ on $\tilde{\Sigma}_{p_1,\dots,p_n}$ and set $z_0 = \pi(\tilde{z}_0)$. For each $z \in \pi^{-1}(z_0)$, there exists a unique deck transformation T such that $T(\tilde{z}_0) = z$. Thus the fundamental group $\pi_1(\Sigma_{p_1,\dots,p_n})$ is identified with the deck transformation group. By (2.1) and (2.2), there exists a representation $\rho_g\colon \pi_1(\Sigma_{p_1,\dots,p_n}) \to \mathrm{PSU}(2)$ such that

$$g \circ T^{-1} = \rho_g(T) \star g, \qquad \bigl(T \in \pi_1(\Sigma_{p_1,\dots,p_n})\bigr). \tag{2.4}$$

Later, we will see that the representation ρ_g can be lifted to an SU(2)-representation.

Metrics in $\mathrm{Met}_1(\Sigma)$ are divided into the three classes defined below.

(1) A metric is called *irreducible* when the image of the representation ρ_g can not be diagonalized.
(2) A metric is called *$\mathcal{H}^1$-reducible* when the image of the representation ρ_g can be diagonalized but non-trivial.
(3) A metric is called *$\mathcal{H}^3$-reducible* when the image of the representation ρ_g is trivial.

If there exists $a \in \mathrm{PSL}(2,\mathbf{C})$ such that the image of the representation $a \cdot \rho_g \cdot a^{-1}$ is also contained in PSU(2), then another metric $d\sigma_a^2 := (a \star g)^* d\sigma_0^2$ has the same divisor and the Schwarzian derivative as $d\sigma^2$. Hence,

$$I_{d\sigma^2} := \bigl\{ d\sigma_a^2 = (a \star g)^* d\sigma_0^2 \mid a \in \mathrm{PSL}(2,\mathbf{C})\,;\ a \cdot \mathrm{Im}\,\rho_g \cdot a^{-1} \subset \mathrm{PSU}(2) \bigr\} \tag{2.5}$$

is the set of the metrics whose divisors and the Schwarzian derivatives coincide with those of $d\sigma^2 = d\sigma_{\mathrm{id}}^2$. Since $d\sigma_a^2 = d\sigma^2$ for $a \in \mathrm{PSU}(2)$, the set $I_{d\sigma^2}$ is identified with the subset I_Γ of the hyperbolic 3-space in Appendix B, where $\Gamma = \mathrm{Im}\,\rho_g$. By

Lemma B in Appendix B, we have the following:

FACT 2.1 [RUY1]. *For an irreducible metric $d\sigma^2$, the set $I_{d\sigma^2}$ consists of one point. For an $\mathcal{H}^1$-reducible (resp. $\mathcal{H}^3$-reducible) metric $d\sigma^2$, the set $I_{d\sigma^2}$ coincides with a totally geodesic subset of dimension one (resp. three) in the hyperbolic 3-space.*

We now determine all the irreducible metrics in $\mathrm{Met}_1(S^2)$ with three conical singularities. The reducible case is discussed in the next section. We identify S^2 with $\mathbf{C}\cup\{\infty\}$ by the stereographic projection, and let z be the canonical complex coordinate of $\mathbf{C}$. Let $d\sigma^2 \in \mathrm{Met}_1(S^2)$ with divisor

$$D := \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 \qquad (p_j \in S^2,\ \beta_j > -1) \tag{2.6}$$

and take a function g as in (2.1). Since the Möbius transformation group acts on the sphere, without loss of generality we may assume

$$p_1 = 0, \qquad p_2 = 1, \qquad \text{and} \qquad p_3 = \infty. \tag{2.7}$$

For each p_j, there exists $a \in \mathrm{PSU}(2)$ such that

$$a \star g = (z - p_j)^{\beta_j+1}\left(g_0 + g_1(z - p_j) + \cdots\right) \qquad (g_0 \neq 0). \tag{2.8}$$

Hence $\tilde{S}(d\sigma^2)$ can be written with the following leading terms in the Laurent expansions at $z = p_j$:

$$\tilde{S}(d\sigma^2) = \left[-\frac{\beta_j(\beta_j+2)}{2}\frac{1}{(z-p_j)^2} + \cdots\right] dz^2, \tag{2.9}$$

and $\tilde{S}(d\sigma^2)$ is holomorphic on $S^2_{p_1,p_2,p_3}$. By (2.9), the Schwarzian derivative $\tilde{S}(d\sigma^2)$ is uniquely determined by D, since the total order of a holomorphic 2-differential on S^2 is 4. By (2.7) and (2.9), we have

$$\tilde{S}(d\sigma^2) = \left[\frac{c_3 z^2 + (c_2 - c_1 - c_3)z + c_1}{z^2(z-1)^2}\right] dz^2, \tag{2.10}$$

where $c_j = -\beta_j(\beta_j+2)/2 \in \mathbf{R}$ $(j = 1, 2, 3)$.

Now we set

$$G := z \quad \text{and} \quad Q := \frac{1}{2}\left(\frac{c_3 z^2 + (c_2 - c_1 - c_3)z + c_1}{z^2(z-1)^2}\right) dz^2, \tag{2.11}$$

where $c_j = -\beta_j(\beta_j+2)/2$. By Fact 1.3, there exists a unique null holomorphic map $F\colon \tilde{S}^2_{p_1,p_2,p_3} \to \mathrm{PSL}(2,\mathbf{C})$ such that $G \circ \pi$ and g are the hyperbolic Gauss map and

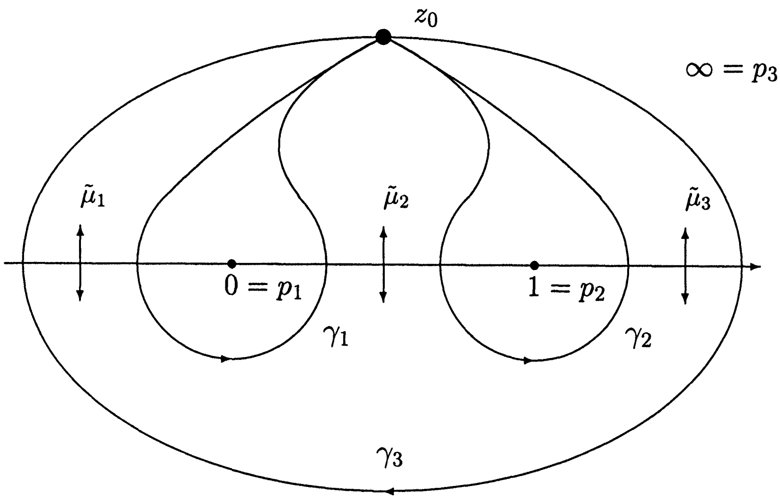

Figure 1. Generator of the fundamental group

the secondary Gauss map respectively. Since $\tilde{S}^2_{p_1,p_2,p_3}$ is simply connected, F can be lifted to a null holomorphic map $\tilde{F}$: $\tilde{S}^2_{p_1,p_2,p_3} \to \mathrm{SL}(2,\mathbf{C})$. By Fact 1.4, we have the following relations:

$$dF \cdot \tilde{F}^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG}, \tag{2.12}$$

$$g = -\frac{d\tilde{F}_{12}}{d\tilde{F}_{11}} = -\frac{d\tilde{F}_{22}}{d\tilde{F}_{21}}, \tag{2.13}$$

where $\tilde{F} = (\tilde{F}_{ij})$. The right-hand side of (2.12) is single-valued and has poles at p_1, p_2 and p_3. Thus, there exists a representation $\rho_{\tilde{F}}$: $\pi_1(S^2_{p_1,p_2,p_3}) \to \mathrm{SL}(2,\mathbf{C})$ such that

$$\tilde{F} \circ T = \tilde{F} \cdot \rho_{\tilde{F}}(T) \qquad \left(T \in \pi_1(S^2_{p_1,p_2,p_3})\right). \tag{2.14}$$

By (1.3), we have

$$\rho_g(T) = \pm\rho_{\tilde{F}}(T) \in \mathrm{PSU}(2) \qquad \left(T \in \pi_1(S^2_{p_1,p_2,p_3})\right). \tag{2.15}$$

Here, we consider $\pm a$ ($a \in \mathrm{SL}(2,\mathbf{C})$) as an element of $\mathrm{PSL}(2,\mathbf{C})$. In particular, $\rho_{\tilde{F}}$ is an SU(2)-representation of $\pi_1(S^2_{p_1,p_2,p_3})$.

Now we describe the converse procedure. Consider the differential equation (2.12) for G and Q as in (2.11). Let $\tilde{F}$ of be a solution of (2.12) with initial data $\tilde{F}(\tilde{z}_0) = \mathrm{id}$, where $\tilde{z}_0$ is a base point on $\tilde{S}^2_{p_1,p_2,p_3}$. Then a representation $\rho_{\tilde{F}}$: $\pi_1(S^2_{p_1,p_2,p_3}) \to \mathrm{SL}(2,\mathbf{C})$ satisfying (2.14) is induced. Let γ_j: $[0,1] \to S^2_{p_1,p_2,p_3}$ be a loop at $z_0 =$

$\pi(\tilde{z}_0)$ surrounding p_j for each $j = 1, 2, 3$, and $\tilde{\gamma}_j$: $[0, 1] \to \tilde{S}^2_{p_1,p_2,p_3}$ the lift of γ_j satisfying $\tilde{\gamma}_j(0) = \tilde{z}_0$. Let T_j be the deck transformation of $\tilde{S}^2_{p_1,p_2,p_3}$ satisfying $T_j(\tilde{z}_0) = \tilde{\gamma}_j(1)$ (see Figure 1). The following lemma holds.

LEMMA 2.2. *Let* $\tilde{F}$: $\tilde{S}^2_{p_1,p_2,p_3} \to \mathrm{SL}(2, \mathbf{C})$ *be any solution of the equation* (2.12) *for G and Q as in* (2.11). *Then the eigenvalues of* $\rho_{\tilde{F}}(T_j)$ *are* $\{-e^{iB_j}, -e^{-iB_j}\}$, *where* $B_j = \pi(\beta_j + 1)$ $(j = 1, 2, 3)$. *In particular,* $\operatorname{Trace} \rho_{\tilde{F}}(T_j) = -2\cos B_j$ *holds. Moreover if* $\rho_{\tilde{F}}(\pi_1(S^2_{p_1,p_2,p_3}))$ *lies in* SU(2), *then the metric defined by* (2.1) *and* (2.13) *belongs to* $\mathrm{Met}_1(S^2)$ *with the divisor D.*

Proof. Let $\tilde{F}_D$ be the solution of the initial value problem

$$d\tilde{F}_D \cdot \tilde{F}_D^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG}, \qquad \tilde{F}_D(\tilde{z}_0) = \mathrm{id} \tag{2.16}$$

for the divisor D, where $\tilde{z}_0 \in \tilde{S}^2_{p_1,p_2,p_3}$ is a base point. By Fact 1.3, the function g defined by (2.13) can be expressed as in (2.8). Since $\tilde{F}_D$ is a solution of the equation (2.12), the monodromy representation ρ_D: $\pi_1(S^2_{p_1,p_2,p_3}) \to \mathrm{SL}(2, \mathbf{C})$ with respect to $\tilde{F}_D$ is conjugate to $\rho_{\tilde{F}}$. Let g be a function defined by (2.13) for given $\tilde{F}$. Then there exists a representation ρ_g: $\pi_1(S^2_{p_1,p_2,p_3}) \to \mathrm{PSL}(2, \mathbf{C})$ as (2.14), which satisfy $\rho_g(T) = \pm\rho_{\tilde{F}}(T)$ for each $T \in \pi_1(S^2_{p_1,p_2,p_3})$. On the other hand, by Fact 1.3, the function g should be expressed as in (2.8), and then the eigenvalues of $\rho_g(T_j)$ are $\pm e^{\pm iB_j}$ $(j = 1, 2, 3)$. Hence $\tau_j := \operatorname{Trace} \rho_D(T_j)$ satisfies

$$\tau_j := \operatorname{Trace} \rho_D(T_j) = \operatorname{Trace} \rho_{\tilde{F}}(T_j) = \pm \operatorname{Trace} \rho_g(T_j) = \pm 2\cos B_j$$

for each $j = 1, 2, 3$. Since (2.16) is real analytic in parameters $(\beta_1, \beta_2, \beta_3)$, the τ_j's are also real analytic functions in β_j. When $\beta_1 = \beta_2 = \beta_3 = 0$, $\tilde{F}_D$ is constant because $Q \equiv 0$, and hence each $\rho_D(T_j)$ is the identity matrix. Thus we have

$$\tau_j|_{(\beta_1,\beta_2,\beta_3)=(0,0,0)} = 2 = -2\cos B_j|_{(\beta_1,\beta_2,\beta_3)=(0,0,0)},$$

and, by real analyticity, $\tau_j = -2\cos B_j$ for all $(\beta_1, \beta_2, \beta_3)$. Hence $\operatorname{Trace} \rho_{\tilde{F}}(T_j) = \operatorname{Trace} \rho_D(T_j) = -2\cos B_j$, and the eigenvalues of $\rho_D(T_j)$ are $-e^{\pm iB_j}$. The final assertion is obtained since the metric $d\sigma^2$ defined by (2.1) and (2.13) is single-valued on $S^2_{p_1,p_2,p_3}$ if and only if $\rho_{\tilde{F}}(\pi_1(S^2_{p_1,p_2,p_3})) \subset \mathrm{SU}(2)$. $\square$

Since $T_1 \circ T_2 \circ T_3 = \mathrm{id}$, Lemma 2.2 and Lemma A in Appendix A imply:

COROLLARY 2.3. *Let* $d\sigma^2 \in \mathrm{Met}_1(S^2)$ *be a metric with divisor D as* (2.6). *Then we have the inequality*

$$\cos^2 B_1 + \cos^2 B_2 + \cos^2 B_3 + 2\cos B_1 \cos B_2 \cos B_3 \leq 1, \tag{2.17}$$

where $B_j = \pi(\beta_j + 1)$ $(j = 1, 2, 3)$. *Moreover,* $d\sigma^2$ *is reducible if and only if the equality of* (2.17) *holds.*

Now we prove the following result.

THEOREM 2.4. *There exists an irreducible metric* $d\sigma^2 \in \mathrm{Met}_1(S^2)$ *with a given divisor*

$$D := \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 \qquad (p_j \in S^2,\ \beta_j > -1) \tag{2.18}$$

if and only if the following inequality holds:

$$\cos^2 B_1 + \cos^2 B_2 + \cos^2 B_3 + 2\cos B_1 \cos B_2 \cos B_3 < 1, \tag{2.19}$$

where $B_j := \pi(\beta_j + 1)$ $(j = 1, 2, 3)$. *Moreover, such a metric* $d\sigma^2$ *is uniquely determined.*

Remark 1. The condition (2.19) implies the inequality (0.2). (See (A.2) in Appendix A.) Moreover, all the β_j $(j = 1, 2, 3)$ are not integers. In fact, suppose one of the β_j's, say β_1, is an integer. Then $\cos B_1 = \pm 1$, and hence (2.19) fails. Conversely, if all the β_j $(j = 1, 2, 3)$ are not integers, the metric is automatically irreducible (Corollary 3.2).

Remark 2. If a metric $d\sigma^2 \in \mathrm{Met}_1(\Sigma)$ with divisor D in (2.6) is reducible, then the equality of (2.17) holds. However, even if D satisfies the equality of (2.17), it does not imply the existence of the metric. In fact, if $\beta_1, \beta_2, \beta_3 \notin \mathbf{Z}$, such a metric never exists (see Lemma 3.1 and Corollary 3.2).

Proof of Theorem 2.4. Assume there exists an irreducible metric $d\sigma^2 \in \mathrm{Met}_1(S^2)$ with the divisor (2.18). Then the Schwarzian derivative $\tilde{S}(d\sigma^2)$ is uniquely determined. Hence irreducibility implies the uniqueness of the metric. Moreover, by Corollary 2.3, (2.19) holds. Hence it is enough to show the existence of the metric under the condition (2.19).

As in (2.7), we identify $S^2_{p_1,p_2,p_3}$ with $\mathbf{C} \setminus \{0, 1\}$. Let μ be the reflection (i.e., conformal transformations reversing orientation) on $\mathbf{C} \setminus \{0, 1\}$ along the real axis. We define three transformations $\tilde{\mu}_k$ $(k = 1, 2, 3)$ on $\tilde{S}^2_{p_1,p_2,p_3}$ as follows: We may assume that the base point $z_0 = \pi(\tilde{z}_0)$ lies on upper half-plane. We choose three points on the real axis such that

$$z_1 \in (-\infty, 0), \qquad z_2 \in (0, 1), \qquad z_3 \in (1, \infty). \tag{2.20}$$

Let $\tau_j\colon [0, 1] \to S^2_{p_1,p_2,p_3}$ be a line segment from z_0 to z_j. Then there exists a unique lift $\tilde{\tau}_j\colon [0, 1] \to \tilde{S}^2_{p_1,p_2,p_3}$ of τ_j such that $\tilde{\tau}_j(0) = \tilde{z}_0$. We set

$$\tilde{z}_j := \tilde{\tau}_j(1) \qquad (j = 1, 2, 3). \tag{2.21}$$

For each point $\tilde{z} \in \tilde{S}^2_{p_1,p_2,p_3}$, we take a path $\tilde{\gamma}_j$: $[0,1] \to \tilde{S}^2_{p_1,p_2,p_3}$ from the base point $\tilde{z}_j$ to $\tilde{z}$. Then $\tilde{\mu}_j(\tilde{z})$ $(j = 1,2,3)$ is defined as the end point of the lift of the loop $\mu \circ \pi \circ \tilde{\gamma}_j$. This definition of $\tilde{\mu}_j(\tilde{z})$ is independent of the choice of the path $\tilde{\gamma}_j$. Moreover, we have

$$\pi \circ \tilde{\mu}_k = \mu \circ \pi \qquad (k = 1,2,3).$$

The deck transformations T_j induced from γ_j as in Figure 1 $(j = 1,2,3)$ are represented as

$$T_1 = \tilde{\mu}_1 \circ \tilde{\mu}_2, \qquad T_2 = \tilde{\mu}_2 \circ \tilde{\mu}_3, \qquad \text{and} \qquad T_3 = \tilde{\mu}_3 \circ \tilde{\mu}_1. \tag{2.22}$$

Let $\tilde{F}$ be a solution of the equation (2.12). Then $\overline{\tilde{F} \circ \tilde{\mu}_k}$ $(k = 1,2,3)$ is also a solution of (2.12) because

$$\overline{Q \circ \mu} = Q, \qquad \text{and} \qquad \overline{G \circ \mu} = G. \tag{2.23}$$

Hence, there exist matrices $\rho_{\tilde{F}}(\tilde{\mu}_k) \in \mathrm{SL}(2, \mathbf{C})$ such that

$$\overline{\tilde{F} \circ \tilde{\mu}_k} = \tilde{F} \cdot \rho_{\tilde{F}}(\tilde{\mu}_k) \qquad (k = 1,2,3).$$

Since $\tilde{\mu}_k \circ \tilde{\mu}_k = \mathrm{id}$, it follows that

$$\rho_{\tilde{F}}(\tilde{\mu}_k) \cdot \overline{\rho_{\tilde{F}}(\tilde{\mu}_k)} = \mathrm{id} \qquad (k = 1,2,3), \tag{2.24}$$

and by (2.22), we have

$$\begin{aligned} \rho_{\tilde{F}}(\tilde{\mu}_1) \cdot \overline{\rho_{\tilde{F}}(\tilde{\mu}_2)} &= \rho_{\tilde{F}}(T_1), \\ \rho_{\tilde{F}}(\tilde{\mu}_2) \cdot \overline{\rho_{\tilde{F}}(\tilde{\mu}_3)} &= \rho_{\tilde{F}}(T_2), \\ \rho_{\tilde{F}}(\tilde{\mu}_3) \cdot \overline{\rho_{\tilde{F}}(\tilde{\mu}_1)} &= \rho_{\tilde{F}}(T_3). \end{aligned} \tag{2.25}$$

If there exists a solution $\tilde{F}$ of (2.12) such that

$$\rho_{\tilde{F}}(\tilde{\mu}_k) \in \mathrm{SU}(2) \qquad (k = 1,2,3), \tag{2.26}$$

it follows from (2.25) that $\rho_{\tilde{F}}(T_j) \in \mathrm{SU}(2)$ for $j = 1,2,3$. For such an $\tilde{F}$, we set g as in (2.13) and $d\sigma^2$ as in (2.1). Then $\rho_g(T) \in \mathrm{PSU}(2)$ for each deck transformation T on $\tilde{S}_{p_1,p_2,p_3}$. By Lemma 2.2, this implies that $d\sigma^2 \in \mathrm{Met}_1(S^2)$. Moreover, the divisor of $d\sigma^2$ is D by (1.6).

Thus, it is enough to show that there exists a solution $\tilde{F}$ of (2.12) which satisfies (2.26). To do this, we use the following argument similar to the proof of Proposition 6.7 and Proposition 5.6 in [RUY1].

Step 1. Let z_1 be a point on the segment $(-\infty, 0)$ of the real axis on $\mathbf{C}$, and $\tilde{z}_1 \in \tilde{S}_{p_1,p_2,p_3}$ the lift of z_1 as in (2.21). Take the solution $\tilde{F}$ of (2.12) satisfying $\tilde{F}(\tilde{z}_1) = \mathrm{id}$. By (2.23), $\overline{\tilde{F} \circ \tilde{\mu}_1}$ is also a solution of (2.12). Moreover, since $\tilde{\mu}_1(\tilde{z}_0) =$

$\tilde{z}_0$, $\overline{\tilde{F}\circ\tilde{\mu}_1}$ has the same initial condition as $\tilde{F}$. Hence we have $\overline{\tilde{F}\circ\tilde{\mu}_1} = \tilde{F}$, and then $\rho_{\tilde{F}}(\tilde{\mu}_1) = \mathrm{id}$.

Step 2. Let $\tilde{F}$ be as in the previous step. By (2.25) and Lemma 2.2, the eigenvalues of $\rho_{\tilde{F}}(\tilde{\mu}_2)$ are $-e^{\pm iB_2}$. Moreover, $\sin B_2 \neq 0$ because β_2 is not an integer. In particular, $\rho_{\tilde{F}}(\tilde{\mu}_2)$ is semi-simple. By Fact 1.5 and (2.24), there exists a matrix $u \in \mathrm{SL}(2,\mathbf{R})$ such that

$$u^{-1}\cdot\rho_{\tilde{F}}(\tilde{\mu}_2)\cdot u = \begin{pmatrix} -e^{iB_2} & 0 \\ 0 & -e^{-iB_2}\end{pmatrix}.$$

Let $\hat{F} := \tilde{F}\cdot u$. Then $\hat{F}$ is also a solution of (2.12) and

$$\begin{aligned}\rho_{\hat{F}}(\tilde{\mu}_1) &= u^{-1}\cdot\rho_{\tilde{F}}(\tilde{\mu}_1)\cdot\bar{u} = u^{-1}\cdot\rho_{\tilde{F}}(\tilde{\mu}_1)\cdot u = \mathrm{id},\\ \rho_{\hat{F}}(\tilde{\mu}_2) &= u^{-1}\cdot\rho_{\tilde{F}}(\tilde{\mu}_2)\cdot\bar{u} = u^{-1}\cdot\rho_{\tilde{F}}(\tilde{\mu}_2)\cdot u = \begin{pmatrix} -e^{iB_2} & 0 \\ 0 & -e^{-iB_2}\end{pmatrix}\end{aligned} \tag{2.27}$$

because u is a real matrix.

Step 3. Let $\hat{F}$ be as in Step 2. By Fact 1.5 and (2.24), $\rho_{\hat{F}}(\tilde{\mu}_3)$ can be written as

$$\rho_{\hat{F}}(\tilde{\mu}_3) = \begin{pmatrix} q & i\delta_1 \\ i\delta_2 & \bar{q}\end{pmatrix} \qquad (\delta_1,\delta_2\in\mathbf{R},\ q\bar{q}+\delta_1\delta_2 = 1).$$

Then by (2.25) and Lemma 2.2, we have

$$q = \frac{i}{\sin B_1}(\cos B_2 + e^{iB_1}\cos B_3).$$

Hence, by the assumption (2.19),

$$\begin{aligned}\delta_1\delta_2 &= 1 - q\bar{q}\\ &= 1 - \frac{\cos^2 B_1 + \cos^2 B_2 + \cos^2 B_3 + 2\cos B_1\cos B_2\cos B_3}{\sin^2 B_1} > 0.\end{aligned}$$

Let

$$\check{F} = \hat{F}\cdot\begin{pmatrix} (\delta_1/\delta_2)^{1/4} & 0 \\ 0 & (\delta_2/\delta_1)^{1/4}\end{pmatrix} \in \mathrm{SL}(2,\mathbf{R}).$$

Then $\check{F}$ is a solution of (2.12) and

$$\rho_{\check{F}}(\tilde{\mu}_1) = \mathrm{id},\quad \rho_{\check{F}}(\tilde{\mu}_2) = \begin{pmatrix} -e^{iB_1} & 0 \\ 0 & -e^{-iB_1}\end{pmatrix},\quad \rho_{\check{F}}(\tilde{\mu}_3) = \begin{pmatrix} q & i\delta \\ i\delta & \bar{q}\end{pmatrix} \in \mathrm{SU}(2),$$

where $\delta = (\delta_1\delta_2)^{1/2}$. Thus, we have a desired metric $d\sigma^2$ induced from $g = -d\check{F}_{12}/d\check{F}_{11}$. □

Metrics with conical singularities are closely related to CMC-1 (constant mean curvature 1) surfaces in the hyperbolic 3-space H^3. In fact, as shown in [UY2, Theorem 2.2], the set $\mathrm{Met}_1(\Sigma)$ corresponds bijectively to the set of branched CMC-1 immersions of Σ excluding a finite number of points, of finite total curvature with prescribed hyperbolic Gauss map. One direction of the correspondence is given as follows: Let $x\colon M := \Sigma \setminus \{p_1, \dots, p_n\} \to H^3$ be a conformal CMC-1 immersion whose induced metric ds^2 is complete and of finite total curvature. We set

$$d\sigma_x^2 := (-K)ds^2,$$

where K is the Gaussian curvature of the induced metric ds^2. Then it can be extended to a pseudometric on Σ and $d\sigma_x^2 \in \mathrm{Met}_1(\Sigma)$ holds (cf. [B]). The converse correspondence is described in [UY2, Section 2].

Definition 2.5. A regular end of a CMC-1 immersion x is called Type I, if the Hopf differential of the immersion has pole of order 2 at the end (cf. [RUY2]).

As seen in [UY1, Section 5], *a regular end of the* CMC-1 *immersion x is asymptotic to a certain catenoid cousin end if and only if the end is of Type I and embedded.*

Using the same argument as in the proof of Theorem 2.4, we can classify the set of irreducible CMC-1 surfaces in the hyperbolic 3-space of genus zero, with three ends asymptotic to the catenoid cousins.

THEOREM 2.6. *Take a triple of real numbers $(\beta_1, \beta_2, \beta_3)$ satisfying* (2.19) *for $B_j = \pi(\beta_j + 1)$ $(j = 1, 2, 3)$ and*

$$c_1^2 + c_2^2 + c_3^2 - 2(c_1c_2 + c_2c_3 + c_3c_1) \neq 0 \qquad \left(c_j = -\frac{1}{2}\beta_j(\beta_j + 2)\right). \tag{2.28}$$

Then there exists a unique irreducible conformal CMC-1 *immersion $x\colon S^2_{p_1,p_2,p_3} \to H^3$ such that all ends p_1, p_2, p_3 are embedded and of Type I, and the order of the pseudometric $d\sigma_x^2$ at p_j is β_j. Conversely, any conformal irreducible immersed* CMC-1 *surface of genus zero with three embedded Type I ends are obtained in such a manner.*

Proof. In fact, such a surface is realized by a conformal CMC-1 immersion $x\colon S^2_{p_1,p_2,p_3} \to H^3$ with the following properties:

(1) Since three ends are of Type I, the Hopf differential Q of x has poles of order 2 at the ends p_1, p_2 and p_3. Then necessarily it has two zeros q_1, q_2 of order 1 on $S^2_{p_1,p_2,p_3}$.
(2) Since all ends are regular and embedded, the hyperbolic Gauss map G of x has two branch points of order 1 at the zeros of Q, and no branch point elsewhere (see [UY3]). (Hence, G is a meromorphic function of degree 2.) The case $q_1 = q_2$ never occurs since G has at least 2 branch points.

(3) $\tilde{S}(d\sigma_x^2) - S(G) = 2Q$ holds (see [UY2, (2.3)]).
(4) Since the order of the pseudometric $d\sigma_x^2$ at the umbilic points q_1, q_2 are equal to the order of zeros of Q (see [UY3]), $d\sigma_x^2$ has a divisor of the form

$$D' := \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + q_1 + q_2 \qquad (\beta_j > -1).$$

The metric $d\sigma_x^2$ is irreducible if and only if (2.19) holds (cf. Appendix A).

We may assume that $p_1 = 0$, $p_2 = 1$ and $p_3 = \infty$, namely, $S^2_{p_1,p_2,p_3} = \mathbf{C} \setminus \{0, 1\}$. By (1), (2) and (3), the top term of $2Q$ of Laurent expansion at $z = p_j$ is the same as that of $\tilde{S}(d\sigma_x^2)$. Thus we have

$$Q = \frac{1}{2}\left(-\frac{\beta_j(\beta_j+2)}{2}\frac{1}{(z-p_j)^2} + \cdots\right)dz^2 \qquad (j = 1, 2, 3).$$

Since the Hopf differential Q is holomorphic on $S^2_{p_1,p_2,p_3}$, we have

$$Q = \frac{1}{2}\left(\frac{c_3 z^2 + (c_2 - c_1 - c_3)z + c_1}{z^2(z-1)^2}\right)dz^2, \tag{2.29}$$

where $c_j = -\beta_j(\beta_j + 2)/2 \in \mathbf{R}$ ($j = 1, 2, 3$). By (2), $G^* d\sigma_0^2$ has the divisor of the form

$$D_G := q_1 + q_2, \tag{2.30}$$

where q_1 and q_2 are zeros of Q:

$$c_3 q_l^2 + (c_2 - c_1 - c_3)q_l + c_1 = 0 \qquad (l = 1, 2). \tag{2.31}$$

Since $q_1 = q_2$ never occurs by (2), (2.28) is a necessary condition. Since the hyperbolic Gauss map G has an ambiguity of Möbius transformations, we may set

$$G = z + \frac{(q_1 - q_2)^2}{2\{2z - (q_1 + q_2)\}}, \tag{2.32}$$

where $q_1 \neq q_2$. Then by [RUY1, Theorem 3.1], we can see the uniqueness of an irreducible CMC-1 immersion x with the hyperbolic Gauss map G and the Hopf differential Q. So it is sufficient to show the existence of such a surface. The following proof is almost the same as that of Theorem 2.4: Let μ be the reflection with respect to the real axis and take the reflections $\tilde{\mu}_k$ ($k = 1, 2, 3$) on the universal cover $\tilde{S}^2_{p_1,p_2,p_3}$ as in the proof of Theorem 2.4 (see Figure 1).

Let $\tilde{F}$ be a solution of the equation

$$d\tilde{F} \cdot \tilde{F}^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \frac{Q}{dG} \tag{2.33}$$

for G in (2.32) and Q in (2.29). Then $\overline{\tilde{F} \circ \tilde{\mu}_k}$ $(k = 1, 2, 3)$ is also a solution of (2.12) because

$$\overline{Q \circ \mu} = Q \qquad \text{and} \qquad \overline{G \circ \mu} = G. \tag{2.34}$$

Hence, there exist matrices $\rho_{\tilde{F}}(\tilde{\mu}_j) \in \mathrm{SL}(2, \mathbf{C})$ such that

$$\overline{\tilde{F} \circ \tilde{\mu}_j} = \tilde{F} \cdot \rho_{\tilde{F}}(\tilde{\mu}_j) \qquad (j = 1, 2, 3). \tag{2.35}$$

Now by the completely same argument as in the proof of Theorem 2.4, there exists a solution $\tilde{F}$ of (2.33) such that $\rho_{\tilde{F}}(T_j) \in \mathrm{SU}(2)$ for $j = 1, 2, 3$. For such an $\tilde{F}$, we set g as in (2.13) and $d\sigma^2$ as in (2.1). Then $d\sigma^2 \in \mathrm{Met}_1(S^2)$ has the divisor D'. By [UY2, Theorem 2.2], there exists a branched CMC-1 immersion $x\colon S^2_{p_1,p_2,p_3} \to H^3$ whose hyperbolic Gauss map and Hopf differential are G and Q, respectively, such that $d\sigma_x^2 = d\sigma^2$. One can easily check that the metric given by

$$ds^{2\sharp} := (1 + |G|^2)^2 \left| \frac{Q}{dG} \right|^2$$

is positive definite and complete. Thus by [RUY1, Lemma 2.3], so is the first fundamental form ds^2. Hence x is the desired immersion. □

PROPOSITION 2.7. *Let $x\colon S^2_{p_1,p_2,p_3} \to H^3$ be a complete* CMC-1 *surface with three ends of Type I. Then the total absolute curvature* TA *of x is greater than or equal to 4π.*

Remark. In [UY1], the authors showed TA $> 2\pi$ for three ended CMC-1 surfaces. The estimate in Proposition 2.7 is sharper than this.

Proof of Proposition 2.7. The associated pseudometric $d\sigma_x^2$ has the divisor $D' := \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + q_1 + q_2$ where q_1, q_2 are umbilic points of x. Then we have

$$\begin{aligned} \frac{1}{2\pi}\mathrm{TA} &= \frac{1}{2\pi}\int_{S^2_{p_1,p_2,p_3}} (-K)\, dA_{ds^2} = \frac{1}{2\pi}\int_{S^2_{p_1,p_2,p_3}} dA_{d\sigma^2} \\ &= \chi(S^2) + |D'| = 4 + \beta_1 + \beta_2 + \beta_3. \end{aligned}$$

On the other hand, the metric $d\sigma_x^2$ induces a monodromy representation $\rho_g\colon \pi_1(S^2_{p_1,p_2,p_3}) \to \mathrm{PSU}(2)$. As seen in the proof of Theorem 2.4, it can be lifted to a representation $\rho_{\tilde{F}_g}\colon \pi_1(S^2_{p_1,p_2,p_3}) \to \mathrm{SU}(2)$. Let T_j $(j = 1, 2, 3)$ be the deck transformation corresponding to the loop surrounding p_j. Then the eigenvalues of $\rho_{\tilde{F}_g}$ are $-e^{\pm i B_j}$ where $B_j := \pi(\beta_j + 1)$, which can be proved by the same method as in Lemma 2.2. By Lemma A in Appendix A, we have

$$\cos^2 B_1 + \cos^2 B_2 + \cos^2 B_3 + 2\cos B_1 \cos B_2 \cos B_3 \le 1.$$

By (A.2) in Appendix A, we have $B_1 + B_2 + B_3 \ge \pi$, which yields $\beta_1 + \beta_2 + \beta_3 \ge -2$. □

3. Reducible metrics with three singularities

In this section, we give a necessary and sufficient condition for the existence of reducible metrics with given divisors. As in the previous section, we identify $S^2 = \mathbf{C} \cup \{\infty\}$, and set $(p_1, p_2, p_3) = (0, 1, \infty)$.

LEMMA 3.1. *Let $d\sigma^2 \in \mathrm{Met}_1(S^2)$ be an $\mathcal{H}^3$-reducible (resp. $\mathcal{H}^1$-reducible) pseudometric with divisor D as in* (2.6). *Then all of the β_j's are integers* (*resp. exactly one of the β_j's is an integer*). *Namely, at least one β_j is an integer for reducible metrics.*

Proof. Let $d\sigma^2 \in \mathrm{Met}_1(S^2)$ be an $\mathcal{H}^3$-reducible metric with divisor D as in (2.6). Then, by definition, the representation ρ_g as in (2.4) is trivial. Hence g satisfying (2.1) is a single-valued meromorphic function on $S^2_{p_1,p_2,p_3}$. Since p_j $(j = 1, 2, 3)$ are conical singularities of $d\sigma^2$, g can be extended to a meromorphic function on S^2. Hence β_j $(j = 1, 2, 3)$ are integers.

Next, assume $d\sigma^2$ is $\mathcal{H}^1$-reducible and all β_j's are non-integral numbers. Since $d\sigma^2$ is reducible, we can choose g as in (2.1) such that $\rho_g(T)$ are diagonal for all $T \in \pi_1(S^2_{p_1,p_2,p_3})$. Then $g \circ T_k = e^{2\pi i(\beta_k+1)}g$ because of (2.8). Hence $g_1 := z^{-\beta_1-1}g$ is single-valued on $\tilde{S}^2_{p_2,p_3}$, and $g_2 := (z-1)^{-\beta_2-1}g_1$ is single-valued on $\tilde{S}^2_{p_3}$. Since $S^2_{p_3}$ is simply-connected, g_2 is single-valued on S^2 and g can be written as

$$g = z^{\mu}(z-1)^{\nu}\frac{a(z)}{b(z)} \qquad (\mu, \nu \in \mathbf{R} \setminus \mathbf{Z}), \tag{3.1}$$

where $a(z)$ and $b(z)$ are mutually prime polynomials whose roots are distinct from 0 and 1. Then we have

$$\begin{aligned} dg &= z^{\mu-1}(z-1)^{\nu-1}\frac{p(z)}{q(z)}\,dz, \\ p(z) &:= \{\nu z + \mu(z-1)\}a(z)b(z) + z(z-1)\{a'(z)b(z) - a(z)b'(z)\}, \\ q(z) &:= \{b(z)\}^2. \end{aligned} \tag{3.2}$$

Since $p(0) = -\mu\, a(0)b(0)$, $p(1) = \nu\, a(0)b(0)$, $q(0) = \{b(0)\}^2$ and $q(1) = \{b(1)\}^2$ are not equal to 0, the roots of p and q are distinct from 0 and 1. Moreover, p and q are mutually prime. In fact, assume there exists a common root ξ of p and q. Then $b(\xi) = 0$, and by assumption, $\xi \neq 0, 1$ and $a(\xi) \neq 0$. Then $0 = p(\xi) = \xi(\xi-1)a(\xi)b'(\xi)$ implies $b'(\xi) = 0$. Hence ξ is a multiple root of b. Let $b(z) = (z-\xi)^m\tilde{b}(z)$, where $m \geq 2$ be an integer and $\tilde{b}(z)$ is a polynomial such that $\tilde{b}(\xi) \neq 0$. Then we have

$$dg = z^{\mu-1}(z-1)^{\nu-1}\frac{(z-\xi)r(z) - z(z-1)\tilde{b}(z)}{(z-\xi)^{m+1}\tilde{b}(z)^2}\,dz,$$

where $r(z) = (\nu z + \mu(z+1))a\tilde{b} + z(z-1)(a'\tilde{b} - a\tilde{b}')$ is a polynomial in z. Since $m \geq 2$, ξ is a ramification point of g, and then $d\sigma^2$ has a conical singularity at ξ, which is a contradiction.

Thus p and q are mutually prime with roots distinct from 0 and 1. If p has a root η, then η is a ramification point of g, and thus a conical singularity of $d\sigma^2$. Hence $p(z)$ must be a constant. By (3.2), p is formally a polynomial of degree $\deg a + \deg b + 1$. Then the highest term must vanish: $\mu + \nu + \deg a - \deg b = 0$. This shows that the order of dg at $z = \infty$ must be an integer, and therefore β_3 is an integer. This is a contradiction, and hence at least one of β_j's must be integer.

On the other hand, let $d\sigma^2 \in \mathrm{Met}_1(S^2)$ be $\mathcal{H}^1$-reducible and suppose exactly one of β_j's is not an integer. Without loss of generality, we assume β_1 is a non-integer. Take g as in (2.1). Since β_2 and β_3 are integers, g is well-defined on the universal cover of $S^2_{p_1}$. Here $S^2_{p_1}$ is simply connected. Then g is single-valued on S^2 itself, and hence g is a meromorphic function on S^2. This shows that β_1 is an integer, a contradiction.

Hence, if $d\sigma^2 \in \mathrm{Met}_1(S^2)$ with divisor D in (2.6) is $\mathcal{H}^1$-reducible, exactly one of the β_j's is an integer. □

As an immediate consequence, we have the following corollary.

COROLLARY 3.2. *Suppose $d\sigma^2 \in \mathrm{Met}_1(S^2)$ has exactly three singularities with orders β_1, β_2 and β_3. Then the following three assertions are true.*

(1) *$d\sigma^2$ is $\mathcal{H}^3$-reducible if and only if all of the β_j's are integers.*
(2) *$d\sigma^2$ is $\mathcal{H}^1$-reducible if and only if exactly one of the β_j's is an integer.*
(3) *$d\sigma^2$ is irreducible if and only if all of the β_j's are non-integers.*

Remark. For a metric in $\mathrm{Met}_1(S^2)$ with more than three singularities, such a simple criterion for reducibility is not expected: There exists a reducible metric $d\sigma^2 \in \mathrm{Met}_1(S^2)$ with divisor

$$D' = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 + q_1 + q_2$$

such that all β_j's are non-integers. In fact,

$$g = c\, z^{\mu}(z-1)^{\nu}(z-a) \qquad (c \in \mathbf{C} \setminus \{0\},\ a \in \mathbf{C} \setminus \{0, 1\})$$

induces such a metric whenever $\mu + \nu$ is not an integer. On the other hand, we can construct an irreducible metric with divisor D' such that $\beta_1, \beta_2, \beta_3 \notin \mathbf{Z}$: The metric $d\sigma_x^2 \in \mathrm{Met}_1(S^2)$ obtained in Theorem 2.6 is the desired one.

$\mathcal{H}^3$-reducible case. First, we consider the case of $\mathcal{H}^3$-reducible. In this case, β_1, β_2 and β_3 are integers and g in (2.1) is single-valued on S^2, i.e., a rational function on $\mathbf{C} \cup \{\infty\}$.

Without loss of generality, we assume

$$\beta_1 \leq \beta_2 \leq \beta_3. \tag{3.3}$$

Let g be a rational function such that $d\sigma^2$ is as in (2.1) with the divisor D in (2.6) satisfying (3.3). Then the ramification points of g are 0, 1 and ∞ whose orders are β_1, β_2 and β_3 respectively. By the Riemann-Hurwicz formula,

$$\deg g = \frac{1}{2}(\beta_1 + \beta_2 + \beta_3) + 1 \le \frac{1}{2}\beta_1 + \beta_3 + 1 < (\beta_1 + 1) + (\beta_3 + 1)$$

holds. Then we have $g(p_1) \neq g(p_3)$, and similarly, $g(p_2) \neq g(p_3)$. Thus, by a suitable change as (2.2), we may assume $g(p_1) = g(0) \neq \infty$, $g(p_2) = g(1) \neq \infty$, and $g(p_3) = g(\infty) = \infty$. Under these assumptions, we can write

$$dg = c\frac{z^{\beta_1}(z-1)^{\beta_2}}{\prod_{j=1}^{N}(z-a_j)^2}\,dz, \qquad \beta_3 = \beta_1 + \beta_2 - 2N, \tag{3.4}$$

where $c \neq 0$ is a constant, and $a_1, \dots, a_N \in \mathbf{C} \setminus \{0, 1\}$ are mutually distinct numbers. Conversely, if there exists a g which satisfies (3.4), then we have $d\sigma^2 \in \mathrm{Met}_1(S^2)$ with the desired singularities. Computing residues at $z = a_1, \dots, a_N$, we have:

THEOREM 3.3. *Suppose β_1, β_2 and β_3 are positive integers satisfying* (3.3). *Then there exists $d\sigma^2 \in \mathrm{Met}_1(S^2)$ with divisor D as in* (2.6) *if and only if there exists a non-negative integer N and mutually distinct complex numbers $a_1, \dots, a_N \in \mathbf{C} \setminus \{0, 1\}$ such that*

$$\beta_1 + \beta_2 - \beta_3 = 2N \tag{3.5}$$

and

$$\frac{\beta_1}{a_j} + \frac{\beta_2}{a_j - 1} - \sum_{k \neq j} \frac{2}{a_j - a_k} = 0 \qquad (j = 1, \dots, N). \tag{3.6}$$

Remark. An $\mathcal{H}^3$-reducible metric admits a three parameter space of deformations $I_{d\sigma^2}$ which preserves the divisor D and the Schwarzian derivative. For a given triple $(\beta_1, \beta_2, \beta_3)$ as in Theorem 3.3, such a deformation space is determined uniquely.

Observing the equation (3.6) for small N, one can easily see the following facts: If N in (3.5) is 0, trivially the metric exists.

When $N = 1$, (3.6) has a unique solution $a_1 = \beta_1/(\beta_1 + \beta_2) \neq 0, 1$.

Assume $N = 2$. Then $\beta_3 = \beta_1 + \beta_2 - 4$. By (3.3), this implies $4 \le \beta_1 \le \beta_2$. In this case, it is easy to show that the system of equations (3.6) has a unique solution up to permutations of the a_j's.

Hence, we have:

COROLLARY 3.4. *Let β_1, β_2 and β_3 are positive integers satisfying* (3.3) *and such that*

$$\beta_1 + \beta_2 - \beta_3 = 2N \qquad (N = 0, 1 \text{ or } 2).$$

Then, there exists an $\mathcal{H}^3$-reducible metric $d\sigma^2 \in \mathrm{Met}_1(S^2)$ with the divisor D as in (2.6). *Moreover, such a metric is unique up to a three parameter family of deformations as in* (2.5).

$\mathcal{H}^1$-reducible case. For the $\mathcal{H}^1$-reducible case, one of the β_j's must be an integer because of Lemma 3.1. We assume that β_1 and β_3 are non-integers and β_2 is a positive integer:

$$\beta_1,\ \beta_3 \notin \mathbf{Z}, \qquad \beta_2 \in \mathbf{Z}.$$

Then we can choose g as in (2.1) such that

$$g = z^{\mu}\varphi(z), \qquad \mu \in \mathbf{R} \setminus \mathbf{Z}, \tag{3.7}$$

where $\varphi(z)$ is a rational function. Here, such a normalization of g is unique up to the change $g \mapsto tg$ or $g \mapsto t/g$ for each non-zero constant t. Moreover,

$$(tg)^* d\sigma_0^2 \qquad (t \in \mathbf{R}^+) \tag{3.8}$$

gives a non-trivial deformation of the metric preserving the divisor. This is the one parameter deformation as in (2.5).

Let g be a function as in (3.7) such that $d\sigma^2$ as in (2.1) has the divisor D in (2.6). Replacing g with $1/g$, we may assume $\varphi(1) \neq \infty$ without loss of generality. Under this assumption, dg has a zero of order β_2 at $z = 1$:

$$dg = cz^{\nu_1}\frac{(z-1)^{\beta_2}}{\prod_{j=1}^{N}(z-a_j)^2}\,dz \qquad (\nu_1 = \mu - 1), \tag{3.9}$$

where $c \neq 0$ is a constant, and $a_1, \dots, a_N \in \mathbf{C} \setminus \{0, 1\}$ are distinct numbers.

We denote the order of dg at $z = \infty$ by ν_3:

$$\nu_3 = -\nu_1 - \beta_2 + 2N - 2. \tag{3.10}$$

So the following four cases occur.

(a) $\nu_1 > -1$ and $\nu_3 > -1$. In this case, $\beta_1 = \nu_1$, $\beta_3 = \nu_3$. Hence $\beta_1 + \beta_2 + \beta_3 = 2N - 2$.

(b) $\nu_1 < -1$ and $\nu_3 < -1$. In this case, $\beta_1 = -\nu_1 - 2$, $\beta_3 = -\nu_3 - 2$. Hence $\beta_1 - \beta_2 + \beta_3 = -2N - 2$.

(c) $\nu_1 > -1$ and $\nu_3 < -1$. In this case, $\beta_1 = \nu_1$, $\beta_3 = -\nu_3 - 2$. Hence $\beta_1 + \beta_2 - \beta_3 = 2N$.

(d) $\nu_1 < -1$ and $\nu_3 > -1$. In this case, $\beta_1 = -\nu_1 - 2$, $\beta_3 = \nu_3$. Hence $\beta_1 - \beta_2 - \beta_3 = -2N$.

For cases (a) and (c), there exists a meromorphic function g on the universal cover of $\mathbf{C} \setminus \{0\}$ satisfying (3.9) if and only if

$$\frac{\beta_1}{a_j} + \frac{\beta_2}{a_j - 1} - \sum_{k \neq j} \frac{2}{a_j - a_k} = 0 \qquad (j = 1, \dots, N) \tag{3.11}$$

holds, and for cases (b) and (d), there exists g satisfying (3.9) if and only if

$$\frac{-\beta_1-2}{a_j}+\frac{\beta_2}{a_j-1}-\sum_{k\neq j}\frac{2}{a_j-a_k}=0 \qquad (j=1,\dots,N) \tag{3.12}$$

holds.

Then we have the next result.

THEOREM 3.5. *Let β_1, β_3 be non-integer real numbers greater than -1 and β_2 a positive integer. Then there exists $d\sigma^2 \in \mathrm{Met}_1(S^2)$ with divisor D as in* (2.6) *if and only if one of the following occurs*:

(1) *There exists a non-negative integer N and distinct complex numbers $a_1, \dots, a_N \in \mathbf{C}\setminus\{0,1\}$ such that $\beta_1+\beta_2+\beta_3=2N-2$ and* (3.11) *holds.*
(2) *There exists a non-negative integer N and distinct complex numbers $a_1, \dots, a_N \in \mathbf{C}\setminus\{0,1\}$ such that $\beta_1-\beta_2+\beta_3=-2N-2$ and* (3.12) *holds.*
(3) *There exists a non-negative integer N and distinct complex numbers $a_1, \dots, a_N \in \mathbf{C}\setminus\{0,1\}$ such that $\beta_1+\beta_2-\beta_3=2N$ and* (3.11) *holds.*
(4) *There exists a non-negative integer N and distinct complex numbers $a_1, \dots, a_N \in \mathbf{C}\setminus\{0,1\}$ such that $\beta_1-\beta_2-\beta_3=-2N$ and* (3.12) *holds.*

Moreover, such a metric is unique up to the change in (3.8).

Just as in the $\mathcal{H}^3$-reducible case, we classify the metrics for $N\leq 2$. It is easy to show the following lemma.

LEMMA 3.6. *Let m and N be positive integers, and ν a non-integer real number. Consider the following equations on $a_1, \dots, a_N$*:

$$\frac{\nu}{a_j}+\frac{m}{a_j-1}-\sum_{\substack{k\neq j\\ 1\leq k\leq N}}\frac{2}{a_j-a_k}=0 \qquad (j=1,\dots,N). \tag{3.13}$$

(1) *If $N=1$, the equation has the unique solution $a_1=\nu/(\nu+m)$ which is different from* 0 *and* 1.
(2) *If $N=2$, the equation has a solution if and only if $m\neq 1$. Both solutions a_1, a_2 are distinct from* 0 *and* 1 *if $m\neq 1$.*

Using this lemma, we have the following non-existence and existence results.

COROLLARY 3.7. *Let $\beta_2=1$, and β_1 and β_3 (>-1) be non-integer real numbers satisfying $\beta_1+\beta_3=1$. Then there exists no metric $d\sigma^2\in\mathrm{Met}_1(S^2)$ with divisor D as in* (2.6).

Proof. This is case (1) in Theorem 3.5 for $N = 1$. Since $\beta_2 = 1$, the equation (3.11) has no solution because of Lemma 3.6. □

COROLLARY 3.8. *Let β_1, β_3 be non-integers (> -1) and β_2 an integer satisfying one of the following cases for some non-negative integer n:*

(1) $\beta_1 + \beta_3 = 2n - 1$, $\beta_2 = 2n + 2N + 1$ $(N = 0, 1, 2)$,
(2) $\beta_1 + \beta_3 = 2n$, $\beta_2 = 2n + 2N + 2$ $(N = 0, 1, 2)$,
(3) $\beta_1 - \beta_3 = 2n + 1$, $\beta_2 = 2n + 2N + 1$ $(N = 0, 1, 2)$,
(4) $\beta_1 - \beta_3 = 2n$, $\beta_2 = 2n + 2N$ $(N = 0, 1, 2)$.

Then there exists an $\mathcal{H}^1$-reducible metric $d\sigma^2 \in \mathrm{Met}_1(S^2)$ with divisor D in (2.6).

Proof. The first two cases follow from (2) of Theorem 3.5, and the others from (4) of Theorem 3.5. □

Finally, we remark on symmetry of reducible metrics. As shown in the proof of Theorem 2.4, an irreducible metric $d\sigma^2$ is invariant under the reflection with respect to the real axis:

$$d\sigma^2 \circ \mu = d\sigma^2. \tag{3.14}$$

We call a (reducible) metric *symmetric* if (3.14) holds.

Let $d\sigma^2 \in \mathrm{Met}_1(S^2)$ be an $\mathcal{H}^1$- (resp. $\mathcal{H}^3$-) reducible metric with divisor D. Then the set of metrics with divisor D coincides with $I_{d\sigma^2}$, which is homeomorphic to $\mathbf{R}$ (resp. $\mathbf{R}^3$). Hence the involution μ: $I_{d\sigma^2} \to I_{d\sigma^2}$ has a fixed point. In other words, there exists a symmetric metric $d\sigma_1^2 \in I_{d\sigma^2}$. Moreover, one can easily determine the set of symmetric metrics.

THEOREM 3.9. *If $d\sigma^2 \in \mathrm{Met}_1(S^2)$ is $\mathcal{H}^1$-reducible with divisor D as in* (2.6), *all metrics in $I_{d\sigma^2}$ are symmetric.*

If $d\sigma^2 \in \mathrm{Met}_1(S^2)$ is $\mathcal{H}^3$-reducible with divisor D as in (2.6), *the subset of symmetric metrics of $I_{d\sigma^2}$ is a two dimensional totally geodesic subset of $I_{d\sigma^2} = \mathcal{H}^3$.*

Appendix A

The following lemma is easy to show.

LEMMA A. *Let $a_j \in \mathrm{SU}(2)$ $(j = 1, 2, 3)$ be matrices satisfying $a_1 \cdot a_2 \cdot a_3 = \mathrm{id}$. Then the following inequality holds:*

$$\cos^2 C_1 + \cos^2 C_2 + \cos^2 C_3 + 2\cos C_1 \cos C_2 \cos C_3 \leq 1, \tag{A.1}$$

where $-e^{\pm iC_j}$ $(C_j \geq 0)$ *are the eigenvalues of the matrices* a_j $(j = 1, 2, 3)$. *Moreover, equality holds if and only if* a_1, a_2 *and* a_3 *are simultaneously diagonalizable. Furthermore,* (*A*.1) *yields the inequality*

$$C_1 + C_2 + C_3 \geq \pi, \tag{A.2}$$

where equality holds only when equality also holds in (*A*.1).

Appendix B

This is the appendix in [RUY1]. We attach it here for the sake of convenience.

Let Γ be a subgroup of $\mathrm{PSU}(2) = \mathrm{SU}(2)/\{\pm 1\}$.

In this appendix, we prove a property of a set of groups conjugate to Γ in $\mathrm{PSL}(2, \mathbf{C})$ defined by

$$C_\Gamma := \{\sigma \in \mathrm{PSL}(2, \mathbf{C}) \mid \sigma \cdot \Gamma \cdot \sigma^{-1} \subset \mathrm{PSU}(2)\}.$$

The authors wish to thank Hiroyuki Tasaki for valuable comments on the first draft of the appendix.

If $\sigma \in C_\Gamma$, it is obvious that $a \cdot \sigma \in C_\Gamma$ for all $a \in \mathrm{PSU}(2)$. So if we consider the quotient space

$$I_\Gamma := C_\Gamma / \mathrm{PSU}(2),$$

the structure of the set C_Γ is completely determined. Define a map $\tilde{\phi}$: $C_\Gamma \to \mathcal{H}^3$ by

$$\tilde{\phi}(\sigma) := \sigma^* \cdot \sigma,$$

where $\mathcal{H}^3$ is the hyperbolic 3-space defined by $\mathcal{H}^3 := \{a \cdot a^* \mid a \in \mathrm{PSL}(2, \mathbf{C})\}$. Then it induces an injective map ϕ: $I_\Gamma \to \mathcal{H}^3$ such that $\phi \circ \pi = \tilde{\phi}$, where π: $C_\Gamma \to I_\Gamma$ is the canonical projection. So we can identify I_Γ with a subset $\phi(I_\Gamma) = \tilde{\phi}(C_\Gamma)$ of the hyperbolic 3-space $\mathcal{H}^3$. The following assertion holds.

LEMMA B. *The subset* $\phi(I_\Gamma)$ *is a point, a geodesic line, or all of* $\mathcal{H}^3$.

Proof. For each $\gamma \in \Gamma$, we set

$$C_\gamma := \{\sigma \in \mathrm{PSL}(2, \mathbf{C}) \mid \sigma \cdot \gamma \cdot \sigma^{-1} \in \mathrm{PSU}(2)\}.$$

Then we have

$$C_\Gamma := \bigcap_{\gamma \in \Gamma} C_\gamma. \tag{B.1}$$

The condition $\sigma \cdot \gamma \cdot \sigma^{-1} \in \mathrm{PSU}(2)$ is rewritten as $\sigma^* \cdot \sigma \cdot \gamma = \gamma \cdot \sigma^* \cdot \sigma$. So we have

$$\tilde{\phi}(C_\gamma) = \mathcal{H}^3 \cap Z_\gamma, \tag{B.2}$$

where Z_γ is the center of $\gamma \in \Gamma$. In the following discussions, Γ can be considered as a subgroup of SU(2) by ignoring the $\pm$-ambiguity.

Assume $\gamma \neq \pm\,\mathrm{id}$. If γ is a diagonal matrix, it can easily be checked that Z_γ consists of diagonal matrices in PSL(2, $\mathbf{C}$). Since any $\gamma \in \Gamma$ can be diagonalized by a matrix in SU(2), we have $Z_\gamma = \{\exp(zT) \mid z \in \mathbf{C}\}$, where $T \in su(2)$ is chosen so that $\gamma = \exp(T)$. Hence we have

$$\tilde{\phi}(C_\gamma) = \mathcal{H}^3 \cap Z_\gamma = \exp(i\mathbf{R}T)\,, \tag{B.3}$$

because $\exp(i\, su(2)) = \mathcal{H}^3$.

Now suppose that Γ is not diagonalizable. Then there exist $\gamma, \gamma' \in \Gamma$ such that $\gamma \cdot \gamma' \neq \gamma' \cdot \gamma$. Set $\gamma = \exp(T)$ and $\gamma' = \exp(T')$, where $T, T' \in su(2)$. Then we have $i\mathbf{R}T \cap i\mathbf{R}T' = \{0\}$. It is well known that the restriction of the exponential map $\exp|_{i\, su(2)} \colon i\, su(2) \to \mathcal{H}^3$ is bijective. Hence we have

$$\tilde{\phi}(C_\gamma) \cap \tilde{\phi}(C_{\gamma'}) = \exp(i\mathbf{R}T) \cap \exp\left(i\mathbf{R}T'\right) = \{\mathrm{id}\}.$$

By (B.1), (B.2) and (B.3), we have

$$\phi(I_\Gamma) = \{\mathrm{id}\} \qquad (\text{if } \Gamma \text{ is not abelian}).$$

Next we consider the case where Γ is diagonalizable. If $\Gamma \subset \{\pm\,\mathrm{id}\}$, then obviously

$$\phi(I_\Gamma) = \mathcal{H}^3.$$

Suppose $\Gamma \not\subset \{\pm\,\mathrm{id}\}$. Then there exists $\gamma \in \Gamma$ such that $\gamma \neq \pm\,\mathrm{id}$. We set $\gamma = \exp T$ $(T \in su(2))$. Since $\exp(\mathbf{R}T)$ is a maximal abelian subgroup containing γ, we have $\Gamma \subset \exp(\mathbf{R}T)$. Then by (B.3), we have

$$\phi(I_\Gamma) \;=\; \exp(i\mathbf{R}T). \qquad \square$$

Added in Proof. After submitting the paper, the authors found a classical work of F. Klein, *Vorlesungen über die hypergeometrische Funktion*, Springer-Verlag, 1933, in which he investigated immersed spherical triangles with given angles $0 < \alpha, \beta, \gamma < \infty$, allowing edges to be circular. Moreover, it was applied to the study of the monodromy of the holomorphic function g satisfying

$$S(g) = (c_3 z^2 + (c_2 - c_1 - c_3)z + c_1)/z^2(z-1)^2\, dz^2.$$

His motivation and formulation are closely related to our work.

Recently the authors received a preprint by M. Furuta and Y. Hattori, *Two-dimensional spherical space forms*, containing an alternative proof of Theorem 2.4 based on the geometry of spherical polytopes. Moreover, they gave a considerably simpler criterion for the reducible metrics which is euqivalent to ours as in Theorem 3.3 and Theorem 3.5.

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