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ON CERTAIN EQUIVALENT NORMS ON TSIRELSON'S SPACE

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ABSTRACT. Tsirelson's space T is known to be distortable but it is open as to whether or not T is arbitrarily distortable. For $n \in \mathbb{N}$ the norm $\|\cdot\|_n$ of the Tsirelson space $T(S_n, 2^{-n})$ is equivalent to the standard norm on T. We prove there exists $K < \infty$ so that for all n, $\|\cdot\|_n$ does not K distort any subspace Y of T.

Introduction

An important and still open question is whether or not there exists a distortable Banach space which is not arbitrarily distortable. The primary candidate for such a space is Tsirelson's space T. While it is not difficult to directly define, for every $1 < \lambda < 2$, an equivalent norm on T which is a λ-distortion, T does not belong to any general class of Banach spaces known to be arbitrarily distortable. In fact (see below) if there does exists a distortable not arbitrarily distortable Banach space X then X must contain a subspace which is very Tsirelson-like in appearance. Thus it is of interest, in particular, to examine all known equivalent norms on T to see if they can arbitrarily distort T (or a subspace of T). We do so in this paper for a previously unstudied fascinating class of renormings.

The renormings we consider here are "natural" in that they pertain to the deep combinatorial nature of the norm of T. Namely, for each n we denote by $\|\cdot\|_n$ the norm of the Tsirelson space $T(S_n, 2^{-n})$, which can easily be seen to be equivalent to the original norm on T. Our main result (Theorem 2.1) is that this family of equivalent norms does not arbitrarily distort T or even any subspace of T. The proof actually introduces a larger family of equivalent norms $(\|\cdot\|_j^n)_{j,n}$ and $(|\cdot|_j^n)_{j,n}$ which are shown to not arbitrarily distort any subspace of T. Quantitative estimates for the stabilizations of these norms are given in Theorem 2.5. It is shown that (up to absolute constants) for all n and subspaces $X \subseteq T$, there is a subspace $Y \subseteq X$ such that $\|y\|_n \sim \frac{1}{n}$ if $y \in Y$ with $\|y\| = 1$.

Some stabilization results for more general norms on T of various classes are also given in Section 3. In Section 4 we raise some problems.

Section 1 contains the relevant terminology and background material. Otherwise our notation is standard as may be found in [LT].

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More detailed information about Tsirelson's space and Tsirelson type spaces can be found in [CS], [OTW], [AD], [AO] and the references therein.

1. Preliminaries

$X, Y, Z, \ldots$ will denote separable infinite-dimensional real Banach spaces. If (x_i) is a basic sequence, $(y_i) \prec (x_i)$ shall mean that (y_i) is a block basis of (x_i). $X = [(x_i)]$ is the closed linear span of (x_i). If X has a basis (x_i), $Y \prec X$ denotes $Y = [(y_i)]$ where $(y_i) \prec (x_i)$. The terminology is imprecise in that "$\prec$" refers to a fixed basis for X but no confusion shall arise. $S_X = \{x \in X\colon \|x\| = 1\}$.

A space $(X, \|\cdot\|)$ is *arbitrarily distortable* if, for all $\lambda > 1$, there exists an equivalent norm $|\cdot|$ on X such that

$$\sup\left\{\frac{|y|}{|z|}\colon y, z \in S_Y\right\} > \lambda \quad \text{for all } Y \subseteq X. \tag{1.1}$$

The norm $|\cdot|$ satisfying (1.1) is said to λ-distort X. X is λ*-distortable* if some norm λ-distorts X. X is *distortable* if it is λ-distortable for some $\lambda > 1$. If X has a basis then "for all $Y \subseteq X$" in (1.1) can be replaced by "for all $Y \prec X$".

Tsirelson's space T (defined below) is known to be $2 - \varepsilon$ distortable for all $\varepsilon > 0$ (e.g., see [OTW]). If a space X exists which is distortable but not arbitrarily distortable then X can be assumed to have an unconditional basis [T], to be asymptotic c_0 or ℓ_p for some $1 \le p < \infty$ [MT] and to contain ℓ_1^n's uniformly [M]. These characteristics in conjunction with others developed in [OTW] show that T is the prime candidate for such a space.

For $n \in \mathbb{N}$, the Schreier classe S_n is a pointwise compact hereditary collection of finite subsets of $\mathbb{N}$ [AA]. For $E, F \subseteq \mathbb{N}$, we write $E < F$ (resp. $E \le F$) if $\max E < \min F$ (resp. $\max E \le \min F$) or if either one is empty.

$$S_0 = \{\{n\}\colon n \in \mathbb{N}\} \cup \{\emptyset\}\,.$$

We inductively define

$$S_{k+1} = \left\{\bigcup_{p=1}^{\ell} E_p\colon \{\ell\} \le E_1 < \cdots < E_\ell \text{ and } E_p \in S_k \text{ for } 1 \le p \le \ell\right\}.$$

$(E_i)_{i=1}^{\ell}$ is k-admissible if $E_1 < \cdots < E_\ell$ and $(\min E_i)_{i=1}^{\ell} \in S_k$. It is easy to see that

$$\begin{aligned} S_k[S_n] &\equiv \left\{\bigcup_{i=1}^{\ell} E_i\colon (E_i)_1^{\ell} \text{ is } k\text{-admissible and } E_i \in S_n \text{ for } 1 \le i \le \ell\right\} \\ &= S_{n+k}. \end{aligned}$$

If (y_i) is a basis then $(x_i)_1^{\ell} \prec (y_i)$ is k*-admissible* (w.r.t. (y_i)) if $(\operatorname{supp} x_i)_{i=1}^{\ell}$ is k-admissible. Here, if $x = \sum_{i \in A} a_i y_i$ and $a_i \neq 0$ for $i \in A$, then $\operatorname{supp} x = A$.

c_{00} denotes the linear space of finitely supported real sequences and (e_i) is the unit vector basis for c_{00}. If $x = \sum_i x(i)e_i \in c_{00}$ and $E \subseteq \mathbb{N}$ then $Ex \in c_{00}$ is defined by $Ex = \sum_{i \in E} x(i)e_i$. Let $\mathcal{F}$ be a pointwise compact *hereditary* (that is, $G \subseteq F \in \mathcal{F} \Rightarrow G \in \mathcal{F}$) family of finite subsets of $\mathbb{N}$ containing S_0 and let $0 < \lambda < 1$. The Tsirelson space $T(\mathcal{F}, \lambda)$ is the completion of c_{00} under the implicit norm

$$\|x\| = \|x\|_\infty \vee \sup \left\{ \lambda \sum_{i=1}^{\ell} \|E_i x\| \colon E_1 < \cdots < E_\ell \text{ and } (\min E_i)_1^\ell \in \mathcal{F} \right\} \tag{1.2}$$

Then (e_i) is a normalized unconditional basis for $T(\mathcal{F}, \lambda)$. Furthermore if $\mathcal{F} \supseteq S_1$ then $T(\mathcal{F}, \lambda)$ does not contain an isomorph of ℓ_1 but is *asymptotically* ℓ_1 (that is, if $(x_i)_1^\ell$ is 1-admissible then $\| \sum_1^\ell x_i \| \geq \lambda \sum_1^\ell \|x_i\|$). The existence of such a norm (1.2) can be found in [AD].

The classical Tsirelson's space is $T \equiv T(S_1, 2^{-1})$ and we write $\| \cdot \|$ $(= \| \cdot \|_1)$ for the norm of T. We also consider the space $T(S_n, 2^{-n})$, for a fixed $n \in \mathbb{N}$, and we denote its norm by $\| \cdot \|_n$. These norms are all equivalent on c_{00} and thus the spaces coincide. Indeed,

$$\|x\|_n \leq \|x\| \leq 2^{n-1} \|x\|_n \quad \text{for } x \in T. \tag{1.3}$$

We explain (1.3) and set some terminology for later use. $\|x\|$ is calculated as follows. If $\|x\| \neq \|x\|_\infty$ then $\|x\| = \frac{1}{2} \sum_1^\ell \|E_i^1 x\|$ for some 1-admissible collection $(E_i^1)_1^\ell$. For $i \leq \ell$ either $\|E_i^1 x\| = \|E_i^1 x\|_\infty$ or $\|E_i^1 x\|$ is calculated by means of a similar decomposition. Ultimately, for some finite $A \subseteq \mathbb{N}$, one obtains

$$\|x\| = \sum_{i \in A} 2^{-n(i)} |x(i)|,$$

where $n(i)$ is the number of decompositions necessary before obtaining a set $E_j^{n(i)}$ for which $\|E_j^{n(i)} x\| = \|E_j^{n(i)} x\|_\infty = |x(i)|$.

Thus the norm in T can be described as follows in terms of trees of sets. By an *admissible tree* $\mathcal{T}$ *of sets* we shall mean $\mathcal{T} = (E_i^n)$ for $1 \leq i \leq i(n)$, $0 \leq n \leq k$ is a tree of finite subsets of $\mathbb{N}$ partially ordered by reverse inclusion with the following properties. E_i^n is said to have level n. $i(0) = 1$, $E_i^n < E_j^n$ if $i < j$, all successors of any E_i^n form a 1-admissible partition of E_i^n and every set E_i^{n+1} is a successor of some E_j^n. Thus all sets of level n form an n-admissible collection. E_i^n is a *terminal* set of $\mathcal{T}$ if it has no successors.

Thus, for $x \in T$, one has

$$\|x\| = \sup \left\{ \sum_{i \in A} 2^{-n(i)} \|E_i x\|_\infty \colon (E_i)_{i \in A} \text{ are terminal sets of an admissible tree with level } E_i = n(i) \right\}. \tag{1.4}$$

Also, (1.4) holds if $\|E_i x\|_\infty$ is replaced by $\|E_i x\|$.

The norm $\|\cdot\|_n$ is calculated in a similar fashion except that terminal sets are allowed only to have levels kn for some $k = 0, 1, 2, \dots$:

$$\|x\|_n = \sup\Big\{\sum_{i\in A} 2^{-nk(i)}\|E_i x\|_\infty\colon (E_i)_{i\in a} \text{ are terminal sets of an admissible tree where } E_i \text{ has level } nk(i) \text{ for some } k(i) = 0, 1, 2, \dots\Big\}. \tag{1.5}$$

From these formulas we see that $\|x\|_n \le \|x\|$. Furthermore if $\mathcal{T}$ is an admissible tree, terminal sets not having levels $0, n, 2n, \cdots$ can be continued to the next such level, an increase of at most $n-1$ levels, yielding $\|x\| \le 2^{n-1}\|x\|_n$.

More exotic *mixed Tsirelson spaces* were introduced in [AD]. We shall not discuss a general definition, but we shall give a formula for the norm in a special case of interest here. For $j \ge 0$ and $n \in \mathbb{N}$ we let $\|\cdot\|_j^n$ be the norm of the mixed Tsirelson space $T\left((S_{j+kn}, 2^{-(j+kn)})_{k=0}^\infty\right)$. One obtains a formula for the norm similar to that in (1.4), except that terminal sets may only have levels $j, j+n, j+2n, \dots$:

$$\|x\|_j^n = \|x\|_\infty \vee \sup\Big\{\sum_{i\in A} 2^{-(j+nk(i))}\|E_i x\|_\infty\colon (E_i)_{i\in A} \text{ are terminal sets of an admissible tree having level } E_i = j + nk(i) \text{ for some } k(i) = 0, 1, 2, \dots\Big\}. \tag{1.6}$$

Thus $\|\cdot\|_0^n = \|\cdot\|_n$. Furthermore $\|\cdot\|_j^n$ is an equivalent norm on T.

We prove in Section 2 that the family of norms $(\|\cdot\|_j^n)_{n,j}$ cannot arbitrarily distort any subspace of T. We do this by introducing a slight variation of $\|\cdot\|_j^n$ (which omits the first term in (1.6)):

$$|x|_j^n = \sup\Big\{\sum_{i\in A} 2^{-(j+nk(i))}\|E_i x\|_\infty\colon (E_i)_{i\in A} \text{ are terminal sets of an admissible tree having level } E_i = j + nk(i),\ k(i) \ge 0\Big\}. \tag{1.7}$$

Thus $|\cdot|_0^n = \|\cdot\|_n$, $|\cdot|_j^n$ is an equivalent norm on T and $|\cdot|_j^n \le \|\cdot\|_j^n$. Our next proposition shows some simple facts about $|\cdot|_j^n$. Statements (a) and (b) are the reason we work with $|\cdot|_j^n$ rather than directly with $\|\cdot\|_j^n$. Moreover (d) shows that $\|\cdot\|_j^n$ and $|\cdot|_j^n$ are nearly the same on some subspace of any given $Y \prec T$. First recall the Schreier space X_m (see [AA], also [CS], for $m = 1$). X_m is the completion of c_{00} under

$$|x|_m = \sup\Big\{\Big|\sum_{i\in E} x(i)\Big|\colon E \in S_m\Big\}.$$

X_m is isometric to a subspace of $C(\omega^{\omega^m})$ and hence is c_0-saturated: if $Y \subseteq X$ then Y contains an isomorph of c_0. For $Z \subseteq T$, S_Z is the unit sphere w.r.t. the Tsirelson norm $\|\cdot\|$.

PROPOSITION 1.1. (a) *Let* $j \geq 0$ *and* $n \in \mathbb{N}$. *For* $x \in T$,

$$|x|_j^n = \frac{1}{2^j} \sup \left\{ \sum_{\ell=1}^{r} \|E_\ell x\|_n \colon (E_\ell)_1^r \textit{ is } j\textit{-admissible} \right\}$$

(b) *Let* $j \geq 0$ *and* $k, n \in \mathbb{N}$. *For* $x \in T$,

$$|x|_{j+k}^n = \frac{1}{2^k} \sup \left\{ \sum_{\ell=1}^{r} |E_\ell x|_j^n \colon (E_\ell)_1^r \textit{ is } k\textit{-admissible} \right\}$$

(c) *Let* $\varepsilon > 0$, $n, k \in \mathbb{N}$ *and* $0 \leq j < n$. *Let* $Y \prec T$. *Then there exists* $Z \prec Y$ *so that for all* $z \in S_Z$,

$$\left| |z|_j^n - |z|_{j+np}^n \right| < \varepsilon \textit{ if } 1 \leq p \leq k. \tag{1.8}$$

(d) *For* $n, j \in \mathbb{N}$, $\varepsilon > 0$ *and* $Y \prec T$ *there exists* $Z \prec Y$ *so that for all* $z \in S_Z$,

$$\left| |z|_j^n - \|z\|_j^n \right| < \varepsilon \quad \textit{and} \quad \left| |z|_n^n - \|z\|_n \right| < \varepsilon.$$

Proof. (a) and (b) follow easily from (1.5)–(1.7) and the fact that $S_{k+j} = S_k[S_j]$. (c) is proved by choosing Z so that the first few levels of the admissible tree used to compute $|z|_{j+nk}^n$ will contribute only a negligible amount. More precisely, we first note that

$$|z|_j^n \geq |z|_{j+np}^n \geq |z|_{j+nk}^n \text{ for } 1 \leq p \leq k\,.$$

Thus we need only achieve (1.8) for $p = k$. Let $|\cdot|_{j+nk}$ be the norm of the Schreier space X_{j+nk}. For $z \in T$ let

$$|z|_j^n = \sum_{\ell \in A} 2^{-(j+nk(\ell))} |z(\ell)|$$

be obtained from (1.7). Thus if

$$E = \{\ell \in A \colon k(\ell) < k\}$$

then $E \in S_{j+nk}$ and so

$$|z|_j^n \leq |Ez|_{j+nk} + |z|_{j+k}^n \leq |z|_{j+nk} + |z|_{j+k}^n.$$

Also $|z|_{j+nk} \leq 2^{j+nk}\|z\|$ for $z \in T$. Since X_{j+nk} is c_0-saturated and T does not contain c_0 it follows that given $Y \prec T$ there exists $Z \prec Y$ so that if $z \in S_Z$ then $|z|_{j+nk} < \varepsilon$. This proves (c), and (d) is proved similarly. The norms in question differ only in that the terminal sets of an admissible tree can differ only in a finite number of levels. $\square$

We shall need a generalized notion of n admissible. For $k, n \in \mathbb{N}$, $(E_r)_1^s$ is n *admissible* (k) if $(kE_r)_1^s$ is n admissible where $kE \equiv \{ke\colon e \in E\}$. Similarly we say $(y_i)_1^\ell \prec (e_i)$ is n admissible (k) if $(\operatorname{supp} y_i)_1^\ell$ is n admissible (k). Also we say a tree $\mathcal{T}$ is admissible (k) if $(kE)_{E\in\mathcal{T}}$ is an admissible tree.

PROPOSITION 1.2. *There exists $K_1 < \infty$ so that if $n, k \in \mathbb{N}$, $1 > \varepsilon > 0$ and $(y_i) \prec (e_i)$ is normalized (in T), then there exists a finite set $A \subseteq \mathbb{N}$ and $(\alpha_\ell)_{\ell\in A} \subset (0,1]$ so that $(y_\ell)_{\ell\in A}$ is n admissible, and setting $z = \sum_{\ell\in A} \alpha_\ell y_\ell$ we have the following*:

(i) $\sum_{\ell\in A} \alpha_\ell = 2^n$.
(ii) *If $B \subseteq A$ and $(y_\ell)_{\ell\in B}$ is $n-1$ admissible (k) then $\sum_{i\in B} \alpha_i < \varepsilon$.*
(iii) $1 \le \|z\| \le K_1$.

We call such a z an (n,ε) *average* (k) of (y_ℓ). This was proved in [OTW] for $k = 1$. The proof uses the following fact (e.g., see [CS], Prop. II.4).

PROPOSITION 1.3. *There exists $K_2 < \infty$ so that if (y_i) is a normalized block basis of (e_i) in T then for all (a_i), if $m_i = \min \operatorname{supp} y_i$ then*

$$\left\|\sum a_i e_{m_i}\right\| \le \left\|\sum a_i y_i\right\| \le K_2 \left\|\sum a_i e_{m_i}\right\|.$$

Proof of Proposition 1.2 By passing to a subsequence of (y_i) we may assume that $m_{i+1} > km_i$ where $m_i = \min \operatorname{supp} y_i$. By [OTW], we can find $z = \sum_{\ell\in A} \alpha_\ell y_\ell$, $(\alpha_\ell)_{\ell\in A} \subseteq \mathbb{R}^+$, $\sum_{\ell\in A} \alpha_\ell = 2^n$ and $\sum_{\ell\in B} \alpha_\ell < \varepsilon/2$ if $(m_\ell)_{\ell\in B} \in S_{n-1}$. Furthermore $1 \le \|z\| \le K_1$. It remains to check that (ii) holds. Suppose that $B \subseteq A$ so that $(km_i)_{i\in B} \in S_{n-1}$. Since $m_{i+1} > km_i$, this shows that $(m_{i+1})_{i\in B} \in S_{n-1}$ and hence $(m_i)_{i\in B\setminus \min B} \in S_{n-1}$. Thus $\sum_{\ell\in B\setminus\min B} \alpha_\ell < \varepsilon/2$. Also $\alpha_{\min B} < \varepsilon/2$ and so (ii) holds. □

2. Stabilizing the norms $(\|\cdot\|_n)$

Our goal is to prove that the norms $(\|\cdot\|_j^n)$ and hence in particular the norms $(\|\cdot\|_n)$ do not arbitrarily distort any subspace of T. In light of Proposition 1.1 it suffices to prove the following:

THEOREM 2.1. *There exists $K > 1$ so that for all $Y \prec T$ and $n \in \mathbb{N}$ there exist $Z \prec Y$ and $d > 0$ such that for all $0 \le j < n$ and $z \in S_Z$,*

$$d \le |z|_j^n \le Kd.$$

Before beginning the proof we recall that there exists $K_3 < \infty$ so that $\|\sum b_i e_{3i}\| \le K_3\|\sum b_i e_i\|$ [CS, Prop. I.12].

LEMMA 2.2. *Let (w_i) be a normalized block basis of (e_i) in T. Suppose that for some $c > 0$ and $L \geq 1$, for all i we have*

$$L^{-1}c \leq |w_i|_j^n \leq Lc \text{ for } 0 \leq j \leq n.$$

Let $w = \sum a_i w_i$, $\|w\| = 1$. Then for $0 \leq j < n$, $c(LK_2)^{-1} \leq |w|_j^n \leq 2LK_3 c$.

Proof. From Proposition 1.3 there exists an admissible tree $\mathcal{T}$ whose terminal sets are all equal to $\operatorname{supp} w_i$ for some i, yielding

$$\|w\| \geq \sum_{i \in A} |a_i| 2^{-n(i)} \|w_i\| = \sum_{i \in A} |a_i| 2^{-n(i)} \geq K_2^{-1}.$$

Let $1 \leq j \leq n$ be fixed. We shall produce a lower estimate for $|w|_j^n$ by extending $\mathcal{T}$ as follows. Fix $i \in A$ and consider the term $|a_i| 2^{-n(i)} \|w_i\|$. Suppose this term resulted from $E = \operatorname{supp} w_i$ where E was terminal in $\mathcal{T}$ of level $n(i)$. First suppose that $n(i) \geq j$ so that $n(i) = j + kn + p$ for some $0 \leq p < n$ and $k \geq 0$; then let $q = n - p$. If $n(i) < j$, let $q = j - n(i)$. If $q \geq 1$ extend $\mathcal{T}$ q-levels below E via the q-admissible family of sets which, by Proposition 1.1, yields

$$|w_i|_q^n = \frac{1}{2^q} \sum_{s=1}^{r} \|E_s^i w_i\|_n \geq cL^{-1}.$$

The new tree has terminal sets only at levels $(j + kn)_{k=0}^{\infty}$. When used in (1.7) it yields

$$|w|_j^n \geq \sum |a_i| 2^{-n(i)} cL^{-1} \geq c(LK_2)^{-1}.$$

For the upper estimate let $\mathcal{T}$ be the admissible tree having terminal sets (which we may assume to be singletons) of levels $j, j+n, j+2n, \ldots$ which produces $|w|_j^n$ in (1.7). We say w_i is *badly split* by some level of $\mathcal{T}$ if there exists $E \neq F$ in $\mathcal{T}$ having the same level with $Ew_i \neq 0$, $Ew_s \neq 0$ for some $s \neq i$ and $Fw_i \neq 0$. If no w_i is badly split by some level of $\mathcal{T}$ then if for some i, $\operatorname{supp} w_i$ contains a terminal set in $\mathcal{T}$ then there exists a 1-admissible family $(E_s^i)_1^{\ell(i)}$ in $\mathcal{T}$ of minimal level having the property that $\bigcup_1^{\ell(i)} E_s^i \subseteq \operatorname{supp} w_i$ and $F \cap \operatorname{supp} w_i = \emptyset$ for all other $F \in \mathcal{T}$ of the same level as the E_s^i's. Thus for some set A,

$$|w|_j^n = \sum_{i \in A} 2^{-n(i)} |a_i| \sum_{s=1}^{\ell(i)} |E_s^i w_i|_{j(i)}^n \tag{2.1}$$

where E_s^i has level $n(i)$ and $j(i) < n$ satisfies $n(i) + j(i) \in \{j, j+n, j+2n, \ldots\}$. Since $\|w\| = \|w_i\| = 1$, $\sum_{i \in A} 2^{-n(i)} |a_i| \leq 1$. Also,

$$\frac{1}{2} \sum_{s=1}^{\ell(i)} |E_s^i w_i|_{j(i)}^n \leq |w_i|_{j(i)+1}^n \leq Lc$$

by our hypothesis. Hence

$$|w|_j^n \leq 2Lc.$$

Of course $\mathcal{T}$ may badly split some w_i's. In this case we alter $\mathcal{T}$ as follows. Starting with the smallest level we check to see if a given level badly splits any w_i's. If it does we split the offending sets at $\min \operatorname{supp} w_i$ and $\max \operatorname{supp} w_i$. Thus, a given $E \in \mathcal{T}$ could be split into at most 3 pieces at this stage. We intersect successors of split sets with each of the at most three new pieces maintaining a tree, but losing admissibility. Then proceed to the next level of the new tree and repeat. We now have a tree $\mathcal{T}'$ that does not badly split any w_i. If we replace each set E in this tree by $3E$ we obtain an admissible tree. Thus $\mathcal{T}'$ is admissible (3). Furthermore, we obtain an expression like (2.1), except that the equality is replaced by the inequality

$$|w|_j^n \leq \sum_{i\in A} 2^{-n(i)}|a_i| \sum_{s=1}^{\ell(i)} |E_s^i w_i|_{j(i)}^n,$$

where the sets (E_s^i) come from our altered tree just as (2.1) was obtained from $\mathcal{T}$.

Letting $m_i = \min \operatorname{supp} w_i$ we have $\|\sum a_i e_{3m_i}\| \leq K_3 \|\sum a_i e_{m_i}\| \leq K_3\|w\| = K_3$. Since $\mathcal{T}'$ is an admissible (3) tree we have

$$\sum_{i\in A} 2^{-n(i)}|a_i| \leq K_3.$$

Thus $|w|_j^n \leq 2K_3Lc$. $\square$

Proof of Theorem 2.1. Fix $0 < \varepsilon < 1$ to be specified later. By Proposition 1.1 we may assume

$$\Big| |y|_j^n - |y|_{j+n}^n \Big| < \varepsilon \quad \text{for} \quad 0 \leq j \leq n \text{ and } y \in S_Y. \tag{2.2}$$

Also we may assume $(y_\ell) \prec (e_\ell)$ is a normalized (in T) basis for Y and that for some $(c_j)_0^{2n} \subseteq (0, 1]$,

$$\Big| |y_\ell|_j^n - c_j \Big| < \varepsilon \quad \text{for all } \ell, 0 \leq j \leq 2n. \tag{2.3}$$

Hence, from (2.2) and (2.3), we also have

$$|c_j - c_{j+n}| < 3\varepsilon \text{ if } 0 \leq j \leq n. \tag{2.4}$$

LEMMA 2.3. *Let $0 < i \leq n$ and let $z = \sum_{\ell\in A} \alpha_\ell y_\ell$ be an (i, ε) average* (3) *of (y_ℓ), $\alpha_\ell > 0$ for $\ell \in A$. Thus $(y_\ell)_{\ell\in A}$ is i admissible and $\sum_{\ell\in A} \alpha_\ell = 2^i$. Then*

$$c_{j-i} - \varepsilon \leq |z|_j^n \leq 2c_{j-i+1} + (K_3 + 1)2\varepsilon, \qquad 0 < i \leq j \leq n \tag{2.5}$$

$$c_{n+j-i} - \varepsilon \leq |z|_j^n \leq 2c_{n+j-i+1} + (K_3 + 1)3\varepsilon, \qquad 0 \leq j < i \leq n \tag{2.6}$$

Proof. For the first inequality in (2.5), let $k = j - i$. From Proposition 1.1, (2.3) and the fact that $S_k[S_i] = S_j$ we have

$$\begin{aligned} |z|_j^n &\geq \frac{1}{2^j} \sum_{\ell \in A} \alpha_\ell \sup \left\{ \sum_{s=1}^r \|E_s y_\ell\|_n \colon (E_s)_1^r \text{is } k \text{ admissible}\} \right\} \\ &= \frac{1}{2^i} \sum_{\ell \in A} \alpha_\ell |y_\ell|_k^n \geq \frac{1}{2^i} \sum_{\ell \in A} \alpha_\ell (c_k - \varepsilon) = c_{j-i} - \varepsilon. \end{aligned}$$

The second inequality in (2.5) is more difficult. By Proposition 1.1, there exist j admissible sets $(E_s)_1^r$ with

$$|z|_j^n = \frac{1}{2^j} \sum_{s=1}^r \|E_s z\|_n. \tag{2.7}$$

The sets $(E_s)_1^r$ are the terminal sets of an admissible tree $\mathcal{T}$, all having level j, and we may assume each $E_s \subseteq \bigcup_{\ell \in A} \operatorname{supp} y_\ell$. We adjust the tree $\mathcal{T}$ by splitting some sets if necessary, as we did in the proof of Lemma 2.2, to obtain a tree $\mathcal{T}'$ which is admissible (3) and which does not badly split any y_ℓ, $\ell \in A$. It may be that for some $E \in \mathcal{T}'$ we have $E \subseteq \operatorname{supp} y_\ell$ for some ℓ and level $E < i$. We remove all such sets from the tree $\mathcal{T}'$ (replace each F by $F \setminus \cup$ such sets and throw out the empty sets thus obtained). This gives us a tree $\mathcal{T}''$ which does not badly split any y_ℓ and for which no set of level $< i$ is contained in $\operatorname{supp} y_\ell$ for any $\ell \in A$. $\mathcal{T}''$ is admissible (3).

Let $(E_s')_{s=1}^{r'}$ and $(E_s'')_{s=1}^{r''}$ be the terminal sets of $\mathcal{T}'$ and $\mathcal{T}''$ respectively. Then (2.7) yields

$$\begin{aligned} |z|_j^n &\leq \frac{1}{2^j} \sum_{s=1}^{r'} \|E_s' z\|_n \\ &= \frac{1}{2^j} \sum_{s=1}^{r''} \|E_s'' z\|_n + \frac{1}{2^j} \sum_{s \in D} \|E_s' z\|_n \end{aligned} \tag{2.8}$$

where $D = \{1 \leq s \leq r' \colon E_s' \text{ was discarded from } \mathcal{T}' \text{ in forming } \mathcal{T}''\}$. Let

$$\begin{aligned} B &\equiv \{\ell \in A \colon E_s' \subseteq \operatorname{supp} y_\ell \text{ for some } s \in D\} \\ &= \{\ell \in A \colon E \subseteq \operatorname{supp} y_\ell \text{ for some } E \in \mathcal{T}' \text{ with level } E \leq i - 1\}. \end{aligned}$$

Thus if $B' \equiv B \setminus \min B$ then $(y_\ell)_{\ell \in B'}$ is $i - 1$ admissible (3). Hence $\sum_{\ell \in B} \alpha_\ell \leq \alpha_{\min B} + \sum_{\ell \in B'} \alpha_\ell < \varepsilon + \varepsilon = 2\varepsilon$. Now $(E_s')_{s \in D}$ is j admissible (3) and so for $\ell \in B$,

$$\frac{1}{2^j} \sum_{s \in D} \|E_s' y_\ell\|_n \leq \|\tilde{y}_\ell\| \leq K_3$$

where $\tilde{y} = \sum a_i e_{3i}$ if $y = \sum a_i e_i$. Thus

$$\frac{1}{2^j} \sum_{s \in D} \|E_s' z\|_n \leq K_3 \sum_{\ell \in B} \alpha_\ell < 2K_3 \varepsilon.$$

From this and (2.8) we obtain

$$|z|_j^n \leq \frac{1}{2^j} \sum_{s=1}^{r''} \|E_s''z\|_n + 2K_3\varepsilon. \tag{2.9}$$

Recall that $k = j - i$. For $\ell \in A$, $\{E_s'': E_s'' \subseteq \operatorname{supp} y_\ell\}$ is $k+1$ admissible. Indeed y_ℓ could first be split into a 1-admissible family only at level i or later by $\mathcal{T}''$. The tree $\mathcal{T}''$ continues from this point in an admissible fashion up to level j. Thus from (2.9),

$$\begin{aligned} |z|_j^n &\leq \frac{1}{2^j} \sum_{\ell \in A} \alpha_\ell \sup \left\{ \sum_{s=1}^{p} \|F_s y_\ell\|_n \colon (F_s)_1^p \text{ is } k+1 \text{ admissible} \right\} + 2K_3\varepsilon \\ &\leq \frac{1}{2^i} \sum_{\ell \in A} \alpha_\ell 2 |y_\ell|_{k+1}^n + 2K_3\varepsilon \\ &\leq 2c_{k+1} + 2K_3\varepsilon + 2\varepsilon. \end{aligned}$$

This completes the proof of (2.5).

For the lower estimate in (2.6) note that

$$\begin{aligned} |z|_j^n &\geq |z|_{j+n}^n \geq \frac{1}{2^{j+n}} \sum_{\ell \in A} \alpha_\ell \sup \left\{ \sum_{s=1}^{r} \|E_s y_\ell\|_n \colon (E_s)_1^r \text{ is } n+j-i \text{ admissible} \right\} \\ &\geq \frac{1}{2^i} \sum_{\ell \in A} \alpha_\ell (c_{n+j-i} - \varepsilon) = c_{n+j-i} - \varepsilon. \end{aligned} \tag{2.10}$$

Furthermore the argument in proving the upper estimate of (2.5) yields that $|z|_{j+n}^n \leq 2c_{n+j-i+1} + (K_3+1)2\varepsilon$ and since $|z|_j^n \leq |z|_{j+n}^n + \varepsilon$ we obtain (2.6). □

We continue the proof of Theorem 2.1 by using Proposition 1.2 to construct a block basis $(z_i)_{i=1}^n$ of (y_ℓ) so that each z_i is an (i, ε) average (3) of $(y_\ell)_{\ell=n}^\infty$. Let $z = \frac{1}{n}\sum_1^n z_i$. Let $c = \frac{1}{n}\sum_1^n c_i$.

LEMMA 2.4. *For* $0 \leq j \leq n$,

$$\frac{1}{2}c - \frac{n+3}{2n}\varepsilon \leq |z|_j^n \leq 2c + 3\varepsilon(K_3+1).$$

Proof. If $1 \leq j < n$ then by Lemma 2.3,

$$\begin{aligned} |z|_j^n &\leq \frac{1}{n} \sum_{i=1}^n |z_i|_j^n \\ &\leq \frac{1}{n} \left[2(c_j + c_{j-1} + \cdots + c_1 + c_n + c_{n-1} + \cdots + c_{j+1}) + (K_3+1)3n\varepsilon \right] \\ &\leq 2c + 3\varepsilon(K_3+1). \end{aligned}$$

Similarly if $j = 0$ or n,

$$|z|_j^n \leq \frac{1}{n}\left[2(c_n + c_{n-1} + \cdots + c_1) + (K_3 + 1)3n\varepsilon\right].$$

Hence the upper estimate is established.

To obtain the lower estimate we note that $(z_i)_1^n$ is 1 admissible hence by Proposition 1.1(b), if $1 \leq j \leq n$

$$\begin{aligned} |z|_j^n &= \frac{1}{n}\left|\sum_{i=1}^n z_i\right|_j^n \geq \frac{1}{2}\frac{1}{n}\sum_{i=1}^n |z_i|_{j-1}^n \\ &\geq \frac{1}{2n}[c_0 + \cdots + c_{n-1} - n\varepsilon]. \end{aligned}$$

Since $c_n > c_0 - 3\varepsilon$,

$$|z|_j^n \geq \frac{1}{2n}[c_1 + \cdots + c_n - (n+3)\varepsilon] = \frac{1}{2}c - \frac{n+3}{2n}\varepsilon.$$

Also $|z|_0^n \geq |z|_n^n$ and so the lemma is proved. □

Note that, by Proposition 1.2, z satisfies $\|z\| \leq \max_{1 \leq i \leq n} \|z_i\| \leq K_1$ and $\|z\| \geq \frac{1}{2n}\sum_1^n \|z_i\| \geq \frac{1}{2}$.

Furthermore, for an arbitrary $y \in T$, $\|y\| = 1$ implies that $|y|_j^n \geq 2^{-n}$ for $0 \leq j \leq n$ and thus we could have chosen $c_j \geq 2^{-n}$ for $0 \leq j \leq n$ and so in particular $c \geq 2^{-n}$. Thus (using Lemma 2.4) we can choose ε above to show that the element z satisfies

$$\frac{1}{3}c \leq |z|_j^n \leq 3c \text{ for } 0 \leq j \leq n.$$

These remarks in conjunction with Lemma 2.2 complete the proof of Theorem 2.1. □

Theorem 2.1 can be restated as saying that there exists an absolute constant K such that for all $Y \prec T$ and $n \in \mathbb{N}$ there exists $Z \prec Y$ such that

$$d = \inf_{0 \leq j < n} \inf_{z \in S_Z} |z|_j^n \leq \sup_{0 \leq j < n} \sup_{z \in S_Z} |z|_j^n \leq Kd. \tag{2.11}$$

It is natural to say that $Z \prec T$ is *n-stable at d* if Z satisfies (2.11). Obvious questions then arise. How does d depend upon Z, how does it depend upon n? Our next result answers these questions.

THEOREM 2.5. *There exists an absolute constant L so that if $Z \prec Y$ is n-stable at d then $(Kn)^{-1} \leq d \leq Ln^{-1}$.*

Proof. The lower estimate is relatively easy. Let $Z \prec Y$ be n stable at d and let $z \in S_Z$. $\|z\|$ is calculated by a tree ultimately yielding $\|z\| = 1 = \sum_{i\in A} 2^{-n(i)}|z(i)|$ as explained previously. The sets in the tree are permitted to stop at any level. If we gather together those which stop at levels $j, j+n, j+2n, \dots$ for $j = 0, 1, \dots, n-1$, we obtain $1 \le \sum_{j=0}^{n-1} |z|_j^n$. Hence for some $j < n$, $|z|_j^n \ge \frac{1}{n}$, and thus $d \ge \frac{1}{Kn}$, by (2.11).

Let $y \in Z \cap [(e_m)]_n^\infty$ with $\|y\| = 1$ and $y = \sum a_j e_j$. For $0 \le j < n-1$, choose y_j^* in the unit ball B_{T^*} of T^* so that

$$y_j^*(y) = |y|_j^n = \sum_{s\in A_j} 2^{-(j+k_j(s)n)}|a_s| \ge d.$$

We may assume that $A_j \subseteq \operatorname{supp} y$. Note that $\sum_{j=0}^{n-1} y_j^*(y) \ge nd$. Partition $\bigcup_{i=0}^{n-1} A_i$ into sets $(E_0, \dots, E_{n-1})$ as follows. $s \in E_j$ if and only if for all $i \ne j$ either $s \notin A_i$ or $j + k_j(s)n < i + k_i(s)n$. Then $(E_j y_j^*)_{j=0}^{n-1}$ is a collection of n disjointly supported vectors in B_{T^*} all having support contained in $[(e_m)]_n^\infty$. Since T and the modified Tsirelson space T_M are naturally isomorphic [CS] there exists an absolute constant L' so that

$$\left\|\sum_{j=0}^{n-1} E_j y_j^*\right\| \le L' \max_{0\le j<n} \|E_j y_j^*\|_{T^*} \le L'.$$

Furthermore

$$\left(\sum_{j=0}^{n-1} E_j y_j^*\right)(y) \ge \frac{nd}{2}.$$

Indeed, for $s \in \bigcup_{j=0}^{n-1} E_j$ pick j_0 such that $s \in E_{j_0}$ and denote by F_s the set of all $0 \le i < n$, $i \ne j_0$, such that $s \in A_i$. Then $\{i + k_i(s)n\colon i \in F_s\}$ is a subset of $\{j_0 + k_{j_0}(s)n + 1, j_0 + k_{j_0}(s)n + 2, \dots\}$. Thus

$$\begin{aligned} nd &\le \sum_{j=0}^{n-1} y_j^*(y) = \sum_{j=0}^{n-1}\sum_{s\in E_j} |a_s| \left(2^{-(j+k_j(s)n)} + \sum_{i\in F_s} 2^{-(i+k_i(s)n)}\right) \\ &\le \sum_{j=0}^{n-1}\sum_{s\in E_j} |a_s| \left(2^{-(j+k_j(s)n)} + 2^{-(j+k_j(s)n)}\right) = 2\sum_{j=0}^{n-1} E_j y_j^*(y). \end{aligned}$$

Hence $nd/2 \le L'$ so $d \le 2L'/n$. □

As an immediate consequence of Theorems 2.1 and 2.5 we get the following.

COROLLARY 2.6. *There exists an absolute constant C so that for every $Y \prec T$ and $n \in \mathbb{N}$ there exists $Z \prec Y$ and $d > 0$ so that $Z \prec Y$ is n-stable at d and $(Cn)^{-1} \le d \le Cn^{-1}$.*

3. Further results

We now turn to some stabilization results for more general norms on T. Given an arbitrary equivalent norm $|\cdot|$ on $Y \prec T$, we describe some procedures on $|\cdot|$, natural in the context of Tsirelson space, which lead to new norms that cannot distort T by too much.

Recall [OTW] that if (y_i) is a basis for Y and $n \in \mathbb{N}$ then

$$\delta_n(y_i) = \inf\left\{\delta \geq 0\colon \quad \left\|\sum_1^k x_i\right\| \geq \delta \sum_1^k \|x_i\| \right.$$
$$\left. \text{whenever } (x_i)_1^k \text{ is } n\text{-admissible w.r.t. } (y_j)\right\}. \quad (3.1)$$

A result of the type we pursue and which we shall need later was proved in [OTW], Theorem 6.2 (in stronger form).

PROPOSITION 3.1. *There exists $D < \infty$ so that if (y_i) is a normalized block basis of (e_i) for $Y \prec T$ and $|\cdot|$ is an equivalent norm on Y with $\delta_1((y_i), |\cdot|) = \frac{1}{2}$ then $|\cdot|$ does not D distort Y.*

Remark 3.2. It was shown in [OTW] that for a block basis (y_i) of (e_i) and any equivalent norm $|\cdot|$ on $Y = [(y_i)]$,

$$\delta_n((y_i), |\cdot|) \leq 2^{-n} \text{ for all } n.$$

If $|\cdot|$ is an equivalent norm on $Y = [(y_j)] \prec T$, for $j \geq 0$ and $x \in Y$, we set

$$|x|_j = \frac{1}{2^j} \sup\left\{\sum_1^\ell |E_i x|\colon (E_i)_1^\ell \text{ is } j \text{ admissible}\right\},$$

(If $x = \sum a_i y_i$, $Ex = \sum_{i \in E} a_i y_i$.) Thus $|z|_0 = |z|$ and $|\cdot|_j$ is an equivalent norm on Y for all j. For $n \in \mathbb{N}$ we let

$$|z|^{(n)} = \frac{1}{n} \sum_{j=0}^{n-1} |z|_j.$$

PROPOSITION 3.3. *There exists $D < \infty$ so that if $n \in \mathbb{N}$ and $|\cdot|$ is an equivalent norm on $Y \prec T$ having basis $(y_i) \prec (e_i)$ and satisfying*

$$\left|\sum_1^k x_i\right| \geq \frac{1}{2} \sum_1^k |x_i|_{n-1}$$

for all 1 admissible $(x_i)_1^k \prec (y_j)$ then $|\cdot|^{(n)}$ cannot D distort Y.

Proof. Let $(x_i)_1^k$ be 1 admissible w.r.t. (y_ℓ). Then for $j \geq 1$,

$$\left|\sum_{i=1}^k x_i\right|_j \geq \frac{1}{2}\sum_{i=1}^k |x_i|_{j-1}$$

since $S_1[S_{j-1}] = S_j$. Thus using the hypothesis,

$$\begin{aligned}\left|\sum_{i=1}^k x_i\right|^{(n)} &= \frac{1}{n}\sum_{j=1}^{n-1}\left|\sum_{i=1}^k x_i\right|_j + \frac{1}{n}\left|\sum_{i=1}^k x_i\right| \\ &\geq \frac{1}{2}\frac{1}{n}\sum_{j=0}^{n-2}\sum_{i=1}^k |x_i|_j + \frac{1}{2}\frac{1}{n}\sum_{i=1}^k |x_i|_{n-1} \\ &= \frac{1}{2}\sum_{i=1}^k\left(\frac{1}{n}\sum_{j=0}^{n-1}|x_i|_j\right) = \frac{1}{2}\sum_{i=1}^k |x_i|^{(n)}.\end{aligned}$$

Thus $\delta_1((y_i), |\cdot|^{(n)}) = \frac{1}{2}$. The proposition follows from Proposition 3.1. □

Remark 3.4. The hypothesis of Proposition 3.3 is satisfied if $\delta_n(|\cdot|) = 2^{-n}$.

If $|\cdot|$ is an equivalent norm on $Y = [(y_i)] \prec T$, we define an equivalent norm on Y by

$$|x|_{\mathrm{Tr}} = \sup\left\{\sum_{i\in A} 2^{-n(i)}|E_i x| : \quad (E_i)_{i\in A} \text{ are the terminal sets of an admissible tree with level } E_i = n(i)\right\}.$$

Clearly $|\cdot| \leq |\cdot|_{\mathrm{Tr}}$ and if $|\cdot| \leq \|\cdot\|$ then $|\cdot|_{\mathrm{Tr}} \leq \|\cdot\|$. Note that $\|\cdot\| = \|\cdot\|_{\mathrm{Tr}}$ if $(y_i) = (e_i)$.

The constant K_2 appearing in several arguments below is the constant from Proposition 1.3.

PROPOSITION 3.5. *There exists K $(= 2K_2M)$ so that if $|\cdot|$ is any equivalent norm on $Y = [(y_i)] \prec T$ then $|\cdot|_{\mathrm{Tr}}$ does not K distort Y.*

Proof. By multiplying $|\cdot|$ by a scalar and passing to $Z \prec Y$ we may assume that $\|\cdot\| \geq |\cdot|$ on Z and Z has a basis $(z_i)_i^\infty$ with $\|z_i\| = 1$ and $|z_i| > \frac{1}{2}$ for all i. Furthermore, by [AO], we may assume that for all j if $(z_i)_{i\in E}$ is j-admissible w.r.t. (e_i) then $(z_{i+1})_{i\in E}$ is j-admissible w.r.t. (y_i).

Let $z = \sum_1^\ell a_i z_i$ with $\|z\| = 1$. Then $\|\sum_1^\ell a_i z_{i-1}\| \geq M^{-1}$ for some absolute constant M [CS]. By Proposition 1.3 there exists an admissible tree w.r.t. (e_i) having terminal sets of the form $\operatorname{supp} z_{i-1}$ and level $n(i)$ for all i in some set A so that

$$\sum_{i\in A} 2^{-n(i)}|a_i| \geq K_2^{-1}\|\sum a_i z_{i-1}\| \geq (K_2M)^{-1}.$$

It follows that

$$|z|_{\mathrm{Tr}} \geq \sum_{i\in A} 2^{-n(i)}|a_i|\,|z_i| > (2K_2M)^{-1},$$

completing the proof. □

Remark 3.6. It follows from Proposition 3.5 that if $|\cdot|$ is an equivalent norm on $Y = [(y_i)] \prec T$ satisfying $|y|_{\mathrm{Tr}} \leq \gamma|y|$ for all $y \in Y$, then $|\cdot|$ does not $K\gamma$ distort Y.

PROPOSITION 3.7. *For all $\gamma > 0$ there exists $D(\gamma) < \infty$ with the following property. Let $Y = [(y_i)] \prec T$. If $|\cdot|$ is an equivalent norm on Y and $n \in \mathbb{N}$ is such that $\delta_n((y_i), |\cdot|) = 2^{-n}$ and $|y|_j \geq \gamma|y|$ for all $y \in Y$ and $j < n$, then $|\cdot|$ does not $D(\gamma)$ distort Y.*

Proof. By Theorem 2.1 we may choose $(z_i) \prec (y_i)$, $Z = [(z_i)]$, so that for some $d > 0$,

$$d \leq \|z\|_n \leq Kd \text{ for all } z \in S_Z.$$

Furthermore, by passing to a block basis of Z and scaling $|\cdot|$ as necessary, we may assume that $\|z\|_n \geq |z|$ for all $z \in Z$ and $1 = \|z_i\| \geq \|z_i\|_n \geq |z_i| \geq \frac{1}{2}\|z_i\|_n$ for all i. Finally, again by [AO] we may assume that if $(z_i)_{i\in E}$ is j-admissible w.r.t. (e_i) then $(z_{i+1})_{i\in E}$ is j-admissible w.r.t. (y_i).

Let $z = \sum a_i z_i$ with $\|z\| = 1$.

As in the proof of Proposition 3.5 there exists an admissible tree w.r.t (y_i) having terminal sets of the form $\operatorname{supp} z_i$ and level $n(i)$, $i \in A$, yielding

$$\sum_{i\in A} 2^{-n(i)}|a_i|\,\|z_i\| \geq (K_2M)^{-1}.$$

Choose $0 \leq j(i) < n$ so that $n(i) + j(i) \in \{0, n, 2n, \ldots\}$. Since $\delta_n((y_i), |\cdot|) = 2^{-n}$ we obtain

$$\begin{aligned} |z| &\geq \sum_{i\in A} 2^{-n(i)}|a_i|\,|z_i|_{j(i)} \geq \gamma \sum_{i\in A} 2^{-n(i)}|a_i|\,|z_i| \\ &\geq \frac{\gamma}{2}\sum_{i\in A} 2^{-n(i)}|a_i|\,\|z_i\|_n \geq \frac{\gamma d}{2K_2M}. \end{aligned}$$

Thus

$$\|z\|_n \geq |z| \geq \frac{\gamma}{2K_2KM}\|y\|_n.$$

Hence $Kd \geq |z| \geq \frac{\gamma}{2K_2KM}d$. The theorem is proved with $D(\gamma) = 2\gamma^{-1}K_2K^2M$. □

Our next result combines the proofs of Proposition 3.7 and the main theorem.

PROPOSITION 3.8. *For $\gamma > 0$ there exists $D(\gamma) < \infty$ so that the following holds. Let $n \in \mathbb{N}$ and let $|\cdot|$ be an equivalent norm on $Y = [(y_i)] \prec T$ with $\delta_n(|\cdot|) = 2^{-n}$. Suppose that for all $y \in Y$, $|y|_n \geq \gamma|y|$ and $|y| \geq \gamma|y|_j$ for $1 \leq j \leq n$. Then $|\cdot|$ does not $D(\gamma)$ distort Y.*

Proof. As in the proof of Proposition 3.7 we may assume that $\|\cdot\|_n \geq |\cdot|$ on Z, Z has a normalized (in T) basis $(z_i) \prec (y_i)$ with $|z_i| \geq \frac{1}{2}\|z_i\|_n$ for all i. In addition, from Theorem 2.1 we may assume

$$d \leq |z|_j^n \leq Kd \quad \text{for } 0 \leq j \leq n \text{ and } z \in S_Z.$$

Finally, we again assume that if $(z_i)_E$ is j-admissible w.r.t. (e_i) then $(z_{i+1})_E$ is j-admissible w.r.t. (y_i).

Note that the hypothesis $\delta_n(|\cdot|) = 2^{-n}$ implies $|\cdot| \geq |\cdot|_n$ and more generally $|\cdot|_j \geq |\cdot|_{n+j}$. $|\cdot|_n \geq \gamma|\cdot|$ implies that (on Y) $|\cdot|_j \leq \gamma^{-1}|\cdot|_{n+j}$.

Furthermore we may assume that, for a suitably small $\varepsilon > 0$, $||z_\ell|_j - c_j| < \varepsilon$ for all $\ell \in \mathbb{N}$ and $0 \leq j \leq n$ for some $(c_j)_0^n \subseteq \mathbb{R}^+$.

Fix $1 \leq i \leq n$ and let $z = \sum \alpha_\ell z_\ell$ be an (i, ε) average (3) of (z_ℓ). Note that $|z|_i \geq \frac{1}{2^i}\sum_{\ell\in A}\alpha_\ell|z_\ell| \geq \frac{d}{2}$ hence

(i) $|z| \geq \gamma|z|_i \geq \frac{d\gamma}{2}$.

The argument of Lemma 2.3 remains valid for estimates on $|z|_j$. The proof of the upper estimate of (2.6) yields

$$|z|_{j+n} \leq 2c_{n+j-i+1} + 2\varepsilon(K_3 + 1),$$

hence

$$|z|_j \leq \gamma^{-1}\Big(2c_{n+j-i+1} + 2\varepsilon(K_3 + 1)\Big).$$

If we set $w = \frac{1}{n}\sum_1^n w_i$ where $(w_i)_1^n \prec (z_\ell)_n^\infty$ and each w_i is an (i, ε) average (3) of (z_i) then, as in Lemma 2.4 (taking ε suitably small),

(ii) $\frac{1}{3}c \leq |w|_j \leq 3\gamma^{-1}c$ $(0 \leq j \leq n)$ where $c = \frac{1}{n}\sum_1^n c_i$.

Also

$$|w|_1 \geq \frac{1}{2n}\sum_1^n |w_i| \geq \frac{d\gamma}{4}$$

from (i) and so

(iii) $|w| \geq \gamma |w|_1 \geq \frac{d}{4}\gamma^2$.

From (ii) and (iii) we have $\frac{d}{4}\gamma^2 \leq 3\gamma^{-1}c$ and so $c \geq \frac{d\gamma^3}{12}$. Thus, from (ii)

(iv) $|w|_j \geq \frac{d\gamma^3}{36}$ for $0 \leq j \leq n$.

We are ready to apply the proof of Proposition 3.7. Let $(w_i) \prec (z_\ell)$ be such that each w_i is constructed as was w above. Let $w = \sum a_i w_i$ with $\|w\| = 1$. Choose an admissible tree having terminal sets $\operatorname{supp} w_i$ for $i \in A$ yielding $\sum 2^{-n(i)}|a_i|\,\|w_i\| \geq (K_2 M)^{-1}$. It follows that if $0 \leq j(i) < n$ satisfies $n(i) + j(i) \in \{0, n, 2n, \ldots\}$ then

$$Kd \geq \|w\|_n \geq |w| \geq \sum 2^{-n(i)}|a_i|\,|w_i|_{j(i)} \geq \frac{d\gamma^3}{36K_2M}.$$

The theorem is proved with $D(\gamma) = \frac{36KK_2M}{\gamma^3}$. □

In comparison with (1.3), it is of interest to consider the mixed Tsirelson space (see [AD]) $T((S_k, c_k 2^{-k})_k)$, where $c_k \uparrow 1$. We then ask whether it also coincides with T, or, at least, whether its norm, $|\cdot|$ say, is an equivalent norm on a subspace of T. The following result gives the positive answer to the latter question. It also indicates that the answer to the former question probably depends upon the asymptotic behavior of (c_k). Finally, it should be compared with Example 5.12 from [OTW] which implies that if $c_k < \delta < 1$ then no subspace of $T((S_k, c_k 2^{-k})_k)$ is isomorphic to a subspace of T.

PROPOSITION 3.9. *There exists a block subspace $X \prec T$ such that $c\|x\| \leq |x| \leq \|x\|$ for $x \in X$, where $c > 0$ is an absolute constant, independent of the choice of $c_k \uparrow 1$.*

Outline of the proof. Clearly, $|x| \leq \|x\|$ for all $x \in T$. Choose $n(i) \uparrow \infty$ such that $\prod_1^\infty c_{n(i)} > \frac{1}{2}$ and $\sum_1^\infty 2^{-n(i)} < 1/4K_2$. Let $m(1) = n(1)$ and inductively choose $m(i) \uparrow \infty$ so that $m(i+1) \geq 2(m(i) + n(i))$ for all $i = 1, 2, \ldots$.

Choose $(x_i) \prec T$ to be a block basis of (e_i) such that each x_i is an $(m(i), 1)$ average (1) of (e_i). In particular, $x_i = \sum_{j \in F_i} \alpha_j^i e_j$, where $\alpha_j^i > 0$ for $j \in F_i$, $F_i \in S_{m(i)}$ and $\sum_{j \in F_i} \alpha_j^i = 2^{m(i)}$. It is easy to check that $1 \leq \|x_i\| \leq 2$.

Let $x = \sum_1^\ell a_i x_i$ with $\|x\| = 1$. By Proposition 1.3, there exists an admissible tree $\mathcal{T}$ having terminal sets of the form $\operatorname{supp} x_i$ and level $p(i)$, yielding

$$\sum_{i \in S} 2^{-p(i)} |a_i| \, \|x_i\| \geq 1/K_2$$

for some $S \subseteq \{1, \ldots, \ell\}$. Set $G = \{i \in S \colon p(i) \leq n(i)\}$. Note that if $B = S \setminus G$ then $\sum_B 2^{-p(i)} \leq \sum_B 2^{-n(i)} < 1/4K_2$, and so

$$\sum_{i \in B} 2^{-p(i)} |a_i| \, \|x_i\| < 2/4K_2 = 1/2K_2.$$

Thus

$$\sum_{i \in G} 2^{-p(i)} |a_i| \, \|x_i\| > 1/2K_2.$$

Prune the tree $\mathcal{T}$ so as to only admit terminal sets of the form $\operatorname{supp} x_i$ for $i \in G$. Extend each of these sets $m(i)$ levels in an admissible fashion, ending at the singletons which form $\operatorname{supp} x_i$, ultimately obtaining an admissible tree $\mathcal{T}'$. Since $\|x_i\| \leq 2$, it follows that

$$\sum_{i \in G} 2^{-p(i)} |a_i| \sum_{j \in F_i} 2^{-m(i)} \alpha_j^i > 1/4K_2,$$

which can be rewritten as

$$\sum_{i \in G} \sum_{j \in F_i} 2^{-p(i)-m(i)} |a_i| \alpha_j^i > 1/4K_2.$$

For $i \in G$, all elements in the support of x_i are terminal sets of $\mathcal{T}'$ having level $j(i) \equiv p(i) + m(i)$. Note that for $i' \in G$, $i' > i$, the definition of G and the growth condition on $m(i)$ imply that

$$j(i') - j(i) = p(i') + m(i') - p(i) - m(i) \geq m(i') - n(i) - m(i) \geq n(i).$$

Let $G = \{i_1, \ldots, i_s\}$ written in the increasing order. The admissible tree $\mathcal{T}'$ has terminal sets of level $j(i_1)$ which together equal the support of x_{i_1}, of level $j(i_2)$ which together equal the support of x_{i_2}, and so on. Also, $j(i_{k+1}) - j(i_k) \geq n(i_k) \geq n(k)$.

By considering all the sets of $\mathcal{T}'$ of level $j(i_1)$ we obtain

$$|x| \geq c_{j(i_1)} \left(2^{j(i_1)} |a_{i_1}| \sum_{j \in F_{i_1}} \alpha_j^{i_1} + 2^{j(i_1)} \sum_{r=1}^{r(1)} |E_r^{i_1} x| \right),$$

where $(E_r^{i_1})$ are the remaining sets in $\mathcal{T}'$ of level $j(i_1)$ which are disjoint from the support of x_{i_1}.

We iterate this estimate next continuing the sets $(E_r^{i_1})$ to level $j(i_2)$ and so on. Ultimately we obtain

$$|x| \geq c_{j(i_1)}\left(2^{j(i_1)}|a_{i_1}|\sum_{j\in F_{i_1}}\alpha_j^{i_1} + c_{j(i_2)-j(i_1)}\left(2^{j(i_2)}|a_{i_2}|\sum_{j\in F_{i_2}}\alpha_j^{i_2} + c_{j(i_3)-j(i_2)}\left(2^{j(i_3)}|a_{i_3}|\sum_{j\in F_{i_3}}\alpha_j^{i_3} + \cdots\right)\right)\right).$$

Since $j(i_{k+1}) - j(i_k) \geq n(k)$, this yields

$$|x| \geq \prod_{k=1}^{\infty} c_{n(k)}\left(\sum_{r=1}^{s} 2^{-j(i_r)}|a_{i_r}|\sum_{j\in F_{i_r}}\alpha_j^{i_r}\right) \geq \frac{1}{2}(1/4K_2) = 1/8K_2,$$

completing the proof. □

Until now we considered the Tsirelson space $T \equiv T(S_1, 2^{-1})$, its subspaces and renormings. Analogous results also hold for Tsirelson spaces $T_\theta \equiv T(S_1, \theta)$, where $0 < \theta < 1$. It should be noted, however, that absolute constants will change to functions depending on θ (typically of the form $c\theta^{-1}$ where c is an absolute constant). In particular, let us recall that the space T_θ admits a $\theta^{-1} - \varepsilon$ distorted norm for every $\varepsilon > 0$ (the proof is exactly the same as for T). In this context a distortion property of the renorming $T(S_n, \theta^n)$ of T_θ might be also of interest.

PROPOSITION 3.10. *Let $n \in \mathbb{N}$ and $0 < \theta < 1$. Let $X = T(S_n, \theta^n)$. Every $Y \prec X$ contains $Z \prec Y$ such that Z is $\theta^{-1} - \varepsilon$ distortable for every $\varepsilon > 0$.*

Outline of the proof. First note that the modulus δ_m defined in (3.1) has the following property: For all $Y \prec X$ and $k \in \mathbb{N}$, $\delta_{nk}(Y) \leq \theta^{n(k-1)+1}$. Indeed, let $Y \prec X$ and let (y_i) be a normalized basis in Y. Let $0 < \varepsilon < 1$ and $y = \sum_{i\in A}\alpha_i y_i$ be an (nk, ε) average (1), satisfying conditions (i) and (ii) of Proposition 1.2. (Observe that these two conditions have a purely combinatorial character, and their validity does not depend on the underlying Banach space.) In particular, $\sum_{i\in A}\alpha_i = 2^{nk}$. Then $\|y\| \geq \delta_{nk}(Y)2^{nk}$. Iterating the definition of the norm $k-1$ times we obtain

$$\|y\| \leq \theta^{n(k-1)}\sum_{j=1}^{\ell}\|E_j y\| + \sum_{i\in B}\alpha_i,$$

where $i \in B$ if supp y_i is split by some set in the tree of sets obtained by iterating the norm definition. Thus $B \in S_{n(k-1)}$ and $E_1 < \cdots < E_\ell$ is $n(k-1)$-admissible, and for $s \leq \ell$ and $i \in A$ one has $E_s \cap \operatorname{supp} y_i = \emptyset$ or $E_j \supseteq \operatorname{supp} y_i$. Thus

$$\|y\| \leq \theta^{n(k-1)} \sum_{i \in A} \alpha_i + \varepsilon \leq \theta^{n(k-1)} 2^{nk} + \varepsilon.$$

Comparing this with the lower estimate for $\|y\|$ yields the required bound for $\delta_{nk}(Y)$.

The supermultiplicativity property $\delta_{nk}(Y) \geq (\delta_1(Y))^{nk}$ ([OTW], Prop. 4.11) and the previous estimate immediately imply that for all $Y \prec X$, $\delta_1(Y) \leq \theta$.

This in turn implies that for every $Y \prec X$ there exists $Z \prec Y$ such that for every $\varepsilon > 0$ there is $k \in \mathbb{N}$ satisfying the following: For all $W \prec Z$ there exist $w_1 < \cdots w_k$ in W such that $\| \sum_{i=1}^k w_i \| = 1$ and $\sum_{i=1}^k \|w_i\| \geq \theta^{-1} - \varepsilon$. If not, then stabilizing suitable quantities for $k = 1, 2, \ldots$ by passing to appropriate subspaces, and using a diagonal argument and the definition of S_1, we would get a subspace Y' with $\delta_1(Y') > \theta$.

Now, given $\varepsilon > 0$, define $|\cdot|$ on Z by

$$|z| = \sup_{E_1 < \cdots < E_k} \sum_{i=1}^k \|E_i z\|,$$

where Ez is the projection with respect to the basis of Z. Clearly, $\|z\| \leq |z| \leq k\|z\|$ for $z \in Z$. Let $W \prec Z$. By the previous claim, there exists $w \in W$ with $\|w\| = 1$ and $|w| \geq \theta^{-1} - \varepsilon$. On the other hand, a standard argument involving long ℓ_1^m averages implies that there exists $x \in W$ with $\|x\| = 1$ and $|x| \leq 1 + \varepsilon$ (e.g., see [OTW], Prop. 2.7). □

4. Problems

Of course the main problem is the following.

PROBLEM 4.1. *Is T arbitrarily distortable? Is any subspace of T arbitrarily distortable?*

Our work in Section 3 suggests the following problems.

PROBLEM 4.2. *Prove that the class of equivalent norms on T for which $\delta_n(|\cdot|) = 2^{-n}$ for some $n > 1$ do not arbitrarily distort T or any $Y \prec T$.*

PROBLEM 4.3. *Prove that for $\gamma > 0$ there exists $K(\gamma) < \infty$ so that if $|\cdot|$ is an equivalent norm on T satisfying $\delta_n(|\cdot|) \geq \gamma 2^{-n}$ for all n then $|\cdot|$ does not $K(\gamma)$ distort any $Y \prec T$.*

PROBLEM 4.4. *Prove there exists $K < \infty$ so that if $|\cdot|$ is an equivalent norm on T and $Y \prec T$ then for some n, $|\cdot|^{(n)}$ does not K distort Y.*

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