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REITER'S CONDITION P_2 AND THE PLANCHEREL MEASURE FOR HYPERGROUPS

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ABSTRACT. In this paper we study the Reiter P_2 condition for commutative hypergroups and give necessary and sufficient conditions for $x \in \operatorname{supp} \pi$, where π is the Plancherel measure. Finally we apply general results to characterize $\operatorname{supp} \pi$ in the case of polynomial hypergroups.

1. Introduction

Let $(p_n)_{n \in \mathbb{N}_0}$ be a sequence of polynomials on the real line satisfying a recurrence formula

$$x\, p_n(x) = \alpha_{n+1}\, p_{n+1}(x) + \beta_n\, p_n(x) + \alpha_n\, p_{n-1}(x), \tag{1.1}$$

where $\alpha_n > 0$, for $n \in \mathbb{N}$, $p_0(x) = 1$ and $p_{-1}(x) = 0$. By the Favard theorem there exists a probability measure π on $\mathbb{R}$ such that the polynomials p_n are orthonormal with respect to π. In general it is rather difficult to derive from properties of $p_n(x)$ and α_n, β_n whether some real number x is contained in $\operatorname{supp} \pi$ or not. If $(p_n)_{n \in \mathbb{N}_0}$ belongs to the Nevai class $\mathcal{M}(b, a)$, i.e., $\lim_{n\to\infty} \alpha_n = \frac{a}{2}$ and $\lim_{n\to\infty} \beta_n = b$, and if $a > 0$ then by a theorem of Blumenthal we have $\operatorname{supp} \pi = [b-a], [b+a] \cup S$, where S is bounded and countable with only possible accumulation points in $\{b \pm a\}$ (see [9], p. 23). If one assumes in addition that the polynomials $(p_n)_{n \in \mathbb{N}_0}$ give rise to a convolution structure on $l^1(\mathbb{N}_0)$ (i.e., they induce a polynomial hypergroup structure on $\mathbb{N}_0$ (see [5], [8]) and if $a = 1$, $b = 0$ one has $\operatorname{supp} \pi = [-1, 1]$ (see [14] and [8]). In the latter case Banach algebra techniques are applied, where the algebra structure is inherited from the hypergroup structure on $\mathbb{N}_0$. In [10] amenability of hypergroups is investigated (a concept of harmonic analysis). Among many other results connected with amenability it is shown that the constant character 1 is contained in the support of the Plancherel measure if, and only if the Reiter condition (P_2) is satisfied. Translating this result to orthogonal polynomials inducing convolution structure on $l^1(\mathbb{N}_0)$ this yields a characterization of $1 \in \operatorname{supp} \pi$. The purpose of this paper is first to initiate a systematic study of a shifted Reiter condition (P_2) on commutative hypergroups and second to apply these results to characterize $\operatorname{supp} \pi$ in the case of orthogonal polynomials that induce a hypergroup on $\mathbb{N}_0$.

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2. Preliminaries

Througout this paper, K will denote a commutative hypergroup, see [1] (same as convo in Jewett [3]). The following notations and basic results of harmonic analysis on K will be applied in the sequel. The convolution of two elements $x, y \in K$ is denoted by $\omega(x, y)$, and the involution by $\tilde{x}$.

Let $C_c(K)$ be the space of all continuous functions with compact support. Given $y \in K$ and $f \in C_c(K)$ the translation $L_y f$ of f is given by

$$L_y f(x) = \omega(y, x)(f).$$

A Haar measure on K is a regular positive Borel measure m on K, $m \neq 0$ such that $\int_K f(x)\,dm(x) = \int_K L_y f(x)\,dm(x)$ for all $f \in C_c(K)$ and $y \in K$. By the commutativity of K the existence of a Haar measure m on K is ensured. A fixed Haar measure on K is denoted throughout by m. The dual space $\widehat{K}$ is defined by

$$\widehat{K} = \left\{\alpha \in C^b(K) \colon \alpha \neq 0,\ \omega(x, y)(\alpha) = \alpha(x)\,\alpha(y), \alpha(\tilde{x}) = \overline{\alpha(x)}\right\},$$

where $C^b(K)$ is the space of all bounded continuous functions on K. For $f \in L^p(K, m)$ let $f^* \in L^p(K, m)$ be given by $f^*(x) = \overline{f(\tilde{x})}$. Similar as for locally compact Abelian groups one can identify $\widehat{K}$ with the symmetric structure space $\Delta_S(L^1(K, m))$ of the Banach* algebra $L^1(K, m)$; see [3] or [1]. The topology of convergence on compacta on $\widehat{K}$ is equal to the topology $\sigma(L^\infty(K, m),\ L^1(K, m))$ restricted to $\widehat{K}$. Equipped with this topology $\widehat{K}$ is a locally compact space. The Fourier transform of $f \in L^1(K, m)$ is defined by

$$\widehat{f}(\alpha) = \int_K f(x)\,\overline{\alpha(x)}\,dm(x), \qquad \alpha \in \widehat{K}.$$

For $f \in L^p(K, m)$, $1 \leq p \leq \infty$, one can define the translation $L_y f$ of f with $y \in K$ by putting

$$L_y f(x) = \omega(y, x)(f).$$

Based on that translation one defines the convolution $f * g$, where $g \in L^1(K, m)$ and $f \in L^p(K, m)$, $1 \leq p < \infty$ by

$$f * g(x) = \int_K f(y)\, L_{\tilde{y}} g(x)\,dm(y).$$

We have $f * g \in L^p(K, m)$ and due to the translation invariance of the Haar measure one has in the case of $f,\ g \in L^1(K, m)$,

$$\int_K L_x f(y)\, g(y)\,dm(y) = \int_K f(y)\, L_{\tilde{x}} g(y)\,dm(y).$$

Every $f \in L^1(K,m)$ defines a bounded linear operator L_f on the Hilbert space $L^2(K,m)$ by $L_f(h) = f * h$, where $h \in L^2(K,m)$. The mapping $f \to L_f$, $L^1(K,m) \to B(L^2(K,m))$ is called regular representation of K. It is an injective mapping and satisfies $\|L_f\| \leq \|f\|_1$, $L_{f*g} = L_f \circ L_g$ and $(L_f)^* = L_{f^*}$.

Now we introduce

$$\mathcal{S} = \left\{\alpha \in \widehat{K}\colon\ |\hat{f}(\alpha)| \ \leq\ \|L_f\| \text{ for every } f \in L^1(K,m)\right\}.$$

$\mathcal{S}$ is a nonvoid closed subset of $\widehat{K}$, and for each $f \in L^1(K,m)$ one has $\|L_f\| = \sup_{\alpha\in\mathcal{S}} |\hat{f}(\alpha)|$. To obtain this one can apply the fact that $\mathcal{S}$ is homeomorphic to the structure space $\Delta(A)$, where A is the commutative C^*–algebra $A = \mathrm{cl}\{L_f\colon f \in L^1(K,m)\}$, the closure taken in the Banach space $B(L^2(K,m))$. One should note that $\widehat{K}$ in general does not bear a dual hypergroup structure, which makes harmonic analysis on K more delicate. The proof of the next result can be found in [3] or [1].

THEOREM 2.1 (Plancherel–Levitan). *Let K be a commutative hypergroup. Then there exists a unique regular positive Borel measure π on $\widehat{K}$ with*

$$\int_K |f(x)|^2\, dm(x) = \int_{\widehat{K}} |\widehat{f}(\alpha)|^2\, d\pi(\alpha)$$

for all $f \in L^1(K,m) \cap L^2(K,m)$. The support of π is equal to $\mathcal{S}$.

The set $\{\widehat{f}\colon f \in C_c(K)\}$ is dense in $L^2(\widehat{K},\pi)$. π is called Plancherel measure.

In the next section we will give several equivalent conditions for $\alpha \in \mathcal{S} = \operatorname{supp}\pi$.

3. Characterization of supp π

We start by recalling some further notions of harmonic analysis. For $f \in L^1(\widehat{K},\pi)$ define the inverse Fourier transform

$$\check{f}(x) \;=\; \int_{\widehat{K}} f(\alpha)\,\alpha(x)\,d\pi(\alpha)$$

for $x \in K$. A function $\varphi \in C^b(K)$ is called positive definite if for all choices of $n \in \mathbb{N}, c_1, c_2, \ldots, c_n \in \mathbb{C}$ and $x_1, x_2, \ldots, x_n \in K$,

$$\sum_{i,j=1}^{n} c_i\,\overline{c_j}\,\omega(x_i,\widetilde{x_j})(\varphi) \geq 0.$$

Important examples of positive definite functions are $\alpha \in \widehat{K}$ or $f * f^*$, where $f \in L^2(K,m)$. We mention that one can prove a Bochner theorem for commutative hypergroups; see [3] or [6]. We apply the following inversion result; see [6].

PROPOSITION 3.1. *If $\varphi \in L^1(K,m) \cap C^b(K)$ is positive definite then $\widehat{\varphi} \in L^1(\widehat{K},\pi)$ and $(\hat{\varphi})^{\vee} = \varphi$.*

We now introduce a concept very useful for investigating $\operatorname{supp}\pi$. In the case of $\alpha = 1$ it has already been studied in the context of hypergroups, see [10], and in the group case closely related to the notion of amenability.

Definition 3.1. Let $\alpha \in \widehat{K}$. We say that the P_2 condition is satisfied in α if for each $\varepsilon > 0$ and every compact subset $C \subset K$ there exists some $g \in C_c(K)$ such that $\|g\|_2 = 1$ and

$$\|L_{\tilde{y}} g - \overline{\alpha(y)}\, g\|_2 < \varepsilon \qquad \text{for all} \qquad y \in C.$$

Now we characterize those $\alpha \in \widehat{K}$ which belong to $\mathcal{S} = \operatorname{supp}\pi$. The equivalence of the conditions (i) and (ii) in the following theorem is already shown by M. Voit; see [1], Corollary 4.1.12.

THEOREM 3.1. *Let $\alpha \in \widehat{K}$. Then the following conditions are equivalent*:

(i) $\alpha \in \mathcal{S} = \operatorname{supp}\pi$.

(ii) *There exists a net $(f_i)_{i\in I} \subseteq C_c(K)$, $\|f_i\|_2 = 1$ such that $f_i * f_i^*$ converges to α uniformly on compact subsets of K.*

(iii) *The P_2 condition is satisfied in α.*

Proof. First we show that (i) implies (ii). Let $\varepsilon > 0$, $C \subseteq K$ be compact. Choose a neighborhood $U \subseteq \widehat{K}$ of α such that $0 < \pi(U) < \infty$ and

$$U \subseteq \{\beta \in \widehat{K}\colon |\alpha(x) - \beta(x)| < \varepsilon/2 \text{ for all } x \in C\}.$$

Define $h = \chi_U/\pi(U) \in L^1(\widehat{K},\pi)$. Then for all $x \in C$ we have

$$|\check{h}(x) - \alpha(x)| = \frac{1}{\pi(U)} \left| \int_U \beta(x)\, d\pi(\beta) - \int_U \alpha(x)\, d\pi(\beta) \right| < \varepsilon/2.$$

For $h^{1/2} = \chi_U/\pi(U)^{1/2}$ there exists some $f \in C_c(K)$ such that $\|\widehat{f} - h^{1/2}\|_2 < \varepsilon/4$; cf. Theorem 2.1. Since $\|h^{1/2}\|_2 = 1$ we can assume that $\|\widehat{f}\|_2 = \|f\|_2 = 1$. Furthermore we get

$$\begin{aligned}\|(f * f^*)^\wedge - h\|_1 &= \|\, |\widehat{f}|^2 - h\|_1 \\ &\le \int_K |\widehat{f}(\beta) - h^{1/2}(\beta)| \cdot \big||\widehat{f}|(\beta) + h^{1/2}(\beta)\big|\, d\pi(\beta) \\ &\le \|\widehat{f} - h^{1/2}\|_2 \left(\|\widehat{f}\|_2 + \|h^{1/2}\|_2\right) = 2\|\widehat{f} - h^{1/2}\|_2 < \varepsilon/2.\end{aligned}$$

Applying Proposition 3.1, for $x \in K$ we obtain

$$|f * f^*(x) - \check{h}(x)| = \left|\left((f * f^*)^\wedge\right)^\vee(x) - \check{h}(x)\right| \le \|(f * f^*)^\wedge - h\|_1 < \varepsilon/2,$$

and hence $|f * f^*(x) - \alpha(x)| < \varepsilon$ for every $x \in C$.

In order to prove that (ii) implies (iii) we again consider a compact set $C \subseteq K$. Then the convolution of C with itself, $C * C := \bigcup_{x,y \in C} \operatorname{supp} \omega(x,y)$, is also a compact subset of K; see [3] or [1]. Let $\varepsilon > 0$. Then by (ii) there is a function $f \in C_c(K)$ with $\|f\|_2 = 1$ and

$$|f * f^*(x) - \alpha(x)| < \varepsilon \qquad \text{for all } x \in C * C.$$

We can assume that $e \in C$ and $C = \widetilde{C}$. Since for all $x,\ y \in C$,

$$|L_y(f * f^*)(x) - \alpha(y)\,\alpha(x)| \le \int_K |f * f^*(z) - \alpha(z)|\, d\omega(y,x)(z) < \varepsilon$$

and $|f * f^*(x)\,\alpha(y) - \alpha(x)\,\alpha(y)| < \varepsilon$ we obtain

$$|L_y(f * f^*)(x) - f * f^*(x)\,\alpha(y)| < 2\varepsilon,$$

and hence

$$\begin{aligned} &\left|\int_K \overline{L_{\tilde{x}} f(z)}\ \left[L_y f(z) - \alpha(y)\ f(z)\right]\, dm(z)\right| \\ &= \left|\int_K \overline{f(z)}\ \left[L_x(L_y f)(z) - \alpha(y)\ L_x f(z)\right]\, dm(z)\right| \\ &= \left|L_y(f * f^*)(x) - \alpha(y)\ (f * f^*)(x)\right| < 2\varepsilon. \end{aligned}$$

In a similar way for $y \in C$ and each $x \in K$ we get

$$\begin{aligned} &\left|\int_K \overline{\alpha(\tilde{x}) f(z)}\ \left[L_y f(z) - \alpha(y) f(z)\right]\, dm(z)\right| \\ &= |\alpha(x)| \cdot |f * f^*(y) - \alpha(y)| < \varepsilon. \end{aligned}$$

For $y = \tilde{x} \in C$ we therefore have

$$\begin{aligned} \left\|L_y f - \alpha(y) f\right\|_2^2 &= \int_K \overline{\left[L_y f(z) - \alpha(y)\ f(z)\right]}\ \left[L_y f(z) - \alpha(y)\ f(z)\right]\, dm(z) \\ &\le 3\varepsilon; \end{aligned}$$

thus the implication is shown.

It remains to show that (iii) implies (i). Assume that the P_2 condition is satisfied in α. We will prove that

$$|\widehat{f}(\alpha)| \le \sup \left\{\|f * g\|_2 \colon g \in L^2(K,m),\ \|g\|_2 = 1\right\} \text{ for every } f \in C_c(K),\ f \ne 0.$$

Since $C_c(K)$ is dense in $L^1(K,m)$, this condition implies that $\alpha \in \mathcal{S}$.

Let $f \in C_c(K)$, $f \neq 0$. By the P_2 condition there exists a function $g \in L^2(K, m)$, $\|g\|_2 = 1$ such that

$$\left\| L_{\tilde{y}} g - \overline{\alpha(y)}\, g \right\|_2 < \varepsilon / \|f\|_1$$

for all $y \in \operatorname{supp} f$. Since

$$f * g(x) - \widehat{f}(\alpha)\, g(x) = \int_K f(y) \left(L_{\tilde{y}} g(x) - \overline{\alpha(y)\, g(x)} \right) dm(y),$$

it follows that

$$\left\| f * g - \widehat{f}(\alpha)\, g \right\|_2 \leq \int_K |f(y)| \cdot \|L_{\tilde{y}} g - \overline{\alpha}(y)\, g\|_2 \, dm(y) < \varepsilon.$$

Thus we have the estimate

$$|\widehat{f}(\alpha)| = |\widehat{f}(\alpha)| \cdot \|g\|_2 \leq \varepsilon + \|f * g\|_2,$$

which obviously implies

$$|\widehat{f}(\alpha)| \leq \sup \left\{ \|f * g\|_2 \colon g \in L^2(K, m),\ \|g\|_2 = 1 \right\}. \qquad \square$$

Remark. In the case of $\alpha = 1 \in \widehat{K}$ we can assume that the functions $g \in C_c(K)$ in Definition 3.1 are nonnegative. In fact one can proceed as in the proof of Lemma 4.4 of [10] to construct nonnegative f_i, $i \in I$, in condition (ii) of our Theorem 3.1, which also yields nonnegative functions for the P_2 condition in $\alpha = 1$.

4. Application to orthogonal polynomials

Now we apply the general result of Section 3 to polynomial hypergroups. In order to do this it seems to be useful to recall some basic facts about polynomial hypergroups.

Consider a polynomial sequence $(P_n)_{n \in \mathbb{N}_0}$ defined by a recurrence relation of the form

$$P_1(x)\, P_n(x) = a_n\, P_{n+1}(x) + b_n\, P_n(x) + c_n\, P_{n-1}(x) \tag{4.1}$$

for $n \in \mathbb{N}$ and starting with

$$P_0(x) = 1, \qquad P_1(x) = \frac{1}{a_0}(x - b_0) \tag{4.2}$$

with $a_n \in \mathbb{R}\backslash\{0\}$ for all $n \in \mathbb{N}_0$, $c_n \in \mathbb{R}\backslash\{0\}$ for all $n \in \mathbb{N}$ and $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0$. A well-known result, usually referred to as Favard's theorem, states that $(P_n)_{n \in \mathbb{N}_0}$ is an orthogonal polynomial sequence with respect to a certain probability measure π on the real line; see [2].

We impose two assumptions on $(P_n)_{n\in\mathbb{N}_0}$. A minor one is

$$P_n(1) = 1 \qquad \text{for all } n \in \mathbb{N}_0 \tag{4.3}$$

and a more restrictive one is

$$g(m, n; k) \geq 0, \tag{4.4}$$

where $g(m, n; k)$ are the linearization coefficients of the products

$$P_m(x)\ P_n(x) = \sum_{k=|n-m|}^{n+m} g(m, n; k)\ P_k(x). \tag{4.5}$$

Note that $P_n(1) = 1$ implies $a_0 + b_0 = 1$ and $\sum_{k=|n-m|}^{n+m} g(m, n; k) = 1$. Furthermore we have

$$\int_{\mathbb{R}} P_n^2(x)\ d\pi(x) = g(n, n; 0).$$

We write $h_n = g(n, n; 0)^{-1}$. Hence $p_n(x) = \sqrt{h(n)}P_n(x)$ is the orthonormal version of $P_n(x)$. There is an abundance of orthogonal polynomial sequences $(P_n)_{n\in\mathbb{N}_0}$ satisfying (4.3) and the crucial nonnegativity condition (4.4); see [5], [6] and [8].

By means of coefficients $g(m, n; k)$ (that are in one-to-one correspondence to $(P_n)_{n\in\mathbb{N}_0}$) we define a convolution ω_P on $\mathbb{N}_0$:

$$\omega_P(m, n) = \sum_{k=|n-m|}^{n+m} g(m, n; k)\ \varepsilon_k,$$

where ε_k is the point measure of $k \in \mathbb{N}_0$. With the identity mapping as involution, i.e., $\tilde{n} = n$, and the discrete topology the natural numbers $\mathbb{N}_0$ are a commutative hypergroup, called polynomial hypergroup; see [5].

The translation now reads as follows:

$$L_n\beta(m) = \sum_{k=|n-m|}^{n+m} g(m, n; k)\ \beta(k).$$

The dual space $\widehat{\mathbb{N}_0}$ can be identified with

$$D_s = \left\{x \in \mathbb{R}\colon\ (P_n(x))_{n\in\mathbb{N}_0} \text{ is a bounded sequence}\right\} \tag{4.6}$$

by the mapping $x \to \alpha_x$, $D_s \to \widehat{\mathbb{N}_0}$, where $\alpha_x(n) = P_n(x)$. Direct consequences (see [5]) are:

(i) $D_s = \{x \in \mathbb{R}\colon\ |P_n(x)| \leq 1 \quad \text{for all} \quad n \in \mathbb{N}_0\}$.

(ii) D_s is compact.
(iii) $D_s \subseteq [1 - 2a_0, 1]$.

A Haar measure m on $\mathbb{N}_0$ is the counting measure on $\mathbb{N}_0$ with weights $h(n)$ on the points $n \in \mathbb{N}_0$. The theorem of Plancherel–Levitan has in that case the form:

THEOREM 4.1. *There exists an unique probability measure π on D_s such that*

$$\sum_{n\in\mathbb{N}_0} |d(n)|^2\, h(n) = \int_{D_s} |\hat{d}(x)|^2\, d\pi(x)$$

for every $d = (d(n))_{n\in\mathbb{N}_0} \in l^1(\mathbb{N}_0, m)$, where $\hat{d}(x) = \sum_{n\in\mathbb{N}_0} P_n(x)\, d(n)\, h(n)$.

Applying the polarization identity it is easy to see that π is in fact the orthogonalization measure for $(P_n)_{n\in\mathbb{N}_0}$, guaranteed by Favard's theorem. In particular, see [5], as a first result we have:

PROPOSITION 4.1. *Let $(P_n)_{n\in\mathbb{N}_0}$ be an orthogonal polynomial sequence satisfying* (4.3) *and* (4.4). *Then*

$$\begin{aligned}\operatorname{supp}\pi = \mathcal{S} \subseteq D_s &= \{x \in \mathbb{R}\colon |P_n(x)| \le 1 \text{ for all } n \in \mathbb{N}_0\}\\ &\subseteq [1 - 2a_0,\ 1].\end{aligned}$$

We will now derive some sufficient conditions for $x \in \operatorname{supp}\pi$. For this the next result plays a fundamental role throughout the remainder of this section.

PROPOSITION 4.2. *Let $(P_n)_{n\in\mathbb{N}_0}$ define a polynomial hypergroup on $\mathbb{N}_0$ and $x \in D_s$. If for every $\varepsilon > 0$ there exists some $\beta = (\beta(n))_{n\in\mathbb{N}_0} \in C_c(\mathbb{N}_0)$ with $\|\beta\|_2 = 1$ such that*

$$\|L_1\beta - P_1(x)\,\beta\|_2 < \varepsilon, \tag{4.7}$$

then the P_2 condition is satisfied in $x \in D_s$. (The $\|\ .\ \|_2$–norm is in $l^2(\mathbb{N}_0, m)$.)

Proof. We show that (4.7) implies the following property: Given $\varepsilon > 0$, $n \in \mathbb{N}$ there exists $\beta \in C_c(\mathbb{N}_0)$ with $\|\beta\|_2 = 1$ such that

$$\|L_k\beta - P_k(x)\,\beta\|_2 < \varepsilon \qquad \text{for each } k = 0, 1, \dots, n. \tag{4.8}$$

We use induction and assume that (4.8) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}\|L_1(L_n\beta) - P_1(x)\,P_n(x)\,\beta\|_2 &\le \|L_1(L_n\beta) - P_n(x)\,L_1\beta\|_2\\ &\qquad + |P_n(x)|\ \|L_1\beta - P_1(x)\beta\|_2\\ &\le 2\varepsilon.\end{aligned}$$

Now we apply the recurrence relation

$$P_1(x)P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x)$$

and obtain the estimate

$$\begin{aligned}\|L_{n+1}\beta - P_{n+1}(x)\beta\|_2 &= \Bigg\| \frac{1}{a_n} L_1(L_n\beta) - \frac{b_n}{a_n} L_n\beta - \frac{c_n}{a_n} L_{n-1}\beta \\ &\quad - \left[\frac{1}{a_n} P_1(x)\, P_n(x)\beta - \frac{b_n}{a_n} P_n(x)\beta - \frac{c_n}{a_n} P_{n-1}(x)\beta\right]\Bigg\|_2 \\ &\leq \frac{1}{a_n}(2\varepsilon + b_n\varepsilon + c_n\varepsilon) = \frac{2 + b_n + c_n}{a_n}\,\varepsilon.\end{aligned}$$

After an appropriate modification of the ε's it is obvious that (4.8) is valid for $n+1$. □

In view of our general result we get for polynomial hypergroups the following theorem.

THEOREM 4.2. *Let* $(P_n)_{n\in\mathbb{N}_0}$ *define a polynomial hypergroup on* $\mathbb{N}_0$, *and let* $x \in D_s$. *Then* $x \in \operatorname{supp}\pi$, *if and only if for every* $\varepsilon > 0$ *there exists* $\beta \in C_c(\mathbb{N}_0)$ *with* $\|\beta_2\| = 1$ *and*

$$\|L_1\beta - P_1(x)\beta\|_2 < \varepsilon.$$

Next we give a sufficient condition for $x \in \operatorname{supp}\pi$. Let $\beta_n \in l^2(\mathbb{N}_0, h)$ be given by

$$\beta_n(k) = \frac{P_k(x)\ \chi_{\{0,\dots,n\}}(k)}{\left(\sum_{j=0}^n P_j^2(x)\ h(j)\right)^{1/2}}. \tag{4.9}$$

It is straightforward to see that $\|\beta_n\|_2 = 1$ and

$$\begin{aligned}&L_1\beta_n(k) - P_1(x)\beta_n(k) \\ &= \underbrace{g(1,k,k+1)}_{a_k}\beta_n(k+1) + \underbrace{g(1,k,k)}_{b_k}\beta_n(k) \\ &\quad + \underbrace{g(1,k,k-1)}_{c_k}\beta_n(k-1) - P_1(x)\beta_n(k) \\ &= 0\end{aligned}$$

for all $k = 0, 1, \dots, n-1$.

For the sake of brevity let $\lambda_n(x) = \left(\sum_{k=0}^{n} P_k^2(x)\, h(k)\right)^{-1}$. Then we have

$$\begin{aligned}
&\|L_1\, \beta_n - P_1(x)\, \beta_n\|_2^2 \\
&\quad = \lambda_n(x)\big(|b_n\, P_n(x) + c_n\, P_{n-1}(x) - P_1(x)\, P_n(x)|^2\, h(n) \\
&\qquad\qquad + |c_{n+1}\, P_n(x)|^2\, h(n+1)\big) \\
&\quad = \lambda_n(x)\big(|a_n\, P_{n+1}(x)|^2\, h(n) + |c_{n+1}\, P_n(x)|^2\, h(n+1)\big) \\
&\quad = \lambda_n(x)\, a_n\, c_{n+1}\, \left(P_{n+1}^2(x)\, h(n+1) + P_n^2(x)\, h(n)\right).
\end{aligned}$$

For the latter equality we used the fact that $c_{n+1}\, h(n+1) = a_n\, h(n)$. Therefore from Theorem 4.2 we obtain:

PROPOSITION 4.3. *Let $(P_n)_{n\in\mathbb{N}_0}$ define a polynomial hypergroup on $\mathbb{N}_0$, and let $x \in D_s$. If*

$$\liminf_{n\to\infty} \frac{P_n^2(x)h(n) + P_{n+1}^2(x)h(n+1)}{\sum_{k=0}^{n} P_k^2(x)h(k)} = 0$$

then $x \in \operatorname{supp} \pi$.

To give an example where this criterion works, consider orthogonal polynomials which are defined by the following reccurrence coefficients in (4.1) and (4.2):

$$a_0 = 1, \quad b_0 = 0$$

and

$$a_n = \begin{cases} \frac{\alpha-1}{\alpha} & \text{for } n \text{ odd,} \\ \frac{\beta-1}{\beta} & \text{for } n \text{ even.} \end{cases}$$

We call the corresponding orthogonal polynomials Karlin-McGregor polynomials, since they were first considered in [4]. Applying the recursion formula of [5] one can determine the linearization coefficients $g(n, m; k)$ explicitly. Here we only state that the nonnegativity of all $g(n, m; k)$ is fulfilled if $\alpha \geq 2$ and $\beta \geq 2$. The weights $h(n)$ are $h(0) = 1$ and for $n \geq 1$,

$$h(n) = \begin{cases} \alpha(\alpha-1)^{(n-1)/2}(\beta-1)^{(n-1)/2} & \text{for } n \text{ odd,} \\ \beta(\alpha-1)^{n/2}(\beta-1)^{n/2-1} & \text{for } n \text{ even.} \end{cases}$$

Furthermore applying methods of [8] (in particular property (T)) one can easily deduce that $D_s = [-1, 1]$. Now we consider some points $x \in [-1, 1]$ for which Proposition 4.3 works. Let $x = 0$. It is easily seen that $P_n(0) = \left(\frac{-1}{\alpha-1}\right)^{n/2}$ for n even and obviously $P_n(0) = 0$ for n odd. Hence

$$\sum_{n=0}^{\infty} P_n^2(0)h(n) = 1 + \frac{\beta}{\beta-1} \sum_{k=1}^{\infty} \left(\frac{\beta-1}{\alpha-1}\right)^k.$$

For $\alpha > \beta \geq 2$ we have

$$\sum_{n=0}^{\infty} P_n^2(0)h(n) = \frac{\alpha}{\alpha - \beta}$$

and hence $0 \in \operatorname{supp} \pi$. Moreover, by Theorem 4.1 we get $\pi(\{0\}) = \frac{\alpha-\beta}{\alpha}$ provided $\alpha > \beta$. In order to determine $P_n(x)$ in general we observe that

$$x^2 P_{2n}(x) = r P_{2n+2}(x) + s P_{2n}(x) + t P_{2n-2}(x)$$

and $P_0(x) = 1$, $P_2(x) = \frac{\alpha}{\alpha-\beta} x^2 - \frac{1}{\alpha-1}$, where $r = \frac{(\alpha-1)(\beta-1)}{\alpha\beta}$, $s = \frac{(\alpha-1)+(\beta-1)}{\alpha\beta}$, $t = \frac{1}{\alpha\beta}$.

Now we can apply the method of difference equations with constant coefficients to first calculate $P_{2n}(x)$ and then $P_{2n+1}(x)$ for fixed $x \in]-1, 1[$. It is well known that

$$P_{2n}(x) = c\lambda_1^n + d\lambda_2^n, \quad \text{where } \lambda_{1,2} = \frac{(x^2 - s) \pm \sqrt{(x^2 - s)^2 - 4rt}}{2r},$$

provided $(x^2 - s)^2 \neq 4rt$. If $(x^2 - s)^2 = 4rt$ we have

$$P_{2n}(x) = \lambda^n(1 + nd), \quad \text{where } \lambda = \frac{x^2 - s}{2r}.$$

To be brief we only discuss the case where $x^2 = s \pm 2\sqrt{rt} = \frac{1}{\alpha\beta}\left(\sqrt{\alpha - 1} \pm \sqrt{\beta - 1}\right)^2$. In that case we get $\lambda = \frac{1}{\sqrt{(\alpha-1)(\beta-1)}}$. Without calculating the constant d explicitly we see that $P_{2n}^2(x)h(2n) \sim n^2$. Inserting $P_{2n}(x)$ into the recurrence system we also obtain $P_{2n+1}^2(x)h(2n+1) \sim n^2$. Therefore Proposition 4.3 implies that $\pm\frac{1}{\sqrt{\alpha\beta}}\left(\sqrt{\alpha - 1} + \sqrt{\beta - 1}\right)$ and $\pm\frac{1}{\sqrt{\alpha\beta}}\left(\sqrt{\alpha - 1} - \sqrt{\beta - 1}\right)$ are elements of $\operatorname{supp} \pi$. As already sketched above we have $x \in \operatorname{supp} \pi$ for those x such that $(x^2 - s)^2 < 4rt$. Hence we see that $[-\frac{1}{\sqrt{\alpha\beta}}(\sqrt{\alpha - 1} + \sqrt{\beta - 1}), -|\frac{1}{\sqrt{\alpha\beta}}(\sqrt{\alpha - 1} - \sqrt{\beta - 1})|]$ and $[\frac{1}{\sqrt{\alpha\beta}}(\sqrt{\alpha - 1} - \sqrt{\beta - 1}), \frac{1}{\sqrt{\alpha\beta}}(\sqrt{\alpha - 1} + \sqrt{\beta - 1})]$ are subsets of $\operatorname{supp} \pi$. If, in addition, $\alpha > \beta$ then $0 \in \operatorname{supp} \pi$.

Choosing $\beta_n(k)$ once more as in (4.9) we can derive a further result.

PROPOSITION 4.4. *Let $(P_n)_{n \in \mathbb{N}_0}$ define a polynomial hypergroup on $\mathbb{N}_0$ and let $x \in D_s$. Assume that* $\liminf_{n\to\infty} a_n = 0$ *or* $\liminf_{n\to\infty} c_n = 0$. *If*

$$\left\{ \frac{P_{n+1}^2(x)h(n+1)}{\sum_{k=0}^{n} P_k^2(x)h(k)} : n \in \mathbb{N}_0 \right\}$$

is bounded, then $x \in \operatorname{supp} \pi$.

We close this paper with an example which shows that the condition

$$\lim_{n\to\infty} \frac{P_n^2(x)h(n)}{\sum_{k=0}^n P_k^2(x)h(k)} = 0$$

is not necessary for having $x \in \operatorname{supp}\pi$. For that we consider the little q-Legendre polynomials $(P_n)_{n\in\mathbb{N}_0}$; see [1], p. 187. To have $P_n(1) = 1$ we have to make a slight modification by putting $1 - x$ for x. For fixed $q \in]0, 1[$ the little q-Legendre polynomial $P_n(x) = P_n(q; x)$ are given by

$$\begin{aligned} &P_1(x)P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n \geq 2 \\ &P_0(x) = 1, \quad P_1(x) = (q+1)x - q \end{aligned}$$

where

$$\begin{aligned} a_n &= q^n \frac{(1+q)(1-q^{n+1})}{(1-q^{2n+1})(1+q^{n+1})} \\ b_n &= \frac{(1-q^n)(1-q^{n+1})}{(1+q^n)(1+q^{n+1})} \\ c_n &= q^n \frac{(1+q)(1-q^n)}{(1-q^{2n+1})(1+q^n)}. \end{aligned}$$

It is known (see [1]) that the $(P_n)_{n\in\mathbb{N}_0}$ define a polynomial hypergroup on $\mathbb{N}_0$ and $\operatorname{supp}\pi = \{1\} \cup \{1 - q^{2k}\colon k \in \mathbb{N}_0\}$. Furthermore $\frac{h(n)}{h(n-1)} \to \frac{1}{q}$. Hence we see that

$$\frac{h(n)}{\sum_{k=0}^n h(k)} \longrightarrow 1 - q,$$

but $1 \in \operatorname{supp}\pi$.

The contributions of Section 4 are strongly connected with results of R. Szwarc; cf. [11], [12], [13].

References

[1] W. R. Bloom and H Heyer, *Harmonic analysis of probability measures on hypergroups*, deGruyter, Berlin, 1995.

[2] T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach, New York, 1978.

[3] R. I. Jewett, *Spaces with an abstract convolution of measures*, Adv. in Math. **18** (1975), 1–101.

[4] S. Karlin and J. McGregor, *Random walks*, Illinois J. Math. **3** (1959) 66–81.

[5] R. Lasser, *Orthogonal polynomials and hypergroups*, Rend. Mat. **3** (1983),185–209.

[6] ———, *Bochner theorems for hypergroups and their application to orthogonal polynomial expansions*, J. Approx. Theory **37** (1983), 311-325.

[7] R. Lasser and J. Obermaier, "On Fejér means with respect to orthogonal polynomials: A hypergroup theoretic approach" in *Progress in Approximation Theory*, Academic Press, Boston, 1991, pp. 551–565.

[8] R. Lasser, *Orthogonal polynomials and hypergroups II - the symmetric case*, Trans. Amer. Math. Soc. **341** (1994), 749–770.
[9] P. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. vol. 213, 1979.
[10] M. Skantharajah, *Amenable hypergroups*, Illinois J. Math. **36** (1992), 15–46.
[11] R. Szwarc, *A lower bound for orthogonal polynomials with an application to polynomials hypergroups*, J. Approx. Theory **81** (1995), 145–150.
[12] ———, *Uniform subexponential growth of orthogonal polynomials*, J. Approx. Theory **81** (1995), 296–302.
[13] ———, *A counterexample to subexponential growth of orthogonal polynomials*, Constr. Approx. **11** (1995), 381–389.
[14] M. Voit, *Factorization of probability measures on symmetric hypergroups*, J. Austral. Math. Soc. Ser. A **50** (1991), 417–467.

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