ILLINOIS JOURNAL OF MATHEMATICS
Volume 44, Number 1, Spring 2000

COMPACT LIE GROUPS ACTING ON PSEUDOMANIFOLDS

RAIMUND POPPER

ABSTRACT. In this paper we introduce the concept of a G-pseudomanifold, which is an equivariant version of the stratified spaces defined by Goresky and Mac Pherson for compact Lie group actions.

Let G be a compact Lie group acting on a topological manifold M. Then the orbit space M/G is a topological pseudomanifold, if the action is locally linear. Recall that a topological pseudomanifold is a space that admits a filtration

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset,$$

such that $X_n - X_{n-1}$ is a dense n-manifold, $X_i - X_{i-1}$ for $i \leq n-1$ is an i-manifold (or empty), and along which the normal structure of X is locally trivial. Moreover if G acts smoothly on M, the above is valid in the context of Thom-Mather stratified spaces [2], [10]. For topological actions on manifolds no such structure exists in general, however progress has recently been made using homotopically stratified sets, developed by Quinn [8].

The objective of this work is to give an answer to the following problem: find a class of compact Lie group actions on topological pseudomanifolds X such that the corresponding orbit space X/G is also a pseudomanifold. A solution is obtained by considering stratified G-spaces having a conical slice at a point in each orbit, which we call G-pseudomanifolds. These spaces extend the notion of locally linear actions to topological pseudomanifolds, and were first introduced by the author in [7], using links without fixed points. In the present work we remove this restriction, obtaining a significant generalization. An example is given by compact Lie group actions on orbit spaces.

The content of this paper is the following.

In Section 1 we define conical slices and G-pseudomanifolds. In particular, given a locally linear G-manifold M, and a closed normal subgroup K of G, we show that M/K is a G/K-pseudomanifold.

Section 2 deals with the orbit type refinement of a G-pseudomanifold. We also prove the existence of principal orbits, observing that their union need not necessarily coincide with the highest dimensional stratum in the orbit type refinement.

In Section 3 we study the corresponding orbit space, showing that it is a topological pseudomanifold.

Smooth G-pseudomanifolds are defined in Section 4, where we prove a generalization of Mostow's equivariant embedding theorem [6].

Received July 30, 1997.
1991 Mathematics Subject Classification. Primary 55N33; Secondary 57N80.
Supported by CDCH of the Central University of Venezuela.

I wish to express my gratitude to Martin Saralegi for his invaluable comments and encouragement.

1. Conical slices and G-pseudomanifolds

In this section we define G-pseudomanifolds and give some examples.

Let G be a compact Lie group.

By a G-space we mean a Hausdorff topological space X, with a continuous action $\Theta\colon G \times X \to X$, such that the orbit space X/G is connected. Let $\Theta(g, x) = g \cdot x$. We denote by $G \cdot x$ the orbit containing the point x, and by $G_x = \{g \in G\colon g \cdot x = x\}$ the stabilizer or isotropy subgroup at x. Also we denote by $X_{(H)}$ the union of all orbits of type(G/H) in X, see [3, p. 42], and by $X^G = X_{(G)}$ the fixed point set of the action.

Recall the definition of a slice; see [3, II.4.1].

Given a G-space X, we say that a subspace S_x is a *slice* at a point x, if $x \in S_x$, S_x is invariant by $H = G_x$, and the canonical map $\Phi\colon G \times_H S_x \to X$, given by $[g, x] \mapsto g \cdot x$, is a G-equivalence onto an open neighborhood Γ of $G \cdot x$, called a *tubular neighborhood.*

Now let Y be a non empty topological space; then its *open cone*, denoted cY, is defined as follows: $cY = Y \times [0, 1)/(y, 0) \sim (y', 0)$. Let $[y, r]$ denote the corresponding equivalence class and $*$ the vertex $[y, 0]$. For $Y = \emptyset$ we let $cY = \{*\}$.

Given a G-space X there is a canonical G-action on cX, which is the following $g \cdot [x, r] = [g \cdot x, r]$ where $g \in G, x \in X, r \in [0, 1)$. Notice that the vertex $*$ is a fixed point. Furthermore any invariant open neighborhood of the vertex is a slice in cX.

A slice S_x at x is said to be *linear* if it is G_x-equivalent to a Euclidean space with an orthogonal action. We say that X is *locally linear* if it admits a linear slice at a point in each orbit. Since each tubular neighborhood of an orbit P in X is a vector bundle over P, see [3], it follows that X is a topological manifold. We also call such a space a *locally linear G-manifold.* For example a smooth (C^∞) G-manifold is locally linear [3, VI.2.4].

Definition 1.1. A G-space X is said to be *stratified*, if it admits a filtration

$$X = X^m \supset X^{m-1} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset$$

by closed invariant subsets, such that the subspace $X^k - X^{k-1}$ is a topological k-manifold (if non empty), for $k = 0, \dots, m$. (Assume that $X^m \neq X^{m-1}$).

We now define the concept of a conical slice.

Definition 1.2. Let X be a stratified G-space. Given an orbit P in $X^k - X^{k-1}$, for some $k = 0, \dots, m$, we say that a slice S_x at a point x in P, is a *conical slice* of P

at x, if the following holds: There is a compact H-space L, where $H = G_x$, called a *link* of P, together with an H-equivalence ϕ: $S_x \to \mathfrak{R}^{i_0} \times cL$, for an integer $i_0 \geq 0$, and the trivial H-action on Euclidean space $\mathfrak{R}^{i_0}$, such that

$$(X^k - X^{k-1})_{(H)} \cap S_x = S_x^0 = \phi^{-1}(\mathfrak{R}^{i_0} \times \{*\}).$$

We also allow L to be empty. (Notice in particular that $x \in S_x^0$).

Let Γ be the tubular neighborhood corresponding to a conical slice $S_x \simeq \mathfrak{R}^{i_0} \times cL$ of P. Then since $(X^k - X^{k-1})_{(H)} \cap \Gamma \simeq^{\Phi^{-1}} G \times_H S_x^0 \simeq G/H \times \mathfrak{R}^{i_0}$, the integer i_0 is independent of the choice of a conical slice for P. We shall write $sd(P) = i_0 + \dim(G/H)$.

Moreover if each orbit in X admits a conical slice, we have shown that the connected components of the subspaces $(X^k - X^{k-1})_{(H)}$ are topological manifolds, since on such a subspace the function $y \mapsto sd(G \cdot y)$ is continuous.

Notice that for each conical slice $S_x^H \simeq \mathfrak{R}^{i_0} \times c(L^H)$. Therefore the fixed points of S_x can be divided into two classes, one Euclidean and the other conical. The separation of these classes is accomplished by the condition $(X^k - X^{k-1}) \cap S_x^H = S_x^0$, which is equivalent to the one given in 1.2. In other words, the fixed points of conical slices behave nicely relative to the stratification of X.

Now we define the concept of a G-pseudomanifold.

Definition 1.3. The definition is by induction.

A (-1)-*dimensional G-pseudomanifold* is the empty set.

An *n-dimensional G-pseudomanifold* ($n \geq 0$) is a stratified G-space X, which satisfies the following conditions:

(C1) Each orbit P in X has a conical slice $S_x \simeq^{\phi} \mathfrak{R}^{i_0} \times cL$ at x, such that L is an $(n - i - 1)$-dimensional H-pseudomanifold, where $H = G_x$ and $i = sd(P)$.

(C2) For each point $y \in S_x - S_x^0$ with $\phi(y) = (t, [l, r])$, we have the relation

$$sd(G \cdot y) = sd(G \cdot x) + sd(H \cdot l) + 1.$$

We shall prove in §2 that n is the topological dimension of X. Condition (C2) is necessary in order to obtain local normal triviality.

Here are some examples.

Examples 1.4.

1. *Locally linear actions.* Let M be an n-dimensional locally linear G-manifold. For $n = -1$ we define M to be the empty set. Claim that M with the trivial stratification, is an n-dimensional G-pseudomanifold.

The proof is by induction on the dimension of M. Assume that $n \geq 0$. Given an orbit P in M, let S_x be a linear slice at x, with $x \in P$ and $G_x = H$. Then S_x is

H-equivalent to a Euclidean space E with an orthogonal H-action. Thus there is an H-equivalence ϕ, given by

$$S_x \simeq E^H \oplus (E^H)^\perp \simeq \Re^{i_0} \times c(S^q),$$

where $\perp$ denotes the orthogonal complement with respect to an H-invariant Riemannian metric on E, $i_0 = \dim(E^H)$, $q+1 = \dim(E^H)^\perp$, and S^q is the standard q-sphere in Euclidean space. (For $q = -1$ we put $S^q = \emptyset$.)

Now if $q \geq 0$, then H acts locally linearly on S^q, since H acts orthogonally on $\Re^{q+1}$ and S^q is a smoothly embedded (C^∞) submanifold. Then we have $M_{(H)} \cap S_x = S_x^H = S_x^0$, since $(S^q)^H = \emptyset$. Therefore S_x is a conical slice of P at x. By the inductive hypothesis, since $q < n$, S^q with the trivial stratification is an $(n-i-1)$-dimensional H-pseudomanifold, where $i = i_0 + \dim(G/H)$. The case $q = -1$ is trivial.

Now given $y \in S_x - S_x^0$ with $\phi(y) = (t, [l, r])$, let $S_l \simeq^\eta \Re^{k_0} \times c(S^p)$ be a linear (conical) K-slice of $H \cdot l$ in S^q, where $K = G_y = H_y = H_l$. Using an equivariant retraction, see [3, II.4.2], it follows that

$$S_y \simeq^\phi \Re^{i_0} \times (0, 1) \times S_l \simeq^{1\times\eta} \Re^{i_0+k_0+1} \times c(S^p)$$

is a linear (conical) K-slice of $H \cdot y$ in the H-manifold S_x. Thus by [3, II.4.6], S_y is a linear (conical) K-slice of $G \cdot y$ in M. It follows that $sd(G \cdot y) = sd(G \cdot x) + sd(H \cdot l) + 1$, and M with the trivial stratification is an n-dimensional G-pseudomanifold.

2. *Actions on orbit spaces.* Let M be an n-dimensional locally linear G-manifold ($n \geq 0$), and K a closed normal subgroup of G such that M/K is connected. Then, with the restricted K-action, M is also a locally linear K-manifold (see argument below). Now $G \times M \to M$ induces an action $G/K \times M/K \to M/K$, such that the canonical projection $\pi\colon M \to M/K$ is $(G, G/K)$-equivariant. Claim that M/K, with the K-orbit type stratification, (see [7])

$$M/K = (M/K)^m \supset (M/K)^{m-1} \supset \cdots \supset (M/K)^0 \supset (M/K)^{-1} = \emptyset,$$

is an m-dimensional G/K-pseudomanifold, where $m = n - k$ and k is the dimension of the principal K-orbits in M.

The proof is by induction on the length of the G-orbit type filtration of M, i.e., the difference between the highest and lowest dimension of the non empty strata in M. For $\operatorname{len}(M) = 0$ it is trivial (see below). Assume that $\operatorname{len}(M) > 0$.

Given an orbit P in M we shall prove as in [3], that each linear H-slice S_x at a point x in P, for $H = G_x$, is contained in a linear J-slice U_x at x, for $J = K \cap H$. In particular, since $U_{g\cdot x} = g \cdot U_x$ is a linear gJg^{-1}-slice at $g \cdot x$,this shows that M is also a locally linear K-manifold.

Consider the smooth (C^∞) action $(K \times H) \times G \to G$, where we have $(k, h) \cdot (k', h') = (kk', h'h)$ and $(k, h) \cdot g = kgh$. Then clearly

$$(K \times H)_e = \{(j, j^{-1}) \in K \times H\colon j \in K \cap H\} \simeq K \cap H = J.$$

Let W be a linear J-slice at e in G; i.e., the canonical map $(K \times H) \times_J W \to G$ is a $(K \times H)$-equivalence onto an open neighborhood of KH in G. Since

$$K \times_J (W \times H) \simeq (K \times H) \times_J W$$

is a canonical K-equivalence (with $j \cdot w = jwj^{-1}$), we obtain a K-equivalence $\tilde{\theta}$: $K \times_J (W \times H) \to G$ onto an open neighborhood of KH in G.

Now let S_x be a linear H-slice at x in M. Then the following map θ is a K-equivalence onto an open neighborhood of $K \cdot x$ in M:

$$\begin{aligned} K \times_J (W \times S_x) &\simeq K \times_J (W \times (H \times_H S_x)) \simeq K \times_J ((W \times H) \times_H S_x) \\ &\simeq (K \times_J (W \times H)) \times_H S_x \to^{[\tilde{\theta} \times 1]} G \times_H S_x \to M. \end{aligned}$$

Since $\theta[k, (w, s)] = kw \cdot s$, it follows that $U_x = \theta(J \times_J (W \times S_x)) = W \cdot S_x$ is a linear J-slice at x in M, which contains the linear H-slice S_x.

We now determine conical slices in M/K.

First we notice that the K-orbit type stratification of M/K is G/K-invariant, since $U_{g \cdot x} = g \cdot U_x$ is a linear gJg^{-1}-slice at $g \cdot x$ in M, and $\mathcal{L}_g$: $U_x \to U_{g \cdot x}$ is a (J, gJg^{-1})-equivalence. In particular, any principal G-orbit in M decomposes into a union of principal K-orbits.

Clearly $(G/K)_{\pi(x)} = HK/K \simeq H/J$, which is a Lie group isomorphism. Also $U_x^* = \pi(K \cdot U_x) \simeq U_x/J$ by [3, II.4.7], hence we have a $(HK/K, H/J)$-equivalence $S_x^* = \pi(K \cdot S_x) \simeq S_x/J$. Then the following diagram commutes:

$$\begin{array}{ccccc} G \times S_x & \xrightarrow{\pi_1} & G \times_H S_x & \xrightarrow{\Phi \simeq} & G \cdot S_x \\ \downarrow{\scriptstyle p \times \pi|} & & \downarrow{\scriptstyle [p \times \pi|]} & & \downarrow{\scriptstyle \pi} \\ G/K \times S_x^* & \xrightarrow{\pi_2} & G/K \times_{HK/K} S_x^* & \xrightarrow{\tilde{\Phi}} & G/K \cdot S_x^*. \end{array}$$

Since $[p \times \pi|]$ and π are open maps, Φ a G-equivalence, and $\tilde{\Phi}$ a bijection, it follows that $\tilde{\Phi}$ is a G/K-equivalence. Also $G/K \cdot S_x^*$ is open in M/K. Therefore by [3, II.4.1], S_x^* is a HK/K-slice at $\pi(x)$ in M/K.

For $j_0 = \dim(U_x^J)$, we also have,

$$\{(M/K)^{j_0} - (M/K)^{j_0 - 1}\}_{(HK/K)} \cap S_x^* = (U_x^J)^* \cap (S_x^*)^{HK/K} = (S_x^H)^*,$$

since $G_x = G_y \iff G_x \supset G_y, \quad G_x \cap K = G_y \cap K, \quad G_x K = G_y K$, and $(k \cdot U_x) \cap U_x \neq \emptyset \implies k \in J$; see [3, II.4.4].

Let $S_x \simeq^{\phi} \mathfrak{R}^{i_0} \times c(S^q)$ be an H-equivalence, as in 1.4.1, with $i_0 = \dim(S_x^H)$ and $q \geq 0$. Then there is an H/J-equivalence $S_x/J \simeq^{[\phi]} \mathfrak{R}^{i_0} \times c(S^q/J)$, and hence an HK/K-equivalence $S_x^* \simeq^{\phi^*} \mathfrak{R}^{i_0} \times c(S^q/J)$, for the induced HK/K-action on S^q/J. Therefore S_x^* is a conical HK/K-slice (in M/K) of $\pi(P)$ at $\pi(x)$, because $(S_x^*)^0 = (S_x^H)^*$.

Then since $\operatorname{len}(S^q) < \operatorname{len}(M)$, (see [7]), it follows from the inductive hypothesis, that S^q/J with the J-orbit type stratification is a compact H/J-pseudomanifold.

Hence S^q/J is an HK/K-pseudomanifold with the required dimension, as can easily be checked. The case $q = -1$ is trivial.

Using the reiterated slice argument of example 1, and the determination of conical slices in M/K given above, we can easily verify condition (C2) in 1.3. Therefore M/K with the K-orbit type stratification, is an m-dimensional G/K-pseudomanifold.

2. The orbit type refinement

In this section we prove that a G-pseudomanifold is a topological pseudomanifold. Moreover we also show the existence of principal orbits.

Recall the definition of a topological pseudomanifold; see [1], [5].

Definition 2.1. The definition is by induction.

A (-1)-*dimensional topological pseudomanifold* is the empty set.

An n-*dimensional topological pseudomanifold* ($n \geq 0$) is a (non empty) topological space Y, which admits a filtration by closed subsets

$$Y = Y_n \supset Y_{n-1} \supset \cdots \supset Y_0 \supset Y_{-1} = \emptyset,$$

satisfying the following conditions.

(C1) The subspace $Y_n - Y_{n-1}$ is dense in Y.

(C2) Local normal triviality. For each point $y \in Y_i - Y_{i-1}$ there exists a distinguished neighborhood N of y in Y, a compact $(n-i-1)$-dimensional topological pseudomanifold

$$L = L_{n-i-1} \supset L_{n-i-2} \supset \cdots \supset L_0 \supset L_{-1} = \emptyset,$$

and a homeomorphism $h\colon N \to \Re^i \times cL$ which takes $N \cap Y_{i+j+1}$ homeomorphically to $\Re^i \times cL_j$ for $j = -1, \ldots, n-i-1$.

Thus, the subspace $Y_i - Y_{i-1}$ is a topological i-manifold (if non empty), for $i = 0, \ldots, n$.

If Y is a topological pseudomanifold then it is locally compact, and a *CS* space in the sense of Siebenmann [9]. It can be shown that n is the topological dimension of Y, and that every compact topological pseudomanifold can be embedded in Euclidean space [5]. Notice that we allow $i = n-1$ in the definition of Y, as in [1, p. 61].

The following types of spaces are topological pseudomanifolds: Whitney stratified sets [4], [5], abstract (or Thom-Mather) stratified sets [2], [10], and piecewise linear spaces [1], [4].

For the rest of this section let X be an n-dimensional G-pseudomanifold,

$$X = X^m \supset X^{m-1} \supset \cdots \supset X^0 \supset X^{-1} = \emptyset.$$

We shall prove that X is an n-dimensional topological pseudomanifold. Hence n is the topological dimension of X and $m = n$. Assume $n \geq 0$.

Recall that in §1 we proved the following: given a class (H) corresponding to orbits in any $X^k - X^{k-1}$, then the connected components of the subspaces $(X^k - X^{k-1})_{(H)}$ are topological manifolds of dimension $i = sd(P)$ for any orbit P intersecting such a component. Therefore $0 \leq i \leq n$, since P has a link L which is an $(n-i-1)$-dimensional H-pseudomanifold. We shall call these manifolds the *strata* of X.

Then there is a canonical refinement of the filtration of X, called *the orbit type refinement of* X,

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset,$$

where each X_i is the union of the strata of X with dimension less than, or equal to i. We shall prove that the connected components of each non empty subspace $X_i - X_{i-1}$ coincide with the i-dimensional strata of X for $i = 0, \ldots, n$.

PROPOSITION 2.2. *Each orbit $P = G \cdot x$ in X has a tubular neighborhood which is a bundle $(\Gamma, \tau, \Gamma_0, cL)$, where $\Gamma \simeq G \times_H S_x$ is the tubular neighborhood of P in X corresponding to a conical slice at x, and $\Gamma_0 \simeq G \times_H S_x^0$. Furthermore τ is equivariant, $\tau|\Gamma_0 = 1$ and $\tau^{-1}(S_x^0) = S_x$.*

Proof. Let S_x be a conical slice of P at x. Consider the homeomorphism q given by the composition

$$S_x \xrightarrow{\phi} \mathfrak{R}^{i_0} \times cL \xrightarrow{\simeq} (\mathfrak{R}^{i_0} \times \{*\}) \times cL \xrightarrow{\phi|^{-1} \times 1} S_x^0 \times cL,$$

where we assume $L \neq \emptyset$. Then there is an equivariant map $\tau\colon \Gamma \to \Gamma_0$ given by

$$\Gamma \xrightarrow{\Phi^{-1}} G \times_H S_x \xrightarrow{[1 \times q]} G \times_H (S_x^0 \times cL) \xrightarrow{[1 \times p]} G \times_H S_x^0 \xrightarrow{\Phi|} \Gamma_0,$$

which is well defined since p and q are H-equivariant. Clearly $\tau^{-1}(S_x^0) = S_x$.

There is a distinguished neighborhood N of x obtained as follows. Let $\sigma\colon \Sigma \to G$ be a local section of $\pi_0\colon G \to G/H$, with Σ a chart of G/H, $eH \in W$ and $\sigma(eH) = e$, i.e., $\pi_0^{-1}(\Sigma) = \sigma(\Sigma)H$.

Let $U = \pi_0^{-1}(\Sigma) \cdot S_x^0$ and $N = \tau^{-1}(U) = \pi_0^{-1}(\Sigma) \cdot S_x$.

Consider the associated bundle $G \times_H S_x$ with projection map π_1. Then a trivialization φ_1 of this bundle is given by the composition [3, II.2.4]

$$\pi_1^{-1}(\Sigma) \simeq \pi_0^{-1}(\Sigma) \times_H S_x \simeq (\Sigma \times H) \times_H S_x \simeq \Sigma \times (H \times_H S_x) \simeq \Sigma \times S_x,$$

where $\varphi_1[g, s] = (gH, \sigma(gH)^{-1}g \cdot s) \quad$ for $g \in \pi_0^{-1}(\Sigma), s \in S_x$.

Hence a trivialization φ of the bundle τ over U is given by the following commutative diagram:

$$\begin{array}{ccccccc}
\tau^{-1}(U) & \overset{=}{\to} & \pi_0^{-1}(\Sigma)\cdot S_x & \overset{\cong}{\to} & \pi_0^{-1}(\Sigma)\times_H S_x & \overset{\varphi_1}{\to} & \Sigma\times S_x \\
\downarrow\varphi & & & & & & \downarrow 1\times q \\
U\times cL & \overset{=}{\to} & U\times cL & \overset{\cong}{\to} & (\pi_0^{-1}(\Sigma)\times_H S_x^0)\times cL & \overset{\cong}{\to} & \Sigma\times S_x^0\times cL \\
\downarrow p & & & & & & \downarrow 1\times p \\
U & \overset{=}{\to} & \pi_0^{-1}(\Sigma)\cdot S_x^0 & \overset{\cong}{\to} & \pi_0^{-1}(\Sigma)\times_H S_x^0 & \overset{\varphi_2}{\to} & \Sigma\times S_x^0.
\end{array}$$

It can easily be shown that $\varphi|S_x = q$. Moreover we have

$$\varphi\mathcal{L}_g\varphi^{-1}(\tau(s),[l,r]) = (g\cdot\tau(s),[\sigma(gH)^{-1}g\cdot l,r])$$

for $g\in\pi_0^{-1}(\Sigma)$, $s\in S_x$, $[l,r]\in cL$.

Notice in particular, that if T is the stratum of X containing x, then U is a chart of T at x. For $L=\emptyset$ we have $\tau = 1$ and the above proof is trivial.

A distinguished neighborhood of $g\cdot x$ for $g\in G$ is given by $g\cdot N = \tau^{-1}(g\cdot U)$, with a trivialization $\tilde{\varphi} = (\mathcal{L}_g\times 1)\varphi\mathcal{L}_{g^{-1}}$. □

It can easily be shown that a basis for the neighborhood system of x is given by the family $\{N_r = \pi_0^{-1}(\Sigma_r)\cdot S_x(r)\colon 0<r\leq 1\}$ for $S_x(r)\simeq^{\phi|} D_r(\mathfrak{R}^{i_0})\times c_r L$ and $\Sigma_r\simeq D_r(\mathfrak{R}^{i-i_0})$, where D_r denotes the standard r-disk in Euclidean space, $i = sd(G\cdot x)$ and $c_rL = L\times[0,r)/(l,0)\sim(l',0)$ for $L\neq\emptyset$. Clearly, using an equivariant retraction [3, II.4.2], $S_x(r)$ is also a conical slice of P at x.

We shall now examine the orbit type refinement locally, using the same notation as in 2.2.

PROPOSITION 2.3. *Let N be a distinguished neighborhood (in X) of the point x. Then the map φ: $N=\tau^{-1}(U)\longrightarrow U\times cL$, for $L\neq\emptyset$, satisfies*

$$\tau^{-1}(U)\cap(X_j - X_{j-1})\simeq^{\varphi}\begin{cases}\emptyset & \text{if } 0\leq j<i,\\ U\times\{*\} & \text{if } j=i,\\ U\times(L_{j-i-1}-L_{j-i-2})\times(0,1) & \text{if } i<j\leq n,\end{cases}$$

where $L = L_{n-i-1}\supset L_{n-i-2}\supset\cdots\supset L_0\supset L_{-1}=\emptyset$ is the orbit type refinement of the H-pseudomanifold L, and $i = sd(G\cdot x)$.

Proof. Put on $\tau^{-1}(U)$ the relative filtration induced by the orbit type refinement of X, and on $U\times cL$ the canonical filtration induced by the orbit type refinement of L.

Then for each $y\in S_x - S_x^0$ with $\varphi(y) = (\tau(y),[l,r])$, by 2.2 we have

$$(G\cdot y)\cap\tau^{-1}(U) = \pi_0^{-1}(\Sigma)\cdot y\simeq^{\varphi}\sigma(\Sigma)\cdot\tau(y)\times(H\cdot l)\times\{r\}.$$

However, $sd(G \cdot y) = sd(G \cdot x) + sd(H \cdot l) + 1$ using 1.3 (C2), and the proof is complete. □

A similar result also holds for the map $\tilde{\varphi}\colon \tau^{-1}(g \cdot U) \to (g \cdot U) \times cL$. For $L = \emptyset$ we have $i = n$ with $\tau = 1$, and the result is trivial.

COROLLARY 2.4. *The subspace $X_i - X_{i-1}$ is a topological i-manifold (if non empty), whose connected components coincide with the i-strata of X, for $i = 0, \ldots, n$. Furthermore $X_n - X_{n-1}$ is non empty.*

Proof. Let T be an i-stratum in X. Then T is open in $X_i - X_{i-1}$ and thus a component, since for a distinguished neighborhood N of any point x in T, we have $x \in U = N \cap (X_i - X_{i-1}) \subset T$, using 2.3. Hence, since U is a chart of T (see 2.2), $X_i - X_{i-1}$ is an i-manifold. Furthermore, if $X_j - X_{j-1}$ is non empty for $0 \le j = i < n$, and empty for $i < j \le n$, then given $x \in X_i - X_{i-1}$, $G \cdot x$ has a non empty link L, since the dimension of L is $n - i - 1 \ge 0$. Hence, using 2.3 again, we obtain a contradiction. Therefore $X_n - X_{n-1}$ is non empty. □

Using 2.2, 2.3 and 2.4, we have the following.

COROLLARY 2.5. *The map $\varphi\colon N = \tau^{-1}(U) \longrightarrow U \times cL$ is a stratum-preserving homeomorphism, and hence* $\mathrm{len}(L) < \mathrm{len}(X)$.

COROLLARY 2.6. *If X is compact, then it has a finite number of orbit types.*

COROLLARY 2.7. *The subsets X_i in the orbit type refinement of X are closed.*

Proof. Assume X is an n-dimensional G-pseudomanifold. Given a stratum T of X with $y \in T \subset X_i - X_{i-1}$, let N be a distinguished neighborhood of y in X. Then by 2.3, N does not intersect strata of dimension strictly smaller than i. Hence $y \in N \cap X_i \subset X_i - X_{i-1}$ and X_{i-1} is closed in X_i for $i = 0, \ldots, n$. Therefore each X_i is closed in X. □

By 2.3 and 2.7, X is an n-dimensional G-pseudomanifold, with the orbit type refinement.

COROLLARY 2.8. *Let X be an n-dimensional G-pseudomanifold with the orbit type refinement, and Y an open invariant subspace such that Y/G is connected. Then Y is also an n-dimensional G-pseudomanifold, with the relative stratification.*

Proof. Let $\pi\colon X \to X^*$ be the canonical projection onto the orbit space of X. Given an orbit P in X, let $S_x \simeq \mathfrak{R}^{i_0} \times cL$ be a conical slice of P at x, where $H = G_x$ and $L \neq \emptyset$. Then (see [3, II.4.7]) we have

$$\Gamma_r^* = \pi(\Gamma_r) \simeq (G \times_H S_x(r))^* \simeq S_x^*(r) \simeq D_r(\mathfrak{R}^{i_0}) \times c_r(L^*)$$

for the tube Γ_r corresponding to the conical slice $S_x(r)$ of P. Hence the family $\{\Gamma_r^*\colon 0 < r \leq 1\}$ is a neighborhood basis for x^* in X^*, since π is open. Therefore, for any orbit P in Y, it is possible to find a tubular neighborhood Γ_r of P, corresponding to some conical slice $S_x(r)$ of P, such that Γ_r is contained in the open subspace Y. Thus $S_x(r)$ is a conical slice of P in Y. For $L = \emptyset$ we have a similar result. □

We now prove the existence of principal orbits in a G-pseudomanifold.

THEOREM 2.9. *There exists a conjugacy class* (H_0) *corresponding to certain orbits in* X*, called principal orbits, with* $\text{type}(G/H_0) \geq \text{type}(P)$ *for all orbits* P *in* X*. Furthermore* $X_{(H_0)}$ *is open, has a connected orbit space, and contains* $X - X_{n-1}$*, the latter being dense in* X*.*

Proof. We use induction on the length of the orbit type refinement of X. For $\text{len}(X) = 0$ it trivially holds locally, since by 2.4 all orbits have an empty link. Therefore it holds globally, using the argument given below.

Now let X be an n-dimensional G-pseudomanifold, with $\text{len}(X) > 0$. Given an orbit P in X, let $S_x \simeq^{\varphi} S_x^0 \times cL$ be a conical slice of P at x, where L is a link of P. Assume $i = sd(G \cdot x) \neq n$. Then L is a compact $(n-i-1)$-dimensional H-pseudomanifold for $H = G_x$, with $\text{len}(L) < \text{len}(X)$ by 2.5.

By the inductive hypothesis, L has principal orbits of $\text{type}(H/K_0)$ such that $L_{(K_0)}$ is open, has a connected orbit space, and contains $L - L_{n-i-2}$, the latter being dense in L.

Let $\Gamma \simeq^{\Phi} G \times_H S_x$ be the tubular neighborhood of P corresponding to the conical slice S_x. Then for any non principal orbit type H/K ocurring in L, we may suppose by conjugating that $K_0 \subset K \subset H$ with $K_0 \neq K$. Then K_0 and K differ either in dimension or number of components and thus cannot be conjugate in G. Since subgroups of H which are conjugate in H are, a fortiori, conjugate in G, it follows that

$$\Gamma_{(K_0)} \simeq^{\Phi} (G \times_H S_x)_{(K_0)} = G \times_H (S_x)_{(K_0)}$$

which is open in Γ, because

$$(S_x)_{(K_0)} \simeq^{\varphi} \begin{cases} S_x^0 \times L_{(K_0)} \times (0,1) & \text{if} \quad K_0 \neq H, \\ S_x^0 \times cL & \text{if} \quad K_0 = H, \end{cases}$$

since for $K_0 = H$ we have $L_{(K_0)} = L^H = L$. Therefore, we obtain

$$\Gamma^*_{(K_0)} = \pi(\Gamma_{(K_0)}) \simeq (G \times_H S_x)^*_{(K_0)} \simeq (S_x)^*_{(K_0)},$$

and this is connected, and open in Γ^* since $\pi : X \to X^*$ is open.

Now S_x has a canonical filtration induced by the H-orbit type refinement of L. Hence by 2.3, for $i_0 = \dim(S_x^0)$ and $j = -1, \ldots, n-i-1$, we have $S_x \cap X_{i+j+1} = (S_x)_{i_0+j+1} \simeq^{\varphi} S_x^0 \times cL_j$, and therefore

$$\Gamma_{i+j+1} = \Gamma \cap X_{i+j+1} \simeq G \times_H (S_x)_{i_0+j+1}.$$

Clearly $\Gamma_{(K_0)} \supset \Gamma - \Gamma_{n-1}$, since $(S_x)_{(K_0)} \supset S_x - (S_x)_{n-i+i_0-1}$, hence $\Gamma - \Gamma_{n-1}$ is dense in Γ, and the theorem is valid locally in X. In particular, it follows that $\Gamma^*_{(K_0)}$ is dense in Γ^*.

If $sd(G \cdot x) = n$ we have $S_x = S_x^H = S_x^0$, and the above statements are trivial.

We now extend the theorem globally, using the following argument given in [3].

By above, for all $x^* \in X^* = X/G$ we have a neighborhood U_x^* of x^* which contains an open, connected, dense set W_x^*, such that all orbits in W_x^* have the same type, and all other orbits in U_x^* have type strictly smaller.

Let H be any closed subgroup of G and $C_{(H)} = \overline{\operatorname{int}(X^*_{(H)})}$. Then $x^* \in C_{(H)} \Longleftrightarrow W_x^*$ consists of orbits of type(G/H) and, in this case, $C_{(H)} \supset U_x^*$. Thus $C_{(H)}$ is both open and closed in X^*. Hence $C_{(H_0)} = X$ for some (H_0) (now fixed), since X^* is connected. Also $C_{(K)} = \emptyset$ if K is not conjugate to H_0.

Then $X^*_{(H_0)}$ is open, since $X^*_{(H_0)} \cap U_x^* = W_x^*$, and is also dense. All other orbits have type strictly smaller than that of G/H_0. If D is a component of $X^*_{(H_0)}$, then, since W_x^* is connected for each x^*, we see that D^- is open (and closed) in X^*. Hence $X^*_{(H_0)} = D$ is connected.

Therefore $X_{(H_0)}$ is open, and has a connected orbit space. Also since $X_{(H_0)}$ is dense in X, it follows that $X_{(H_0)}$ contains $X - X_{n-1}$, the latter being dense in X. □

We now prove the result stated at the beginning of this section.

THEOREM 2.10. *Let X be an n-dimensional G-pseudomanifold. Then X is an n-dimensional topological pseudomanifold.*

Proof. Use induction on the dimension of X. For $n = -1$, X is empty and both concepts coincide.

Let X be an n-dimensional G-pseudomanifold ($n \geq 0$) and P a given orbit in $X_i - X_{i-1}$ for some $i = 0, \ldots, n$. If N is a distinguished neighborhood of a point x in P (see 2.2), then for a given trivialization $N = \tau^{-1}(U) \simeq^{\varphi} U \times cL$, by 2.3 we have

$$N \cap X_{i+j+1} \simeq^{\varphi} U \times cL_j \simeq \mathfrak{R}^i \times cL_j \quad \text{for} \quad \mathrm{j} = -1, \ldots, \mathrm{n} - \mathrm{i} - 1,$$

since U is a chart of the i-manifold $X_i - X_{i-1}$ (see 2.4).

However $n - i - 1 < n$, hence by the inductive hypothesis L is an $(n - i - 1)$-dimensional, compact topological pseudomanifold. Then using 2.7 and 2.9, X is an n-dimensional topological pseudomanifold. □

COROLLARY 2.11. *The union of all principal orbits $X_{(H_0)}$ is an n-dimensional topological pseudomanifold embedded in X.*

COROLLARY 2.12. *Let T be a stratum of X with* $\dim(T) \neq n$, *intersecting orbits of dimension t. Then* $\dim(T) \leq n - h + t - 1$, *where h is the dimension of the* *principal orbits in X. Equality holds, if and only if there is an orbit P intersecting T, and a conical slice $S_x \simeq \mathfrak{R}^{i_0} \times cL$ of P at x, such that $H = G_x$ acts transitively on L.*

Proof. Let P be an orbit in X intersecting T and S_x a conical slice of P at x. There is no loss in generality in assuming $x \in T$.

Then $\Gamma \simeq^{\Phi} G \times_H S_x$ is a tubular neighborhood of P and by 2.9, there is a point $y \in S_x$ with $sd(G \cdot y) = n$. Let T' be the stratum of X which contains y. Then for $H_0 = G_y$, using 1.3 (C2) we have

$$n = \dim(T') = \dim(T) + (k_0 + \dim(H/H_0)) + 1 \geq \dim(T) + (h - t) + 1. \quad \square$$

3. Stratification of the orbit space

In this section we study the orbit space of a G-pseudomanifold.

Let $B = X/G$ be the orbit space of an n-dimensional G-pseudomanifold X, with the orbit type refinement, for $n \geq 0$. Denote by $\pi\colon X \to B$ the canonical projection, and let $\pi(A) = A^*$ for $A \subset X$.

Given an orbit P in $X_i - X_{i-1}$, for $i = 0, \ldots, n$, then if $\Gamma \simeq G \times_H S_x$ is the tubular neighborhood corresponding to a conical slice $S_x \simeq \mathfrak{R}^{i_0} \times cL$ of P at x, where $H = G_x$, we have

$$(X_i - X_{i-1})^*_{(H)} \cap \Gamma^* \simeq (G \times_H S_x^0)^* \simeq \mathfrak{R}^{i_0}.$$

Therefore the connected components of the subspaces $(X_i - X_{i-1})^*_{(H)}$ are topological manifolds, called the *strata* of B. Clearly each stratum T in X projects onto a stratum T^* of B with $\dim(T^*) = \dim(T) - t$, where t is the dimension of the orbits in X intersecting T.

This leads to the canonical filtration of B, induced by the orbit type refinement of X,

$$B \supset \cdots \supset B_k \supset B_{k-1} \supset \cdots \supset B_{-1} = \emptyset,$$

where each B_k is the union of the strata of B with dimension less than, or equal to k. We shall prove that the connected components of each non empty subspace $B_k - B_{k-1}$, coincide with the k-dimensional strata of B, for all k.

LEMMA 3.1. *The above filtration has the following properties.*

(a) $B = B_m$ *for $m = n - h$, where h is the dimension of the principal orbits in the n-dimensional G-pseudomanifold X.*

(b) $B - B_{m-1}$ *is a dense in B.*

Proof. For an n-dimensional stratum T of X, it follows from 2.9 that T^* is an m-dimensional stratum of B. If T is a stratum of X with $\dim(T) \neq n$, then by 2.12,

$\dim(T^*) = \dim(T) - t \leq n - h - 1 = m - 1$, where t is the dimension of the orbits in X intersecting T. This proves (a).

For (b) note that

$$B = \pi(X) = \pi(\overline{X - X_{n-1}}) \subset \overline{\pi(X - X_{n-1})} = \overline{B - B_{m-1}}. \qquad \square$$

Now given an an orbit P in X, let $(\Gamma, \tau, \Gamma_0, cL)$ be a tubular neighborhood of P corresponding to a conical slice $S_x \simeq S_x^0 \times cL$ of P at x, as in 2.2. Then there is a map ρ: $\Gamma^* \to \Gamma_0^*$, given by $\rho(z^*) = \tau(z)^*$ for $z \in \Gamma$, which is well defined since τ is equivariant.

Let $N = \tau^{-1}(U)$ be a distinguished neighborhood of a point y in P, with $V = \pi(U)$. Clearly [3, II.4.7], $\pi(N) = \Gamma^* \simeq S_x/H$ and $\pi(U) = \Gamma_0^* \simeq S_x^0$, for $H = G_x$. Then $\pi(N) = \rho^{-1}(V)$ is a distinguished neighborhood of y^* in B.

Now assume $L \neq \emptyset$. Put on $\rho^{-1}(V)$ the relative filtration in B, and on $V \times c(L/H)$ the canonical filtration induced by the projection π': $L \to L/H$.

PROPOSITION 3.2. *There is a map ψ: $\rho^{-1}(V) \longrightarrow V \times c(L/H)$, which is a stratum-preserving homeomorphism, commuting with the projection to V.*

Proof. Given a trivialization φ over U, let $\psi = \alpha \circ [\varphi|] \circ \beta^{-1}$ in the following diagram, which commutes by 2.2:

$$\begin{array}{ccccc}
\tau^{-1}(U) & \xrightarrow{\pi} & \rho^{-1}(V) & \xrightarrow{\alpha \simeq} & S_x/H \\
\downarrow \varphi & & \downarrow \psi & & \downarrow [\varphi|] \\
U \times cL & \xrightarrow{\pi \times c\pi'} & V \times c(L/H) & \xrightarrow{\beta \simeq} & S_x^0 \times c(L/H).
\end{array}$$

Clearly ψ is a stratum-preserving homeomorphism; see proof of 2.3. $\square$

For $L = \emptyset$ we have $\rho^{-1}(V) = V \subset B - B_{m-1}$.

COROLLARY 3.3. *The subset B_k in the filtration of B is closed, and the subspace $B_k - B_{k-1}$ is a topological k-manifold (if non empty), whose connected components coincide with the k-strata of B for $k = 0, \ldots, m$.*

We now state our main result, which shows that both X and X/G belong to the same class of spaces, namely topological pseudomanifolds.

THEOREM 3.4. *Let X be an n-dimensional G-pseudomanifold. Then the orbit space X/G is an m-dimensional topological pseudomanifold.*

Proof. By induction on length of the orbit type refinement of X. For $\mathrm{len}(X) = 0$ we have $B = B - B_{m-1}$ using 2.9 and 3.1, and the proof is trivial.

Now let X be an n-dimensional G-pseudomanifold with $\operatorname{len}(X) > 0$. If $y^* \in B_k - B_{k-1}$ for some $k \neq m$, we consider the neighborhood $\rho^{-1}(V)$ of y^*, as given in'3.2.

Then we have

$$\rho^{-1}(V) \cap B_{k+j+1} \simeq^{\psi} V \times c(L/H)_j \simeq \Re^k \times c(L/H)_j,$$

for $j = -1, \ldots, m-k-1$, since V is a chart of $B_k - B_{k-1}$.

However since $\operatorname{len}(L) < \operatorname{len}(X)$, it follows by the inductive hypothesis that L/H is an $(m-k-1)$-dimensional, compact topological pseudomanifold. For $k = m$, the proof is trivial.

Therefore, by 3.1 and 3.3, B is an m-dimensional topological pseudomanifold. □

4. The embedding theorem

In this section we define smooth G-pseudomanifolds. Moreover we prove a generalization of Mostow's smooth equivariant embedding theorem, see [6], for compact smooth G-pseudomanifolds.

Definition 4.1. A Hausdorff topological space is said to be *smoothly stratified* if it admits a filtration

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

by closed subsets such that $X_i - X_{i-1}$ is a smooth (C^∞) i-manifold (if non empty) for $i = 0, \ldots, n$. In this case we say that X is a *smooth stratified space*. In addition, if X is also a G-space, and each non empty $X_i - X_{i-1}$ is invariant, with a smooth restricted G-action, we say that X is a *smooth stratified G-space*. We call the connected components of each $X_i - X_{i-1}$ the *strata* of X. (Assume that $X_n \neq X_{n-1}$.)

Definition 4.2. Let X, Y be smooth stratified spaces and $f\colon X \to Y$ a continuous map.

(i) f is *smooth* if it is stratum-preserving (i.e., maps each stratum of X in a stratum of Y), and smooth when restricted to the strata of X.

(ii) f is a *smooth embedding* if it is a topological embedding which is also stratum-preserving, and a smooth embedding when restricted to the strata of X.

(iii) f is a *submersion* if it is open and surjective, a stratum-preserving projection (i.e., maps each stratum of X onto a stratum of Y), and a smooth submersion when restricted to the strata of X.

(iv) f is a *diffeomorphism* if it is a homeomorphism and f, f^{-1} are smooth.

Now let H be a closed subgroup of G, and S a smooth stratified H-space. Then we show that $G \times_H S$ is canonically a smooth stratified G-space.

Notice that the twisted product is the orbit space of a free action on the Cartesian product. Then the projection $p\colon G \times S \to G \times_H S$ induces a smooth structure on the strata of $G \times_H S$ such that p is an open submersion, since $G \times_H (S_k - S_{k-1})$ is the orbit space of the smooth H-manifold $G \times (S_k - S_{k-1})$ for any non empty $S_k - S_{k-1}$ in S. In particular, the left action of G on each manifold $G \times_H (S_k - S_{k-1})$ is also smooth, hence $G \times_H S$ is a smooth stratified G-space.

Now let $\sigma\colon \Sigma \to G$ be a local section at eH of $\pi_0\colon G \to G/H$ with $\sigma(eH) = e$. Then the following diagram commutes:

$$\begin{array}{ccccccc}
\pi_0^{-1}(\Sigma) \times S & \overset{\approx}{\to} & (\Sigma \times H) \times S & \overset{\approx}{\to} & \Sigma \times (H \times S) & \overset{=}{\to} & \Sigma \times (H \times S) \\
\downarrow p & & \downarrow p' & & \downarrow 1\times p'' & & \downarrow 1\times\Phi_H \\
\pi_0^{-1}(\Sigma) \times_H S & \overset{\approx}{\to} & (\Sigma \times H) \times_H S & \overset{\approx}{\to} & \Sigma \times (H \times_H S) & \overset{\approx}{\to} & \Sigma \times S.
\end{array}$$

Therefore, since the action map Φ_H is also a submersion, it follows that the trivialization $\pi_0^{-1}(\Sigma) \times_H S \simeq \Sigma \times S$ is a diffeomorphism. In particular, the canonical map $S \to G \times_H S$ given by $s \mapsto [e, s]$ is a smooth embedding.

We now define the concept of a smooth G-pseudomanifold.

Definition 4.3. We use induction.

A (-1)*-dimensional smooth G-pseudomanifold* is the empty set.

An n-dimensional G-pseudomanifold X, $n \geq 0$, is said to be *smooth* if it satisfies the following conditions.

(C1) With the orbit type refinement, X is a smooth stratified G-space.

(C2) Each orbit P in X has a conical slice $S_x \simeq^{\phi} \mathfrak{R}^{i_0} \times cL$ at x, with L an $(n-i-1)$-dimensional smooth H-pseudomanifold, where $H = G_x$ and $i = sd(P)$.

(C3) The canonical map $\Phi\colon G \times_H S_x \to G \cdot S_x$ is a diffeomorphism.

Examples 4.4.

1. *Smooth actions.* Let M be an n-dimensional smooth G-manifold. For $n = -1$ we define M to be the empty set. Claim that M is an n-dimensional smooth G-pseudomanifold.

The proof is by induction on the dimension of M.

Assume that $n \geq 0$ and choose a G-invariant Riemannian metric on M. Given an orbit P in M, choose a point $x \in P$ with $H = G_x$. Then there is a Riemannian normal coordinate system S_x at x of radius $r > 0$, which is the union of all geodesic segments of length less than r, starting from x in a direction orthogonal to P. Then S_x is H-equivalent to $N_x = T_x(G \cdot x)^{\perp}$, the orthogonal complement in $T_x(M)$, called the normal space to P at x, which has an orthogonal H-action given by the slice representation; see [3, p. 174]. It follows, see [3], that $S_x = S_x(t)$ is a linear slice at x for some $t \leq r$, and M is a locally linear G-manifold.

Therefore it follows from example 1.4 (1) that M is also an n-dimensional G-pseudomanifold. Moreover since the canonical map $\Phi\colon G \times_H S_x \to G \cdot S_x$ is a diffeomorphism of smooth G-manifolds, see [3, p. 308], there is a natural smooth structure on each non empty manifold $M_i - M_{i-1}$ such that the inclusion in M is a smooth embedding. In particular, G acts smoothly on each $M_i - M_{i-1}$.

Using the same notation as in 1.4 (1), we have a conical slice of P at x,

$$S_x \simeq N_x \simeq N_x^H \oplus (N_x^H)^{\perp} \simeq \mathfrak{R}^{i_0} \times c(S^q),$$

where $i_0 = \dim(N_x^H)$ and $q + 1 = \dim(N_x^H)^{\perp} = n - sd(P)$. If $q \geq 0$, then since S^q is a smooth H-manifold and $q < n$, it follows from the inductive hypothesis that S^q is an $(n - i - 1)$-dimensional smooth H-pseudomanifold for $i = sd(P)$. Moreover the map Φ: $G \times_H S_x \to G \cdot S_x$ is a stratum-preserving homeomorphism; see proof of 2.9. Then it follows that Φ is a diffeomorphism of smoothly stratified G-spaces, since the strata of S^q are (C^∞) embedded submanifolds. The case $q = -1$ is trivial.

Therefore M is an n-dimensional smooth G-pseudomanifold.

2. *Actions on orbit spaces (smooth case).* Let M be an n-dimensional smooth G-manifold, $n \geq 0$, and K a closed normal subgroup of G such that M/K is connected. Since M is locally linear it follows from 1.4 (2) that M/K is a G/K pseudomanifold, which we claim is also smooth.

The proof is by induction on the length of the G-orbit type refinement of M. For $\text{len}(M) = 0$, it is trivial. Assume that $\text{len}(M) > 0$.

Since the invariant G-strata in M (i.e., $G \cdot S$, for a stratum S in M) have a local product structure, we can put a smooth structure on the invariant G/K-strata in M/K such that π: $M \to M/K$ is a submersion; see 2.4. For simplicity we call the map π a $(G, G/K)$-submersion. In particular, G/K acts smoothly on each non empty $(M/K)_i - (M/K)_{i-1}$.

Using the same notation as in 1.4 (2), let $S_x \simeq \mathfrak{R}^{i_0} \times c(S^q)$ be a conical H-slice of an orbit P (in M) at x, with $q \geq 0$, where $H = G_x$. Then $S_x^* \simeq \mathfrak{R}^{i_0} \times c(S^q/J)$ is a conical HK/K-slice of $\pi(P)$ at $\pi(x)$, where $J = K \cap H$. Since $\text{len}(S^q) < \text{len}(M)$, it follows from the inductive hypothesis that S^q/J is a smooth HK/K-pseudomanifold.

Consider again the diagram given in 1.4 (2). Then clearly $\pi|$: $S_x \to S_x^*$ is an $(H, HK/K)$-submersion, because π': $S^q \to S^q/J$ is a $(H, H/J)$-submersion. Hence the map $p \times \pi|$ is also a submersion. In addition, π_1 and π_2 are submersions; therefore the map $[p \times \pi|]$ is also a submersion and since Φ is a diffeomorphism, see 4.4 (1), we conclude that $\tilde{\Phi}$ is a diffeomorphism. The case $q = -1$ is trivial.

Therefore M/K is a smooth G/K-pseudomanifold.

Moreover we can also define smooth pseudomanifolds similarly to 4.3, and prove that the orbit space of a smooth G-pseudomanifold is a smooth pseudomanifold. In particular, it can easily be shown that the canonical map $(M/K)/(G/K) \simeq M/G$ is a diffeomorphism of smooth pseudomanifolds.

We shall now formulate the equivariant embedding theorem.

This theorem is a generalization of the classical smooth equivariant embedding theorem of Mostow, see [6], which is valid for G-manifolds.

Our result is the following.

THEOREM 4.5. *Let X be a compact smooth G-pseudomanifold. Then there is a Euclidean space $\mathfrak{R}^m$ with an orthogonal G-action, together with an equivariant smooth embedding $\theta\colon X \to \mathfrak{R}^m$.*

Proof. By induction on the length of the orbit type refinement of X. For $\operatorname{len}(X) = 0$, the statement follows from the smooth equivariant embedding theorem of Mostow, see [6], since by 2.4 all orbits in X have an empty link.

Let X be an n-dimensional smooth G-pseudomanifold with $\operatorname{len}(X) > 0$. If P is an orbit in X, let S_x be a conical slice of P at x, as given by 4.3 (C2). Then, there is an H-equivalence $\phi\colon S_x \to D(\mathfrak{R}^{i_0}, 1) \times cL$, with $G_x = H$, where L is a compact smooth H-pseudomanifold, and H acts trivially on $D(\mathfrak{R}^{i_0}, 1)$ the open unit disk of $\mathfrak{R}^{i_0}$. Assume that $L \neq \emptyset$. Clearly S_x with the canonical stratification induced by L, is a smoothly stratified H-space.

Now if Γ is the tubular neighborhood of P corresponding to the slice S_x, then the map $\Phi^{-1}\colon \Gamma = G \cdot S_x \to G \times_H S_x$ is a G-equivariant diffeomorphism, where the domain has the relative stratification, and the range the canonical stratification induced by S_x, using 4.3 (C3). In particular, it follows that S_x is smoothly embedded in X.

Since $\operatorname{len}(L) < \operatorname{len}(X)$, by the inductive hypothesis there is a Euclidean space $\mathfrak{R}^{m_0}$ with an orthogonal H-action, together with an equivariant smooth embedding $\theta_1\colon L \to \mathfrak{R}^{m_0}$.

Therefore there is an H-equivariant smooth embedding $\theta_2\colon S_x \to V$, where $\theta_2 = \phi \circ (1 \times c\theta_1)$ and $V = \mathfrak{R}^{i_0} \oplus \mathfrak{R} \oplus \mathfrak{R}^{m_0}$, which has an orthogonal H-action given by the sum of these representations, with H acting trivially on $\mathfrak{R}^{i_0}$ and $\mathfrak{R}$. Here $c\theta_1\colon cL \to c\mathfrak{R}^{m_0} \subset \mathfrak{R} \oplus \mathfrak{R}^{m_0}$.

Let $D(V, r) = \{v \in V\colon \|v\| < r\}$, and $c(L, r) = L \times [0, r)/(l, 0) \sim (l', 0)$ for $0 < r \leq 1$. Using a suitable homothety, we may assume that there is an H-equivariant smooth embedding of S_x into $D(V, \sqrt{3})$.

By symmetry there is an H-equivariant smooth embedding of $S_x(r) = \phi^{-1}\{D(\mathfrak{R}^{i_0}, r) \times c(L, r)\}$ into $D(V, r\sqrt{3})$. It can easily be shown using an equivariant retraction, see [3, II.4.2], that $S_x(r)$ is also a conical slice of P at the point x.

Hence the following composition is a G-equivariant smooth embedding:

$$\Gamma \xrightarrow{\Phi^{-1}} G \times_H S_x \xrightarrow{[1\times\theta_2]} G \times_H D(V, \sqrt{3}) \xrightarrow{[1\times i]} G \times_H V.$$

For $L = \emptyset$, the above is trivially satisfied.

To conclude the proof, we shall give an equivariant smooth embedding of $G \times_H V$ into Euclidean space.

Using [3, 0.5.2], it follows that there exists an orthogonal representation of G on some Euclidean space V_0 and a point $v_0 \in V_0$ with $G_{v_0} = H$. Now by [3, 0.4.2], the orthogonal representation of H on the Euclidean space V given above may be extended to an orthogonal representation of G on some Euclidean space $V' \supset V$ such that this inclusion is H-equivariant. Then, G acts orthogonally on $W = V_0 \oplus V'$ via the sum of these two representations (i.e., diagonally).

Consider the subspace $v_0 + V$ in W, then by a similar argument to the one given in the proof of [3, II.4.4], there is an equivariant map $G \cdot (v_0 + V) \to G/H$, whose fiber over eH is $v_0 + V$. Since H is closed, it follows from [3, II.3.2] that the canonical map α given by $G \times_H V \simeq G \times_H (v_0 + V) \to G \cdot (v_0 + V)$, is a G-equivalence.

Clearly $G \times_H V$ is a smooth G-manifold, see the remark after 4.2, and since the canonical projection $G \times V \to G \times_H V$ is a submersion, it follows that α is smooth into W. Now let σ: $\Sigma \to G$ be a local section of π_0: $G \to G/H$ at eH, with $\sigma(eH) = e$. Consider the following trivialization $\tilde{\alpha}$, (see proof of 2.2):

$$\Sigma \times V \xrightarrow{\varphi_1^{-1}} G \times_H V \xrightarrow{\simeq} G \times_H (v_0 + V) \xrightarrow{\Phi} W \xrightarrow{-v_0 \oplus 1} W.$$

Then $\tilde{\alpha}(gH, 0) = (g \cdot v_0 - v_0, 0)$ and $\tilde{\alpha}(eH, v) = (0, v)$ for $gH \in \Sigma$, $v \in V$. Thus $\tilde{\alpha}_*$ is injective at $(eH, 0)$, and hence α_* is injective at $[e, 0]$.

Moreover given $v \in V$ with $H_v = K$, let S_1 be a linear K-slice at v in V, i.e., $S_1 = v + V_1$ for some K-invariant linear subspace V_1 inside V.

Then the map

$$G \times_K V_1 \xrightarrow{\simeq} G \times_H (H \times_K S_1) \xrightarrow{\simeq} G \times_H (H \cdot S_1) \xrightarrow{\alpha|} W,$$

coincides with the canonical map $G \times_K V_1 \simeq G \times_K (v_0 + S_1) \to G \cdot (v_0 + S_1)$. Therefore, using an argument similar to the one given previously, α_* is injective at $[e, v]$. Hence by equivariance, α_* is everywhere injective, and consequently α is a smooth equivariant embedding of $G \times_H V$ into W.

Then the following map β is a G-equivariant smooth embedding:

$$\Gamma \xrightarrow{\Phi^{-1}} G \times_H S_x \xrightarrow{[1 \times \theta_2]} G \times_H D(V, \sqrt{3}) \xrightarrow{[1 \times i]} G \times_H V \xrightarrow{\alpha} W.$$

Now given $1 > s > t > 0$, let f: $\mathfrak{R} \to \mathfrak{R}$ be a smooth function such that

$$\begin{cases} f(r) = 1 & \text{for} \quad r \leq t, \\ f(r) \neq 0 & \text{for} \quad r < s, \\ f(r) = 0 & \text{for} \quad r \geq s. \end{cases}$$

Let ρ: $\Gamma \to [0, 1)$ be the smooth invariant function obtained from the radius r of cL, which is well defined by 2.2, since H acts trivially on r. Then we can define a smooth equivariant map ψ: $\Gamma \to W$ by $y \mapsto f(\rho(y)) \cdot \beta(y)$, for $y \in \Gamma$. Since X/G is Hausdorff and $(G \times_H \overline{S_x(s)})/G \simeq \overline{S_x(s)}/H$ is compact, where the closure is taken in S_x, it follows that $\rho^{-1}([0, s])$ is closed in X. Therefore ψ extends to a smooth equivariant map on X.

Also the smooth invariant function γ: $\Gamma \to \mathfrak{R}$, given by $y \mapsto f(\rho(y) \cdot s/t)$ for $y \in \Gamma$, extends to a smooth invariant function on X.

Thus for each orbit P in X we have an orthogonal representation of G on an Euclidean space W_x, and a smooth equivariant map ψ_x: $X \to W_x$, which is a smooth embedding on the tubular neighborhood Γ_x corresponding to the conical slice $S_x(t)$ of P at x.

Additionally, we have a smooth invariant function $\gamma_x \colon X \to \mathfrak{R}$, which is non zero exactly on Γ_x.

Since X is compact, it can be covered by finitely many tubular neighborhoods $\Gamma_{x_1}, \dots, \Gamma_{x_k}$. Let $\theta \colon X \to W_{x_1} \oplus \cdots \oplus W_{x_k} \oplus \mathfrak{R}^k \simeq \mathfrak{R}^m$ be given as follows:

$$\theta(x) = (\psi_{x_1}(x), \dots, \psi_{x_k}(x), \gamma_{x_1}(x), \dots, \gamma_{x_k}(x)) \quad \text{for} \quad x \in X.$$

This map is clearly smooth and equivariant.

If $x, y \in \bigcup \Gamma_{x_p}$ and $\theta(x) = \theta(y)$, then for some $p = 1, \dots, k$ we have $\gamma_{x_p}(x) = \gamma_{x_p}(y) \neq 0$, which implies that $x, y \in \Gamma_{x_p}$ and hence that $x = y$, since ψ_{x_p} is injective on Γ_{x_p}. Therefore θ is injective and a topological embedding. Because ψ_{x_p} is a smooth embedding on Γ_{x_p} for all p, it follows that θ is a smooth equivariant embedding. □

References

[1] A. Borel, *Intersection cohomology*, Progr. Math., no. 50, Birkhäuser, 1984.

[2] J. Brasselet, G. Hector, and M. Saralegi, *Théorème de De Rham pour les variétés stratifiées*, Ann. Global Anal. Geom. **9** (1991), 211–243.

[3] G. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

[4] M. Goresky and R. Mac Pherson, *Intersection homology theory*, Topology **19** (1980), 135–162.

[5] ———, *Intersection homology II*, Invent. Math. **71** (1983), 77–129.

[6] G. Mostow, *Equivariant embeddings in Euclidean space*, Ann. of Math. **65** (1957), 432–446.

[7] R. Popper, *G-pseudomanifolds*, Acta Math. Hungar. **73** (1996), 235–245.

[8] F. Quinn, *Homotopically stratified sets*, J. Amer. Math. Soc. **1-2** (1988), 441–499.

[9] L. Siebenmann, *Deformations of homeomorphisms on stratified sets*, Com. Math. Helvetici **47** (1972), 123–163.

[10] A. Verona, Stratified mappings- structure and triangulability, Lecture Notes in Math., no. 1102, Springer, 1984.

Department of Mathematics, Central University of Venezuela, Caracas, Venezuela
rpopper@euler.ciens.ucv.ve