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COMPACT HERMITIAN SURFACES AND ISOTROPIC CURVATURE

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ABSTRACT. It is shown that on a Kähler surface the non-negativity (resp. non-positivity) of the isotropic curvature is implied by the non-negativity (resp. non-positivity) of the holomorphic bisectional curvature. The compact Hermitian surfaces of non-negative isotropic curvature are described. The full list of compact half conformally flat Hermitian surfaces of non-positive isotropic curvature is also given.

1. Introduction

The notion of curvature on totally isotropic two-planes has been introduced and successfully used by M. Micallef and J. Moore [20] to study the topology of Riemannian manifolds. Given such a manifold (M, g), extend the Riemannian metric g to a complex bilinear symmetric form on the complexification $T^{\mathbb{C}}M$ of the tangent bundle of M. The complex linear extension $\mathcal{R}$: $\Lambda^2 T^{\mathbb{C}}M \to \Lambda^2 T^{\mathbb{C}}M$ of the curvature operator is then a Hermitian operator with respect to the Hermitian metric $\langle z, w\rangle = g(z, \overline{w})$ on $T^{\mathbb{C}}M$. For any 2-dimensional complex subspace $\sigma = \operatorname{span}\{z, w\}$ of $T_p^{\mathbb{C}}M$, $p \in M$,

$$K(\sigma) = \frac{\langle \mathcal{R}(z \wedge w), z \wedge w\rangle}{\|z \wedge w\|^2}$$

is a real number which does not depend on the choice of the basis $\{z, w\}$. A complex subspace V of $T^{\mathbb{C}}M$ is said to be *totally isotropic* if $g(v, v) = 0$ for every $v \in V$. If σ is a totally isotropic two-plane, we shall say that $K(\sigma)$ is the *isotropic curvature* at σ. Since the dimension of any totally isotropic subspace does not exceed $\frac{1}{2}\dim M$, the notion of isotropic curvature is non-vacuous only if $\dim M \geq 4$.

Micallef and Moore [20] have been able to prove that every compact simply connected Riemannian manifold of dimension ≥ 4 which has positive isotropic curvature is homeomorphic to the unit sphere. Their proof suggests that the condition "positive isotropic curvature" could be useful in studying minimal surfaces in Riemannian manifolds (in this connection see [22]). Using a different approach, R. Hamilton [15] has recently proved that any compact simply connected four-manifold of positive isotropic curvature is, in fact, diffeomeorphic to the unit sphere. This is a consequence of a more general result of his on four-mainifolds of positive isotropic

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curvature which implies a diffeomorphic classification of the manifolds with this property under the assumption that their fundamental group is trivial, $\mathbb{Z}_2$, or $\mathbb{Z}$. Other interesting classification results for compact Riemannian manifolds of non-negative isotropic curvature satisfying some additional topological or curvature restrictions have been also obtained in [21], [25], [26], [27], [31], [33].

Let us note that either of the following commonly used curvature conditions implies non-negative (positive) isotropic curvature [20]: (1) M has non-negative (positive) curvature operator; (2) the sectional curvature of M is point-wise (strictly) quarter-pinched. In dimension four, there is a simple necessary and sufficient condition for positive (negative) isotropic curvature [20], [22]: Denote by τ and $\mathcal{W}$: $\Lambda^2 TM \to \Lambda^2 TM$ the scalar curvature and the Weyl curvature operator of M. Then M has non-negative (positive, non-positive, negative) isotropic curvature if and only if the operator $\mathcal{P} = \frac{\tau}{6}\mathrm{Id} - \mathcal{W}$ is non-negative (positive, non-positive, negative). In particular, since $\mathcal{W}$ is a traceless operator, every four-manifold of non-negative (positive) isotropic curvature has non-negative (positive) scalar curvature τ; moreover if $\tau = 0$, then $\mathcal{W} = 0$. (In fact this holds in any dimension [21].) The topological significance of the operator $\mathcal{P}$ on four-manifolds can be seen by means of the Weitzenböck formula for 2-forms: for any 2-form ϕ we have $\Delta\phi = \nabla^*\nabla\phi + 2\mathcal{P}(\phi)$. In particular, if $\mathcal{P}$ is strictly positively defined, i.e., if M is of *positive* isotropic curvature, then there are no harmonic two forms on M, i.e., the second Betti number $b_2(M)$ of M vanishes. In the case that $\mathcal{P}$ is non-negative, so that M is of *non-negative* isotropic curvature, any non-zero harmonic 2-form on M is in fact parallel and thus defines a *Kähler structure* on M (see Section 4).

A simple example of a four-manifold of non-negative (non-positive) isotropic curvature is the product of two Riemann surfaces: it can be easily seen that the product metric has non-negative (non-positive) isotropic curvature if and only if the sum of the Gauss curvatures is non-negative (non-positive). Moreover, on any Kähler surface, the condition of non-negative (non-positive) isotropic curvature can be nicely expressed in terms of the scalar curvature and holomorphic bisectional curvature: Denoting by $\mathcal{P}_\pm$ the restriction of the operator $\mathcal{P} = \frac{\tau}{6}\mathrm{Id} - \mathcal{W}$ on the bundles $\Lambda^2_\pm TM$ of self-dual and anti-self dual 2-vectors, we have the following result.

PROPOSITION 1. *Let (M, g, J) be a Kähler surface. Then:*

(a) *The operator $\mathcal{P}_+$ is non-negative (non-positive) if and only if so is the scalar curvature τ. Moreover, $\mathcal{P}_+(\phi) = 0$ for a $\phi \in \Lambda^2_+ T_pM$, $\phi \neq 0$, if and only if $\tau(p) = 0$ or ϕ is a multiple of the Kähler form of (M, g, J).*

(b) *The operator $\mathcal{P}_-$ is non-negative (non-positive) if and only if the holomorphic bisectional curvature $H(\sigma', \sigma'')$ of any two mutually perpendicular J-invariant planes σ' and σ'' is non-negative (resp. non-positive). Moreover, $\mathcal{P}_-(\phi) = 0$ for a $\phi \in \Lambda^2_- T_pM$, $\phi \neq 0$, if and only if $H(\sigma', \sigma'') = 0$ for any perpendicular J-invariant planes σ' and σ'' which, considered as elements of $\Lambda^2_- T_pM$, are perpendicular to ϕ.*

It thus follows that compact Kähler surfaces of non-negative *holomorphic bisectional* curvature are all of non-negative isotropic curvature. The former have been described by A. Horward and B. Smyth [16] and, partially motivated by this fact, we consider more generally compact four-dimensional Riemannian manifolds (M, g) of non-negative (or non-positive) isotropic curvature under the condition that the conformal class of g admits a compatible complex structure J, i.e., (M, g, J) is a Hermitian surface. We use the topological meaning of the non-negativity of the operator $\mathcal{P}$ explained above together with the Kodaira classification of the complex surfaces to describe the compact Hermitian surfaces of non-negative isotropic curvature. More precisely, we prove the following result.

THEOREM 1. *Any compact Hermitian surface of non-negative isotropic curvature is either biholomorphically isometric to*

(1) *a flat Kählerian torus or*
(2) *a flat Kählerian hyper-elliptic surface or*
(3) $(\mathbb{CP}^1, g_1) \times (\mathbb{CP}^1, g_2)$ *for some Hermitian metrics* g_1 *and* g_2 *on* $\mathbb{CP}^1$ *such that the sum of their Gauss curvatures is non-negative or*
(4) *a unitary flat* $\mathbb{CP}^1$*-bundle over a compact Riemann surface* Σ *of genus* ≥ 1 *with a metric* g *for which there exist Hermitian metrics* g_1 *on* Σ *and* g_2 *on* $\mathbb{CP}^1$ *such that* g *locally is the product* $g_1 \times g_2$ *and the sum of the Gauss curvatures of* g_1 *and* g_2 *is non-negative,*

or the surface is biholomorphic to

(5) *the complex projective space* $\mathbb{CP}^2$ *and the metric is Kähler, or*
(6) *a Hopf surface.*

The next remarks are to show that Theorem 1 is the optimal result that can be obtained.

Remarks.

1. There are a lot of Hermitian metrics $g = g_1 \times g_2$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ of non-negative isotropic curvature. Indeed, let K_1 and K_2 be two smooth functions on $\mathbb{CP}^1$ such that $K_i(x) = K_i(-x)$ for $x \in \mathbb{CP}^1 \cong S^2$, K_i is positive at some point x_i, $i = 1, 2$, and $K_1 + K_2 \geq 0$ everywhere. By a result of J. Moser [23], there exist metrics g_1 and g_2 in the conformal class of the standard metric of $\mathbb{CP}^1$ whose Gauss curvatures are equal to K_1 and K_2, respectively. Then $g_1 \times g_2$ is a Hermitian metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$ of non-positive isotropic curvature.
2. Suppose M is a unitary flat $\mathbb{CP}^1$-bundle over a compact Riemann surface Σ and g_1, g_2 are Hermitian metrics on Σ and $\mathbb{CP}^1$, respectively. Then M admits a Hermitian metric g which locally is the product $g_1 \times g_2$. This follows

from the fact that the group $PSU(2)$ is conjugate in $PSL(2)$ to a (maximal) compact subgroup which contains the group of holomorphic isometries of g_2. It is well-known that every $\mathbb{CP}^1$-bundle over Σ is the projectivization $P(E)$ of a holomorphic vector bundle E of rank 2 over Σ (e.g., see [2]). By the Narasimhan-Seshadri theorem [24], M is unitary flat if and only if $M = P(E)$ for a bundle E which is either stable or else the sum of two line bundles over Σ of the same degree. Note that, by the Gauss-Bonnet theorem, the Gauss curvature of the metric g_2 on $\mathbb{CP}^1$ is everywhere non-negative (positive if Σ is of genus > 1).

3. The Fubini-Study metric on $\mathbb{CP}^2$ has non-negative isotropic curvature which vanishes at a point p only on the two-plane associated with the Kähler form at p. According to Proposition 1, any Kähler metric on $\mathbb{CP}^2$ sufficiently close to the Fubini-Study metric is of non-negative isotropic curvature.
4. It is well-known [7], [8], [28] that some of the Hopf surfaces admit a canonical conformally flat metric of constant positive scalar curvature [35]. Its isotropic curvature is positive and suitably deforming this metric and the complex structure as in [13] one provides plenty of Hopf surfaces with Hermitian metrics of positive isotropic curvature.

We observe finally that the proof of Theorem 1 in fact shows slightly more: *If a compact complex surface M admits a metric (not necessary compatible with the complex structure) whose isotropic curvature is non-negative, then M is diffeomorphic to one of the smooth manifolds underlying the complex surfaces described in* (1)–(6) *of the Theorem* 1.

In Section 5 we consider Hermitian metrics of non-positive isotropic curvature. A simple observation shows that a compact Hermitian surface of non-positive isotropic curvature must be Kähler (Section 5, Lemma 2). A necessary and sufficient condition for non-positivity of the isotropic curvature is then given by Proposition 1; it follows in particular that the isotropic curvature of a compact Hermitian surface can not be strictly negative. Moreover, *Einstein* surfaces of non-positive isotropic curvature can be related via Lemma 2 and Proposition 1 to a still open conjecture of Siu which states that any Kähler-Einstein surface of non-positive bisectional curvature is a compact quotient of the complex hyperbolic space (see [32] and [27] for some results in this direction). Finally, observing that compact *anti-self-dual* Hermitian surfaces of non-positive isotropic curvature are in fact conformally-flat (Section 5, Lemma 3), we use the classification of compact *self-dual* Kähler surfaces [9], [18], [6], [10], [1] to give the full list of compact half-conformally-flat Hermitian surfaces on *non-positive* isotropic curvature (Section 5 Corollary 1).

Classification results concerning Einstein or half-conformally-flat 4-manifolds of *non-negative* isotropic curvature have been given in [21], [33], [27] and [25], respectively. In the particular case of compact Hermitian surfaces, the corresponding lists can be also easily derived from Theorem 1.

2. Preliminaries

Let (M, g) be an oriented Riemannian manifold of dimension four. The Riemannian metric g induces a metric on the bundle $\Lambda^2 TM$ of 2-vectors on M by $g(X_1 \wedge X_2, X_3 \wedge X_4) = \det(g(X_i, X_j))$. Identifying $\Lambda^2 TM$ with the dual bundle $(\Lambda^2 TM)^*$ via this metric, we shall consider any operator on $\Lambda^2 TM$ as an operator on the 2-forms.

The curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of $\Lambda^2 TM$ defined by

$$g(\mathcal{R}(X \wedge Y), Z \wedge W) = g(R(X, Y)Z, W)$$

(for the curvature tensor we adopt the following definition: $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$). Let r: $TM \to TM$ be the Ricci operator and τ the scalar curvature of M. Then the operator $\mathcal{B}$: $\Lambda^2 TM \to \Lambda^2 TM$ defined by

$$\mathcal{B}(X \wedge Y) = \frac{1}{2}\left(r(X) \wedge Y + X \wedge r(Y) - \frac{\tau}{2} X \wedge Y\right) \tag{1}$$

is self-adjoint and traceless. The manifold M is *Einstein* if and only if $\mathcal{B} = 0$. The Weyl curvature operator $\mathcal{W}$: $\Lambda^2 TM \to \Lambda^2 TM$ given by

$$\mathcal{W}(X \wedge Y) = \mathcal{R}(X \wedge Y) - \frac{1}{2}\left(r(X) \wedge Y + X \wedge r(Y) - \frac{\tau}{3} X \wedge Y\right),$$

is also self-adjoint and traceless and we have the splitting

$$\mathcal{R} = \frac{\tau}{12} + \mathcal{B} + \mathcal{W}. \tag{2}$$

The Hodge star operator $*$ defines an involution on $\Lambda^2 TM$ and

$$\mathcal{B} = \frac{1}{2}(\mathcal{R} - *\mathcal{R}*); \quad \mathcal{W} = \frac{1}{2}(\mathcal{R} + *\mathcal{R}*) - \frac{\tau}{12}\mathrm{Id}.$$

The 2-forms of the eigen-spaces $\Lambda^2_+ TM$ and $\Lambda^2_- TM$ of $*$ corresponding to the ± 1-eigenvalues are called self-dual and anti-self-dual forms, respectively. As a consequence of the orthogonal decomposition $\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM$, we have the splitting of $\mathcal{W}$ into two (self-adjoint and traceless operators) $\mathcal{W}_\pm = \frac{1}{2}(\mathcal{W} \pm * \mathcal{W})$. The (2,1)-tensors corresponding to these operators are invariant under conformal changes of the metric and reversing the orientation of the manifold interchanges their roles. Also, observe that the operator $\mathcal{B}$ interchanges $\Lambda^2_+ TM$ and $\Lambda^2_- TM$ while $\mathcal{W}_\pm$ vanishes on $\Lambda^2_\mp TM$.

An oriented Riemannian four manifold is said to be *self-dual* (resp. *anti-self-dual*) if $\mathcal{W}_- = 0$ (resp. $\mathcal{W}_+ = 0$).

If $\{E_1, E_2, E_3, E_4\}$ is a local oriented orthonormal frame of T_pM, $p \in M$, then

$$\phi^1_\pm = \frac{1}{\sqrt{2}}(E_1 \wedge E_2 \pm E_3 \wedge E_4), \; \phi^2_\pm = \frac{1}{\sqrt{2}}(E_1 \wedge E_3 \pm E_4 \wedge E_2),$$

$$\phi_\pm^3 = \frac{1}{\sqrt{2}}(E_1 \wedge E_4 \pm E_2 \wedge E_3), \tag{3}$$

is a local oriented orthonormal frame of $\Lambda_\pm^2 TM$. Moreover, the complex two-plane $\sigma = \mathrm{span}_{\mathbb{C}}\{E_1 + iE_2, E_3 + iE_4\}$ is totally isotropic and any totally isotropic two-plane σ possesses a basis of this form (just take a complex basis $z = E_1 + iE_2$, $w = E_3 + iE_4$ of σ such that $|z| = |w| = \sqrt{2}$ and $< z, w >= 0$). Assume we are given such a basis of an isotropic two-plane σ which yields the orientation of T_pM. The isotropic curvature at σ is then given by

$$2K(\sigma) = g(\mathcal{R}(\phi_+^2), \phi_+^2) + g(\mathcal{R}(\phi_+^3), \phi_+^3) = g((\frac{\tau}{6}\mathrm{Id} - \mathcal{W})(\phi_+^1), \phi_+^1).$$

We also observe that for any non-vanishing self-dual 2-form $\phi \in \Lambda_+^2 T_pM$, there is a positively oriented orthonormal basis of T_pM for which $\frac{\phi}{|\phi|}$ can be written as ϕ_+^1 in (3). It thus follows that the isotropic curvature is non-negative (positive) if and only if the operator

$$\mathcal{P} = \frac{\tau}{6}\mathrm{Id} - \mathcal{W} \tag{4}$$

is non-negative (positive) [20], [22]. In particular, the traceless Ricci operator $\mathcal{B}$ does not effect the isotropic curvature. Since $\mathrm{Trace}(\mathcal{W}) = 0$, the isotropic curvature vanishes identically exactly when $\tau = 0$ and $\mathcal{W} = 0$. Also, observe that if the operator $\mathcal{P}$ is non-negative, positive, non-positive, negative or vanishes, then the scalar curvature τ is also non-negative, positive and so on. It is well-known that the operator $\mathcal{P}$ is noting but the curvature term of the the Weitzenböck formula for 2-forms on M: For any 2-form ϕ we have

$$\Delta\phi = \nabla^*\nabla\phi + 2\mathcal{P}(\phi) \tag{5}$$

where ∇ is the Levi-Civita connection, ∇^* is the formal adjoint operator of ∇ and Δ is the Laplacian of (M, g).

Finally, recall that the so-called *$*$-scalar curvature* τ^* of a Hermitian surface (M, g, J) is defined by

$$\tau^* = \sum_{k,l=1}^{4} g(R(JE_k, E_l)JE_l, E_k)$$

where $\{E_j\}$ is an orthonormal basis. By the first Bianchi identity we have

$$\tau^* = \frac{1}{2}\sum_{k,l=1}^{4} g(R(E_k, JE_k)E_l, JE_l)$$

Denote by $\Omega(X, Y) = g(JX, Y)$ the Kähler form of M. It is well-known (e.g., see [36]) that $d\Omega = \omega \wedge \Omega$ where $\omega = -\delta\Omega \circ J$ is the Lee form of M and the scalar and $*$-scalar curvatures are related by

$$\tau - \tau^* = 2\delta\omega + \|\omega\|^2 \tag{6}$$

Note that on a Kähler surface we have $\tau^* = \tau$.

3. Proof of Proposition 1

(a) It is well-known that the eigenvalues of the operator $\mathcal{W}_+$ on a Kähler surface M are equal to $\frac{\tau}{6}, -\frac{\tau}{12}, -\frac{\tau}{12}$. This fact, which can be traced back to [11], can be seen as follows: Let J be the complex structure of M and $\{E_1, E_2 = JE_1, E_3, E_4 = JE_3\}$ an orthonormal frame. Define $\phi_+^1, \phi_+^2, \phi_+^3$ by (3). Then the decomposition (2), the Kähler curvature identities and the first Bianchi identity imply that $\phi_+^1, \phi_+^2, \phi_+^3$ are eigenvectors of $\mathcal{W}_+$ corresponding to the eigenvalues $\frac{\tau}{6}, -\frac{\tau}{12}, -\frac{\tau}{12}$. Thus $\phi_+^1, \phi_+^2, \phi_+^3$ are eigenvectors of the operator $\mathcal{P}_+$ corresponding to the eigenvalues $0, \frac{\tau}{4}, \frac{\tau}{4}$. In our notation $\sqrt{2}\phi_+^1$ is the the Kähler form of M. In particular, on a Kähler surface, the operator $\mathcal{P}_+$ is non-negative (non-positive) if and only if the scalar curvature τ is non-negative (non-positive); moreover, $\mathcal{P}_+(\sigma) = 0$ for a $\sigma \in \Lambda_+^2 T_pM$, $\sigma \neq 0$, if and only if $\tau(p) = 0$ or σ is a multiple of the Kähler form, which completes the proof of Proposition 1 (a).

(b) Let σ' and σ'' be two perpendicular J-invariant tangent planes at a point $p \in M$. Take an orthonormal basis $\{E_1, E_2 = JE_1, E_3, E_4 = JE_3\}$ of T_pM such that $\sigma' = \mathrm{span}_{\mathbb{R}}\{E_1, E_2\}$, $\sigma'' = \mathrm{span}_{\mathbb{R}}\{E_3, E_4\}$. Let $\phi_\pm^i$, $i = 1, 2, 3$, be define by (3). By the first Bianchi identity and the Kähler curvature identities we have

$$\begin{aligned} 2H(\sigma', \sigma'') &= 2R_{1234} \\ &= 2(R_{13} + R_{14}) \\ &= g(\mathcal{R}(\phi_+^2 + \phi_-^2), \phi_+^2 + \phi_-^2) + g(\mathcal{R}(\phi_+^3 + \phi_-^3), \phi_+^3 + \phi_-^3) \end{aligned}$$

where $H(.,.)$ is the holomorphic sectional curvature, $R_{ijkl} = g(R(E_i, E_j)E_k, E_l)$ and $R_{ij} = R_{ijij}$. Since the Ricci tensor of a Kähler surface is J-invariant, formula (1) implies

$$g(\mathcal{B}(\phi_+^2), \phi_-^2) = 0, \;\; g(\mathcal{B}(\phi_+^3), \phi_-^3) = 0.$$

Now, using (2) and the fact that ϕ_+^2 and ϕ_+^3 are eigenvectors of $\mathcal{W}_+$ corresponding to the eigenvalue $-\frac{\tau}{12}$, we obtain

$$\begin{aligned} 2H(\sigma', \sigma'') &= \frac{\tau}{6} + g(\mathcal{W}_-(\phi_-^2), \phi_-^2) + g(\mathcal{W}_-(\phi_-^3), \phi_-^3) \\ &= \frac{\tau}{6} - g(\mathcal{W}_-(\phi_-^1), \phi_-^1) \\ &= g(\mathcal{P}_-(\phi_-^1), \phi_-^1) \end{aligned}$$

Thus if $\mathcal{P}_-$ is non-negative (non-positive), so is $H(\sigma', \sigma'')$. To prove the converse, take $\phi \in \Lambda^2_- T_pM$, $\phi \neq 0$. Then the formula

$$g(KX, Y) = \frac{\sqrt{2}}{|\phi|}\phi(X, Y), \quad X, Y \in T_pM$$

defines a complex structure K on the vector space T_pM compatible with the metric g and the opposite orientation of T_pM. Since J yields the orientation of T_pM, the operators K and J commute. Then $Q = KJ$ is a symmetric operator whose eigenvalues are equal to ± 1. Let E_1 and E_3 be unit eigenvectors corresponding to the eigenvalues -1 and $+1$, respectively. Thus $JE_1 = KE_1$ and $JE_3 = -KE_3$. Put $E_2 = JE_1$, $E_4 = JE_3$ and define ϕ^i_-, $i = 1, 2, 3$, by (3). Then $\frac{\phi}{|\phi|} = \phi^1_-$. Taking the planes $\sigma' = \mathrm{span}_{\mathbb{R}}\{E_1, JE_1\}$ and $\sigma'' = \mathrm{span}_{\mathbb{R}}\{E_3, JE_3\}$, we have

$$g(\mathcal{P}_-(\phi), \phi) = |\phi|^2 g(\mathcal{P}_-(\phi^1_-), \phi^1_-) = 2|\phi|^2 H(\sigma', \sigma'')$$

and Proposition 1 follows. □

4. Proof of Theorem 1

Here and in the sequel we shall need the following result.

LEMMA 1. *Let (M, h) be a compact four-dimensional Riemannian manifold and g a Riemannian metric in the conformal class of h with (everywhere) non-negative or non-positive isotropic curvature. Then*:

(a) *The metrics h and g cannot have isotropic curvatures of opposite signs.*
(b) *If (M, h) is a Kähler surface, the metric g is homothetic to h.*

Proof of Lemma 1. Let $g = \varphi^2 h$ for a smooth positive function φ. Denote by τ_g and τ_h the scalar curvatures of g and h. Then (e.g., see [3])

$$\varphi^3 \tau_g = 6\Delta_h \varphi + \tau_h \varphi \tag{7}$$

where Δ_h is the Laplacian of h. Since the Weyl tensors of g and h coincide and $g = \varphi^4 h$ on $\Lambda^2 TM$, the corresponding Weyl operators $\mathcal{W}_g$ and $\mathcal{W}_h$ are related by $\mathcal{W}_g = \varphi^{-2}\mathcal{W}_h$. Therefore if $\mathcal{P}_g$ and $\mathcal{P}_h$ are the operators defined by (4), then

$$\mathcal{P}_g = \varphi^{-3}\Delta_h\varphi \cdot \mathrm{Id} + \varphi^{-2}\mathcal{P}_h.$$

Thus for $\phi \in \Lambda^2 TM$, $\phi \neq 0$, we have

$$\Delta_h \varphi = \varphi^3 g(\phi, \phi)^{-1}(g(\mathcal{P}_g(\phi), \phi) - \varphi^2 h(\mathcal{P}_h(\phi), \phi)).$$

(a) If h and g have isotropic curvatures of opposite signs, then $\Delta_h\varphi \geq 0$ or $\Delta_h\varphi \leq 0$, hence $\varphi \equiv \mathrm{const}$ by the maximum principle for sub-harmonic functions.

(b) If (M, h) is a Kähler surface, then $\mathcal{P}_h(\phi) = 0$ for the Kähler form ϕ of (M, h). It again follows that $\Delta_h\varphi \geq 0$ or $\Delta_h\varphi \leq 0$, hence $\varphi \equiv \mathrm{const}$. □

Proof of Theorem 1. Let (M, g, J) be a compact Hermitian surface of nonnegative isotropic curvature. Since $\mathcal{P} \geq 0$ and $\mathrm{Trace}(\mathcal{W}) = 0$, we have $\tau \geq 0$; moreover if $\tau \equiv 0$, then $\mathcal{W} = 0$, i.e., M is conformally flat.

Suppose $\tau \equiv 0$. Then the classification of compact conformally flat Hermitian surfaces due to C. Boyer [7], [8] and M. Pontecorvo [28], and Lemma 1 imply that M is biholomorphically isometric to one of the surfaces listed in the cases (1), (2), (4).

Now assume $\tau \not\equiv 0$. It is well-known that the sign of the scalar curvature is an invariant of the conformal class of g, called its type (this follows from (7)). So, in our case, (M, g) is of positive type. Then, according to [12] (also, see [1, Lemma 3.3]), all plurigenera of M vanish, i.e., M has Kodaira dimension $-\infty$. As usual, denote by $b_2^+(M)$, resp. $b_2^-(M)$, the dimension of the space of self-dual, resp. anti-self-dual, harmonic two-forms on M. We have to consider the following cases:

(1) $b_2^+(M) = 0$, $b_2^-(M) = 0$. It is a well-known result of H.Grauert that every compact complex surface can be obtained from a minimal one by successive blowing ups (e.g., see [2]). It is also well-known that a blowing up at a point decreases $b_2^-(M)$ by one and does not affect $b_1(M)$, $b_2^+(M)$ and the Kodaira dimension. Since $b_2^-(M) = 0$, we infer that M is minimal. Then, by the Kodaira classification of minimal surfaces (e.g., see [2]), M is rational, of class VII or ruled of genus ≥ 1. But, among the latter surfaces, only those of class VII have $b_2^+(M) = 0$. Thus M is a minimal surface of class VII. Hence $b_1(M) = 1$, so $c_2(M) = \chi(M) = 2 - 2b_1(M) - b_2(M) = 0$ and $c_1^2(M) = 3(b_2^+(M) - b_2^-(M)) + 2c_2(M) = 0$. Therefore, by the Bogomolov theorem [4], [5] (e.g., also see [34]), M is biholomorphic either to a Hopf surface or to an Inoue surface. Suppose M is biholomorphic to an Inoue surface. It is a consequence of the considerations in [37] that every Inoue surface is finitely covered (at most doubly) by a solvmanifold S, i.e., a compact quotient of a solvable Lie group by a discrete subgroup. The lift on S of the metric g has non-negative scalar curvature. By a result of M.Gromov and H.B.Lawson [14, Theorem A (and Corollary 4.4)] any metric of non-negative scalar curvature on a solvmanifold is Ricci flat, in particular is scalar flat. Thus, if our surface M is biholomorphic to an Inoue surface, we have $\tau \equiv 0$ while we have assumed $\tau \not\equiv 0$. Therefore M is biholomorphic to a Hopf surface.

(2) $b_2^+(M) = 0$, $b_2^-(M) > 0$. Since $b_2^-(M) > 0$, M admits a non-zero harmonic anti-self-dual form ϕ. By the Weitzenböck formula (5),

$$0 = \int_M g(\Delta\phi, \phi)dV = \int_M g(\nabla\phi, \nabla\phi)dV + 2\int_M g(\mathcal{P}(\phi), \phi)dV.$$

By hypothesis, $g(\mathcal{P}(\phi), \phi) \geq 0$; therefore $\nabla\phi = 0$. Then the formula

$$g(KX, Y) = \frac{\sqrt{2}}{\|\phi\|}\phi(X, Y), \quad X, Y \in TM,$$

defines a complex structure which is Kählerian with respect to g (and yields the opposite orientation of M). Hence $b_1(M)$ is an even number, i.e., (M, J) is of Kähler type and then $b_2^+(M) = 2p_g + 1 > 0$ where $p_g = h^{0,1}$ is the geometric genus, a contradiction.

(3) $b_2^+(M) > 0$, $b_2^-(M) = 0$. In this case (M, g) admits an orthogonal Kähler structure I compatible with the orientation of M. We claim that $I = \pm J$. This fact is an immediate consequence of the more general result [29, Prop. 3.1] and the assumption $\tau \not\equiv 0$, but we prefer to re-prove it in our particular situation. As we have already mentioned, the operator $\mathcal{W}_+$ has eigenvalues $\frac{\tau}{6}, -\frac{\tau}{12}, -\frac{\tau}{12}$. Thus, at each point $p \in M$, either $\mathrm{Spect}(\mathcal{W}_+)_p = 2$ or $\tau(p) = 0$. In particular, the set U of points where $\mathrm{Spect}(\mathcal{W}_+) = 2$ is open. It has been observed by A. Derdzinski [10] and S. Salamon [30] that if $\mathrm{Spect}(\mathcal{W}_+)_p = 2$ at a point p, then, up to sign, there is only one complex structure on T_pM compatible with the metric. Hence $J_p = \pm I_p$. It follows that, on each connected component of the open set U, either $J = I$ or $J = -I$. Thus (g, J) is a Kähler structure on U, hence $\tau = \tau^*$ on U. If $p \in M \setminus U$, then $\tau(p) = 0$ and we have $(\mathcal{W}_+)_p = 0$ since $\mathcal{P}_+ = \frac{\tau}{6}\mathrm{Id} - \mathcal{W}_+$ is a non-negative operator and $\mathrm{Trace}(\mathcal{W}_+) = 0$. It is well known that $g(\mathcal{W}_+(\Omega), \Omega) = \frac{3\tau^* - \tau}{6}$, where Ω is the Kähler form of (M, g, J) ([7, Lemma 2, 2)]) and (6), Section 2). Therefore $\tau^* = \tau = 0$ on $M \setminus U$. Thus $\tau - \tau^* = 0$ at each point of M and, by (6), we have $2\delta\omega + \|\omega\|^2 = 0$ where ω is the Lee form of (M, g, J). Integrating the latter identity on M, we see that $\omega = 0$, so (M, g, J) is a Kähler surface. Since the Kähler forms of both I and J are parallel, we conclude that $I = \pm J$ everywhere on M. Now, since $b_2^-(M) = 0$, the signature $\sigma(M)$ of M is positive, hence M is biholomorphic to $\mathbb{CP}^2$ by the Kodaira classification.

(4) $b_2^+(M) > 0$, $b_2^-(M) > 0$. In this case (M, g) admits two Kähler structures I and K one of which, say I, yields the orientation of M and the other one yields the opposite orientation. The latter property of I and K implies that the operators I and K commute. Since we have assumed that $\tau(p) \neq 0$ for at least one point, we can then see as above that (g, J) is a Kähler structure and $J = \pm I$. For any Kähler surface of Kodaira dimension $-\infty$, we have $b_2^+(M) = 1$. Similarly (M, g, K) is a Kähler surface and we infer that $b_2^-(M) = 1$. Thus the signature $\sigma(M) = 0$ and, by the Kodaira classification, M is biholomorphic to a ruled surface. Moreover, since J commutes with K, it follows that M is locally biholomorphically isometric to the product $(M, g_1) \times (M, g_2)$ of two Riemann surfaces ("folklore"). Indeed, the endomorphism $Q = JK$ is parallel, $Q^2 = \mathrm{Id}$ and $Q \neq \pm\mathrm{Id}$ at each point. Therefore the (real) tangent bundle of M splits into two orthogonal vector sub-bundles of rank 2 corresponding to the eigenvalues ± 1 of Q. They are J-invariant and are invariant under parallel transport which implies our claim. If M is a ruled surface of genus 0 (i.e., a minimal rational surface), then M is simply connected, hence it is globally biholomorphically isometric to the product of two Riemann surface. Each of these surfaces is compact and simply connected, hence biholomorphic to $\mathbb{CP}^1$. Now assume that M is a ruled surface over a compact Riemann surface Σ of genus ≥ 1. Then M is the projectivization $P(E)$ of a holomorphic vector bundle E of rank 2 over Σ (e.g., see [2]). It is well-known that the universal covering of M can be constructed in the following way: Let $\pi\colon X \to \Sigma$ be the (holomorphic) universal covering of Σ. Then X is either the complex plane $\mathbb{C}$ or the unit disk Δ in $\mathbb{C}$, hence the pull-back bundle π^*E over X is (holomorphically) trivial. Thus we have

a trivialization map Φ: $X \times \mathbb{CP}^1 \to \pi^*P(E)$ and composing Φ with the canonical map π^*: $\pi^*P(E) \to P(E)$ we get a covering map $X \times \mathbb{CP}^1 \to P(E) = M$. Denote by $\tilde{g}$ the pull-back on $X \times \mathbb{CP}^1$ of the metric g on M. By the preceding considerations, there is a biholomorphic isometry $F = (F_1, F_2)$ of $(X \times \mathbb{CP}^1, \tilde{g})$ onto the product $(M_1, g_1) \times (M_2, g_2)$ of two Riemann surfaces. Both of M_1 and M_2 are simply connected, so each of them is the unit disk Δ, the complex plane $\mathbb{C}$ or the complex projective line $\mathbb{CP}^1$. Obviously at least one of M_1 and M_2, say M_1, is not compact, so $M_1 = \Delta$ or $M_1 = \mathbb{C}$. Then F_1 does not depend on the $\mathbb{CP}^1$-variable by the maximum principle. It follows that $M_2 = \mathbb{CP}^1$. We also have $M_1 = \Delta$ if $X = \Delta$ and $M_1 = \mathbb{C}$ if $X = \mathbb{C}$ in view of the Liouville theorem. Therefore $X = M_1$ and $(X \times \mathbb{CP}^1, \tilde{g})$ is biholomorphically isometric to $(X, g_1) \times (\mathbb{CP}^1, g_2)$ where g_1 and g_2 are Hermitian metrics. The sum of their Gauss curvatures is non-negative since $\tilde{g}$ is of non-negative isotropic curvature. Every holomorphic automorphism F of $X \times \mathbb{CP}^1$ has the form $F(z, w) = (\varphi(z), \psi(z, w))$. If F is a deck-transformation of the covering $(X, g_1) \times (\mathbb{CP}^1, g_2) \to (M, g)$, then F leaves the metric $g_1 \times g_2$ invariant. It is easy to see that this implies that $\psi(z, w)$ does not depend on z. Hence φ leaves g_1 invariant. If h is a deck-transformation of the covering $X \to \Sigma$, then $\tilde{h}(z, w) = (h(z), m)$, $(z, m) \in X \times P(E)$, is a biholomorphism of $\pi^*P(E)$ and $\Phi \circ \tilde{h} \circ \Phi^{-1}$ is a deck-transformation of $X \times \mathbb{CP}^1$ of the form $(z, w) \mapsto (h(z), \psi(w))$. Therefore g_1 is invariant under every deck-transformation h of the covering $X \to \Sigma$. Thus g_1 descends to a (Hermitian) metric on Σ.

To show that $M = P(E)$ is a unitary flat $\mathbb{CP}^1$-bundle, we shall use arguments suggested in essence by C. LeBrun. The homotopy sequence for the bundle $M = P(E) \to \Sigma$ gives an isomorphism $\pi_1(M) \cong \pi_1(\Sigma)$, so $\pi_1(\Sigma)$ can be considered as the group of deck-transformations of the covering $X \times \mathbb{CP}^1 \to M$. Thus $\pi_1(\Sigma)$ acts on $X \times \mathbb{CP}^1$ by holomorphic isometries $(z, w) \mapsto (\varphi(z), \psi(w))$ of $g_1 \times g_2$ where $z \mapsto \varphi(z)$ is the action on X of $\pi_1(\Sigma)$ considered as the group of deck-transformations of $X \to \Sigma$. Denote by ρ the representation of $\pi_1(\Sigma)$ defined by $w \mapsto \psi(w)$. Then M is isomorphic to the bundle $X \times_\rho \mathbb{CP}^1 \to \Sigma$ associated to the principal bundle π: $X \to \Sigma$ by means of the representation ρ, the holomorphic isomorphism being given by $[x, w] \mapsto \pi^* \circ \Phi(x, w)$, $(x, w) \in X \times \mathbb{CP}^1$. The group of holomorphic isometries of g_2 is a compact subgroup of $PSL(2)$, hence it is contained in a maximal compact subgroup of $PSL(2)$. This subgroup is conjugate to $PSU(2)$ by a biholomorphism f of $\mathbb{CP}^1$. Thus the representation $f\rho f^{-1}$ of $\pi_1(\Sigma)$ takes its values in the group $PSU(2)$; therefore the associated bundle $X \times_{f\rho f^{-1}} \mathbb{CP}^1$ admits a flat projective unitary connection. But the latter bundle is (biholomorphically) isomorphic to the bundle $X \times_\rho \mathbb{CP}^1$ by the map $[x, w] \mapsto [x, f^{-1}(w)]$. $\square$

5. Non-positive isotropic curvature and half conformally flat surfaces

In this section we describe compact half conformally flat Hermitian surfaces of non-negative isotropic curvature.

We start with the following general observations:

LEMMA 2. *Any compact Hermitian surface whose operator* $\mathcal{P}_+ = \frac{\tau}{6}\mathrm{Id} - \mathcal{W}_+$ *is non-positive is Kähler.*

Proof. Denote by Ω the Kähler form of such a surface. As already mentioned in the preceding section, we have $g(\mathcal{W}_+(\Omega), \Omega) = \frac{3\tau^* - \tau}{6}$, hence $g(\mathcal{P}_+(\Omega), \Omega) = \frac{\tau - \tau^*}{2}$. Thus $\tau - \tau^* \leq 0$ and the identity (6) implies $\omega = 0$, i.e., the metric is Kähler. □

LEMMA 3. *Any compact anti-self-dual Hermitian surface of everywhere non-positive isotropic curvature is conformally flat.*

Proof. If the isotropic curvature is non-positive, then the surface is Kähler according to Lemma 2, hence the scalar curvature τ identically vanishes (since the metric is anti-self-dual). Therefore $\mathcal{P} \leq 0$ implies $\mathcal{W} = 0$. □

As a direct consequence of Lemmas 1, 2 and 3 and the classification of compact self-dual Kähler surfaces [9], [18], [6], [10], [1], we obtain the following result.

COROLLARY 1. *Let* M *be a compact half conformally flat Hermitian surface of non-positive isotropic curvature. Then it is biholomorphically isometric to one of the following surfaces*:

(i) *a flat Kähler torus*;
(ii) *a flat Kähler hyper-elliptic surface*;
(iii) *a unitary flat* $\mathbb{CP}^1$*-bundle over a Riemann surface* Σ *of genus* ≥ 2 *with the conformally flat Kähler metric which locally is the product of* $(+1)$*-curvature metric on* $\mathbb{CP}^1$ *and* (-1)*-curvature metric on* Σ;
(iv) *a compact quotient of the unit ball in* $\mathbb{C}^2$ *with the Bergman metric.*

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