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L^2-VON NEUMANN MODULES, THEIR RELATIVE TENSOR PRODUCTS AND THE SPATIAL DERIVATIVE

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ABSTRACT. We develop a theory of L^2-von Neumann modules, which encompasses a reformulation of Connes' Spatial Derivative, and the Relative Tensor Product of Sauvageot. We demonstrate the naturality of the relative tensor product construction in the category of L^2-von Neumann bimodules. Finally, we give evidence for the claim that the relative tensor product is essentially the only tensor product which should be used when considering this tensor category.

1. Introduction

It should come as no surprise (due to their origins) that von Neumann algebras play a role in current Conformal Field Theories. In particular, the tensor category of bimodules over one or several von Neumann algebras is fundamental in their exposition. Hence, it is important to understand the special nuances that arise in considering tensor products of von Neumann algebra bimodules. In general, a purely algebraic approach to their theory is insufficient. Sauvageot [1] outlined a construction for the tensor product (the Relative Tensor Product) of two bimodules which is not canonical, but depends on the choice of a faithful, normal and semi-finite weight. (Hereafter to be referred to as an *fns* weight.) Note that, in the case where the weight is actually a vector state, this choice of weight corresponds, in Field Theory, to fixing a so-called "vacuum vector". Some work subsequent to Sauvageot's in this area has at times neglected the extreme care which is required when dealing with weights. However, it is possible to show that, given a bimodule $\mathfrak{H}$ over a fixed von Neumann algebra, if the existence of another bimodule $\mathfrak{K}$ having certain "universal, tensor product-like" properties is assumed, then $\mathfrak{K}$ is (i.e., is isomorphic to, as $\mathcal{M}$-$\mathcal{M}$ bimodules) the relative tensor product $\mathfrak{H} \otimes_\tau \mathfrak{H}$ *with respect to a trace* τ *on* $\mathcal{M}$. Therefore, we see that the existence of such a bimodule implies that the von Neumann algebra must be semi-finite. Moreover, it turns out that the existence of such a $\mathfrak{K}$, in which the tensor product of any two arbitrary elements is defined, forces the algebra $\mathcal{M}$ to be atomic.

Originally, substantial inroads into the theory were made by Sauvageot. We show herein that the relative tensor product introduced by Sauvageot is, in a sense, the only bimodule tensor product which encompasses the intricacies present when dealing

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with infinite von Neumann algebras. Naively, one would expect the "tensor product" of the $\mathcal{M}$-$\mathcal{M}$ bimodule $L^2(\mathcal{M})$ with itself to also be an $\mathcal{M}$-$\mathcal{M}$ bimodule, possessing the usual universal property of tensor products, viz., that any (continuous) "$\mathcal{M}$-bilinear" map on the Cartesian product should induce an $\mathcal{M}$-bimodule morphism on the tensor product. If we require that an $\mathcal{M}$-bilinear map I include the property that $I(\xi x, \eta) = I(\xi, x\eta)$, $\forall x \in \mathcal{M}$, then we will show that the only $\mathcal{M}$-bimodule tensor product which exhibits the universality described above is the relative tensor product $L^2(\mathcal{M}) \otimes_\tau L^2(\mathcal{M})$, where τ is a trace on the *atomic* von Neumann algebra $\mathcal{M}$. (Once again, recall that the relative tensor product is not canonical, but rather depends on a choice of fns weight.) This result demonstrates that no such universal object can exist when $\mathcal{M}$ is not simply of the form

$$\mathcal{M} = \bigoplus_\alpha \mathcal{L}(\mathfrak{H}_\alpha),$$

where each $\mathfrak{H}_\alpha$ is an arbitrary Hilbert space. Since Type II and Type III algebras can (and often do) arise in physical theories, it is obviously important to be able to decide whether one may assume the existence of a tensor product having the aforementioned characteristics. If the algebra is non-atomic, then it is impossible, in general, to define $\xi \otimes \eta$ for arbitrary ξ, η. This implies that any strictly algebric approach to the theory will necessarily be incomplete. Hence, a satisfactory resolution of this issue is needed.

Interestingly, in formulating a theory of L^2-von Neumann modules, a serendipitous by-product emerges: a clear exposition of the Spatial Derivative, originally introduced by Connes [6]. Suppose we are given a right L^2-module $\mathfrak{H}$ over a von Neumann algebra $\mathcal{N}$, and we denote by $\mathcal{M}$ the von Neumann Algebra $\mathcal{L}(\mathfrak{H}_\mathcal{N})$, i.e., the set of (bounded) operators on $\mathfrak{H}$ which commute with the right action of $\mathcal{N}$. Then, given an fns weight ψ on $\mathcal{N}$ (which induces an fns weight ψ' on $\mathcal{N}^\circ \cong \mathcal{L}_\mathcal{M}(\mathfrak{H})$, the commutant of $\mathcal{M}$ in $\mathcal{L}(\mathfrak{H})$), and a normal, semi-finite weight ϕ on $\mathcal{M}$, the spatial derivative $\frac{\mathrm{d}\phi}{\mathrm{d}\psi'}$ arises naturally in the L^2-module context: it appears as the relative modular operator $\Delta_{\phi,\psi}$. Hence, the L^2-von Neumann module theory incorporates the theory of the spatial derivative.

The Hilbert spaces on which von Neumann algebras act from both the left and right have been referred to as "L^2-von Neumann modules". What should be inferred from this usage is that there exist other types of modules. Indeed, following the work of Lance [3] on "Hilbert C^*-modules", it is possible to define a notion of an *L^∞-von Neumann module.* $\mathcal{E}$ is an L^∞-von Neumann module if it is the dual of a Banach space $\mathcal{E}_*$, and if a von Neumann algebra $\mathcal{M}$ acts on $\mathcal{E}$ (from either the left or the right); additionally, $\mathcal{E}$ should be equipped with an "$\mathcal{M}$-valued inner product". Proceeding in a fashion analagous to the methods used in the L^2 theory, we can characterize L^∞-modules as "sitting in" von Neumann algebras. Moreover, we may develop a tensor product of these modules which respects their module structure. This leads directly to a theory regarding the preduals, which may well be termed an L^1 theory. We may then proceed to a tensor product of these modules. The ultimate goal is to arrive at a satisfactory L^p theory, which would of course encompass all previous results.

Additionally, it is important to note that many of the results contained herein were presaged by Connes; what we refer to as L^2-von Neumann bimodules he called "correspondences". [2] Throughout this work we will sometimes refer to the work of Sauvageot, but not to that of Connes. This is in no way to be interpreted as a diminution of Connes' contribution; it simply reflects the fact that the author was first introduced to the subject via the work of Sauvageot, and his terminology reflects this historical bias.

Finally, the author acknowledges a large debt of graditude to Masamichi Takesaki of UCLA. Most of the results contained herein were obtained in the years 1995–1996, during graduate school. The approach to bimodules, etc., adopted in this paper was inspired by that of Takesaki, and is encapsulated in [5]. Throughout, the influence of his vision is indisputable; the unity of purpose that this vision offers was without doubt one of the most important lessons that the author absorbed during his five years as a graduate student.

2. L^2-von Neumann modules

2.1. *Modules over a von Neumann algebra.* It is commonplace to think about von Neumann algebras presented spatially, i.e., we consider the pair $\{\mathcal{M}, \mathfrak{H}\}$. Then, $\mathfrak{H}$ has a natural stucture as a *left* $\mathcal{M}$-module. We now want to consider a *right* action of a von Neumann algebra on a Hilbert space. Hence, we are motivated to the following:

Definition 2.1.

(i) Given a von Neumann algebra $\mathcal{N}$, the *opposite* von Neumann algebra $\mathcal{N}^\circ$ means the von Neumann algebra obtained by reversing the product in $\mathcal{N}$, i.e., as a linear space equipped with *-operation we take $\mathcal{N}^\circ$ to be $\mathcal{N}$, denote by x° the element in $\mathcal{N}^\circ$ corresponding to $x \in \mathcal{N}$, and then define the product in $\mathcal{N}^\circ$ via

$$x^\circ y^\circ \triangleq (yx)^\circ, \quad \forall\, x, y \in \mathcal{N}. \tag{1}$$

(ii) A *right $\mathcal{N}$-module* is a Hilbert space $\mathfrak{H}$ on which $\mathcal{N}$ acts from the right, i.e., $\mathfrak{H}$ equipped with a normal anti-representation, $\pi'_{\mathfrak{H}}$, of $\mathcal{N}$ on $\mathfrak{H}$; equivalently a Hilbert space equipped with a normal representation of $\mathcal{N}^\circ$. To avoid uninteresting notational complexity, we consider only *faithful* right $\mathcal{N}$-modules $\mathfrak{H}$, in the sense that $\pi'_{\mathfrak{H}}(x) \neq 0$ for every non-zero $x \in \mathcal{N}$. We denote the right $\mathcal{N}$-module $\mathfrak{H}$ by $\mathfrak{H}_{\mathcal{N}}$ to emphasize that $\mathfrak{H}$ is being viewed as a right $\mathcal{N}$-module.

(iii) For a pair $\mathcal{M}$, $\mathcal{N}$ of von Neumann algebras, an $\mathcal{M}$-$\mathcal{N}$ bimodule means a Hilbert space $\mathfrak{H}$, (often denoted ${}_{\mathcal{M}}\mathfrak{H}_{\mathcal{N}}$ to emphasize its bimodule structure), equipped with a normal representation π of $\mathcal{M}$ on $\mathfrak{H}$ and a normal anti-representation π' of $\mathcal{N}$ on $\mathfrak{H}$ such that $\pi(\mathcal{M})$ and $\pi'(\mathcal{N})$ commute. We write

$$x\xi y = \pi(x)\pi'(y)\xi, \quad \forall\, x \in \mathcal{M},\ y \in \mathcal{N}. \tag{2}$$

The commutativity of $\pi(\mathcal{M})$ and $\pi'(\mathcal{N})$ is equivalent to associativity: $x(\xi y) = (x\xi)y$, $x \in \mathcal{M}$ and $y \in \mathcal{N}$. Once again, we will consider only faithful bimodules.

Now, let's fix von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$. If $\mathfrak{H}$ is an $\mathcal{M}$-$\mathcal{N}$ bimodule, then its Banach space dual $\overline{\mathfrak{H}}$ is canonically an $\mathcal{N}$-$\mathcal{M}$ bimodule by the action

$$x\overline{\xi}y \overset{\Delta}{=} \overline{y^*\xi x^*}, \quad x \in \mathcal{M},\ y \in \mathcal{N} \tag{3}$$

where $\overline{\xi}$ denotes the vector in $\overline{\mathfrak{H}}$ corresponding to $\xi \in \mathfrak{H}$ by the pairing $\langle \eta, \overline{\xi}\rangle = (\eta \mid \xi)$, with $\eta \in \mathfrak{H}$ and $\overline{\xi} \in \overline{\mathfrak{H}}$. This left $\mathcal{N}$-module $\overline{\mathfrak{H}}$ will be called the *conjugate* bimodule or the *bimodule dual* to the original bimodule $\mathfrak{H}$.

Of special interest is a von Neumann algebra in standard form. Let us fix an fns weight ψ on $\mathcal{N}$ (so we can and will write $\psi \in \mathfrak{W}_0(\mathcal{N})$), and consider the standard form, which we will denote by $\{L^2(\mathcal{N}), L^2(\mathcal{N})_+, \mathrm{J}\}$. The right action of $\mathcal{N}$ is given by

$$\xi x \overset{\Delta}{=} \mathrm{J}x^*\mathrm{J}\xi, \quad x \in \mathcal{N}. \tag{4}$$

Thus we obtain an $\mathcal{N}$-$\mathcal{N}$ bimodule $L^2(\mathcal{N})$, which will be called the *standard bimodule*. Sometimes, we write ξ^* for $\mathrm{J}\xi$, $\xi \in L^2(\mathcal{N})$. We state here the following easy but important propostion:

PROPOSITION 2.2. *For a von Neumann algebra $\mathcal{N}$, the standard bimodule $L^2(\mathcal{N})$ is self-dual under the correspondence: $\xi^* \leftrightarrow \overline{\xi}, \xi \in L^2(\mathcal{N})$.*

The proof is straightforward, so will be omitted.

With $\psi \in \mathfrak{W}_0(\mathcal{N})$, the left action on $L^2(\mathcal{N})$ is nothing but the semi-cyclic representation π_ψ on $\mathfrak{H}_\psi$. The right action π'_ψ of $\mathcal{N}$ is then given by

$$\pi'_\psi(x) = \mathrm{J}_\psi \pi_\psi(x^*)\mathrm{J}_\psi, \quad x \in \mathcal{N}. \tag{5}$$

Then it follows from the theory of the cocycle Radon–Nikodym derivative (see [5]) that the right action of $\mathcal{N}$ is also given by

$$\eta_\psi(x)b = \eta_\psi(x\sigma^\psi_{-i/2}(b)), \quad x \in \mathfrak{n}_\psi,\ b \in \mathcal{D}(\sigma^\psi_{-i/2}). \tag{6}$$

This twist on the right action suggests that we write $x\psi^{1/2}$ for $\eta_\psi(x)$, $x \in \mathfrak{n}_\psi$, viewing $\psi^{1/2}$ as a vector of infinite magnitude "in" $L^2(\mathcal{N})$. Then (6) can be written more suggestively as

(6') $x\psi^{1/2}b = (x\psi^{1/2}b\psi^{-1/2})\psi^{1/2} = (x\sigma^\psi_{-i/2}(b))\psi^{1/2}, \quad x \in \mathfrak{n}_\psi,\ b \in \mathcal{D}(\sigma^\psi_{-i/2}).$

We now introduce a new notation,

$$\eta'_\psi(x) \stackrel{\Delta}{=} \mathrm{J}_\psi \eta_\psi(x^*), \quad x \in \mathfrak{n}_\psi^*, \tag{7}$$

which can be written as $\psi^{1/2}x$, $x \in \mathfrak{n}_\psi^*$. This new map η'_ψ: $x \in \mathfrak{n}_\psi^* \mapsto \eta'_\psi(x) \in L^2(\mathcal{N})$ allows us to write (5) as simply

$$\pi'_\psi(b)\eta'_\psi(x) = \eta'_\psi(xb) = \eta'_\psi(x)b, \quad x \in \mathfrak{n}_\psi^*,\ b \in \mathcal{N}. \tag{8}$$

We now consider a general right $\mathcal{N}$-module $\mathfrak{H}$. First, given a pair $\{\mathfrak{H}_1, \mathfrak{H}_2\}$ of right $\mathcal{N}$-modules, we define

$$\mathcal{L}((\mathfrak{H}_1)_\mathcal{N}, (\mathfrak{H}_2)_\mathcal{N}) \stackrel{\Delta}{=} \{t \in \mathcal{L}(\mathfrak{H}_1, \mathfrak{H}_2)\colon t(\xi y) = (t\xi)y, \quad y \in \mathcal{N}\}, \tag{9}$$

and for $\mathcal{L}(\mathfrak{H}_\mathcal{N}, \mathfrak{H}_\mathcal{N})$, we shall write $\mathcal{L}(\mathfrak{H}_\mathcal{N})$. With this notation, the right $\mathcal{N}$-module $\mathfrak{H}$ becomes canonically an $\mathcal{L}(\mathfrak{H}_\mathcal{N})$-$\mathcal{N}$ bimodule. Also, we note that $\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}) = \mathcal{N}$ (a direct consequence of Tomita-Takesaki theory)—a fact that will be used throughout. For the pair $\{\mathfrak{H}_1, \mathfrak{H}_2\}$, we shall also consider the direct sum right $\mathcal{N}$-module $\mathfrak{H}_\mathcal{N} = (\mathfrak{H}_1)_\mathcal{N} \oplus (\mathfrak{H}_2)_\mathcal{N}$; if we let e_1 and e_2 denote the projections of $\mathfrak{H}$ down to $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively, then we have $\mathcal{L}((\mathfrak{H}_1)_\mathcal{N}, (\mathfrak{H}_2)_\mathcal{N}) = e_2\mathcal{L}(\mathfrak{H}_\mathcal{N})e_1$.

Now let $\{\mathcal{M}, \mathfrak{H}\}$ be a von Neumann algebra. We want to study the relation between a semi-finite, normal weight φ on $\mathcal{M}$ and an fns weight ψ' on $\mathcal{M}'$. Set $\mathcal{N} = (\mathcal{M}')^\circ$, which allows us to view $\mathfrak{H}$ as an $\mathcal{M}$-$\mathcal{N}$ bimodule. Let ψ be the weight on $\mathcal{N}$ defined by

$$\psi(y) \stackrel{\Delta}{=} \psi'(y^\circ), \quad y \in \mathcal{N}_+.$$

We first pair the von Neumann algebra $\{\mathcal{M}, \mathfrak{H}\}$ with one in standard form, in the following manner: let $\tilde{\mathfrak{H}} = L^2(\mathcal{N}) \oplus \mathfrak{H}$ as a right $\mathcal{N}$-module. Then, set $\mathcal{R} = \mathcal{L}(\tilde{\mathfrak{H}}_\mathcal{N})$. It is easy to verify that $\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}) = f\mathcal{R}e$, where e and f are the projections of $\tilde{\mathfrak{H}}$ onto $L^2(\mathcal{N})$ and $\mathfrak{H}$, respectively. The semi-finite, normal weights ψ on $\mathcal{N}$ and φ on $\mathcal{M}$ give rise to a semi-finite, normal weight ρ on $\mathcal{R}$ given by

$$\rho(x) \stackrel{\Delta}{=} \psi(exe) + \varphi(fxf), \quad x \in \mathcal{R}.$$

We set

$$\begin{aligned} \mathfrak{n}_\psi(\mathfrak{H}) &= f\mathfrak{n}_\rho e = \{t \in \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})\colon \psi(t^*t) < +\infty\}; \\ \mathfrak{D}(\mathfrak{H}, \psi) &= \{\xi \in \mathfrak{H}\colon \|\xi x\| \le C_\xi \|\eta'_\psi(x)\|, \quad x \in \mathfrak{n}_\psi^* \text{ for some } C_\xi \ge 0\}. \end{aligned} \tag{10}$$

Observe that each $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$ gives rise to an operator, denoted $\mathrm{L}_\psi(\xi)$, which belongs to $\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})$; it is defined by the equation

$$\mathrm{L}_\psi(\xi)\eta'_\psi(x) \stackrel{\Delta}{=} \xi x, \quad x \in \mathfrak{n}_\psi^*,\ \xi \in \mathfrak{D}(\mathfrak{H}, \psi). \tag{11}$$

LEMMA 2.3.

(i)

$$\mathfrak{n}_\psi(\mathfrak{H}) = \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})\mathfrak{n}_\psi$$

and

$$\mathfrak{D}(\mathfrak{H}, \psi) = \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})\mathfrak{B}_\psi,$$

where $\mathfrak{B}_\psi = \eta_\psi(\mathfrak{n}_\psi) \subset L^2(\mathcal{N})$.

(ii) *The map* $t \otimes_\mathcal{N} y \in \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}) \bigodot_\mathcal{N} \mathfrak{n}_\psi \mapsto t\eta_\psi(y) \in \mathfrak{D}(\mathfrak{H}, \psi)$ *gives rise to a map, denoted* $\tilde{\eta}_\psi$, *from* $\mathfrak{n}_\psi(\mathfrak{H})$ *onto* $\mathfrak{D}(\mathfrak{H}, \psi)$ *such that*

$$\begin{aligned} \tilde{\eta}_\psi(at) &= a\tilde{\eta}_\psi(t), \quad a \in \mathcal{M},\ t \in \mathfrak{n}_\psi(\mathfrak{H}); \\ \tilde{\eta}_\psi(t\sigma^\psi_{-i/2}(b)) &= \tilde{\eta}_\psi(t)b, \quad t \in \mathfrak{n}_\psi(\mathfrak{H}), \quad b \in \mathcal{D}(\sigma^\psi_{-i/2}). \end{aligned} \tag{12}$$

Here, $\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}) \bigodot_\mathcal{N} \mathfrak{n}_\psi$ *represents the algebraic tensor product of the (algebraic) right* $\mathcal{N}$*-module* $\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})$ *and the (algebraic) left* $\mathcal{N}$*-module* $\mathfrak{n}_\psi$.

(iii) $\mathfrak{D}(\mathfrak{H}, \psi)$ *is dense in* $\mathfrak{H}$.

(iv) *The maps* $L_\psi\colon \xi \in \mathfrak{D}(\mathfrak{H}, \psi) \mapsto L_\psi(\xi) \in \mathfrak{n}_\psi(\mathfrak{H})$ *and* $\tilde{\eta}_\psi\colon t \in \mathfrak{n}_\psi(\mathfrak{H}) \mapsto \tilde{\eta}_\psi(t) \in \mathfrak{D}(\mathfrak{H}, \psi)$ *are the inverse of each other.*

(v)

$$\text{(12')} \qquad L_\psi(\xi\sigma^\psi_{i/2}(b)) = L_\psi(\xi)b, \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi),\ b \in \mathcal{D}(\sigma^\psi_{i/2}).$$

(vi) *With the semi-finite, normal weight* $\overline{\psi}$ *on* $\mathcal{R}$ *defined by* $\overline{\psi}(x) \triangleq \psi(exe) = \rho(exe)$, $x \in \mathcal{R}_+$, *we have* $\mathfrak{n}_{\overline{\psi}} = \mathfrak{n}_\psi \oplus \mathfrak{n}_\psi(\mathfrak{H}) \oplus \mathcal{R}f$, *with* $\mathcal{R}f \subset N_{\overline{\psi}}$, *where* $N_{\overline{\psi}}$ *means the left kernel of* $\overline{\psi}$ *(i.e.,* $\{y \in \mathcal{N}\colon \overline{\psi}(y^*y) = 0\}$*). Moreover, the action of* $\mathcal{R}$ *on* $\tilde{\mathfrak{H}}$ *is semi-cyclic relative to the semi-finite normal weight* $\overline{\psi}$ *under the identification* $\eta_{\overline{\psi}}((x, t, 0)) \in \mathfrak{H}_{\overline{\psi}} \leftrightarrow (\eta_\psi(x), \tilde{\eta}_\psi(t)) \in L^2(\mathcal{N}) \oplus \mathfrak{H}$, $\quad x \in \mathfrak{n}_\psi, t \in \mathfrak{n}_\psi(\mathfrak{H})$.

Proof.

(i) If t is in $\mathfrak{n}_\psi(\mathfrak{H})$, then the absolute value $|t|$ belongs to $\mathfrak{n}_\psi$ by definition; let us consider the polar decomposition of t, viz., $t = u|t|$. Then it becomes clear that $t \in \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})\mathfrak{n}_\psi$. Conversely, if $a \in \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})$ and $x \in \mathfrak{n}_\psi$, then the inequality $x^*a^*ax \leq \|a\|^2x^*x$ implies that $ax \in \mathfrak{n}_\psi(\mathfrak{H})$.

If $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$, then $a = L_\psi(\xi)$ belongs to $\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})$, and with the polar decomposition $a = u|a|$ it is clear that $|a|$ belongs to $\mathfrak{n}_\psi$. Now we make use of the fact that, for any $x \in \mathfrak{n}_\psi$, $y \in \mathfrak{n}_\psi^*$, we have $x\eta'_\psi(y) = \eta_\psi(x)y$. (Note that this follows since $\eta_\psi(x)$ and $\eta'_\psi(y)$ are, respectively, left and right bounded vectors in the left Hilbert algebra which is the completion of $\eta_\psi(\mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*)$.) Hence, for any $y \in \mathfrak{n}_\psi^*$, $u\eta_\psi(|a|)y = u|a|\eta'_\psi(y) = a\eta'_\psi(y) = \xi y$; hence, we may conclude $u\eta_\psi(|a|) = \xi$. Similarly, the reverse argument shows that any element of the form $t\eta_\psi(x)$, with $t \in \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})$, $x \in \mathfrak{n}_\psi$ is in fact in $\mathfrak{D}(\mathfrak{H}, \psi)$.

(ii) In order to show that the map $t \otimes_\mathcal{N} y \in \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}) \bigodot_\mathcal{N} \mathfrak{n}_\psi \mapsto t\eta_\psi(y) \in \mathfrak{D}(\mathfrak{H}, \psi)$ factors through $\mathfrak{n}_\psi(\mathfrak{H})$, we must show that whenever

$$\sum_i t_i x_i = 0, \tag{13}$$

with $\{t_i\} \subset \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})$, $\{x_i\} \subset \mathfrak{n}_\psi$, we have

$$\sum_i t_i \eta_\psi(x_i) = 0. \tag{14}$$

To see that this is indeed so, suppose (13) held but (14) did not. Then, there must exist a $y \in \mathfrak{n}_\psi^*$, such that

$$\left(\sum_i t_i \eta_\psi(x_i)\right) y \neq 0.$$

But

$$\left(\sum_i t_i \eta_\psi(x_i)\right) y = \sum_i t_i x_i \eta'_\psi(y) = 0,$$

a contradiction. Thus, the map $\tilde{\eta}_\psi$ from $\mathfrak{n}_\psi(\mathfrak{H}) \to \mathfrak{D}(\mathfrak{H}, \psi)$ is well-defined. The rest follows easily by calculation.

(iii) From (i) it follows that

$$[\mathfrak{D}(\mathfrak{H}, \psi)] = [\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})\mathfrak{B}_\psi] = [\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})L^2(\mathcal{N})].$$

Let $\xi \in \mathfrak{H}$. Consider $\omega = \omega_\xi$ as a functional over $\mathcal{N}$, and let $\xi(\omega)$ be the representing vector in $L^2(\mathcal{N})_+$ of ω for the *right* action of $\mathcal{N}$ on $L^2(\mathcal{N})$, i.e., $\langle \omega, x\rangle = (\xi(\omega)x \mid \xi(\omega))$, for $x \in \mathcal{N}$. Then we have a partial isometry u in $\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})$ such that $u\xi(\omega) = \xi$. Hence, $\mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})L^2(\mathcal{N}) = \mathfrak{H}$; this implies that $\mathfrak{D}(\mathfrak{H}, \psi)$ is dense in $\mathfrak{H}$.

(iv) This fact actually follows quite quickly from (i) and (ii): let t be an element of $\mathfrak{n}_\psi(\mathfrak{H})$, and let $\xi = \tilde{\eta}_\psi(t)$. Once again, we take $t = u|t|$ its polar decomposition. Then for any $y \in \mathfrak{n}_\psi^*$, we have

$$\begin{aligned} \mathrm{L}_\psi(\xi)\eta'_\psi(y) = \tilde{\eta}_\psi(t)y &= u\eta_\psi(|t|)y \\ &= u|t|\eta'_\psi(y) = t\eta'_\psi(y), \end{aligned}$$

which says that $\mathrm{L}_\psi(\tilde{\eta}_\psi(t)) = t$. Conversely, we may use the same argument to conclude that, given any ξ in $\mathfrak{D}(\mathfrak{H}, \psi)$, by defining $t = \mathrm{L}_\psi(\xi)$, we get $\tilde{\eta}_\psi(t) = \xi$.

(v) This follows from (ii), (iv) and results from the theory of cocycle derivatives. (See [5].)

(vi) This assertion follows from a routine calculation of the actions of $\mathcal{R}$ on $\tilde{\mathfrak{H}}$ and $\mathfrak{H}_{\overline{\psi}}$. □

Now, it is a fundamental fact of Tomita-Takesaki theory that $\mathfrak{a}_\psi = \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$, or more precisely its image $\eta_\psi(\mathfrak{a}_\psi)$, forms a left Hilbert algebra. Likewise, $\mathcal{A} = \mathcal{D}(\sigma_{-i/2}^\psi) \cap \mathcal{D}(\sigma_{i/2}^\psi) = \mathcal{D}(\sigma_{i/2}^\psi) \cap \mathcal{D}(\sigma_{i/2}^\psi)^*$ is a self-adjoint subalgebra of $\mathcal{N}$ which multiplies $\mathfrak{a}_\psi$ and $\mathfrak{n}_\psi^*$ from both sides. We then have the following tautological statement:

LEMMA 2.4. *The anti-representation π_ψ' of $\mathcal{A}$ defined by*

$$\pi_\psi'(b)\tilde{\eta}_\psi(t) \stackrel{\Delta}{=} \tilde{\eta}_\psi(t\sigma_{-i/2}^\psi(b)), \quad t \in \mathfrak{n}_\psi(\mathfrak{H}),\ b \in \mathcal{A}, \tag{12''}$$

extends to the original right action of $\mathcal{N}$ on $\mathfrak{H}$.

Proof. This assertion follows directly from (6), (12) and (12'). □

We now continue our investigation of the action of $\mathcal{R}$ on $\tilde{\mathfrak{H}}$. The direct sum decomposition, $\tilde{\mathfrak{H}} = L^2(\mathcal{N}) \oplus \mathfrak{H}$, yields the following matrix representation of $\mathcal{R}$:

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

with

$$x_{11} \in \mathcal{N},\ x_{12} \in \mathcal{L}(\mathfrak{H}_\mathcal{N}, L^2(\mathcal{N})_\mathcal{N}),$$
$$x_{21} \in \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}),\ x_{22} \in \mathcal{M} = \mathcal{L}(\mathfrak{H}_\mathcal{N})$$

for each $x \in \mathcal{R}$.

Notice that we have not yet made use of the semi-finite, normal weight $\varphi \in \mathfrak{W}(\mathcal{M})$; all our considerations thus far have involved only $\psi \in \mathfrak{W}_0(\mathcal{N})$. We recall that the "balanced" weight $\rho = \psi \oplus \varphi$ on $\mathcal{R}$ gives a semi-cyclic representation $\{\pi_\rho, \mathfrak{H}_\rho\}$ of $\mathcal{R}$. We wish to characterize the representation π_ρ in terms of $\mathfrak{H}$ and π_φ. To do this, we consider the weights $\overline{\psi}$ and $\overline{\varphi}$ on $\mathcal{R}$ given by $\overline{\psi}(x) \stackrel{\Delta}{=} \psi(exe)$ and $\overline{\varphi}(x) \stackrel{\Delta}{=} \varphi(fxf)$, $x \in \mathcal{R}_+$. We then obtain the decomposition

$$\mathfrak{H}_\rho = \left[\begin{pmatrix} \eta_\rho(e\mathfrak{n}_\rho e) & \eta_\rho(e\mathfrak{n}_\rho f) \\ \eta_\rho(f\mathfrak{n}_\rho e) & \eta_\rho(f\mathfrak{n}_\rho f) \end{pmatrix}\right] \cong \begin{pmatrix} L^2(\mathcal{N}) & \oplus & [\eta_\rho(e\mathfrak{n}_\rho f)] \\ \mathfrak{H} & \oplus & \mathfrak{H}_\varphi \end{pmatrix}, \tag{15}$$

where $[\cdots]$ stands for the closure in the Hilbert space of the linear span, as usual. We have already seen that $\eta_\rho(f\mathfrak{n}_\rho e) = \tilde{\eta}_\psi(\mathfrak{n}_\psi(\mathfrak{H})) = \mathfrak{D}(\mathfrak{H}, \psi)$. In addition, we know $f\mathcal{R}e = \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N})$ and $e\mathcal{R}f = \mathcal{L}(\mathfrak{H}_\mathcal{N}, L^2(\mathcal{N})_\mathcal{N})$. We now want to investigate $e\mathfrak{n}_\rho f$ and its image under the map η_ρ. As we did not assume the faithfulness of φ, we don't have complete symmetry between φ and ψ. At any rate, we do have

$$\begin{aligned} e\mathfrak{n}_\rho f &= \{s \in \mathcal{L}(\mathfrak{H}_\mathcal{N}, L^2(\mathcal{N})_\mathcal{N})): \varphi(s^*s) < +\infty\} \\ &= \{t^*: t \in \mathcal{L}(L^2(\mathcal{N})_\mathcal{N}, \mathfrak{H}_\mathcal{N}), \varphi(tt^*) < +\infty\}. \end{aligned} \tag{10'}$$

Given the decomposition described by (15), it is natural to define

$$\begin{matrix} \mathfrak{H}_{11} = L^2(\mathcal{N}), & \mathfrak{H}_{12} = [\eta_\rho(e\mathfrak{n}_\rho f)] \\ \mathfrak{H}_{21} = \mathfrak{H}, & \mathfrak{H}_{22} = \mathfrak{H}_\varphi \end{matrix} .$$

We conclude this section with a lemma which indicates the relationship between the weights φ and ψ, at least on the level of their semi-cyclic representation spaces.

LEMMA 2.5.

(i) *The restriction of π_ρ to the second column space of (15), $\mathfrak{H}_{12} \oplus \mathfrak{H}_{22}$, is semi-cyclic relative to the weight $\overline{\varphi}$.*
(ii) *The Hilbert space $\mathfrak{H}_{12}$ is isomorphic to $\overline{s(\varphi)\mathfrak{H}_{21}}$ (and hence $\cong \overline{s(\varphi)\mathfrak{H}}$) as an $\mathcal{N}$-$\mathcal{M}_{s(\varphi)}$ bimodule under the natural map.*

Proof. First consider the case when φ is faithful. Then with $\mathfrak{a}_\rho = \mathfrak{n}_\rho \cap \mathfrak{n}_\rho^*$, $\mathfrak{A}_\rho = \eta_\rho(\mathfrak{a}_\rho)$ is a left Hilbert algebra. Furthermore, the $\mathcal{R}$-$\mathcal{R}$ bimodule $L^2(\mathcal{R})$ can be naturally identified with $\mathfrak{H}_\rho$. Under this identification, the components of $\mathfrak{H}_\rho$ defined in (15) allow us to write

$$\begin{aligned} \mathfrak{H}_{11} &= eL^2(\mathcal{R})e, \quad \mathfrak{H}_{12} = eL^2(\mathcal{R})f, \\ \mathfrak{H}_{21} &= fL^2(\mathcal{R})e, \quad \mathfrak{H}_{22} = fL^2(\mathcal{R})f. \end{aligned}$$

The modular conjugation J implements the desired isomorphism between $\overline{\mathfrak{H}_{21}}$ and $\mathfrak{H}_{12}$. This gives assertion (ii). Assertion (i) follows from the symmetry between ψ on $\mathcal{N}$ and φ° on $\mathcal{M}^\circ$.

In the general case, (i.e., if φ is not faithful), we consider an auxiliary semi-finite, normal weight φ' on $\mathcal{M}$ with $s(\varphi') = \mathbf{1}_\mathcal{M} - s(\varphi)$. We then define

$$q \triangleq \begin{pmatrix} 0 & 0 \\ 0 & s(\varphi) \end{pmatrix}, \qquad p \triangleq \begin{pmatrix} 0 & 0 \\ 0 & s(\varphi') \end{pmatrix},$$

i.e., $p = f - q$. We can now form an fns weight $\rho' \triangleq \rho + \overline{\varphi'}$ on $\mathcal{R}$, where $\overline{\varphi'}$ is defined via $\overline{\varphi'}(x) \triangleq \varphi'(pxp)$, $x \in \mathcal{R}_+$. Observe that η_ρ and $\eta_{\rho'}$ agree on $e\mathcal{R}e$ and $f\mathcal{R}e$ and that $\eta_\rho(x) = \eta_{\rho'}(xq)$, $x \in \mathfrak{n}_\rho q$. Hence we get $\mathfrak{H}_{11} = eL^2(\mathcal{R})e$, $\mathfrak{H}_{21} = fL^2(\mathcal{R})e$, $\mathfrak{H}_{12} = eL^2(\mathcal{R})q$ and $\mathfrak{H}_{22} = fL^2(\mathcal{R})q$. So, we may conclude $\mathrm{J}q\mathfrak{H}_{21} = \mathrm{J}qL^2(\mathcal{R})e = eL^2(\mathcal{R})q = \mathfrak{H}_{12}$. This completes the proof of (i). For (ii), we have $\mathfrak{H}_{12} \oplus \mathfrak{H}_{22} = L^2(\mathcal{R})q$ as an $\mathcal{N}$-$\mathcal{M}_q$ bimodule. Therefore, the representation $\{\pi_\rho, \mathfrak{H}_{12} \oplus \mathfrak{H}_{22}\}$ is precisely the semi-cyclic representation $\{\pi_{\overline{\varphi}}, \mathfrak{H}_{\overline{\varphi}}, \eta_{\overline{\varphi}}\}$. □

2.2. *The Spatial Derivative.* If we examine the details of the preceding proof, we recognize that the conjugation operator J: $L^2(\mathcal{R}) \to L^2(\mathcal{R})$ restricts to the conjugate linear operator $S_{\varphi,\psi}$: $\tilde{\eta}_\psi(t) \in f(\mathfrak{n}_\rho \cap \mathfrak{n}_\rho^*)e \mapsto \eta_\rho(t^*) \in \mathfrak{H}_{12}$, which can also be viewed as the restriction of the map $S_{\varphi+\varphi',\psi}$ for ρ' to the smaller domain $qL^2(\mathcal{R})e$. Hence, $S_{\varphi,\psi}$ can be defined directly as the closure of the operator given by $\tilde{\eta}_\psi(t) \mapsto \eta_\rho(t^*)$ for $t \in \mathfrak{n}_\psi(\mathfrak{H}) \cap \mathfrak{n}_\varphi(\mathfrak{H})^*$, where $\mathfrak{n}_\varphi(\mathfrak{H}) \overset{\Delta}{=} \{s \in \mathcal{L}(\mathfrak{H}_\mathcal{N}, L^2(\mathcal{N})_\mathcal{N})\colon \varphi(s^*s) < +\infty\}$. Thus we make the following definition:

Definition 2.6. The absolute value $\Delta_{\varphi,\psi}$ of $S_{\varphi,\psi}$ is called the *spatial derivative* of the semi-finite, normal weight φ on $\mathcal{M}$ relative to the fns weight ψ' on the commutant $\mathcal{M}'$, and is denoted $\frac{d\varphi}{d\psi'}$, since it is determined by φ on $\mathcal{M}$ and ψ' on $\mathcal{M}'$.

Dualizing (10), we set

$$(10'') \qquad \mathfrak{D}'(\mathfrak{H}, \varphi) \overset{\Delta}{=} \{\xi \in \mathfrak{H}\colon \|x\xi\|^2 \leq C_\xi \varphi(x^*x), x \in \mathfrak{n}_\varphi, \text{ for some } C_\xi \geq 0\}.$$

To each $\xi \in \mathfrak{D}'(\mathfrak{H}, \varphi)$ there corresponds an operator $R_\varphi(\xi)$ defined by

$$R_\varphi(\xi)\eta_\varphi(x) \overset{\Delta}{=} x\xi, \quad x \in \mathfrak{n}_\varphi,$$

which belongs to $\mathcal{L}({}_\mathcal{M}L^2(\mathcal{M}), {}_\mathcal{M}\mathfrak{H})$. As φ is not assumed to be faithful, ψ and φ are not symmetric. In fact, we have the following:

LEMMA 2.7. *The closure of* $\mathfrak{D}'(\mathfrak{H}, \varphi)$ *is the range of the projection* $s(\varphi)$, *i.e.*, $[\mathfrak{D}'(\mathfrak{H}, \varphi)] = s(\varphi)\mathfrak{H}$.

Proof. If $\xi \in \mathfrak{D}'(\mathfrak{H}, \varphi)$, then we have $\|(1 - s(\varphi))\xi\|^2 \leq C_\xi \varphi((1 - s(\varphi))) = 0$. Hence $\mathfrak{D}'(\mathfrak{H}, \varphi) \subset s(\varphi)\mathfrak{H}$. Conversely, suppose $\xi \perp \mathfrak{D}'(\mathfrak{H}, \varphi)$. With $\Phi = \{\omega \in \mathcal{M}_*^+\colon \omega \leq \varphi\}$, we know $\varphi(x) = \sup_{\omega \in \Phi} \omega(x)$, $x \in \mathcal{M}_+$, and

$$\eta \in \mathfrak{D}'(\mathfrak{H}, \varphi) \iff \omega_\eta \in \bigcup_{C>0} C\Phi.$$

Also every $\omega \in \Phi$ can be written as a countable sum of ω_{η_n} with $\eta_n \in \mathfrak{D}'(\mathfrak{H}, \varphi)$, so that $s(\varphi) = \sup\{s(\omega_\eta)\colon \eta \in \mathfrak{D}'(\mathfrak{H}, \varphi)\}$; thus we may conclude $s(\varphi)\xi = 0$. □

We now state the main result of this section. Note that the spatial derivative was originally defined by Connes [6]; however, his approach did not use (explicitly) the notions of von Neumann bimodules.

THEOREM 2.8. *Let* $\{\mathcal{M}, \mathfrak{H}\}$ *be a von Neumann algebra,* φ *a semi-finite, normal weight on* $\mathcal{M}$ *and* ψ' *an fns weight on the commutant* $\mathcal{M}'$. *Then the spatial derivative*

$\frac{d\varphi}{d\psi'}$ *has the following properties*:

(i) *The support* $s(\frac{d\varphi}{d\psi'})$ *of the spatial derivative* $\frac{d\varphi}{d\psi'}$ *is equal to* $s(\varphi)$. *Note that here what is meant by the* support of a self-adjoint operator *is the projection to the closure of its range.*

(ii) *On the reduced von Neumann algebra* $\{\mathcal{M}_{s(\varphi)}, s(\varphi)\mathfrak{H}\}$ *and its commutant* $\mathcal{M}'_{s(\varphi)}$, *we have*

$$\left.\begin{aligned}\left(\tfrac{d\varphi}{d\psi'}\right)^{it} x \left(\tfrac{d\varphi}{d\psi'}\right)^{-it} &= \sigma_t^{\varphi}(x), \quad x \in \mathcal{M}_{s(\varphi)},\\ \left(\tfrac{d\varphi}{d\psi'}\right)^{it} y \left(\tfrac{d\varphi}{d\psi'}\right)^{-it} &= \sigma_{-t}^{\psi'}(y), \quad y \in \mathcal{M}'_{s(\varphi)}.\end{aligned}\right\} \tag{16}$$

(iii) *If* φ_1 *and* φ_2 *are fns weights on* $\mathcal{M}$, *then*

$$\left(\frac{d\varphi_2}{d\psi'}\right)^{it} = (D\varphi_2 \colon D\varphi_1)_t \left(\frac{d\varphi_1}{d\psi'}\right)^{it}. \tag{17}$$

(iv) *If* φ *is faithful, then*

$$\frac{d\psi'}{d\varphi} = \left(\frac{d\varphi}{d\psi'}\right)^{-1}. \tag{18}$$

(v) *With* $\mathcal{N} = (\mathcal{M}')^{\circ}$ *and* $\psi = (\psi')^{\circ}$, *the square root of the spatial derivative,* $(\frac{d\varphi}{d\psi'})^{\frac{1}{2}}$, *is essentially self-adjoint on*

$$\mathfrak{D}_{\varphi,\psi}(\mathfrak{H}) \stackrel{\Delta}{=} \{\xi \in \mathfrak{D}(\mathfrak{H}, \psi) \colon L_\psi(\xi)^* \in \mathfrak{n}_\varphi(\mathfrak{H})\}$$

and is determined by

$$\left(\left(\frac{d\varphi}{d\psi'}\right)^{\frac{1}{2}} \xi \mid \left(\frac{d\varphi}{d\psi'}\right)^{\frac{1}{2}} \eta\right) = \varphi(L_\psi(\xi)L_\psi(\eta)^*), \quad \xi,\ \eta \in \mathfrak{D}_{\varphi,\psi}(\mathfrak{H}). \tag{19}$$

Therefore, the spatial derivative $\frac{d\varphi}{d\psi'}$ *of* φ *relative to* ψ' *is directly computable from* φ *and* ψ'. (*Again, see* [6])

Proof. From the previous arguments involving $\mathcal{M}$, $\mathcal{N}$, φ and ψ, we know that the spatial derivative $\frac{d\varphi}{d\psi'}$ is precisely the relative modular operator $\Delta_{\varphi,\psi}$ on the subspace $s(\varphi)\mathfrak{H}$, when we replace $\mathfrak{H}$ by $s(\varphi)\mathfrak{H}$ and assume that φ is faithful. Then the assertions (i) through (v) are really statements about the relative modular operator; all of these are standard results in the theory of the cocycle derivative. (For example, see [4].) □

Again, we wish to emphasize that, via (19), the spatial derivative $\frac{d\varphi}{d\psi'}$ is completely determined by the weights φ and ψ', without making use of the auxiliary von Neumann algebra $\mathcal{R}$. We will now investigate additional properties of the spatial derivative; we begin with a lemma.

LEMMA 2.9. *The linear span* $\mathcal{J}_\psi$ *of* $\{L_\psi(\xi)L_\psi(\eta)^*\colon\ \xi,\ \eta \in \mathfrak{D}(\mathfrak{H}, \psi)\}$ *is a* σ*-weakly dense ideal of* $\mathcal{M}$*; moreover, we have*

$$\mathcal{J}_\psi^+ = \left\{ \sum_{i=1}^n L_\psi(\xi_i)L_\psi(\xi_i)^*\colon\ \xi_i \in \mathfrak{D}(\mathfrak{H}, \psi),\ i = 1, \ldots, n \right\}.$$

Proof. It is easy to see that $\mathrm{L}_\psi(a\xi) = a\mathrm{L}_\psi(\xi)$ for any $a \in \mathcal{M}$ and $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. Hence $\mathcal{J}_\psi$ is an ideal of $\mathcal{M}$. The characterization of the positive cone is accomplished by using polariztion, which is a standard technique, so we omit that portion of the argument.

To demonstrate the σ-weak density of $\mathcal{J}_\psi$, it is sufficient to prove that, if $a\mathfrak{n}_\psi(\mathfrak{H}) = \{0\}$, $a \in \mathcal{M}$, then $a = 0$, since $\mathrm{L}_\psi(\mathfrak{D}(\mathfrak{H}, \psi)) = \mathfrak{n}_\psi(\mathfrak{H})$. So, suppose $a\mathfrak{n}_\psi(\mathfrak{H}) = \{0\}$ for some $a \in \mathcal{M}$. This implies $a\mathrm{L}_\psi(\xi) = 0$ for every $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. Thus for every $x \in \mathfrak{n}_\psi^*$ we have

$$0 = a\mathrm{L}_\psi(\xi)\eta'_\psi(x) = a(\xi x) = (a\xi)x.$$

Since $\mathfrak{n}_\psi^*$ is σ-weakly dense in $\mathcal{N}$, we have $a\xi = 0$. The density of $\mathfrak{D}(\mathfrak{H}, \psi)$ in $\mathfrak{H}$ then gives $a = 0$. □

PROPOSITION 2.10. *The spatial derivative* $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}$ *has the following additional properties*:

(i)

$$\varphi_1 \le \varphi_2 \iff \frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi'} \le \frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi'}.$$

(ii) *If* φ_1 *and* φ_2 *are both finite, then*

$$\frac{\mathrm{d}(\varphi_1 + \varphi_2)}{\mathrm{d}\psi'} = \frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi'} + \frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi'}, \tag{20}$$

where the above sum should be interpreted as a form sum.

(iii) *If* $a \in \mathcal{M}$ *is invertible, then*

$$\frac{\mathrm{d}(a\varphi a^*)}{\mathrm{d}\psi'} = a\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}\right)a^*. \tag{21}$$

(iv) *The support of* $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'}$, $s(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi'})$, *is equal to the support*, $s(\varphi)$, *of* φ.

Proof.

(i) Suppose $\varphi_1 \le \varphi_2$. Then we have

$$\left\| \left(\frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi'}\right)^{1/2} \xi \right\|^2 = \varphi_1(\mathrm{L}_\psi(\xi)\mathrm{L}_\psi(\xi)^*) \le \varphi_2(\mathrm{L}_\psi(\xi)\mathrm{L}_\psi(\xi)^*) = \left\| \left(\frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi'}\right)^{1/2} \xi \right\|^2$$

for every $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. Hence $\frac{d\varphi_1}{d\psi'} \le \frac{d\varphi_2}{d\psi'}$.

Conversely, suppose $\frac{d\varphi_1}{d\psi'} \le \frac{d\varphi_2}{d\psi'}$. This means that we must have $\varphi_1(a) \le \varphi_2(a)$ for every $a \in \mathcal{M}_+$ of the form $a = \sum_{i=1}^n \mathrm{L}_\psi(\xi_i)\mathrm{L}_\psi(\xi_i)^*$. Our assertion then follows from Lemma 2.9.

(ii) Suppose that $\varphi_1, \varphi_2 \in \mathcal{M}_*^+$, and set $\varphi \triangleq \varphi_1 + \varphi_2$. The boundedness of φ_1 and φ_2 of course imply that φ is bounded; we also have seen that the square roots of all the spatial derivatives $\frac{d\varphi_1}{d\psi'}$, $\frac{d\varphi_2}{d\psi'}$ and $\frac{d\varphi}{d\psi'}$ are essentially self-adjoint on $\mathfrak{D}(\mathfrak{H}, \psi)$. Let $H_1 \triangleq \frac{d\varphi_1}{d\psi'}$, $H_2 \triangleq \frac{d\varphi_2}{d\psi'}$ and $H \triangleq \frac{d\varphi}{d\psi'}$. Then we have $\|H^{\frac{1}{2}}\xi\|^2 = \|H_1^{\frac{1}{2}}\xi\|^2 + \|H_2^{\frac{1}{2}}\xi\|^2$, $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$. Hence our assertion follows.

(iii) Again, take $H \triangleq \frac{d\varphi}{d\psi'}$. It follows that aHa^* is a positive, self-adjoint operator with domain $(a^*)^{-1}\mathfrak{D}(H)$, and that for each $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$,

$$\begin{aligned}\|H^{\frac{1}{2}}a^*\xi\|^2 &= \varphi(\mathrm{L}_\psi(a^*\xi)\mathrm{L}_\psi(a^*\xi)^*) = \varphi(a^*\mathrm{L}_\psi(\xi)\mathrm{L}_\psi(\xi)^*a)\\ &= (a\varphi a^*)(\mathrm{L}_\psi(\xi)\mathrm{L}_\psi(\xi)^*).\end{aligned}$$

(Note that $\|H^{\frac{1}{2}}a^*\xi\|^2$ can be $+\infty$ if φ is not finite. In fact, $\|H^{\frac{1}{2}}a^*\xi\|^2 < +\infty \iff a^*\xi \in \mathfrak{D}(H^{\frac{1}{2}})$.)

Since a is invertible, $\mathfrak{D}((aHa^*)^{\frac{1}{2}}) = \mathfrak{D}(H^{\frac{1}{2}}a^*)$ and the absolute value of $H^{\frac{1}{2}}a^*$ is precisely $(aHa^*)^{\frac{1}{2}}$. Hence we may conclude (21).

(iv) Let p be the support of $H = \frac{d\varphi}{d\psi'}$ and $q = s(\varphi)$. Then p is characterized by the fact that $\mathbf{1} - p$ is the projection of $\mathfrak{H}$ onto the null space of H, i.e., onto the subspace $\mathfrak{K} = \{\xi \in \mathfrak{H}\colon H\xi = 0\}$. Let $(\mathfrak{A}_{\rho'})_0$ be the maximal Tomita algebra associated with the left Hilbert algebra $\mathfrak{A}_{\rho'} = \eta_{\rho'}(\mathfrak{n}_{\rho'} \cap \mathfrak{n}_{\rho'}^*)$. (Recall that we defined ρ' in the proof of Lemma 2.5 by adding an auxiliary weight to our original weight ρ in order to make ρ' faithful.) Because $f \in \mathcal{R}$, with $f' = JfJ$ we have $f'(\mathfrak{A}_{\rho'})_0 = J(\mathfrak{A}_{\rho'})_0 \subset (\mathfrak{A}_{\rho'})_0$. If $\xi \in \mathfrak{K}$, then there exists a sequence $\{\xi_n\} \subset \mathfrak{D}(\mathfrak{H}, \psi)$ such that $\xi_n \to \xi$ and $H\xi_n \to 0$, as $\mathfrak{D}(\mathfrak{H}, \psi)$ is a core for H. For each $\eta \in f'(\mathfrak{A}_{\rho'})_0$, we have $\pi_r(\eta)\xi_n \to \pi_r(\eta)\xi$ and

$$\begin{aligned}\|H^{\frac{1}{2}}\pi_r(\eta)\xi\|^2 &= \lim_n \|H^{\frac{1}{2}}\pi_r(\eta)\xi_n\|^2 = \lim_n \|H^{\frac{1}{2}}\pi_\ell(\xi_n)\eta\|^2\\ &= \lim_n \varphi(\pi_\ell(\xi_n\eta)\pi_\ell(\xi_n\eta)^*)\\ &\le \|\pi_\ell(\eta)\|^2 \lim_n \varphi(\pi_\ell(\xi_n)\pi_\ell(\xi_n)^*) = 0,\end{aligned}$$

which gives $\pi_r(\eta)\xi \in \mathfrak{K}$. Since $\{\pi_r(\eta)_e\colon \eta \in f'(\mathfrak{A}_{\rho'})_0\}$ is σ-weakly dense in $\mathcal{N}$, the projection p belongs to $\mathcal{M} \cong \mathcal{N}'$ (in $\mathcal{R}$).

Now, if $\xi \in (\mathbf{1}-q)\mathfrak{D}(\mathfrak{H}, \psi)$, then $\varphi(\mathrm{L}_\psi(\xi)\mathrm{L}_\psi(\xi)^*) = \varphi(q\mathrm{L}_\psi(\xi)\mathrm{L}_\psi(\xi)^*q) = \varphi(\mathrm{L}_\psi(q\xi)\mathrm{L}_\psi(q\xi)^*) = 0$, so $H^{\frac{1}{2}}\xi = 0$. If $\xi \in (\mathbf{1} - q)\mathfrak{H}$, then we choose a sequence $\xi_n \in \mathfrak{D}(\mathfrak{H}, \psi)$ with $\xi_n \to \xi$. It follows that $(\mathbf{1} - q)\xi_n \to \xi$, and since $H^{\frac{1}{2}}(\mathbf{1} - q)\xi_n = 0 \quad \forall n$, we see that $\xi \in \mathfrak{D}(H^{\frac{1}{2}})$, and $H^{\frac{1}{2}}\xi = 0$. Thus $\mathbf{1}-q \le \mathbf{1}-p$, i.e., $p \le q$. On the other hand, φ is a faithful weight on $\mathcal{M}_q$. It is then easy to check that we can view the weight ψ' as one on $\mathcal{M}'_{s(\varphi)}$ without changing $H = \frac{d\varphi}{d\psi'}$ other than to change the underlying Hilbert space from $\mathfrak{H}$ to $s(\varphi)\mathfrak{H}$. By Theorem 2.8, $\frac{d\varphi}{d\psi'}$ is non-singular on $s(\varphi)\mathfrak{H}$, which means $p = s(\varphi)$. □

We continue our investigation of the properties of the spatial derivative. In particular, we are interested in answering this question: Given a positive, self-adjoint operator H in $\mathfrak{H}$, and an fns weight ψ' on $\mathcal{M}'$ (or equivalently an fns weight ψ on $\mathcal{N}$), when is there a weight $\varphi \in \mathfrak{W}(\mathcal{M})$ such that $\frac{d\varphi}{d\psi'} = H$?

THEOREM 2.11. *Let $\{\mathcal{M}, \mathfrak{H}\}$ and $\mathcal{N} = (\mathcal{M}')^\circ$ be as before, and fix an fns weight ψ on $\mathcal{N}$. For a positive self-adjoint operator H in $\mathfrak{H}$, the following three conditions are equivalent*:

(i) *There exists a semi-finite, normal weight φ on $\mathcal{M}$ with $\frac{d\varphi}{d\psi'} = H$.*

(ii) *For every $y \in \mathcal{N}$, $H^{it}\sigma_t^\psi(y) = yH^{it}$, $t \in \mathbb{R}$, where H^{it} is considered only on the closure of the range of H. (Note that we are not assuming that H is non-singular.)*

(iii) *$\mathfrak{D}(\mathfrak{H}, \psi) \cap \mathfrak{D}(H^{\frac{1}{2}})$ is a core for $H^{\frac{1}{2}}$, and the scalar $\sum_{i=1}^n \|H^{\frac{1}{2}}\xi_i\|^2$ depends only on the operator $\sum_{i=1}^n L_\psi(\xi_i)L_\psi(\xi_i)^*$ for $\{\xi_1, \dots, \xi_n\} \subset \mathfrak{D}(\mathfrak{H}, \psi) \cap \mathfrak{D}(H^{\frac{1}{2}})$.*

Proof. (i) $\Longleftrightarrow$ (ii). Let p be the support of H. Each of conditions (i) and (ii) implies $p \in \mathcal{M}$. Hence we may and do assume the non-singularity of H. (We need only consider the reduced algebra $\mathcal{M}_p$.) The implication (i) $\Rightarrow$ (ii) follows from Theorem 2.8. Conversely, assume (ii); take any faithful weight φ' on $\mathcal{M}$. Put $K = \frac{d\varphi'}{d\psi'}$, and define $u_t \triangleq H^{it}K^{-it}$, $t \in \mathbb{R}$. It then follows that u_t is a $\sigma_t^{\varphi'}$-cocycle in $\mathcal{M}$. Again, the result of Connes-Masuda [7] guarantees the existence of a faithful weight φ on $\mathcal{M}$ with $(D\varphi : D\varphi')_t = u_t$, $t \in \mathbb{R}$, and hence $H = \frac{d\varphi}{d\psi'}$.

(i) $\Rightarrow$ (iii). By construction, we have

$$\sum_{i=1}^n \|H^{\frac{1}{2}}\xi_i\|^2 = \varphi\left(\sum_{i=1}^n \mathrm{L}_\psi(\xi_i)\mathrm{L}_\psi(\xi_i)^*\right), \quad \{\xi_1, \dots, \xi_n\} \subset \mathfrak{D}(\mathfrak{H}, \psi),$$

so that the assertion follows.

(iii) $\Rightarrow$ (i). We need only construct a semi-finite normal weight φ on $\mathcal{M}$ such that

$$\|H^{\frac{1}{2}}\xi\|^2 = \varphi(\mathrm{L}_\psi(\xi)\mathrm{L}_\psi(\xi)^*), \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi) \cap \mathfrak{D}(H^{\frac{1}{2}}).$$

Any weight with this property is automatically semi-finite because $\mathfrak{D}(H^{\frac{1}{2}}) \cap \mathfrak{D}(\mathfrak{H}, \psi)$ is dense in $\mathfrak{H}$, which means that $\{\mathrm{L}_\psi(\xi)\mathrm{L}_\psi(\xi)^* : \xi \in \mathfrak{D}(H^{\frac{1}{2}}) \cap \mathfrak{D}(\mathfrak{H}, \psi)\}$ is non-degenerate. The rest of the proof then follows from the next lemma. □

LEMMA 2.12.

(i) *Let H be as in Theorem* 2.11(*iii*). *Then there exists a* preweight φ_1 *on* $\mathcal{J}_\psi$ *such that*

$$\varphi_1(L_\psi(\xi)L_\psi(\xi)^*) = \|H^{\frac{1}{2}}\xi\|^2, \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi), \tag{22}$$

where $\|H^{\frac{1}{2}}\xi\|^2 = +\infty$ *if* $\xi \notin \mathfrak{D}(H^{\frac{1}{2}})$. *The preweight* φ_1 *has the property that for any net* $\{x_\alpha\} \subset \mathcal{M}$ *converging strongly to* $\mathbf{1}$,

$$\liminf_\alpha \varphi_1(x_\alpha y x_\alpha^*) \geq \varphi_1(y), \quad y \in \mathcal{J}_\psi^+. \tag{23}$$

Here, when we say that φ_1 *is a* preweight *on* $\mathcal{J}_\psi$, *we mean that it is an extended real valued map on* $\mathcal{J}_\psi$, *satisfying the usual requirements of positive homogeneity and (finite) additivity. In this case, however, the domain is not the positive cone of a von Neumann algebra (recall that* $\mathcal{J}_\psi$ *is merely a* σ*-weakly dense ideal in* $\mathcal{M}$*), and so we refrain from calling* φ_1 *a weight. However we do have the following.*

(ii) *Any preweight* φ_1 *on* $\mathcal{J}_\psi$ *with the property given by* (23) *extends to a normal weight* φ *on* $\mathcal{M}$.

Proof.

(i) By assumption, φ_1 defined by (19) on $\mathrm{L}_\psi(\xi)\mathrm{L}_\psi(\xi)^*$ extends to a preweight on $\mathcal{J}_\psi^+$ by Theorem 2.9, which we will continue to denote by φ_1. Suppose $y = \sum_{k=1}^n \mathrm{L}_\psi(\xi_k)\mathrm{L}_\psi(\xi_k)^* \in \mathcal{J}_\psi^+$. We then have

$$x_\alpha y x_\alpha^* = \sum_{k=1}^n \mathrm{L}_\psi(x_\alpha\xi_k)\mathrm{L}_\psi(x_\alpha\xi_k)^*,$$

so that

$$\varphi(x_\alpha y x_\alpha^*) = \sum_{k=1}^n \|H^{\frac{1}{2}}x_\alpha\xi_k\|^2.$$

Hence inequality (23) follows from the lower semi-continuity of the positive quadratic form associated with H.

(ii) Since $\mathcal{J}_\psi$ is a σ-weakly dense ideal of $\mathcal{M}$, every element of $\mathcal{M}_+$ can be approximated by $\mathcal{J}_\psi^+$ from below. So we put

$$\varphi(x) \stackrel{\Delta}{=} \sup\{\varphi_1(y) \colon y \in \mathcal{J}_\psi^+,\ y \leq x\}, \quad x \in \mathcal{M}_+.$$

It follows that φ agrees with φ_1 on $\mathcal{J}_\varphi^+$. Since $x_\alpha \nearrow x$ and $y_\alpha \nearrow y \Rightarrow (x_\alpha + y_\alpha) \nearrow (x+y)$, the additivity of φ follows from that of φ_1. We need only check the normality of φ. Suppose that $x_\alpha \nearrow x$ in $\mathcal{M}_+$. Then $x_\alpha^{1/2} = a_\alpha x^{1/2}$

for a unique $a_\alpha \in \mathcal{M}$, with $s_r(a_\alpha) \leq s(x)$. Put $b_\alpha = a_\alpha + (\mathbf{1} - s(x))$. Then $x_\alpha = b_\alpha x b_\alpha^*$ and $\{b_\alpha\}$ converges strongly to $\mathbf{1}$. For any $y \in \mathcal{J}_\psi^+$ with $y \leq x$ we must show that $\sup_\alpha \varphi(x_\alpha) \geq \varphi_1(y)$. But by (23), we have,

$$\varphi_1(y) \leq \liminf_\alpha \varphi_1(b_\alpha y b_\alpha^*) \leq \liminf \varphi(b_\alpha x b_\alpha^*),$$

and we have seen $b_\alpha x b_\alpha^* = x_\alpha$. □

We conclude this section with a corollary which will relate convergence in $\mathfrak{W}(\mathcal{M})$, convergence (in the strongly resolvent sense) amongst positive, self-adjoint and non-singular operators in $\mathfrak{H}$, and convergence in $\mathrm{Aut}(\mathcal{M})$.

COROLLARY 2.13. *Let $\mathcal{M}, \mathcal{N}$ and $\mathfrak{H}$ be as before, and fix ψ' an fns weight on $\mathcal{M}'$. If $\{\varphi_n\}$ is an increasing sequence of fns weights $\mathcal{M}$ and if $\varphi = \sup_n \varphi_n$ is semi-finite, then $\{\frac{d\varphi_n}{d\psi'}\}$ is increasing, and converges to $\frac{d\varphi}{d\psi'}$ in the strongly resolvent sense; hence, $\{\sigma_t^{\varphi_n}\}$ converges to σ_t^φ in* $\mathrm{Aut}(\mathcal{M})$ *uniformly on any finite interval (of $\mathbb{R}$).*

Proof. Let $H_n \triangleq \frac{d\varphi_n}{d\psi'}$. By Proposition 2.10, $\{H_n\}$ is increasing and bounded by $H \triangleq \frac{d\varphi}{d\psi'}$ from above. Hence $\{H_n\}$ converges to a positive self-adjoint operator K in the strongly resolvent sense. Since $H_n^{it} y H_n^{-it} = \sigma_{-t}^{\psi}(y)$ for every $y \in \mathcal{N}$, $K^{it} y K^{it} = \sigma_{-t}^{\psi}(y)$, $y \in \mathcal{N}$. By Theorem 2.11, there exists a unique weight μ on $\mathcal{M}$ with $K = \frac{d\mu}{d\psi'}$. The inequalities

$$H_n \leq K \leq H$$

show that $\varphi_n \leq \mu \leq \varphi$. Hence $\mu = \varphi$ and $K = H$. The rest follows from general facts about monotone convergence. □

3. The relative tensor product

3.1. *Definition of the relative tensor product.* We now proceed to define the *relative tensor product* of a right module and a left module over the same von Neumann algebra $\mathcal{N}$. Unlike the ordinary (i.e., spatial) tensor product, the construction of the relative tensor product depends on the choice of an fns weight on $\mathcal{N}$—hence, the use of the adjective *relative*. Furthermore, given a right $\mathcal{N}$-module $\mathfrak{H}$ and a left $\mathcal{N}$-module $\mathfrak{K}$, the tensor product of an arbitrary pair of vectors from $\mathfrak{H}$ and $\mathfrak{K}$ cannot, in general, be defined. (In fact, as we shall see in the next Section, the existence of the tensor product for all possible pairs of vectors severely limits the possible type of the von Neumann algebra $\mathcal{N}$.) The formation of the relative tensor product is restricted to a subset of vectors from $\mathfrak{H}$ and $\mathfrak{K}$ which depends on the choice of weight. It is interesting to note that the tensor product actually behaves like the product of closed, unbounded operators. We shall begin our discussion by introducing some notation and terminology to be used throughout the chapter.

As in the case of right modules, for two left $\mathcal{N}$-modules ${}_{\mathcal{N}}\mathfrak{K}_1$ and ${}_{\mathcal{N}}\mathfrak{K}_2$ we consider $\mathcal{L}({}_{\mathcal{N}}\mathfrak{K}_1, {}_{\mathcal{N}}\mathfrak{K}_2) = \{t \in \mathcal{L}(\mathfrak{K}_1, \mathfrak{K}_2)\colon ta\eta = at\eta,\ \eta \in \mathfrak{K}_1,\ a \in \mathcal{N}\}$. For $\mathcal{L}({}_{\mathcal{N}}\mathfrak{K}, {}_{\mathcal{N}}\mathfrak{K})$ we write $\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})$. Throughout the remainder of this chapter, $\mathfrak{H}$ will denote a right $\mathcal{N}$-module, $\mathfrak{K}$ a left $\mathcal{N}$-module. Observe that a right $\mathcal{N}$-module $\mathfrak{H}$ is also canonically an $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$-$\mathcal{N}$ bimodule, while a left $\mathcal{N}$-module $\mathfrak{K}$ can always be considered an $\mathcal{N}$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})^\circ$ bimodule in a canonical way. We are now going to construct the relative tensor product $\mathfrak{H} \otimes_\psi \mathfrak{K}$ of a right $\mathcal{N}$-module $\mathfrak{H}$ and a left $\mathcal{N}$-module $\mathfrak{K}$, which will depend on the choice of a fns weight ψ on $\mathcal{N}$.

So, we fix a von Neumann algebra $\mathcal{N}$, a right $\mathcal{N}$-module $\mathfrak{H}$ and a left $\mathcal{N}$-module $\mathfrak{K}$. We also fix a faithful, normal and semi-finite weight ψ on $\mathcal{N}$. We have seen (Lemma 2.3) that the right module $\mathfrak{H}$ can be recovered from $\mathfrak{D}(\mathfrak{H}, \psi)$, and that the left module $\mathfrak{K}$ is also recoverable from $\mathfrak{D}'(\mathfrak{K}, \psi)$. (Observe that in this case, the roles of ψ and ψ° are symmetric, as they are both faithful). We state here a few facts about $\mathfrak{D}(\mathfrak{H}, \psi)$ (resp., $\mathfrak{D}'(\mathfrak{K}, \psi)$) and L_ψ (resp., R_ψ) which have been implicit in our previous results.

$$\begin{aligned}
(\xi_1 \mid \xi_2) &= \psi(\mathrm{L}_\psi(\xi_2)^*\mathrm{L}_\psi(\xi_1)), \quad \xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi);\\
(\eta_1 \mid \eta_2) &= \psi(\mathrm{J}_\psi\mathrm{R}_\psi(\eta_1)^*\mathrm{R}_\psi(\eta_2)\mathrm{J}_\psi), \quad \eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi);\\
\eta_\psi(\mathrm{L}_\psi(\xi_2)^*\mathrm{L}_\psi(\xi_1)) &= \mathrm{L}_\psi(\xi_2)^*\xi_1, \quad \xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi);\\
\eta_\psi(\mathrm{J}_\psi\mathrm{R}_\psi(\eta_1)^*\mathrm{R}_\psi(\eta_2)\mathrm{J}_\psi) &= \mathrm{R}_\psi(\eta_2)^*\eta_1, \quad \eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi). \qquad (24)
\end{aligned}$$

It is also easy to see that if we consider $\mathfrak{H} = L^2(\mathcal{N}, \psi)$, with the standard right action of $\mathcal{N}$ (resp., $\mathfrak{K} = L^2(\mathcal{N}, \psi)$, with the usual left action of $\mathcal{N}$), as a right (resp., left) module, then

$$\begin{aligned}
&\mathfrak{D}(\mathfrak{H}, \psi) = \eta_\psi(\mathfrak{n}_\psi) = \mathfrak{B}_\psi, \quad \mathrm{L}_\psi(\xi) = \pi_\ell(\xi), \quad \xi \in \mathfrak{B}_\psi,\\
&(\text{resp.,} \quad \mathfrak{D}'(\mathfrak{K}, \psi) = \mathfrak{B}'_\psi, \quad \mathrm{R}_\psi(\eta) = \pi_r(\eta), \quad \eta \in \mathfrak{B}'_\psi),
\end{aligned}$$

where $\mathfrak{B}_\psi$ (resp., $\mathfrak{B}'_\psi$) means the algebra of all left (resp., right) bounded vectors in $L^2(\mathcal{N}, \psi)$.

PROPOSITION 3.1.

(i) *The sesquilinear form* $\mathrm{B}\colon \mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{K} \to \mathbb{C}$ *determined by*

$$\mathrm{B}(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) \stackrel{\Delta}{=} (\pi_{\mathfrak{K}}(L_\psi(\xi_2)^*L_\psi(\xi_1))\eta_1 \mid \eta_2) \qquad (25)$$

is positive semi-definite, and so defines an inner product on $\mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{K}$, *which, in many cases, is degenerate.*

(ii) *If* $\xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi)$ *and* $\eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi)$, *then*

$$(\pi_{\mathfrak{K}}(L_\psi(\xi_2)^*L_\psi(\xi_1))\eta_1 \mid \eta_2) = (\pi'_{\mathfrak{H}}(J_\psi R_\psi(\eta_1)^*R_\psi(\eta_2)J_\psi)\xi_1 \mid \xi_2). \qquad (26)$$

(i') *Dual to (i), the sesquilinear form* B' *defined on* $\mathfrak{H} \odot \mathfrak{D}'(\mathfrak{K}, \psi)$ *and determined by*

$$(25') \qquad \mathrm{B}'(\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) \stackrel{\Delta}{=} (\pi'_{\mathfrak{H}}(J_\psi R_\psi(\eta_1)^* R_\psi(\eta_2) J_\psi)\xi_1 \mid \xi_2),$$

is also positive semi-definite, and agrees with B *on* $\mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{D}'(\mathfrak{K}, \psi)$

Proof.

(i) Suppose that $\{\xi_1, \dots, \xi_n\} \subset \mathfrak{D}(\mathfrak{H}, \psi)$. Let $a_{kj} \stackrel{\Delta}{=} \mathrm{L}_\psi(\xi_k)^* \mathrm{L}_\psi(\xi_j)$, with $j, k \in \{1, \dots, n\}$; then $a = (a_{kj})$ is an $n \times n$ matrix over $\mathcal{N}$. If $\{x_1, \dots, x_n\} \subset \mathcal{A}$, where, as before, $\mathcal{A} = \mathcal{D}(\sigma^\psi_{i/2}) \cap \mathcal{D}(\sigma^\psi_{-i/2})$, then, by (6'), we have

$$\begin{aligned} \sum_{j,k=1}^n x_k^* a_{kj} x_j &= \sum_{j,k=1}^n x_k^* \mathrm{L}_\psi(\xi_k)^* \mathrm{L}_\psi(\xi_j) x_j \\ &= \sum_{j,k=1}^n \mathrm{L}_\psi(\xi_k \sigma^\psi_{i/2}(x_k))^* \mathrm{L}_\psi(\xi_j \sigma^\psi_{i/2}(x_j)) \\ &= \left(\sum_{k=1}^n \mathrm{L}_\psi(\xi_k \sigma^\psi_{i/2}(x_k)) \right)^* \left(\sum_{j=1}^n \mathrm{L}_\psi(\xi_j \sigma^\psi_{i/2}(x_j)) \right) \geq 0. \end{aligned}$$

Because $\mathcal{A}$ is σ-weakly dense in $\mathcal{N}$, the matrix a is positive in $M_n(\mathcal{N}) = \mathcal{N} \otimes M_n(\mathbb{C})$; hence there exists a $b = (b_{\ell k}) \in M_n(\mathcal{N})$ such that $a = b^* b$, i.e., $a_{kj} = \sum_{\ell=1}^n b^*_{\ell k} b_{\ell j}$, $j, k \in \{1, \dots, n\}$. We then have, for any $\{\eta_1, \dots, \eta_n\} \subset \mathfrak{K}$,

$$\mathrm{B}\left(\sum_{j=1}^n \xi_j \otimes \eta_j, \sum_{k=1}^n \xi_k \otimes \eta_k \right) = \sum_{j,k=1}^n (a_{kj} \eta_j \mid \eta_k) = \sum_{k=1}^n \| \sum_{j=1}^n b_{kj} \eta_j \|^2 \geq 0.$$

Hence the sesquilinear form B is positive.

(ii) Suppose $\xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi)$ and $\eta_1, \eta_2 \in \mathfrak{D}'(\mathfrak{K}, \psi)$. Then, as both $\mathrm{L}_\psi(\xi_2)^* \mathrm{L}_\psi(\xi_1)$ and $\mathrm{J}_\psi \mathrm{R}_\psi(\eta_1)^* \mathrm{R}_\psi(\eta_2) \mathrm{J}_\psi$ are elements of $\mathfrak{m}_\psi \subset \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$,

$$\begin{aligned} (\pi_{\mathfrak{K}}(\mathrm{L}_\psi(\xi_2)^* \mathrm{L}_\psi(\xi_1))\eta_1 \mid \eta_2) &= (\mathrm{R}_\psi(\eta_1)\eta_\psi(\mathrm{L}_\psi(\xi_2)^* \mathrm{L}_\psi(\xi_1)) \mid \eta_2) \\ = (\mathrm{L}_\psi(\xi_2)^* \xi_1 \mid \mathrm{R}_\psi(\eta_1)^* \eta_2) &= (\mathrm{L}_\psi(\xi_2)^* \xi_1 \mid \eta'_\psi(\mathrm{J}_\psi \mathrm{R}_\psi(\eta_2)^* \mathrm{R}_\psi(\eta_1) \mathrm{J}_\psi)) \\ &= (\xi_1 \mid \mathrm{L}_\psi(\xi_2)\eta'_\psi(\mathrm{J}_\psi \mathrm{R}_\psi(\eta_2)^* \mathrm{R}_\psi(\eta_1) \mathrm{J}_\psi)) \\ &= (\xi_1 \mid \pi'_{\mathfrak{H}}(\mathrm{J}_\psi \mathrm{R}_\psi(\eta_2)^* \mathrm{R}_\psi(\eta_1) \mathrm{J}_\psi)\xi_2) \\ &= (\pi'_{\mathfrak{H}}(\mathrm{J}_\psi \mathrm{R}_\psi(\eta_1)^* \mathrm{R}_\psi(\eta_2) \mathrm{J}_\psi)\xi_1 \mid \xi_2). \end{aligned}$$

(i') The positive semi-definiteness follows from (i) by symmetry. The second assertion follows from (ii). □

Definition 3.2. Let $\mathfrak{N}$ be the subspace of $\mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{K}$ comprising those vectors ζ with $\mathrm{B}(\zeta, \zeta) = 0$. The Hilbert space obtained as the completion of the quotient space $\mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{K} / \mathfrak{N}$ relative to the inner product induced by the positive-definite sequilinear form B will be called the *relative tensor product* of the right $\mathcal{N}$-module $\mathfrak{H}$ and the left $\mathcal{N}$-module $\mathfrak{K}$ with respect to the fns weight ψ and will be written $\mathfrak{H} \otimes_\psi \mathfrak{K}$. The image of $\xi \otimes \eta$ will similarly be denoted $\xi \otimes_\psi \eta$ for $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$, $\eta \in \mathfrak{K}$. By Proposition 3.1, the relative tensor product $\mathfrak{H} \otimes_\psi \mathfrak{K}$ can also be obtained as the completion of the quotient space of the algebraic tensor product $\mathfrak{H} \odot \mathfrak{D}'(\mathfrak{K}, \psi)$ by the subspace $\mathfrak{N}'$, where $\mathfrak{N}'$ consists of null vectors with respect to the positive-definite sequilinear form B'. In this way, we can consider the tensor product $\xi \otimes_\psi \eta$ for a pair $\xi \in \mathfrak{H}$, $\eta \in \mathfrak{D}'(\mathfrak{K}, \psi)$.

THEOREM 3.3. *Let $\mathcal{N}$ be a von Neumann algebra equipped with an fns weight ψ, $\mathfrak{H}$ a right $\mathcal{N}$-module and $\mathfrak{K}$ a left $\mathcal{N}$-module. Set $\mathcal{P} \triangleq \mathcal{L}(\mathfrak{H}_\mathcal{N})$ and $\mathcal{Q} \triangleq \mathcal{L}({}_\mathcal{N}\mathfrak{K})$. We construct the direct sum $\tilde{\mathfrak{H}} \triangleq L^2(\mathcal{N}, \psi) \oplus \mathfrak{H} \oplus \overline{\mathfrak{K}}$ as a right $\mathcal{N}$-module and then consider $\mathcal{R} \triangleq \mathcal{L}(\tilde{\mathfrak{H}}_\mathcal{N})$, together with the "balanced" fns weight $\rho = \psi \oplus \varphi \oplus \nu$, where φ is a faithful, normal and semi-finite weight on $\mathcal{P}$ and ν is an fns weight on $\mathcal{Q}$. Let e, f and g be, respectively, the projections of $\tilde{\mathfrak{H}}$ onto $L^2(\mathcal{N}, \psi)$, $\mathfrak{H}$ and $\overline{\mathfrak{K}}$, which, we note, belong to $\mathcal{R}$. Represent the standard Hilbert space $\mathfrak{H}_\rho$ as the space of 3×3 matrices:*

$$\mathfrak{H}_\rho \cong \begin{pmatrix} L^2(\mathcal{N}, \psi) & \overline{\mathfrak{H}} & \mathfrak{K} \\ \mathfrak{H} & L^2(\mathcal{P}, \varphi) & \mathfrak{H}_{23} \\ \overline{\mathfrak{K}} & \mathfrak{H}_{32} & L^2(\mathcal{Q}, \nu) \end{pmatrix}. \tag{15'}$$

Then there exists a natural isomorphism between $\mathfrak{H} \otimes_\psi \mathfrak{K}$ and $\mathfrak{H}_{23}$.

Proof. Let $\mathfrak{A}$ $(= \mathfrak{A}_\rho)$ be the left Hilbert algebra associated with ρ, $\mathfrak{B}$ $(= \mathfrak{B}_\rho)$ the algebra of left bounded elements in $L^2(\mathcal{R}, \rho)$ and, as usual, $\mathfrak{n}_\rho = \{x \in \mathcal{R}\colon \rho(x^*x) < +\infty\}$. Since e, f and g are all in $\mathcal{R}$, $\mathfrak{A}$ and $\mathfrak{B}$ can each be decomposed into the matrix direct sum relative to (15'), i.e.,

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{A}_{11} & \mathfrak{A}_{12} & \mathfrak{A}_{13} \\ \mathfrak{A}_{21} & \mathfrak{A}_{22} & \mathfrak{A}_{23} \\ \mathfrak{A}_{31} & \mathfrak{A}_{32} & \mathfrak{A}_{33} \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} \mathfrak{B}_{11} & \mathfrak{B}_{12} & \mathfrak{B}_{13} \\ \mathfrak{B}_{21} & \mathfrak{B}_{22} & \mathfrak{B}_{23} \\ \mathfrak{B}_{31} & \mathfrak{B}_{32} & \mathfrak{B}_{33} \end{pmatrix}. \tag{15''}$$

It follows from Lemma 2.3 that $\mathfrak{B}_{21} = \mathfrak{D}(\mathfrak{H}, \psi)$ and $\mathfrak{B}_{31} = \mathfrak{D}(\overline{\mathfrak{K}}, \psi)$. Also, we note that $\mathrm{L}_\psi(\xi) = \pi_\ell(\xi)\,|_{\mathfrak{H}_{11}}$, $\xi \in \mathfrak{D}(\mathfrak{H}, \psi) = \mathfrak{B}_{21}$, and $\mathrm{L}_\psi(\overline{\eta}) = \pi_\ell(\overline{\eta})\big|_{\mathfrak{H}_{12}}$, $\overline{\eta} \in \mathfrak{D}(\bar{\mathfrak{K}}, \psi) = \mathfrak{B}_{31}$, where π_ℓ means the left multiplication representation of $\mathfrak{B}$ on $\mathfrak{H}_\rho$. At this point, one can see (through symmetry) that the right Hilbert algebra $\mathfrak{A}'$, and the algebra $\mathfrak{B}'$ of right bounded vectors, admit similar matrix decompositions; we can use these to obtain $\mathfrak{B}'_{21} = \mathfrak{D}'(\overline{\mathfrak{H}}, \psi)$, $\mathfrak{B}'_{31} = \mathfrak{D}'(\mathfrak{K}, \psi)$ and $\mathrm{R}_\psi(\eta) = \pi_r(\eta)\,|_{\mathfrak{H}_{11}}$, $\eta \in \mathfrak{D}'(\mathfrak{K}, \psi)$.

We claim that $\xi \otimes_\psi \eta$, with $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$ and $\eta \in \mathfrak{K}$, can be naturally identified with $\pi_\ell(\xi)\eta \in \mathfrak{H}_{23}$. Let U_0 be the map from $\mathfrak{D}(\mathfrak{H}, \psi) \odot \mathfrak{K}$ into $\mathfrak{H}_\rho$ determined by $U_0(\xi \otimes \eta) \triangleq \pi_\ell(\xi)\eta$ for $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$, $\eta \in \mathfrak{K}$. Since $\xi \in \mathfrak{B}_{21}$ and $\eta \in \mathfrak{K} \cong \mathfrak{H}_{13}$, $\pi_\ell(\xi)\eta$ belongs to $\mathfrak{H}_{23}$. Now, for $\xi_1, \xi_2 \in \mathfrak{D}(\mathfrak{H}, \psi)$ and $\eta_1, \eta_2 \in \mathfrak{K}$, we have

$$\begin{aligned}(U_0(\xi_1 \otimes \eta_1) \mid U_0(\xi_2 \otimes \eta_2)) &= (\pi_\ell(\xi_1)\eta_1 \mid \pi_\ell(\xi_2)\eta_2) = (\pi_\ell(\xi_2)^*\pi_\ell(\xi_1)\eta_1 \mid \eta_2) \\ &= (\pi_{\mathfrak{K}}(\mathrm{L}_\psi(\xi_2)^*\mathrm{L}_\psi(\xi_1))\eta_1 \mid \eta_2) = (\xi_1 \otimes_\psi \eta_1 \mid \xi_2 \otimes_\psi \eta_2).\end{aligned}$$

Therefore, the map U_0 gives rise to an isometry U of $\mathfrak{H} \otimes_\psi \mathfrak{K}$ into $\mathfrak{H}_{23}$. Let $\mathfrak{M} = U(\mathfrak{H} \otimes_\psi \mathfrak{K}) = [\pi_\ell(\mathfrak{B}_{21})\mathfrak{K}]$. First, we observe that $\mathfrak{H}_{23} = fL^2(\mathcal{R}, \psi)g$, $\mathcal{P} = \mathcal{R}_f$ and $\mathcal{Q} = \mathcal{R}_g$. Hence $\pi_{\mathfrak{H}_{23}}(\mathcal{P})' = \pi'_{\mathfrak{H}_{23}}(\mathcal{Q})$. We know that $\mathfrak{M}$ is invariant under the right action of $\mathcal{Q}$. Thus, the projection $\overline{p}$ of $\mathfrak{H}_{23}$ onto $\mathfrak{M}$ belongs to $\pi_{\mathfrak{H}_{23}}(\mathcal{P})$, i.e., $\overline{p}$ can be identified with left multiplication by a projection in $\mathcal{P}$, which we shall call p. This implies that $\mathfrak{M} = p\mathfrak{H}_{23}$, with $p \in \mathrm{Proj}(\mathcal{P})$. But as $\pi_\ell(a\xi) = a\pi_\ell(\xi)$ for $a \in \mathcal{P}$ and $\xi \in \mathfrak{D}(\mathfrak{H}, \psi) = \mathfrak{B}_{21}$, $\mathfrak{B}_{23}$ is invariant under left multiplication by elements of $\mathcal{P}$, which in turn implies the invariance of $\mathfrak{M}$ under the left action of $\mathcal{P}$. We may therefore conclude that the projection p belongs to the center $\mathcal{Z}(\mathcal{P})$ of $\mathcal{P}$, which is of the form $\mathcal{Z}(\mathcal{P}) = \mathcal{Z}(\mathcal{R})_f$. So p may be viewed as a projection in $\mathcal{Z}(\mathcal{R})$. When we view p as an element in $\mathrm{Proj}(\mathcal{Z}(\mathcal{R}))$, we see that we may write $(f - p)\mathfrak{M} = \{0\}$, so that $0 = (f - p)\pi_\ell(\xi)\eta = \pi_\ell((f - p)\xi)\eta$ for every $\xi \in \mathfrak{B}_{21}$ and $\eta \in \mathfrak{K}$. Thus, $\pi_{\mathfrak{K}}(\pi_\ell((f - p)\xi)^*\pi_\ell((f - p)\xi)) = 0$. Because $\mathfrak{K}$ is a faithful left $\mathcal{N}$-module, we must have $\pi_\ell((f-p)\xi) = 0$, $\xi \in \mathfrak{D}(\mathfrak{H}, \psi)$, which means that $f - p = 0$. Therefore, we see that $f = p$, i.e., as an element of $\mathrm{Proj}(\mathcal{M})$, $p = \mathbf{1}_\mathcal{M}$, which in turn implies $\mathfrak{M} = \mathfrak{H}_{23}$

Thus, via the isometry U, we may conclude that $\mathfrak{H} \otimes_\psi \mathfrak{K}$ can be identified with $\mathfrak{H}_{23}$. □

Using the preceding theorem, it is not difficult to arrive at the following corollary, which is presented without proof.

COROLLARY 3.4.

(i) *If $\mathfrak{H}$ and $\mathfrak{K}$ are, respectively, right and left $\mathcal{N}$-modules, with $\mathcal{N}$ a von Neumann algebra equipped with a fns weight ψ, then the relative tensor product $\mathfrak{H} \otimes_\psi \mathfrak{K}$ is naturally an $\mathcal{L}(\mathfrak{H}_\mathcal{N})$-$\mathcal{L}(_\mathcal{N}\mathfrak{K})^\circ$ bimodule, whose bimodule structure is given by*

$$a(\xi \otimes_\psi \eta)b \triangleq (a\xi) \otimes_\psi (\eta b), \qquad a \in \mathcal{L}(\mathfrak{H}_\mathcal{N}),\ b \in \mathcal{L}(_\mathcal{N}\mathfrak{K})^\circ,\ \xi \in \mathfrak{D}(\mathfrak{H}, \psi),\ \eta \in \mathfrak{K}. \tag{27}$$

(ii) *In terms of operators acting from the left (as usual), if $x \in \mathcal{L}(\mathfrak{H}_\mathcal{N})$ and $y \in \mathcal{L}(_\mathcal{N}\mathfrak{K})$, then there exists a unique operator $x \otimes_\psi y \in \mathcal{L}(\mathfrak{H} \otimes_\psi \mathfrak{K})$ defined by*

$$(x \otimes_\psi y)(\xi \otimes_\psi \eta) \triangleq (x\xi) \otimes_\psi (y\eta), \quad \xi \in \mathfrak{D}(\mathfrak{H}, \psi),\ \eta \in \mathfrak{K}. \tag{28}$$

The map $(x, y) \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}}) \times \mathcal{L}({}_{\mathcal{N}}\mathfrak{K}) \mapsto x \otimes_\psi y \in \mathcal{L}(\mathfrak{H} \otimes_\psi \mathfrak{K})$ *extends canonically to an injective *-homomorphism from the algebraic tensor product,* $\mathcal{L}(\mathfrak{H}_{\mathcal{N}}) \odot \mathcal{L}({}_{\mathcal{N}}\mathfrak{K})$, *into* $\mathcal{L}(\mathfrak{H} \otimes_\psi \mathfrak{K})$.

(iii) *Although* $\mathcal{N}$ *does not act on the relative tensor product* $\mathfrak{H} \otimes_\psi \mathfrak{K}$, *we have*

$$(\xi b) \otimes_\psi \eta = \xi \otimes_\psi (\sigma^\psi_{-i/2}(b)\eta), \quad b \in \mathcal{D}(\sigma^\psi_{-i/2}),\ \xi \in \mathfrak{D}(\mathfrak{H}, \psi),\ \eta \in \mathfrak{K}. \tag{29}$$

In order to summarize the preceding arguments, we restate the matrix decomposition of $\mathfrak{H}_\rho$ in the following form, making explicit use of our results up to this point:

$$L^2(\mathcal{R}, \rho) = \begin{pmatrix} L^2(\mathcal{N}, \psi) & \overline{\mathfrak{H}} & \mathfrak{K} \\ \mathfrak{H} & L^2(\mathcal{P}, \varphi) & \mathfrak{H} \otimes_\psi \mathfrak{K} \\ \overline{\mathfrak{K}} & \overline{\mathfrak{K}} \otimes_\psi \overline{\mathfrak{H}} & L^2(\mathcal{Q}, \nu) \end{pmatrix}. \tag{30}$$

PROPOSITION 3.5.

(i) *Viewing* $L^2(\mathcal{N}, \psi)$ *as a right* $\mathcal{N}$*-module, the map*

$$V^\psi_{\mathfrak{K}}:\ \eta_\psi(y) \otimes_\psi \eta \in L^2(\mathcal{N}, \psi) \otimes_\psi \mathfrak{K} \mapsto y\eta \in \mathfrak{K}, \quad y \in \mathfrak{n}_\psi,\ \eta \in \mathfrak{K}$$

gives rise to an isomorphism of $L^2(\mathcal{N}, \psi) \otimes_\psi \mathfrak{K}$ *onto* $\mathfrak{K}$ *as* $\mathcal{N}$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})^\circ$ *bimodules.*

(ii) *Similarly, if we regard* $L^2(\mathcal{N}, \psi)$ *as a left* $\mathcal{N}$*-module, then the map*

$$U^\psi_{\mathfrak{H}}:\ \xi \otimes_\psi \eta'_\psi(y) \in \mathfrak{H} \otimes_\psi L^2(\mathcal{N}, \psi) \mapsto \xi y \in \mathfrak{H}, \quad \xi \in \mathfrak{H},\ y \in \mathfrak{n}^*_\psi$$

extends to an isomorphism of $\mathfrak{H} \otimes_\psi L^2(\mathcal{N}, \psi)$ *onto* $\mathfrak{H}$ *as* $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$-$\mathcal{N}$ *bimodules.*

The proof of the preceding proposition is entirely routine, and is omitted. Note that in light of this propositon, it is reasonable to refer to $L^2(\mathcal{N}, \psi)$ as a sort of "identity" (both right and left) amongst $\mathcal{N}$ modules, relative to the weight ψ.

It is also easy to verify, using our previous technique, viz., the 3×3 matrix decompostion, that after making the necessary (implicit) identifications, we have the following identities:

$$\left.\begin{aligned} (\mathfrak{H}_1 \oplus \mathfrak{H}_2) \otimes_\psi \mathfrak{K} &\simeq (\mathfrak{H}_1 \otimes_\psi \mathfrak{K}) \oplus (\mathfrak{H}_2 \otimes_\psi \mathfrak{K}), \\ \mathfrak{H} \otimes_\psi (\mathfrak{K}_1 \oplus \mathfrak{K}_2) &\simeq (\mathfrak{H} \otimes_\psi \mathfrak{K}_1) \oplus (\mathfrak{H} \otimes_\psi \mathfrak{K}_2), \end{aligned}\right\} \tag{31}$$

where $\mathfrak{H}$, $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are all right $\mathcal{N}$-modules, while $\mathfrak{K}$, $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are left $\mathcal{N}$-modules.

Given the distributivity evidenced above, it is natural to inquire about the associativity of the relative tensor product. This issue is dealt with in the next Theorem.

THEOREM 3.6. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be two von Neumann algebras equipped with fns weights* φ *and* ψ, *respectively. If* $\mathfrak{H}$ *is a right* $\mathcal{M}$*-module,* $\mathfrak{K}$ *an* $\mathcal{M}$-$\mathcal{N}$ *bimodule and* $\mathfrak{M}$ *a right* $\mathcal{N}$*-module, then after natural identifications we have*

$$(\mathfrak{H} \otimes_\varphi \mathfrak{K}) \otimes_\psi \mathfrak{M} \simeq \mathfrak{H} \otimes_\varphi (\mathfrak{K} \otimes_\psi \mathfrak{M}), \tag{32}$$

as $\mathcal{L}(\mathfrak{H}_{\mathcal{M}})$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{M})^\circ$ *bimodules.*

Proof. For each $\xi \in \mathfrak{D}(\mathfrak{H}, \varphi)$, $\eta \in \mathfrak{K}$ and $\zeta \in \mathfrak{D}'(\mathfrak{M}, \psi)$, set

$$U((\xi \otimes_\varphi \eta) \otimes_\psi \zeta) \stackrel{\Delta}{=} \xi \otimes_\varphi (\eta \otimes_\psi \zeta).$$

Let ξ_i, η_i and ζ_i, $i = 1, 2$, denote elements in $\mathfrak{D}(\mathfrak{H}, \varphi)$, $\mathfrak{K}$ and $\mathfrak{D}'(\mathfrak{M}, \psi)$. We want to show

$$(U((\xi_1 \otimes_\varphi \eta_1) \otimes_\psi \zeta_1) \mid U((\xi_2 \otimes_\varphi \eta_2) \otimes_\psi \zeta_2)) = ((\xi_1 \otimes_\varphi \eta_1) \otimes_\psi \zeta_1 \mid (\xi_2 \otimes_\varphi \eta_2) \otimes_\psi \zeta_2),$$

as this will demonstrate that U is well-defined and a unitary. We compute

$$\begin{aligned}
(U((\xi_1 \otimes_\varphi \eta_1) \otimes_\psi \zeta_1) \mid U((\xi_2 \otimes_\varphi \eta_2) \otimes_\psi \zeta_2)) &= (\xi_1 \otimes_\varphi (\eta_1 \otimes_\psi \zeta_1) \mid \xi_2 \otimes_\varphi (\eta_2 \otimes_\psi \zeta_2)) \\
&= (\pi_{\mathfrak{K} \otimes_\psi \mathfrak{M}}(\mathrm{L}_\varphi(\xi_2)^* \mathrm{L}_\varphi(\xi_1))(\eta_1 \otimes_\psi \zeta_1) \mid \eta_2 \otimes_\psi \zeta_2) \\
&= ((\pi_{\mathfrak{K}}(\mathrm{L}_\varphi(\xi_2)^* \mathrm{L}_\varphi(\xi_1))\eta_1) \otimes_\psi \zeta_1 \mid \eta_2 \otimes_\psi \zeta_2) \\
&= (\pi'_{\mathfrak{K}}(\mathrm{J}_\psi \mathrm{R}_\psi(\zeta_1)^* \mathrm{R}_\psi(\zeta_2) \mathrm{J}_\psi) \pi_{\mathfrak{K}}(\mathrm{L}_\varphi(\xi_2)^* \mathrm{L}_\varphi(\xi_1))\eta_1 \mid \eta_2) \qquad \text{(by (26))} \\
&= (\pi_{\mathfrak{K}}(\mathrm{L}_\varphi(\xi_2)^* \mathrm{L}_\varphi(\xi_1)) \pi'_{\mathfrak{K}}(\mathrm{J}_\psi \mathrm{R}_\psi(\zeta_1)^* \mathrm{R}_\psi(\zeta_2) \mathrm{J}_\psi)\eta_1 \mid \eta_2) \\
&= (\xi_1 \otimes_\varphi (\pi'_{\mathfrak{K}}(\mathrm{J}_\psi \mathrm{R}_\psi(\zeta_1)^* \mathrm{R}_\psi(\zeta_2) \mathrm{J}_\psi)\eta_1 \mid \xi_2 \otimes_\varphi \eta_2) \\
&= (\pi'_{\mathfrak{H} \otimes_\varphi \mathfrak{K}}(\mathrm{J}_\psi \mathrm{R}_\psi(\zeta_1)^* \mathrm{R}_\psi(\zeta_2) \mathrm{J}_\psi)(\xi_1 \otimes_\varphi \eta_1) \mid \xi_2 \otimes_\varphi \eta_2) \\
&= ((\xi_1 \otimes_\varphi \eta_1) \otimes_\psi \zeta_1 \mid (\xi_2 \otimes_\varphi \eta_2) \otimes_\psi \zeta_2).
\end{aligned}$$

Now, for each $a \in \mathcal{L}(\mathfrak{H}_{\mathcal{M}})$ and $b \in \mathcal{L}({}_{\mathcal{N}}\mathfrak{M})^\circ$, we also have

$$\begin{aligned}
U(a((\xi \otimes_\varphi \eta) \otimes_\psi \zeta)b) &= U(((a\xi) \otimes_\varphi \eta) \otimes_\psi (\zeta b)) \\
&= (a\xi) \otimes_\varphi (\eta \otimes_\psi (\zeta b)) = a(\xi \otimes_\varphi (\eta \otimes_\psi \zeta))b \\
&= a(U((\xi \otimes_\varphi \eta) \otimes_\psi \zeta))b.
\end{aligned}$$

Hence, we see that U is indeed an isomorphism of $(\mathfrak{H} \otimes_\varphi \mathfrak{K}) \otimes_\psi \mathfrak{M}$ onto $\mathfrak{H} \otimes_\varphi (\mathfrak{K} \otimes_\psi \mathfrak{M})$ as $\mathcal{L}(\mathfrak{H}_{\mathcal{M}})$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{M})^\circ$ bimodules. □

It is natural at this point to ask what happens to the relative tensor product $\mathfrak{H} \otimes_\psi \mathfrak{K}$ when we change the fns reference weight ψ. In order to investigate this issue, let us first recall some notation: $\mathfrak{W}(\mathcal{N})$ is the set of semi-finite, normal weights on the von Neumann algebra $\mathcal{N}$, while $\mathfrak{W}_0(\mathcal{N})$ represents the set of all faithful such. Once again, we fix $\mathcal{N}$, the right $\mathcal{N}$-module $\mathfrak{H}$, and the left $\mathcal{N}$-module $\mathfrak{K}$.

THEOREM 3.7. *Let $\mathcal{N}$ be a von Neumann algebra, and let $\mathfrak{H}$ and $\mathfrak{K}$ be, respectively, right and left $\mathcal{N}$-modules. To each pair $(\psi_1, \psi_2) \in \mathfrak{W}_0(\mathcal{N}) \times \mathfrak{W}_0(\mathcal{N})$, there corresponds a unique $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})^\circ$ bimodule isomorphism, $U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}}$, from $\mathfrak{H} \otimes_{\psi_1} \mathfrak{K}$ onto $\mathfrak{H} \otimes_{\psi_2} \mathfrak{K}$, which makes the following diagram commute:*

$$\begin{array}{ccc}
\mathfrak{H} \otimes_{\psi_1} \mathfrak{K} & \xrightarrow{U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}}} & \mathfrak{H} \otimes_{\psi_2} \mathfrak{K} \\
\downarrow{\scriptstyle a_1 \otimes_{\psi_1} b_1^*} & & \downarrow{\scriptstyle a_2 \otimes_{\psi_2} b_2^*} \\
L^2(\mathcal{N}, \psi_1) \otimes_{\psi_1} L^2(\mathcal{N}, \psi_1) & & L^2(\mathcal{N}, \psi_2) \otimes_{\psi_2} L^2(\mathcal{N}, \psi_2) \\
\downarrow{\scriptstyle U^{\psi_1}_{L^2(\mathcal{N},\psi_1)}} & & \downarrow{\scriptstyle U^{\psi_2}_{L^2(\mathcal{N},\psi_2)}} \\
L^2(\mathcal{N}, \psi_1) & \xrightarrow{U_{\psi_2,\psi_1}} & L^2(\mathcal{N}, \psi_2)
\end{array} \qquad (33)$$

Here, we take the pair $(a_i, b_i) \in \mathcal{L}(\mathfrak{H}_{\mathcal{N}}, L^2(\mathcal{N}, \psi_i)_{\mathcal{N}}) \times \mathcal{L}({}_{\mathcal{N}}L^2(\mathcal{N}, \psi_i), {}_{\mathcal{N}}\mathfrak{K})$, *for* $i = 1, 2$, *such that* $a_2 = U_{\psi_2,\psi_1}a_1$, $b_2 = b_1 U_{\psi_2,\psi_1}$, *with* U_{ψ_2,ψ_1} *representing the canonical unitary which implements the equivalence of the standard forms, i.e.,*

$$U_{\psi_2,\psi_1}\colon \{\mathcal{N}, L^2(\mathcal{N}, \psi_1), \mathfrak{P}_{\psi_1}, J_{\psi_1}\} \to \{\mathcal{N}, L^2(\mathcal{N}, \psi_2), \mathfrak{P}_{\psi_2}, J_{\psi_2}\}.$$

Moreover, the correspondence

$$(\psi_1, \psi_2) \in \mathfrak{W}_0(\mathcal{N}) \times \mathfrak{W}_0(\mathcal{N}) \mapsto U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}$$

satisfies the chain rule, viz.,

$$U_{\mathfrak{H},\mathfrak{K}}^{\psi_3,\psi_2} U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1} = U_{\mathfrak{H},\mathfrak{K}}^{\psi_3,\psi_1}, \quad \psi_1, \psi_2, \psi_3 \in \mathfrak{W}_0(\mathcal{N}). \tag{34}$$

Proof. We start with existence: we will use the notation established in Theorem 3.3. Choose fns weights $\varphi \in \mathfrak{W}_0(\mathcal{P})$ and $\nu \in \mathfrak{W}_0(\mathcal{Q})$ and set $\rho_i = \psi_i \oplus \varphi \oplus \nu$, for $i = 1, 2$. We observe that the construction of $\mathcal{R}$ does not depend on the choice of the ψ's: there is a canonical isometry U_{ρ_2,ρ_1} from $L^2(\mathcal{R}, \rho_1)$ onto $L^2(\mathcal{R}, \rho_2)$. Moreover, this isometry implements an $\mathcal{R}$-$\mathcal{R}$ bimodule isomorphism. As the projections e, f and g commute with the fns weights ρ_1 and ρ_2 (by their definitions), it is easy to check that U_{ρ_2,ρ_1} preserves the matrix decompositions in of $L^2(\mathcal{R}, \rho_i)$, $i = 1, 2$, which were given by (15'). With J the conjugation operator $\overline{\eta} \in \overline{\mathfrak{K}} \mapsto \eta \in \mathfrak{K}$, set $b_i^\circ \stackrel{\Delta}{=} J b_i^* J \in \mathcal{L}(\overline{\mathfrak{K}}_{\mathcal{N}}, L^2(\mathcal{N}, \psi_i))$, $i = 1, 2$. We then have

$$\tilde{a}_i = \begin{pmatrix} 0 & a_i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \tilde{b}_i = \begin{pmatrix} 0 & 0 & b_i^\circ \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{R}, \ i = 1, 2.$$

Hence, we see that the restriction of the operators $\pi_{\rho_i}(\tilde{a}_i)\pi'_{\rho_i}(\tilde{b}_i)^*$ to the (2,3)-component of $L^2(\mathcal{R}, \rho_i)$ is equal to $U^{\psi_i}_{L^2(\mathcal{N},\psi_i)}(a \otimes_{\psi_i} b_i^*)$, with π'_{ρ_i} the semi-cyclic anti-representation of $\mathcal{R}$ defined by $\pi'_{\rho_i}(x) \stackrel{\Delta}{=} \mathrm{J}\pi_{\rho_i}(x)^*\mathrm{J}$, $x \in \mathcal{R}$. Since U_{ρ_2,ρ_1} is an $\mathcal{R}$-$\mathcal{R}$ bimodule isomorphism of $L^2(\mathcal{R}, \rho_1)$ onto $L^2(\mathcal{R}, \rho_2)$, and carries the matrix decomposition (15') of $L^2(\mathcal{R}, \rho_1)$ onto that of $L^2(\mathcal{R}, \rho_2)$, by restricting to the (1,1)-component, we obtain U_{ψ_2,ψ_1}; similarly, by considering the restriction of U_{ρ_2,ρ_1} to the (2,3)-component, we get $U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}$.

Now, let us turn to unicity. Let $\mathfrak{A}_i$ $(= \mathfrak{A}_{\psi_i})$, for $i = 1, 2$, be the left Hilbert algebras associated with $\{\mathcal{N}, \psi_i\}$, and $(\mathfrak{A}_i)_0$ be the corresponding Tomita algebras. Set $\mathfrak{a}_i = \mathfrak{n}_{\psi_i} \cap \mathfrak{n}_{\psi_i}^* = \pi_\ell(\mathfrak{A}_i)$, and $(\mathfrak{a}_i)_0 = \pi_\ell((\mathfrak{A}_i)_0)$, $i = 1, 2$. For each $\xi \in \mathfrak{D}(\mathfrak{H}, \psi_1)$, $\eta \in \mathfrak{D}'(\mathfrak{K}, \psi_1)$ and y_1, $y_2 \in (\mathfrak{a}_1)_0$, if we take $a_1 = \mathrm{L}_{\psi_1}(\xi)^*$ and $b_1 = \mathrm{R}_{\psi_1}(\eta)$, then by (33) we obtain

$$\begin{aligned} U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}(\xi y_1 \otimes_{\psi_1} y_2\eta) &= U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}(\mathrm{L}_{\psi_1}(\xi)\eta'_{\psi_1}(y_1) \otimes_{\psi_1} \mathrm{R}_{\psi_1}(\eta)\eta_{\psi_1}(y_2)) \\ &= U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}(\mathrm{L}_{\psi_1}(\xi) \otimes_{\psi_1} R_{\psi_1}(\eta))(\eta'_{\psi_1}(y_1) \otimes_{\psi_1} \eta_{\psi_1}(y_2)) \\ &= U_{\mathfrak{H},\mathfrak{K}}^{\psi_2,\psi_1}(a_1^* \otimes_{\psi_1} b_1)(U^{\psi_1}_{L^2(\mathcal{N},\psi_1)})^*(\sigma^{\psi_1}_{-i/2}(y_1)\eta_{\psi_1}(y_2)) \\ &= (a_2^* \otimes_{\psi_2} b_2)(U^{\psi_2}_{L^2(\mathcal{N},\psi_2)})^* U_{\psi_2,\psi_1}(\sigma^{\psi_1}_{-i/2}(y_1)\eta_{\psi_1}(y_2)). \end{aligned}$$

This means that $U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}}$ is uniquely determined on the vectors of the form

$$\{(\xi y_1)\otimes_{\psi_1}(y_2\eta)\colon\ \xi\in\mathfrak{D}(\mathfrak{H},\psi_1),\ \eta\in\mathfrak{D}'(\mathfrak{K},\psi_1),\ y_1,\ y_2\in(\mathfrak{a}_1)_0\},$$

which is dense in $\mathfrak{H}\otimes_{\psi_1}\mathfrak{K}$. Hence, $U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}}$ is uniquely determined by the commutative diagram of (33).

The chain rule (34) follows from the uniqueness of $U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}}$. □

At this point it is necessary to make the following remark: *The bimodule isomorphism $U^{\psi_2,\psi_1}_{\mathfrak{H},\mathfrak{K}}$ does not send $\xi\otimes_{\psi_1}\eta$ into $\xi\otimes_{\psi_2}\eta$ for $\xi\in\mathfrak{H}$ and $\eta\in\mathfrak{K}$.* One must always be careful not to make this mistake when performing calculations involving the relative tensor product.

Before concluding this section, we wish to address (briefly) the following: is it possible to construct the "tensor product" $\mathfrak{H}\otimes_{\mathcal{N}}\mathfrak{K}$ directly from the right $\mathcal{N}$-module $\mathfrak{H}$ and the left $\mathcal{N}$-module $\mathfrak{K}$, i.e., without recourse to a reference weight? It is, in fact, possible to do so if one abandons the notion of the tensor product $\xi\otimes_{\mathcal{N}}\eta$ of the vectors themselves. To see this, suppose we are given a von Neumann algebra $\mathcal{N}$, a right $\mathcal{N}$-module $\mathfrak{H}$, and a left $\mathcal{N}$-module $\mathfrak{K}$. We define $\tilde{\mathfrak{H}}\triangleq L^2(\mathcal{N})\oplus\mathfrak{H}\oplus\overline{\mathfrak{K}}$, recognizing that $L^2(\mathcal{N})$ has meaning independent of any choice of fns weight on $\mathcal{N}$. Then $\tilde{\mathfrak{H}}$ is a right $\mathcal{N}$-module in the obvious way, and we may view $L^2(\mathcal{N})$, $\mathfrak{H}$ and $\overline{\mathfrak{K}}$ as closed subspaces, with e, f and g the projections down to these; note once again that e, f and g are all projections in $\mathcal{R}=\mathcal{L}(\tilde{\mathfrak{H}}_{\mathcal{N}})$. Then we have seen that we have $\mathfrak{H}=fL^2(\mathcal{R})e$ and $\overline{\mathfrak{K}}=gL^2(\mathcal{R})e$, which implies that $\mathfrak{K}=eL^2(\mathcal{R})g$. We may then *define* the "relative tensor product of $\mathfrak{H}$ and $\mathfrak{K}$ over $\mathcal{N}$", $\mathfrak{H}\otimes_{\mathcal{N}}\mathfrak{K}$, to be $fL^2(\mathcal{R})g$. It is clear that, when defined in such a way, $\mathfrak{H}\otimes_{\mathcal{N}}\mathfrak{K}$ has a natural $\mathcal{L}(\mathfrak{H}_{\mathcal{N}})$-$\mathcal{L}({}_{\mathcal{N}}\mathfrak{K})^\circ$ bimodule structure. In fact, it is a straightforward exercise to show that there exists a bimodule isomorphism $\mathfrak{H}\otimes_{\mathcal{N}}\mathfrak{K}\to\mathfrak{H}\otimes_{\psi}\mathfrak{K}$ for any $\psi\in\mathfrak{W}_0(\mathcal{N})$.

3.2. *An example and a unicity theorem.* In order to make the ideas presented in the previous section more concrete, we begin this section with an example of the relative tensor product. While this example will deal exclusively with matrix algebras (and hence the spaces in question will be finite dimensional), all the essential notions regarding the relative tensor product will be evident. In particular, it is not the finite dimensionality which distinguishes this example; we will have more to say about this later.

Example. Let $\mathcal{M}$ be $M_n(\mathbb{C})$, $\mathfrak{H}=\{M_n(\mathbb{C}),(\cdot\mid\cdot)_{\mathfrak{H}}\}$, and $\psi=\mathrm{Tr}(H\cdot)$, $H\in\mathcal{M}_+$, non-sinugular. As any (faithful) positive linear functional on $\mathcal{M}$ is of this form, this, in fact, is the general case. We take $\mathfrak{H}$ to be the Hilbert space which arises, using ψ, via the GNS construction. In order to differentiate between elements of $\mathcal{M}$ and those in $\mathfrak{H}$, we will denote the latter using the usual $\eta_\psi(\cdot)$ notation, e.g., $(\eta_\psi(X)\mid\eta_\psi(Y))_{\mathfrak{H}}\triangleq\mathrm{Tr}(HY^*X)$.

$\mathfrak{H}$ has an $\mathcal{M}$-$\mathcal{M}$ bimodule structure, in which the left and right actions of $\mathcal{M}$ on $\mathfrak{H}$ are given by

$$A\eta_\psi(X) \triangleq \eta_\psi(AX), \qquad \eta_\psi(X)B \triangleq \eta_\psi(XH^{\frac{1}{2}}BH^{-\frac{1}{2}}), \tag{35}$$

where $A,\ B \in \mathcal{M}$. It is important to realize that, while the left action of $\mathcal{M}$ on $\mathfrak{H}$ coincides with the usual matrix multiplication, the right action is "twisted" via conjugation by $H^{\frac{1}{2}}$. This defintion of the right action is necessary in order to have $(\eta_\psi(X)B \mid \eta_\psi(Y))_{\mathfrak{H}} = (\eta_\psi(X) \mid \eta_\psi(Y)B^*)_{\mathfrak{H}}$.

We also note the following:

$$\begin{aligned} \mathrm{J}_\psi\eta_\psi(X) &= \eta_\psi(H^{\frac{1}{2}}X^*H^{-\frac{1}{2}}), \quad \Delta_\psi^{it}\eta_\psi(X) = \eta_\psi(H^{it}XH^{-it}) \\ &\Rightarrow \sigma_t^\psi(A) = H^{it}AH^{-it}. \end{aligned} \tag{36}$$

Now, define φ by $\varphi \triangleq \mathrm{Tr}(K\cdot)$, where K too is a positive, non-singular element of $\mathcal{M}$. How can we give a realization of $\mathfrak{H} \otimes_\varphi \mathfrak{H}$? More precisely, by combining the results of Theorem 3.7 with Proposition 3.5, we see that $\mathfrak{H} \otimes_\varphi \mathfrak{H} \simeq \mathfrak{H}$. (Note that $\mathfrak{H}$ is really just $L^2(\mathcal{M}, \psi)$.) What we would like to do is to exhibit this $\mathcal{M}$-$\mathcal{M}$ bimodule isomorphism explicitly.

We know that

$$\begin{aligned} &(\eta_\psi(X_1) \otimes_\varphi \eta_\psi(Y_1) \mid \eta_\psi(X_2) \otimes_\varphi \eta_\psi(Y_2))_{\mathfrak{H} \otimes_\varphi \mathfrak{H}} \\ &\quad = (\eta_\psi(X_1)\mathrm{J}_\psi \mathrm{R}_\psi(Y_1)^*\mathrm{R}_\psi(Y_2)\mathrm{J}_\psi \mid \eta_\psi(X_2))_{\mathfrak{H}} \end{aligned}$$

from (26). Using (36), we can compute $\mathrm{J}_\psi \mathrm{R}_\psi(Y_1)^*\mathrm{R}_\psi(Y_2)\mathrm{J}_\psi$; we obtain

$$\mathrm{J}_\psi \mathrm{R}_\psi(Y_1)^*\mathrm{R}_\psi(Y_2)\mathrm{J}_\psi = K^{-\frac{1}{2}}Y_1HY_2^*K^{-\frac{1}{2}}. \tag{37}$$

Now, using (35), we know how $K^{-\frac{1}{2}}Y_1HY_2^*K^{-\frac{1}{2}} \in \mathcal{M}$ acts (from the right) on $\mathfrak{H}$. Hence, we can calculate

$$\begin{aligned} &(\eta_\psi(X_1)K^{-\frac{1}{2}}Y_1HY_2^*K^{-\frac{1}{2}} \mid \eta_\psi(X_2))_{\mathfrak{H}} \\ &\quad = (\eta_\psi(X_1H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_1HY_2^*K^{-\frac{1}{2}}H^{-\frac{1}{2}}) \mid \eta_\psi(X_2))_{\mathfrak{H}} \\ &\quad = \mathrm{Tr}(HX_2^*X_1H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_1HY_2^*K^{-\frac{1}{2}}H^{-\frac{1}{2}}) \\ &\quad = \mathrm{Tr}(HY_2^*K^{-\frac{1}{2}}H^{\frac{1}{2}}X_2^*X_1H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_1) \\ &\quad = \mathrm{Tr}(H(X_2H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_2)^*X_1H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_1) \\ &\quad = (\eta_\psi(X_1H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_1) \mid \eta_\psi(X_2H^{\frac{1}{2}}K^{-\frac{1}{2}}Y_2))_{\mathfrak{H}}. \end{aligned}$$

So we see that the $\mathcal{M}$-$\mathcal{M}$ bimodule isomorphism is implemented by the map $\mathfrak{H} \otimes_\varphi \mathfrak{H} \to \mathfrak{H}$ given by

$$\eta_\psi(X) \otimes_\varphi \eta_\psi(Y) \ \mapsto\ \eta_\psi(XH^{\frac{1}{2}}K^{-\frac{1}{2}}Y). \quad \square$$

Let's examine our example further. Suppose we were interested in formulating a theory of "bimodule tensor products", and proceeded naively: then we would expect the elements of $\mathcal{M}$ to merely "move through" the $\otimes_\varphi$, i.e., we anticipate

$$\eta_\psi(X)A \otimes_\varphi \eta_\psi(Y) = \eta_\psi(X) \otimes_\varphi A\eta_\psi(Y). \tag{38}$$

Using calculations found in the example, we have

$$\begin{aligned}\eta_\psi(X)A \ \otimes_\varphi \ \eta_\psi(Y) &= \eta_\psi(XH^{\frac{1}{2}}AH^{-\frac{1}{2}}) \otimes_\varphi \eta_\psi(Y) \\ &\leftrightarrow \ \eta_\psi(XH^{\frac{1}{2}}AH^{-\frac{1}{2}}H^{\frac{1}{2}}K^{-\frac{1}{2}}Y) = \eta_\psi(XH^{\frac{1}{2}}AK^{-\frac{1}{2}}Y),\end{aligned}$$

while

$$\eta_\psi(X) \otimes_\varphi A\eta_\psi(Y) \leftrightarrow \eta_\psi(XH^{\frac{1}{2}}K^{-\frac{1}{2}}AY).$$

So, if we want (38), we must have $XH^{\frac{1}{2}}AK^{-\frac{1}{2}}Y = XH^{\frac{1}{2}}K^{-\frac{1}{2}}AY$, or equivalently $AK^{-\frac{1}{2}} = K^{-\frac{1}{2}}A$. This in turn yields $K^{\frac{1}{2}}AK^{-\frac{1}{2}} = A$, $\forall A \in \mathcal{M}$, which says that $\sigma^\varphi_{-i/2}(A) = A$. Hence we see that the modular automorphism group comprises only the identity automorphism, which says $\varphi = \mathrm{Tr}$, so $K = I$, the identity matrix in $\mathcal{M} = M_n(\mathbb{C})$.

Of course, we could have obtained the above directly from (29), which told us what happens to elements when they move through $\otimes_\varphi$. However, it was our intention to illustrate the theory derived in the previous section directly.

The example presented above is not entirely unmotivated. We now present a theorem which demonstrates that the relative tensor product is really the most natural product construction possible in the category of von Neumann bimodules. As the theorem will show, attempts to formulate a theory motivated solely by algebraic construction can succeed only under restrictive circumstances.

THEOREM 3.8. *Let $\mathcal{M}$ be a σ-finite von Neumann algebra. Take ψ a faithful state on $\mathcal{M}$, and let $\mathfrak{H}_\psi$ denote, as usual, $L^2(\mathcal{M}, \psi)$. Suppose there exists a $\mathbb{C}$-linear map* I: $\mathfrak{H}_\psi \times \mathfrak{H}_\psi \to \mathfrak{K}$, *where $\mathfrak{K}$ (like $\mathfrak{H}_\psi$) is a faithful $\mathcal{M}$-$\mathcal{M}$ bimodule. We also assume* I *is continuous in each variable separately, and satisfies the following conditions*:

(i)

$$\begin{aligned} a\,\mathrm{I}(\xi, \eta) &= \mathrm{I}(a\xi, \eta), \\ \mathrm{I}(\xi b, \eta) &= \mathrm{I}(\xi, b\eta), \\ \mathrm{I}(\xi, \eta c) &= \mathrm{I}(\xi, \eta)c, \end{aligned}$$

where a, b and c are in $\mathcal{M}$, $\xi,\ \eta \in \mathfrak{H}_\psi$.

(ii) $\mathrm{Span}_{\mathbb{C}}\{\mathrm{I}(\xi, \eta)\colon \xi,\ \eta \in \mathfrak{H}_\psi\}$ *is dense in $\mathfrak{K}$.*

(iii) I *is non-degenerate, i.e., for any $0 \neq \xi \in \mathfrak{H}_\psi$, there exists $\eta \in \mathfrak{H}_\psi$ such that* $\mathrm{I}(\xi, \eta) \neq 0$.

Then $\mathcal{M}$ is an atomic von Neumann algebra (hence semi-finite), and $\mathfrak{K} \simeq \mathfrak{H}_\psi \otimes_\tau \mathfrak{H}_\psi$, where τ is a faithful, normal and semi-finite trace on $\mathcal{M}$. (Note that τ may be a tracial weight; $\mathcal{M}$ may possess no tracial state.)

Proof. We will prove the above assertion in stages. We begin with an observation, viz., that the usual appeal to Uniform Boundedness allows us to conclude that $\exists\, 0 < C < +\infty$ such that

$$\| \mathrm{I}(\xi, \eta) \|_{\mathfrak{K}} \le C \|\xi\| \|\eta\| \quad \forall\, \xi, \eta \in \mathfrak{H}_\psi; \tag{39}$$

hence, I is actually jointly continuous.

Now, define $\xi_\psi \triangleq \eta_\psi(\mathbf{1}_{\mathcal{M}})$; then we know that ξ_ψ is both cyclic and separating for $\mathfrak{H}_\psi$ – so, $\{x\xi_\psi\colon x \in \mathcal{M}\}$ and $\{\xi_\psi y\colon y \in \mathcal{M}\}$ are both dense in $\mathfrak{H}$. Now, we have

$$\begin{aligned}
\mathfrak{K} &= \left[\{\mathrm{I}(\xi, \eta)\colon \xi, \eta \in \mathfrak{H}_\psi\}\right] = \left[\{\mathrm{I}(\xi_\psi x, y\xi_\psi)\colon x, y \in \mathcal{M}\}\right] \\
&= \left[\{\mathrm{I}(\xi_\psi, xy\xi_\psi)\colon x, y \in \mathcal{M}\}\right] = \left[\{\mathrm{I}(\xi_\psi, a\xi_\psi)\colon a \in \mathcal{M}\}\right] \\
&= \overline{\{\mathrm{I}(\xi_\psi, \xi_\psi)b\colon b \in \mathcal{M}\}}.
\end{aligned}$$

If we define $\eta_0 \in \mathfrak{K}$ as $\eta_0 \triangleq \mathrm{I}(\xi_\psi, \xi_\psi)$, then the preceding calculation shows that η_0 is separating for $\mathcal{M}$ in $\mathfrak{K}$ (since it is cyclic for $\mathcal{L}_{\mathcal{M}}(\mathfrak{K})$). A similar argument shows that $\overline{\{a\eta_0\colon a \in \mathcal{M}\}} = \mathfrak{K}$; hence η_0 is both cyclic and separating for $\mathcal{M}$ in $\mathfrak{K}$.

Certainly, we lose nothing if we assume that $\|\eta_0\|_{\mathfrak{K}} = 1$. Then, by introducing $\varphi \triangleq (\cdot\, \eta_0 \mid \eta_0)_{\mathfrak{K}}$, we see that $\varphi \in \mathfrak{S}_*(\mathcal{M})$, and $\mathfrak{K} \cong \mathfrak{H}_\varphi$. (Here, $\mathfrak{S}_*(\mathcal{M})$ represents the set of normal states of $\mathcal{M}$.) Now, we can compute

$$\begin{aligned}
\varphi(x^*x) &= (x^*x\eta_0 \mid \eta_0)_{\mathfrak{K}} = \|x\eta_0\|_{\mathfrak{K}}^2 \\
&= \|x\,\mathrm{I}(\xi_\psi, \xi_\psi)\|_{\mathfrak{K}}^2 = \|\,\mathrm{I}(x\xi_\psi, \xi_\psi)\|_{\mathfrak{K}}^2 \\
&\le C^2\|x\xi_\eta\|_{\mathfrak{H}_\psi}^2 = C^2\psi(x^*x), \quad \forall\, x \in \mathcal{M}.
\end{aligned} \tag{40}$$

Note that we have used (39). Hence, we see that $\varphi \le C^2\psi$. From the theory of the cocycle derivative (see [4]), this allows us to infer the following:

(i) The map $t \mapsto (D\varphi\colon D\psi)_t = u_t$ extends to a map $(z \mapsto u_z) \in \mathcal{A}_{\mathcal{M}}(\mathbb{D}_{1/2})$.
(ii) $\varphi(x^*x) = \psi(u_{-i/2}^* x^* x u_{-i/2}), \quad \forall\, x \in \mathcal{M}$.

Here,

$$\mathcal{A}_{\mathcal{M}}(\mathbb{D}_{1/2}) \triangleq \big\{f\colon \mathbb{D}_{1/2} \to \mathcal{M}\colon \text{f is analytic on the interior of } \mathbb{D}_{1/2}, \text{ and continuous and bounded on all } \mathbb{D}_{1/2}\big\},$$

where

$$\mathbb{D}_r \triangleq \begin{cases} \{z \in \mathbb{C}\colon -r \le \Im(z) \le 0\}, & \text{if } r \ge 0 \\ \{z \in \mathbb{C}\colon 0 \le \Im(z) \le -r\}, & \text{otherwise.} \end{cases}$$

We now note that $\varphi \leq C^2\psi$ tells us that $\mathcal{M}_a^\psi \subset \mathcal{M}_a^\varphi$, i.e., the σ-weakly dense *-subalgebra of ψ-analytic elements of $\mathcal{M}$ is actually contained in the set of φ-analytic elements. We can therefore compute as follows: $\forall\, a \in \mathcal{M}_a^\psi$, we have

$$a\xi_\psi = \xi_\psi \sigma_{i/2}^\psi(a),$$

while

$$a\eta_0 = \eta_0 \sigma_{i/2}^\varphi(a).$$

However,

$$\begin{aligned} a\eta_0 &= a\,\mathrm{I}(\xi_\psi, \xi_\psi) = \mathrm{I}(a\xi_\psi, \xi_\psi) = \mathrm{I}(\xi_\psi \sigma_{i/2}^\psi(a), \xi_\psi) \\ &= \mathrm{I}(\xi_\psi, \sigma_{i/2}^\psi(a)\xi_\psi) = \mathrm{I}(\xi_\psi, \xi_\psi \sigma_i^\psi(a)) = \mathrm{I}(\xi_\psi, \xi_\psi)\sigma_i^\psi(a) = \eta_0 \sigma_i^\psi(a). \end{aligned}$$

Hence we are forced to conclude

$$\sigma_i^\psi(a) = \sigma_{i/2}^\varphi(a), \quad \forall\, a \in \mathcal{M}_a^\psi. \tag{41}$$

Now, it is a fundamental property of the cocycle derivative $(D\varphi\colon D\psi)_t = u_t$ that

$$\sigma_t^\varphi(x) = u_t \sigma_t^\psi(x) u_t^*, \quad \forall\, x \in \mathcal{M},$$

or, equivalently,

$$\sigma_t^\varphi(x) u_t = u_t \sigma_t^\psi(x). \tag{42}$$

Therefore, $\forall\, a \in \mathcal{M}_a^\psi$, we obtain

$$\sigma_{-i/2}^\varphi(a) u_{-i/2} = u_{-i/2} \sigma_{-i/2}^\psi(a),$$

since the product of the analytic maps is again analytic, and these agree on all of $\mathbb{R}$, by virtue of (42). This gives us

$$\sigma_{i/2}^\varphi(a^*)^* u_{-i/2} = u_{-i/2} \sigma_{i/2}^\psi(a^*)^* \Longrightarrow u_{-i/2}^* \sigma_{i/2}^\varphi(a^*) = \sigma_{i/2}^\psi(a^*) u_{-i/2}^*.$$

Now, using (41), we obtain

$$u_{-i/2}^* \sigma_i^\psi(a^*) = \sigma_{i/2}^\psi(a^*) u_{-i/2}^*;$$

setting $b = \sigma_{i/2}^\psi(a^*)$, we then have

$$u_{-i/2}^* \sigma_{i/2}^\psi(b) = b u_{-i/2}^*, \quad \forall\, b \in \mathcal{M}_a^\psi. \tag{43}$$

Furthermore, we see from (43) that

$$u_{-i/2} b^* = \sigma_{i/2}^\psi(b)^* u_{-i/2} \Longrightarrow u_{-i/2} b^* = \sigma_{-i/2}^\psi(b^*) u_{-i/2};$$

hence we also obtain

$$\sigma^{\psi}_{-i/2}(b)u_{-i/2} = u_{-i/2}b, \quad \forall b \in \mathcal{M}^{\psi}_a. \tag{43'}$$

By combining (43) and (43'), we observe

$$u_{-i/2}u^*_{-i/2}b = u_{-i/2}\sigma^{\psi}_{-i/2}(b)u^*_{-i/2} = \sigma^{\psi}_{-i}(b)u_{-i/2}u^*_{-i/2}.$$

So, we define $h \triangleq u_{-i/2}u^*_{-i/2}$, and rewrite the above:

$$hb = \sigma^{\psi}_{-i}(b)h, \quad \forall b \in \mathcal{M}^{\psi}_a. \tag{44}$$

Now, we make use of a result from [8], which states that any positive, non-singular h which satisfies (44) on the set of analytic elements for the one parameter automorphism group σ^{ψ} must in fact be an *analytic generator* for σ^{ψ}, i.e., we must have

$$\sigma^{\psi}_t(x) = h^{it}xh^{-it}, \quad \forall x \in \mathcal{M}. \tag{45}$$

In particular, this means that σ^{ψ}_t is inner, and hence $\mathcal{M}$ is semi-finite.

Thus, we are lead to define $\tau \triangleq \psi_{h^{-1}}$, where

$$\psi_{h^{-1}}(x) \triangleq \lim_{\varepsilon \searrow 0} \psi(h^{-1}(\mathbf{1} + \varepsilon h)x), \quad x \in \mathcal{M}_+;$$

note that this makes sense, since, once again, $h \in \mathcal{M}_{\psi}$. Then, this gives $(D\tau\colon D\psi)_t = h^{-it}$, and $\sigma^{\tau}_t = \mathrm{id}$; so τ is a trace on $\mathcal{M}$. (However, notice that τ may be a tracial weight.) Now, let k be such that $(D\varphi\colon D\tau)_t = k^{it}$; we remark that, due to the fact that φ is a state, k is a non-singular, positive element in $\mathcal{M}$. From the chain rule for cocycle derivatives,

$$(D\varphi\colon D\tau)_t = (D\varphi\colon D\psi)_t\,(D\psi\colon D\tau)_t,$$

we may conclude that $k^{it} = u_t h^{it}$. Using (41), and the fact that $h \in \mathcal{M}_{\psi} \subset \mathcal{M}^{\psi}_a$, we have

$$\sigma^{\psi}_i(h) = \sigma^{\phi}_{i/2}(h) = k^{-\frac{1}{2}}hk^{\frac{1}{2}},$$

or

$$k^{\frac{1}{2}}h = hk^{\frac{1}{2}} \Longrightarrow kh = hk, \tag{46}$$

i.e., h and k commute.

So, we may write $u_t = k^{it}h^{-it} = (kh^{-1})^{it}$, which yields $u_{-i/2} = (kh^{-1})^{\frac{1}{2}}$, i.e., $u_{-i/2}$ is a positive element in $\mathcal{M}$. (Strictly speaking, $u_{-i/2} = (kh^{-1})^{\frac{1}{2}}$ is valid only on $h\mathfrak{K}$, the range of h; however, this set is dense in $\mathfrak{K}$, and since we already know that $u_{-i/2}$ is a bounded operator, its positivity follows by continuity.) But recall that, by definition, $h^{\frac{1}{2}} = |u^*_{-i/2}|$; this means that we must have $h^{\frac{1}{2}} = u^*_{-i/2} = u_{-i/2}$, and, from the above calculations we may conclude

$$k = h^2. \tag{47}$$

We will now demonsrate that $\mathfrak{K}$ is actually isomorphic, as an $\mathcal{M}$-$\mathcal{M}$ bimodule, to $\mathfrak{H}_\psi \otimes_\tau \mathfrak{H}_\psi$. We define the map V via

$$V\colon \mathrm{I}(x\xi_\psi, y\xi_\psi) \mapsto x\xi_\psi \otimes_\tau y\xi_\psi, \quad \forall x, y \in \mathcal{M},$$

noting that, given

$$\begin{aligned} \|xy\xi_\psi\|^2_{\mathfrak{H}_\psi} &= \psi(y^*x^*xy) = \tau(hy^*x^*xy) \\ &= \tau(xyhy^*x^*) \le \|yhy^*\|\,\tau(xx^*) = \|yhy^*\|\,\tau(x^*x), \end{aligned}$$

we have $\eta_\psi(\mathfrak{n}_\psi) \subset \mathcal{D}'(\mathfrak{H}_\psi, \tau)$. (In fact, such a fact characterizes τ as a trace.) This makes V well-defined.

This map is an isometry: first, we compute

$$\begin{aligned} \|\,\mathrm{I}(x\xi_\psi, y\xi_\psi)\|^2_{\mathfrak{K}} &= \|\,\mathrm{I}(x\sigma^\psi_{-i/2}(y)\xi_\psi, \xi_\psi)\|^2_{\mathfrak{K}} = \|x\sigma^\psi_{-i/2}(y)\eta_0\|^2_{\mathfrak{K}} \\ &= \|xh^{\frac{1}{2}}yh^{-\frac{1}{2}}\eta_0\|^2_{\mathfrak{K}} = \varphi(h^{-\frac{1}{2}}y^*h^{\frac{1}{2}}x^*xh^{\frac{1}{2}}yh^{-\frac{1}{2}}) \\ &= \tau(kh^{-\frac{1}{2}}y^*h^{\frac{1}{2}}x^*xh^{\frac{1}{2}}yh^{-\frac{1}{2}}) = \tau(h^2h^{-\frac{1}{2}}y^*h^{\frac{1}{2}}x^*xh^{\frac{1}{2}}yh^{-\frac{1}{2}}) \\ &= \tau(hy^*h^{\frac{1}{2}}x^*xh^{\frac{1}{2}}y) \\ &= \psi((xh^{\frac{1}{2}}y)^*xh^{\frac{1}{2}}y), \quad x \in \mathcal{M},\ y \in \mathcal{D}(\sigma^\psi_{-i/2}). \end{aligned}$$

Now, we calculate

$$\begin{aligned} \|x\xi_\psi \otimes_\tau y\xi_\psi\|^2_{\mathfrak{H}_\psi \otimes_\tau \mathfrak{H}_\psi} &= (x\xi_\psi \mathrm{J}_\tau \mathrm{R}_\tau(y\xi_\psi)^*\mathrm{R}_\tau(y\xi_\psi)\mathrm{J}_\tau \mid x\xi_\psi)_{\mathfrak{H}_\psi} \\ &= (x\xi_\psi yhy^* \mid x\xi_\psi)_{\mathfrak{H}_\psi} \\ &= (xh^{\frac{1}{2}}yhy^*h^{-\frac{1}{2}}\xi_\psi \mid x\xi_\psi)_{\mathfrak{H}_\psi} = \psi(x^*xh^{\frac{1}{2}}yhy^*h^{-\frac{1}{2}}) \\ &= \tau(hx^*xh^{\frac{1}{2}}yhy^*h^{-\frac{1}{2}}) = \tau(hy^*h^{\frac{1}{2}}x^*xh^{\frac{1}{2}}y) \\ &= \psi((xh^{\frac{1}{2}}y)^*xh^{\frac{1}{2}}y), \quad x \in \mathcal{M},\ y\xi_\psi \in \mathcal{D}'(\mathfrak{H}_\psi, \tau). \end{aligned}$$

Hence, V is an isometry, and can be extended to a map from $\mathfrak{K}$ onto $\mathfrak{H}_\psi \otimes_\tau \mathfrak{H}_\psi$. It is also immediate that $\mathfrak{K}$ and $\mathfrak{H}_\psi \otimes_\tau \mathfrak{H}_\psi$ have the same $\mathcal{M}$-$\mathcal{M}$ bimodule sturcture. Thus, V is the desired $\mathcal{M}$-$\mathcal{M}$ bimodule isomorphism.

Finally, we wish to demonstrate that $\mathcal{M}$ must be an atomic von Neumann algebra. To see this, we simply note that the map $x\xi_\psi \mapsto \eta_\tau(xh^{\frac{1}{2}})$ implements the standard isometry between $\mathfrak{H}_\psi$ and $\mathfrak{H}_\tau = L^2(\mathcal{M}, \tau)$. Because of the preceding argument, we may view I as a map from $\mathfrak{H}_\tau \times \mathfrak{H}_\tau \to \mathfrak{H}_\tau \otimes_\tau \mathfrak{H}_\tau$, via

$$\mathrm{I}\colon (\eta_\tau(xh^{\frac{1}{2}}), \eta_\tau(yh^{\frac{1}{2}})) \mapsto \eta_\tau(xh^{\frac{1}{2}}) \otimes_\tau \eta_\tau(yh^{\frac{1}{2}}).$$

However, $\mathfrak{H}_\tau \otimes_\tau \mathfrak{H}_\tau$ is isometrically isomorphic to $\mathfrak{H}_\tau$ as an $\mathcal{M}$-$\mathcal{M}$ bimodule under the map

$$\eta_\tau(xh^{\frac{1}{2}}) \otimes_\tau \eta_\tau(yh^{\frac{1}{2}}) \mapsto \eta_\tau(xh^{\frac{1}{2}} yh^{\frac{1}{2}}). \tag{48}$$

If we restrict ourselves to $\eta_\tau(\mathfrak{n}_\tau^* \cap \mathfrak{n}_\tau)$, i.e., the left Hilbert algebra $\mathfrak{A}_\tau$, then (48) is just telling us that the usual multiplication operation "lifts" to the relative tensor product $\mathfrak{H}_\tau \otimes_\tau \mathfrak{H}_\tau$. However, when we combine this with our previous results regarding the map I, specifically (39), we must conclude that (left or right) multiplication by *any* element of $\mathfrak{H}_\tau$ acts as a *bounded* operator on $\mathfrak{H}_\tau$. This can only be the case when $\mathcal{M}$ is atomic. □

We now see that the example with which we began this section is actually quite general: we have discovered that if we wish to define a "naive" tensor product of L^2-von Neumann modules we may do so only under very restrictive conditions, viz., when the von Neumann algebra is essentially a "matrix algebra".

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