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A DIFFERENTIAL COMPLEX FOR LOCALLY CONFORMAL CALIBRATED G_2-MANIFOLDS

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ABSTRACT. We characterize G_2-manifolds that are locally conformally equivalent to a calibrated one as those G_2-manifolds M for which the space of differential forms annihilated by the fundamental 3-form of M becomes a differential subcomplex of de Rham's complex. Special properties of the cohomology of this subcomplex are exhibited when the holonomy group of M can be reduced to a subgroup of G_2. We also prove a theorem of Nomizu type for this cohomology which permits its computation for compact calibrated G_2-nilmanifolds.

1. Introduction

G_2-manifolds are 7-dimensional Riemannian manifolds with a two-fold vector cross product ([BG], [Ca], [G1]–[G4]) identifying each tangent space with the pure imaginary Cayley numbers. Such a manifold M has a nowhere vanishing differential 3-form φ, called the *fundamental* 3-form of M. If φ is *closed*, then M is a *calibrated* G_2-manifold [HL1]–[HL2], a G_2 analog of a symplectic manifold; examples of compact calibrated G_2-manifolds are given in [F1]–[F2]. In particular, if φ is *closed* and *coclosed* then M has a subgroup of G_2 as holonomy group [G2]. Examples of complete G_2-manifolds with holonomy group G_2 have been constructed by Bryant and Salamon [BS]; the first examples of such manifolds in the *compact* case have been given by Joyce [J1], [J2].

In [FG] the first author of the present paper and Gray gave a classification of G_2-manifolds; there are 16 classes. According to this classification, in the present paper we consider the class $\mathcal{W}_2 \oplus \mathcal{W}_4$, that is, the class of all G_2-manifolds for which $d\varphi = \theta \wedge \varphi$, where θ is the differential 1-form on M which can be defined [C] by $\theta = -\frac{1}{4} * (*d\varphi \wedge \varphi)$, where $*$ denotes the Hodge star operator. The class $\mathcal{W}_2 \oplus \mathcal{W}_4$ contains all the calibrated G_2-manifolds; moreover, it is the class of all G_2-manifolds which are locally conformal calibrated.

Let $\Lambda^q(M)$ be the space of differential q-forms on M. Our main object here is the study of those manifolds in $\mathcal{W}_2 \oplus \mathcal{W}_4$ for which the sequence

$$(1) \qquad \cdots \longrightarrow \mathcal{B}^{q-1}(M) \xrightarrow{\hat{d}} \mathcal{B}^q(M) \xrightarrow{\hat{d}} \mathcal{B}^{q+1}(M) \longrightarrow \cdots$$

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is a differential complex (see in Section 2, Corollary 4). Here $\mathcal{B}^q(M)$ is the subspace of $\Lambda^q(M)$ defined by

$$\mathcal{B}^q(M) = \{\beta \in \Lambda^q(M) \mid \beta \wedge \varphi = 0\},$$

and $\hat{d}$ denotes the restriction to $\mathcal{B}^q(M)$ of the exterior differential d of M.

The complex (1) is called G_2-*coeffective complex*, because it is the analog of the coeffective complex for symplectic manifolds [Bou].

In Section 2 we show that if M is a G_2-manifold for which the sequence (1) is a differential complex, then M must be locally conformal to a calibrated G_2-manifold. Therefore, the manifolds in the class $\mathcal{W}_2 \oplus \mathcal{W}_4$ are characterized by the existence of the G_2-coeffective complex.

In Section 3 we study the ellipticity of the coeffective complex. In Proposition 3.3 we prove that such a complex is elliptic for any degree $q \neq 3$. Moreover, for any locally conformal calibrated G_2-manifold M, we obtain the relations between the coeffective cohomology groups $\hat{H}^q(\mathcal{B}(M))$ and the de Rham cohomology groups $H^q(M)$ of M.

In Section 4 we restrict our attention to the particular case of compact calibrated G_2-manifolds. We prove special properties of the cohomology groups of the complementary complex of (1) (see Theorem 4.1). In Theorem 4.2 these properties allow us to prove that, for $q \neq 3$, the coeffective cohomology groups $\hat{H}^q(\mathcal{B}(M))$ are completely determined by the de Rham cohomology $H^*(M)$ when the holonomy group of M is a subgroup of G_2 (see also Corollary 4.3 for $q = 3$). In other words, these groups become invariants of the topology of compact manifolds with Hol $\subseteq G_2$. Therefore, Theorem 4.2 and Corollary 4.3 provide obstructions for a compact G_2-manifold M to have a subgroup of G_2 as its holonomy group. In particular, these results imply the well-known topological conditions $b_3(M) \geq b_1(M)$ and $b_3(M) \geq b_0(M)$ proved by Bonan in [Bo].

The aim of Section 6 is to exhibit an example of compact calibrated G_2-manifold for which the isomorphisms in Theorem 4.2 and Corollary 4.3 fail. The main problem in constructing such an example is the difficulty of computing the G_2-coeffective cohomology.

For any compact nilmanifold $\Gamma\backslash K$, a well-known theorem of Nomizu [N], asserts that the Chevalley-Eilenberg cohomology $H^*(\mathfrak{K}^*)$ of the Lie algebra $\mathfrak{K}$ of K is isomorphic to the de Rham cohomology $H^*(\Gamma\backslash K)$. Hattori has extended Nomizu's theorem for compact completely solvable manifolds (see [H]). The goal of Section 5 is to obtain a similar result for the G_2-coeffective cohomology. In fact, in Theorem 5.3 we prove that, for $q \neq 3$, there exists a canonical isomorphism,

$$\hat{H}^q(\mathcal{B}(\Gamma\backslash K)) \cong \hat{H}^q(\mathcal{B}(\mathfrak{K}^*)),$$

between the coeffective cohomology of a compact calibrated G_2-nilmanifold $\Gamma\backslash K$ and the coeffective cohomology of the Lie algebra $\mathfrak{K}$ of K (see Corollary 5.4 for $q = 3$). This result permits us to compute in a very simple way the coeffective cohomology for a large family of calibrated G_2-manifolds. Moreover, Theorem 5.3 and Corollary 5.4 also hold for compact completely solvable calibrated G_2-manifolds.

In Section 6 we exhibit two examples of compact calibrated G_2-nilmanifolds, for which we study the G_2-coeffective cohomology. The first was given in [F1]. Using Nomizu's theorem proved in Section 5, we show that Theorem 4.2 and Corollary 4.3 hold for this manifold. However, we prove that for the second example (see Theorem 6.10 and Corollary 6.11) such results fail. Therefore, Theorem 4.2 and Corollary 4.3 do not hold for arbitrary calibrated G_2-manifolds.

2. The coeffective complex for locally conformal calibrated G_2-manifolds

Let M be a C^∞ Riemannian manifold of dimension 7 with metric $\langle\ ,\ \rangle$. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M and by $\mathfrak{F}(M)$ the algebra of C^∞ functions on M. A *2-fold vector cross product* on M is a tensor field P: $\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ satisfying the following axioms:

(i) $\langle P(X, Y), X\rangle = \langle P(X, Y), Y\rangle = 0$,
(ii) $\|P(X, Y)\|^2 = \|X\|^2\|Y\|^2 - \langle X, Y\rangle^2$,

for $X, Y \in \mathfrak{X}(M)$. A 7-dimensional Riemannian manifold M with a 2-fold vector cross product P is called a *G_2-manifold.* There is a representation of G_2 on each tangent space of M defined by means of the vector cross product P ([BG], [G2], [G4], [S]). The *fundamental 3-form* of M is given by

$$\varphi(X, Y, Z) = \langle P(X, Y), Z\rangle\,,$$

for $X, Y, Z \in \mathfrak{X}(M)$.

The inner product on $\Lambda^q(M)$ is given by

$$\langle \alpha, \beta\rangle\, \Omega_M = \alpha \wedge *\beta, \tag{2}$$

for $\alpha, \beta \in \Lambda^q(M)$, where Ω_M denotes the volume form on M. In [FG] it is proved that $\Lambda^q(M)$ splits orthogonally into G_2-irreducible components $\Lambda^q_l(M)$ of dimension l. The representation of G_2 on $\Lambda^1(M)$ is the irreducible 7-dimensional representation, and the representations of G_2 on $\Lambda^q(M)$ and $\Lambda^{7-q}(M)$ are the same because the Hodge star $*$: $\Lambda^q(M) \longrightarrow \Lambda^{7-q}(M)$ is an isometry. Therefore, it suffices to describe the representations of G_2 on $\Lambda^2(M)$ and $\Lambda^3(M)$. They are (see [Br], [C], [CMS], [FG], [J1], [J2], [S])

$$\begin{aligned}
\Lambda^2_7(M) &= \{*(\alpha \wedge *\varphi) \mid \alpha \in \Lambda^1(M)\},\\
\Lambda^2_{14}(M) &= \{\beta \in \Lambda^2(M) \mid \beta \wedge *\varphi = 0\},\\
\Lambda^3_1(M) &= \{f\varphi \mid f \in \mathfrak{F}(M)\},\\
\Lambda^3_7(M) &= \{*(\alpha \wedge \varphi) \mid \alpha \in \Lambda^1(M)\},\\
\Lambda^3_{27}(M) &= \{\gamma \in \Lambda^3(M) \mid \gamma \wedge \varphi = \gamma \wedge *\varphi = 0\}.
\end{aligned} \tag{3}$$

Now, from (2) and (3), it is easy to get

$$\Lambda^3_1(M) \oplus \Lambda^3_{27}(M) = \{\gamma \in \Lambda^3(M) \mid \gamma \wedge \varphi = 0\}, \tag{4}$$

$$\Lambda^4_7(M) \oplus \Lambda^4_{27}(M) = \{\lambda \in \Lambda^4(M) \mid \lambda \wedge \varphi = 0\}. \tag{5}$$

Recall ([Br], [C], [FG], [S]) that the G_2-manifold M is said to be *parallel* if $\nabla\varphi = 0$ (or equivalently, $d\varphi = d * \varphi = 0$); *calibrated* (or *almost parallel*) if $d\varphi = 0$; *locally conformal calibrated* if $d\varphi = \theta \wedge \varphi$, where θ is the differential 1-form on M given by $\theta = -\frac{1}{4} * (*d\varphi \wedge \varphi)$.

We need also the following:

LEMMA 2.1. *Let M be a G_2-manifold with fundamental 3-form φ. Then:*

(i) *For any differential 1-form α on M,*

$$*(*(\alpha \wedge \varphi) \wedge \varphi) = -4\alpha.$$

(ii) *If there is a differential 1-form μ on M such that $d\varphi = \mu \wedge \varphi$, then $\mu = -\frac{1}{4} * (*d\varphi \wedge \varphi)$ and M is locally conformal calibrated.*

Proof. Part (i) follows by a straightforward computation. Suppose that μ is a differential 1-form on M such that $d\varphi = \mu \wedge \varphi$. Then $*d\varphi = *(\mu \wedge \varphi)$. In this identity we take the wedge product by φ, obtaining

$$* d\varphi \wedge \varphi = *(\mu \wedge \varphi) \wedge \varphi. \tag{6}$$

Applying $*$ to both sides of (6) and using (i), we get

$$*(*d\varphi \wedge \varphi) = *(*(\mu \wedge \varphi) \wedge \varphi) = -4\mu,$$

which implies (ii). □

Definition 2.2. Let M be a G_2-manifold with fundamental 3-form φ. For each q with $0 \leq q \leq 7$, the space $\mathcal{B}^q(M)$ is defined by

$$\mathcal{B}^q(M) = \{\beta \in \Lambda^q(M) \mid \beta \wedge \varphi = 0\}.$$

Also, let $\mathcal{A}^q(M)$ be the orthogonal complement of $\mathcal{B}^q(M)$ in $\Lambda^q(M)$.

LEMMA 2.3. *Let M be a G_2-manifold. Then we have*

$$\begin{aligned} \mathcal{B}^q(M) &= \{0\} \quad \textit{for}\ 0 \leq q \leq 2, \\ \mathcal{B}^3(M) &= \Lambda^3_1(M) \oplus \Lambda^3_{27}(M), \\ \mathcal{B}^4(M) &= \Lambda^4_7(M) \oplus \Lambda^4_{27}(M), \\ \mathcal{B}^q(M) &= \Lambda^q(M) \quad \textit{for}\ 5 \leq q \leq 7. \end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{A}^q(M) &= \Lambda^q(M) \quad \textit{for}\, 0 \le q \le 2,\\ \mathcal{A}^3(M) &= \Lambda^3_7(M),\\ \mathcal{A}^4(M) &= \Lambda^4_1(M),\\ \mathcal{A}^q(M) &= \{0\} \quad \textit{for}\, 5 \le q \le 7.\end{aligned}$$

Proof. These formulas are consequences of (3)–(5). □

PROPOSITION 2.4. *Let M be a G_2-manifold with fundamental 3-form φ. Then M is locally conformal calibrated if and only if for any differential 3-form $\gamma \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$, the exterior differential $d\gamma$ belongs to $\Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$.*

Proof. Suppose that M is a locally conformal calibrated G_2-manifold. Then $d\varphi = \theta \wedge \varphi$. Let $\gamma \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$. From (4) it follows that

$$\begin{aligned}d\gamma \wedge \varphi &= d(\gamma \wedge \varphi) - \gamma \wedge d\varphi\\ &= -\gamma \wedge \theta \wedge \varphi\\ &= 0;\end{aligned}$$

using (5), this proves that $d\gamma \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$.

To show the converse, we observe that $d\varphi \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$ because $\varphi \in \Lambda^3_1(M)$. Consequently, we have

$$d\varphi = \theta \wedge \varphi + *\gamma, \tag{7}$$

where $\theta \wedge \varphi \in \Lambda^4_7(M)$ and $\gamma \in \Lambda^3_{27}(M)$. Thus $d\gamma \wedge \varphi = 0$, and we deduce that

$$\gamma \wedge d\varphi = d\gamma \wedge \varphi - d(\gamma \wedge \varphi) = 0. \tag{8}$$

Taking the wedge product by γ in (7), and using (8), we get

$$\begin{aligned}0 = \gamma \wedge d\varphi &= \gamma \wedge \theta \wedge \varphi + \gamma \wedge *\gamma\\ &= \gamma \wedge *\gamma,\end{aligned}$$

which implies that $\gamma = 0$. Then (7) becomes

$$d\varphi = \theta \wedge \varphi,$$

which, by Lemma 2.1, proves that M is locally conformal calibrated. □

COROLLARY 2.5. *Let M be a G_2-manifold. Then M is locally conformal calibrated if and only if there exists the complex*

$$
\begin{aligned}
0 \longrightarrow \mathcal{B}^3(M) &= \Lambda^3_1(M) \oplus \Lambda^3_{27}(M) \xrightarrow{\hat{d}} \mathcal{B}^4(M) \\
&= \Lambda^4_7(M) \oplus \Lambda^4_{27}(M) \xrightarrow{\hat{d}} \Lambda^5(M) \\
&\xrightarrow{d} \Lambda^6(M) \xrightarrow{d} \Lambda^7(M) \longrightarrow 0, \qquad (9)
\end{aligned}
$$

where $\hat{d}$ denotes the restriction to $\mathcal{B}^q(M)$ ($q = 3, 4$) of the exterior differential d of M.

Proof. From Proposition 2.4 it is clear that (9) is a complex if M is locally conformal calibrated. To prove the converse, let us first show that for any $f \in \mathfrak{F}(M)$ and $\gamma \in \mathcal{B}^3(M) = \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$ we have

$$\pi_4 \circ d(f\gamma) = f\pi_4 \circ d(\gamma), \qquad (10)$$

that is, the operator $\pi_4 \circ d$: $\mathcal{B}^3(M) \longrightarrow \mathcal{A}^4(M)$ is tensorial, where π_4 denotes the orthogonal projection of $\Lambda^4(M)$ onto $\mathcal{A}^4(M) = \Lambda^4_1(M)$. In fact, since $\gamma \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$, from (4) and (5) it follows that $df \wedge \gamma \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$, that is, $\pi_4(df \wedge \gamma) = 0$; thus $\pi_4 \circ d(f\gamma) = \pi_4(df \wedge \gamma) + \pi_4(fd\gamma) = f\pi_4(d\gamma)$, which shows (10).

Now suppose that (9) is a complex, that is, $d(\hat{d}\gamma) = 0$, for any $\gamma \in \mathcal{B}^3(M)$. Since $d\gamma = \pi_4 \circ d(\gamma) + \hat{d}\gamma$, applying d to this equality we get

$$d(\pi_4 \circ d(\gamma)) = 0, \qquad (11)$$

for any $\gamma \in \mathcal{B}^3(M)$. Therefore, if f is any function on M, from (10) and (11) we get

$$0 = d(\pi_4 \circ d(f\gamma)) = d(f\,\pi_4 \circ d(\gamma)) = df \wedge \pi_4 \circ d(\gamma).$$

Since $\pi_4 \circ d(\gamma) \in \Lambda^4_1(M)$, there is $h_\gamma \in \mathfrak{F}(M)$ such that $\pi_4 \circ d(\gamma) = h_\gamma * \varphi$ and thus $h_\gamma(df \wedge *\varphi) = 0$, for any $f \in \mathfrak{F}(M)$. But $\alpha \wedge *\varphi = 0$ iff $\alpha = 0$, for $\alpha \in \Lambda^1(M)$, which implies that the function h_γ must be zero.

Therefore, $\pi_4 \circ d(\gamma) = 0$ for any $\gamma \in \mathcal{B}^3(M)$, that is, $d(\mathcal{B}^3(M)) \subset \mathcal{B}^4(M)$, and Proposition 2.4 implies that M is locally conformal calibrated. $\square$

For a locally conformal calibrated G_2-manifold M, we denote by $\hat{H}^*(\mathcal{B}(M))$ the cohomology of the complex (9). Then $\hat{H}^q(\mathcal{B}(M)) = H^q(M)$ for $q = 6, 7$. Therefore, to find the cohomology of the complex (9) it suffices to find the cohomology groups $\hat{H}^q(\mathcal{B}(M))$ for $3 \leq q \leq 5$. We need to consider another complex.

Definition 2.6. Let M be a G_2-manifold. For $0 \leq q \leq 3$, the map $\check{d}_q$: $\mathcal{A}^q(M) \longrightarrow \mathcal{A}^{q+1}(M)$ is defined by

$$\check{d}_q = \pi_{q+1} \circ d, \qquad (12)$$

where π_{q+1}: $\Lambda^{q+1}(M) \longrightarrow \mathcal{A}^{q+1}(M)$ is the orthogonal projection of $\Lambda^{q+1}(M)$ onto $\mathcal{A}^{q+1}(M)$.

From Lemma 2.3 it follows that $\check{d}_q = d$ for $q = 0, 1$. It will be convenient to make no distinction to denote the maps $\check{d}_2$ and $\check{d}_3$. In fact, unless clarity is required, we write $\check{d}$ for each of these maps.

PROPOSITION 2.7. *Let M be a G_2-manifold with fundamental 3-form φ. Then M is locally conformal calibrated if and only if the sequence*

$$(13)\quad 0 \longrightarrow \Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \xrightarrow{\check{d}_2} \Lambda^3_7(M) \xrightarrow{\check{d}_3} \Lambda^4_1(M) \longrightarrow 0$$

is a complex.

Proof. Consider $\alpha \in \Lambda^1(M)$. From (12) we see that $\check{d}_2(d\alpha) = \pi_3 \circ d(d\alpha) = 0$. This proves that $\check{d}_2 \circ d = 0$. Now, let us suppose that M is locally conformal calibrated, and let $\beta \in \Lambda^2(M)$. Using the fact that $\Lambda^3(M) = \Lambda^3_1(M) \oplus \Lambda^3_7(M) \oplus \Lambda^3_{27}(M)$, we have

$$(14)\qquad d\beta = \check{d}_2\beta + \gamma,$$

where $\check{d}_2\beta \in \mathcal{A}^3(M) = \Lambda^3_7(M)$ and $\gamma \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$. Proposition 2.4 implies that $d\gamma \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. Then taking in (14) the exterior differential d of M, we obtain

$$0 = d(\check{d}_2\beta) + d\gamma,$$

which means that $d(\check{d}_2\beta) \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. Thus $\check{d}_3(\check{d}_2\beta) = 0$ because $\check{d}_3(\check{d}_2\beta)$ is the image of $d(\check{d}_2\beta)$ by the orthogonal projection π_4: $\Lambda^4(M) \longrightarrow \mathcal{A}^4(M) = \Lambda^4_1(M)$.

To prove the converse, let β be a 2-form on M. Therefore, the exterior differential $d\beta$ of β is

$$(15)\qquad d\beta = \check{d}_2\beta + \gamma,$$

where $\check{d}_2\beta \in \Lambda^3_7(M)$ and $\gamma \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$. Applying in (15) the exterior differential d of M, we get

$$(16)\qquad 0 = d(\check{d}_2\beta) + d\gamma.$$

Applying the projection π_4 to (16), and using (12) together with the hypothesis $\check{d}_3 \circ \check{d}_2 = 0$, we obtain

$$\begin{aligned} 0 &= \pi_4(d(\check{d}_2\beta)) + \pi_4(d\gamma) \\ &= \check{d}_3 \circ \check{d}_2(\beta) + \pi_4(d\gamma) \\ &= \pi_4(d\gamma), \end{aligned}$$

which means that $d\gamma \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. Moreover, using (10) we conclude that $d(\Lambda^3_1(M) \oplus \Lambda^3_{27}(M)) \subset \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. From Proposition 2.4 it follows that M is locally conformal calibrated. $\square$

For a locally conformal calibrated G_2-manifold M, we denote by $\check{H}^*(\mathcal{A}(M))$ the cohomology of the complex (13). Then $\check{H}^q(\mathcal{A}(M)) = H^q(M)$ for $q = 0, 1$. Therefore, to find the cohomology of the complex (13) it is sufficient to find the cohomology groups $\check{H}^q(\mathcal{A}(M))$ for $2 \leq q \leq 4$.

3. Ellipticity of the coeffective complex

In this section, we suppose that M is a locally conformal calibrated G_2-manifold with fundamental 3-form φ. First we study the ellipticity of the complex $(\mathcal{A}^*(M), \check{d})$.

PROPOSITION 3.1. *The complex* $(\mathcal{A}^*(M), \check{d})$ *given by* (13) *is elliptic in degree* q *for any* $q \neq 2$.

Proof. It is obvious that the complex $(\mathcal{A}^*(M), \check{d})$ is elliptic in degrees 0 and 1, because the de Rham complex $(\Lambda^*(M), d)$ of M is elliptic. The complex $(\mathcal{A}^*(M), \check{d})$ is elliptic in degrees 3 and 4 if for any point $m \in M$ and for any 1-form μ non-zero at m, the complex

$$\Lambda^2(T_m^*M) \xrightarrow{\sigma_\mu(\check{d}_2)} \Lambda_7^3(T_m^*M) \xrightarrow{\sigma_\mu(\check{d}_3)} \Lambda_1^4(T_m^*M) \longrightarrow 0$$

is exact in the steps 3 and 4, where T_m^*M is the cotangent space of M at m, and

$$\sigma_\mu(\check{d}_2)(\beta) = \pi_3(\mu \wedge \beta), \tag{17}$$

$$\sigma_\mu(\check{d}_3)(\gamma) = \pi_4(\mu \wedge \gamma), \tag{18}$$

for $\beta \in \Lambda^2(T_m^*M)$ and $\gamma \in \Lambda_7^3(T_m^*M)$. Therefore, to prove that the complex $(\mathcal{A}^*(M), \check{d})$ is elliptic in degree $q = 3$ it is sufficient to prove that

$$\operatorname{Ker}(\sigma_\mu(\check{d}_3)) \subset \operatorname{Im}(\sigma_\mu(\check{d}_2)). \tag{19}$$

Let $\gamma \in \Lambda_7^3(T_m^*M)$ be such that $\gamma \in \operatorname{Ker}(\sigma_\mu(\check{d}_3))$, or equivalently $\pi_4(\mu \wedge \gamma) = 0$. This implies that $\mu \wedge \gamma \in \Lambda_7^4(T_m^*M) \oplus \Lambda_{27}^4(T_m^*M)$, and so $\mu \wedge \gamma \wedge \varphi_m = 0$. Since $\gamma \wedge \varphi_m \in \Lambda^6(T_m^*M)$, from the ellipticity of the de Rham complex it follows that there is $\eta \in \Lambda^5(T_m^*M)$ satisfying

$$\gamma \wedge \varphi_m = \mu \wedge \eta. \tag{20}$$

Now, we use the isomorphism $\wedge\varphi_m \colon \Lambda^2(T_m^*M) \longrightarrow \Lambda^5(T_m^*M)$ given by $\wedge\varphi_m(\beta) = \beta \wedge \varphi_m$, for $\beta \in \Lambda^2(T_m^*M)$. This isomorphism implies that there is $\nu \in \Lambda^2(T_m^*M)$ such that $\eta = \nu \wedge \varphi_m$. Thus (20) becomes

$$\gamma \wedge \varphi_m = \mu \wedge \nu \wedge \varphi_m = \pi_3(\mu \wedge \nu) \wedge \varphi_m.$$

Therefore, we have

$$(\gamma - \pi_3(\mu \wedge \nu)) \wedge \varphi_m = 0. \tag{21}$$

But the wedge product by φ_m is also an isomorphism $\wedge\varphi_m\colon \Lambda^3_7(T^*_mM) \longrightarrow \Lambda^6(T^*_mM)$ and so, from (21), it follows that $\gamma - \pi_3(\mu\wedge\nu) = 0$, or equivalently using (17),

$$\gamma = \pi_3(\mu\wedge\nu) = \sigma_\mu(\check{d}_2)(\nu),$$

which proves (19).

To prove the ellipticity of the complex $(\mathcal{A}^*(M), \check{d})$ in degree $q = 4$, we show

$$\Lambda^4_1(T^*_mM) \subset \operatorname{Im}(\sigma_\mu(\check{d}_3)).$$

Let $\lambda \in \Lambda^4_1(T^*_mM)$. Then $\lambda\wedge\varphi_m \in \Lambda^7(T^*_mM)$. Now, from the ellipticity of the de Rham complex of M, we conclude that

$$\mu\wedge\omega = \lambda\wedge\varphi_m, \tag{22}$$

for some $\omega \in \Lambda^6(T^*_mM)$. Using the isomorphism $\wedge\varphi_m\colon \Lambda^3_7(T^*_mM) \longrightarrow \Lambda^6(T^*_mM)$ again, we obtain $\omega = \gamma\wedge\varphi_m$ for some $\gamma \in \Lambda^3_7(T^*_mM)$. Then (22) becomes

$$\lambda\wedge\varphi_m = \mu\wedge\gamma\wedge\varphi_m = \pi_4(\mu\wedge\gamma)\wedge\varphi_m,$$

which implies that

$$(\lambda - \pi_4(\mu\wedge\gamma))\wedge\varphi_m = 0. \tag{23}$$

But $\wedge\varphi_m\colon \Lambda^4_1(T^*_mM) \longrightarrow \Lambda^7(T^*_mM)$ is an isomorphism, and hence, from (23), we have

$$\lambda = \pi_4(\mu\wedge\gamma) = \sigma_\mu(\check{d}_3)(\gamma).$$

Thus $\lambda \in \operatorname{Im}(\sigma_\mu(\check{d}_3))$. This completes the proof. □

Remark 3.2. We note that the complex $(\mathcal{A}^*(M), \check{d})$ is not elliptic in degree $q = 2$, because

$$\sum_{q=0}^{4}(-1)^q \dim(\mathcal{A}^q(T^*_mM)) = 1 - 7 + 21 - 7 + 1 = 9 \neq 0.$$

PROPOSITION 3.3. *The complex $(\mathcal{B}^*(M), \hat{d})$ given by* (9) *is elliptic in degree q for any $q \neq 3$.*

Proof. It is obvious that the complex $(\mathcal{B}^*(M), \hat{d})$ is elliptic in degrees 6 and 7, because it is the de Rham complex of M. To show that $(\mathcal{B}^*(M), \hat{d})$ is elliptic in degree $q = 4$, we must prove that for $m \in M$ and for non-zero $\mu \in T^*_mM$, the complex

$$\Lambda^3_1(T^*_mM) \oplus \Lambda^3_{27}(T^*_mM) \xrightarrow{\mu\wedge\cdot} \Lambda^4_7(T^*_mM) \oplus \Lambda^4_{27}(T^*_mM) \xrightarrow{\mu\wedge\cdot} \Lambda^5(T^*_mM) \tag{24}$$

is exact in degree 4. Let $\omega \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$ satisfy $\mu \wedge \omega = 0$. We must show that there is $\gamma \in \Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M)$ such that $\omega = \mu \wedge \gamma$. From the ellipticity of the de Rham complex we know that there exists $\gamma_1 \in \Lambda^3(T^*_m M)$ such that

$$\omega = \mu \wedge \gamma_1. \tag{25}$$

Moreover, $\gamma_1 = \gamma_1' + \gamma_1''$ with $\gamma_1' \in \Lambda^3_7(T^*_m M)$ and $\gamma_1'' \in \Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M)$. Now (25) becomes

$$\omega = \mu \wedge \gamma_1 = \mu \wedge \gamma_1' + \mu \wedge \gamma_1''. \tag{26}$$

But ω and $\mu \wedge \gamma_1'' \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$; hence $\pi_4(\mu \wedge \gamma_1') = 0$, that is, $\gamma_1' \in \operatorname{Ker}(\sigma_\mu(\check{d}_3))$. From Proposition 3.1 it follows that $\gamma_1' \in \operatorname{Im}(\sigma_\mu(\check{d}_2))$. This means that there exists $\beta \in \Lambda^2(T^*_m M)$ such that $\gamma_1' = \pi_3(\mu \wedge \beta)$.

Let $\nu \in \Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M)$ be the image of $\mu \wedge \beta$ by the orthogonal projection of $\Lambda^3(T^*_m M)$ onto $\Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M)$. Then we get

$$0 = \mu \wedge (\mu \wedge \beta) = \mu \wedge \gamma_1' + \mu \wedge \nu,$$

or equivalently

$$\mu \wedge \gamma_1' = -\mu \wedge \nu. \tag{27}$$

From (25), (26) and (27) we obtain

$$\omega = \mu \wedge (-\nu + \gamma_1''). \tag{28}$$

Now (28) implies that the form $\gamma = -\nu + \gamma_1''$ is such that $\gamma \in \Lambda^3_1(T^*_m M) \oplus \Lambda^3_{27}(T^*_m M)$ and $\omega = \mu \wedge \gamma$. This proves that (24) is exact in degree 4.

Finally, we must prove that the complex

$$\Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M) \xrightarrow{\mu\wedge\cdot} \Lambda^5(T^*_m M) \xrightarrow{\mu\wedge\cdot} \Lambda^6(T^*_m M)$$

is exact in degree 5. Let $\xi \in \Lambda^5(T^*_m M)$ satisfy $\mu \wedge \xi = 0$. We must find a 4-form $\lambda \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$ such that

$$\xi = \mu \wedge \lambda. \tag{29}$$

By the ellipticity of the de Rham complex of M we see that there is $\nu \in \Lambda^4(T^*_m M)$ such that

$$\xi = \mu \wedge \nu. \tag{30}$$

Because $\Lambda^4(T^*_m M) = \Lambda^4_1(T^*_m M) \oplus \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$ and $\nu \in \Lambda^4(T^*_m M)$ we have

$$\nu = \nu' + \nu'', \tag{31}$$

where $\nu' \in \Lambda^4_1(T^*_m M)$ and $\nu'' \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$. Using Proposition 3.1, we deduce that there exists $\gamma \in \Lambda^3_7(T^*_m M)$ such that

$$\nu' = \pi_4(\mu \wedge \gamma). \tag{32}$$

From (32) it follows that

$$0 = \mu \wedge (\mu \wedge \gamma) = \mu \wedge \nu' + \mu \wedge \tau, \tag{33}$$

where τ is the image of $\mu \wedge \gamma$ by the orthogonal projection of $\Lambda^4(T^*_m M)$ onto the subspace $\Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$. The identity (33) implies that $\mu \wedge \nu' = -\mu \wedge \tau$. Thus from (30) and (31) we conclude that

$$\xi = \mu \wedge (-\tau + \nu'').$$

Consider $\lambda = -\tau + \nu''$. Then $\lambda \in \Lambda^4_7(T^*_m M) \oplus \Lambda^4_{27}(T^*_m M)$, and moreover $\xi = \mu \wedge \lambda$. This proves (29) and completes the proof. □

Remark 3.4. We note that the complex $(\mathcal{B}^*(M), \hat{d})$ is not elliptic in degree $q = 3$, because

$$\sum_{q=3}^{7} (-1)^q \dim(\mathcal{B}^q(T^*_m M)) = -28 + 34 - 21 + 7 - 1 = -9 \neq 0.$$

From Proposition 3.1 and Proposition 3.3 we have the following result.

COROLLARY 3.5. *For any compact locally conformal calibrated G_2-manifold M, the cohomology groups $\check{H}^3(\mathcal{A}(M))$, $\check{H}^4(\mathcal{A}(M))$, $\hat{H}^4(\mathcal{B}(M))$ and $\hat{H}^5(\mathcal{B}(M))$ are of finite dimension.*

In order to obtain the first relations among the groups $H^q(M)$, $\hat{H}^q(\mathcal{B}(M))$ and $\check{H}^q(\mathcal{A}(M))$, we proceed as follows. From (9) and (13) we can consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \Lambda^3_1 \oplus \Lambda^3_{27} \arrow[r, "\hat{d}"] \arrow[d, "i"] & \Lambda^4_7 \oplus \Lambda^4_{27} \arrow[r, "\hat{d}"] \arrow[d, "i"] & \Lambda^5 \arrow[r, "d"] \arrow[d, equal] & \Lambda^6 \arrow[r, "d"] \arrow[d, equal] & \Lambda^7 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda^0 \arrow[r, "d"] \arrow[d, equal] & \Lambda^1 \arrow[r, "d"] \arrow[d, equal] & \Lambda^2 \arrow[r, "d"] \arrow[d, equal] & \Lambda^3 \arrow[r, "d"] \arrow[d, "p"] & \Lambda^4 \arrow[r, "d"] \arrow[d, "p"] & \Lambda^5 \arrow[r, "d"] & \Lambda^6 \arrow[r, "d"] & \Lambda^7 \arrow[r] & 0 \\
0 \arrow[r] & \Lambda^0 \arrow[r, "d"] & \Lambda^1 \arrow[r, "d"] & \Lambda^2 \arrow[r, "\check{d}"] & \Lambda^3_7 \arrow[r, "\check{d}"] & \Lambda^4_1 \arrow[r] & 0,
\end{tikzcd} \tag{34}$$

where i and p denote the natural inclusion and the orthogonal projection, respectively. Using the definitions of $\hat{d}$ and $\check{d}$, given by (9) and (12), respectively, it follows that i and p are cochain maps, that is,

$$d \circ i = i \circ \hat{d} \quad \text{and} \quad \check{d} \circ p = p \circ d. \tag{35}$$

From (34) and (35) we get the *short exact sequence* of differential complexes

$$0 \longrightarrow \mathcal{B}^*(M) \xrightarrow{i} \Lambda^*(M) \xrightarrow{p} \mathcal{A}^*(M) \longrightarrow 0.$$

Therefore, there is the long exact sequence of cohomology groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(M) & \overset{H^2(p)}{\longrightarrow} & \check{H}^2(\mathcal{A}(M)) & \overset{H^2(d)}{\longrightarrow} \\ \hat{H}^3(\mathcal{B}(M)) & \overset{H^3(i)}{\longrightarrow} & H^3(M) & \overset{H^3(p)}{\longrightarrow} & \check{H}^3(\mathcal{A}(M)) & \overset{H^3(d)}{\longrightarrow} \\ \hat{H}^4(\mathcal{B}(M)) & \overset{H^4(i)}{\longrightarrow} & H^4(M) & \overset{H^4(p)}{\longrightarrow} & \check{H}^4(\mathcal{A}(M)) & \overset{H^4(d)}{\longrightarrow} \\ \hat{H}^5(\mathcal{B}(M)) & \overset{H^5(i)}{\longrightarrow} & H^5(M) & \longrightarrow & 0, & \end{array} \tag{36}$$

where $H^j(i)$ and $H^q(p)$ $(3 \leq j \leq 5, 2 \leq q \leq 4)$ are the naturally induced maps, and $H^q(d)$ $(2 \leq q \leq 4)$ is the connecting homomorphism given by

$$H^q(d)([\alpha]_{\mathcal{A}}) = [d\alpha]_{\mathcal{B}} \in \hat{H}^{q+1}(\mathcal{B}(M))$$

for $[\alpha]_{\mathcal{A}} \in \check{H}^q(\mathcal{A}(M))$.

PROPOSITION 3.6. *Let M be a locally conformal calibrated G_2-manifold. Then*:

(i) $H^2(M) \cong \check{H}^2(\mathcal{A}(M))$ *if and only if* $H^2(d) = 0$;
(ii) $H^3(M) \cong \check{H}^3(\mathcal{A}(M)) \oplus \hat{H}^3(\mathcal{B}(M))$ *if and only if* $H^2(d) = H^3(d) = 0$;
(iii) $H^4(M) \cong \check{H}^4(\mathcal{A}(M)) \oplus \hat{H}^4(\mathcal{B}(M))$ *if and only if* $H^3(d) = H^4(d) = 0$;
(iv) $H^5(M) \cong \hat{H}^5(\mathcal{B}(M))$ *if and only if* $H^4(d) = 0$.

Proof. These relations are an easy consequence of the exactness of the sequence (36). □

Let M be a compact locally conformal calibrated G_2-manifold; and let δ denote the coderivative of M. If η is a differential q-form on M we have $\delta\eta = (-1)^q * d * \eta$. On the space $\Lambda^q(M)$ of differential q-forms on M we consider the inner product $(\ ,\ )$ given by

$$(\eta, \mu) = \int_M \eta \wedge *\mu$$

for $\eta, \mu \in \Lambda^q(M)$.

LEMMA 3.7. *The operator δ is the adjoint of $\check{d}$, that is, for $\beta \in \Lambda^2(M)$, $\gamma \in \Lambda^3_7(M)$, $\lambda \in \Lambda^4_1(M)$*

$$(\check{d}_2\beta, \gamma) = (\beta, \delta\gamma), \tag{37}$$

and

$$(\check{d}_3\gamma, \lambda) = (\gamma, \delta\lambda). \tag{38}$$

Proof. Because δ is the adjoint of d we have

$$(d\eta, \mu) = (\eta, \delta\mu) \tag{39}$$

for $\eta, \mu \in \Lambda^*(M)$. To prove (37), we use (39) and the decomposition $d\beta = \check{d}_2\beta + \widetilde{\beta}$, where $\widetilde{\beta} \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$. Thus $(\widetilde{\beta}, \gamma) = 0$, and we obtain

$$(\beta, \delta\gamma) = (d\beta, \gamma) = (\check{d}_2\beta, \gamma),$$

which implies (37). Also (38) is an easy consequence of (39) and $d\gamma = \check{d}_3\gamma + \widetilde{\gamma}$, where $\widetilde{\gamma} \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. $\square$

Definition 3.8. We let

$$\check{\mathcal{H}}^3(\mathcal{A}(M)) = \{\gamma \in \Lambda^3_7(M) \mid \check{d}\gamma = \delta\gamma = 0\}, \tag{40}$$

$$\check{\mathcal{H}}^4(\mathcal{A}(M)) = \{f * \varphi \in \Lambda^4_1(M) \mid \delta(f * \varphi) = 0\}. \tag{41}$$

The spaces $\check{\mathcal{H}}^q(\mathcal{A}(M))$ can also be defined as follows:

PROPOSITION 3.9. *Let M be a compact locally conformal calibrated G_2-manifold with fundamental 3-form φ. Then*

$$\check{\mathcal{H}}^3(\mathcal{A}(M)) = \{\gamma \in \Lambda^3_7(M) \mid d\gamma \wedge \varphi = 0 \quad \text{and} \quad d * \gamma = 0\}, \tag{42}$$

$$\check{\mathcal{H}}^4(\mathcal{A}(M)) = \{f * \varphi \in \Lambda^4_1(M) \mid d(f\varphi) = 0\}. \tag{43}$$

Proof. From (40) and the fact that $\check{d}\gamma = 0$ if and only if $d\gamma \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$ (or equivalently, $d\gamma \wedge \varphi = 0$) we obtain (42). Finally, (43) is an easy consequence of (41). $\square$

Moreover, from Hodge theorem and Proposition 3.1 we have

$$\check{H}^q(\mathcal{A}(M)) \cong \check{\mathcal{H}}^q(\mathcal{A}(M)), \qquad q = 3, 4. \tag{44}$$

4. Calibrated G_2-manifolds

In this section we give more details about the groups $\check{H}^q(\mathcal{A}(M))$ of a calibrated G_2-manifold M, and also details about the groups $\hat{H}^q(\mathcal{B}(M))$ of a G_2-manifold M whose holonomy group is a subgroup of G_2.

THEOREM 4.1. *Let M be a compact calibrated G_2-manifold with fundamental 3-form φ. The cohomology groups $\check{H}^q(\mathcal{A}(M))$ satisfy*

(i) $\check{H}^3(\mathcal{A}(M)) \cong H^1(M)$,
(ii) $\check{H}^4(\mathcal{A}(M)) \cong H^0(M)$.

Proof. Because $d\varphi = 0$, the space $\check{\mathcal{H}}^4(\mathcal{A}(M))$ given by (43) is

$$\check{\mathcal{H}}^4(\mathcal{A}(M)) = \{f * \varphi \mid df = 0\},$$

which is naturally isomorphic to $H^0(M)$. Now, from (44) it follows that $\check{H}^4(\mathcal{A}(M)) \cong \check{\mathcal{H}}^4(\mathcal{A}(M)) \cong H^0(M)$, which proves (ii).

We define the linear map $F\colon \Lambda^1(M) \longrightarrow \Lambda^3_7(M)$ by

$$F(\alpha) = *(\alpha \wedge \varphi)$$

for $\alpha \in \Lambda^1(M)$. From the description (3) of $\Lambda^3_7(M)$ we know that F is an isomorphism. Moreover, it follows that F induces the isomorphism $F^*\colon \mathcal{H}^1(M) \longrightarrow \check{\mathcal{H}}^3(\mathcal{A}(M))$ defined by

$$F^*(\alpha) = F(\alpha) = *(\alpha \wedge \varphi) \tag{45}$$

for $\alpha \in \mathcal{H}^1(M)$. In fact, let us first show that $F^*(\mathcal{H}^1(M)) \subset \check{\mathcal{H}}^3(\mathcal{A}(M))$. Let $\alpha \in \mathcal{H}^1(M)$ and put $\beta = *(\alpha \wedge \varphi)$. Then we obtain

$$\begin{aligned} d * (\beta) &= d(\alpha \wedge \varphi) \\ &= d\alpha \wedge \varphi - \alpha \wedge d\varphi \\ &= 0, \end{aligned} \tag{46}$$

because $d\alpha = d\varphi = 0$. Moreover, using $d * \alpha = d\varphi = 0$, we have

$$\begin{aligned} d\beta \wedge \varphi &= d(\beta \wedge \varphi) \\ &= d(*(\alpha \wedge \varphi) \wedge \varphi) \\ &= -4d(*\alpha) \\ &= 0. \end{aligned} \tag{47}$$

From (46), (47) and (42) we conclude that $\beta \in \check{\mathcal{H}}^3(\mathcal{A}(M))$. Furthermore, because F is injective, it follows that F^* is injective. Now, to prove that F^* is surjective, let us suppose that $\beta \in \check{\mathcal{H}}^3(\mathcal{A}(M))$. Consider the 1-form $\alpha \in \Lambda^1(M)$ defined by

$$\alpha = -\frac{1}{4} * (\beta \wedge \varphi).$$

Using that $d * \beta = 0$ and $d\varphi = 0$, we have

$$d\alpha \wedge \varphi = d(\alpha \wedge \varphi) = -\frac{1}{4}d(*(\beta \wedge \varphi) \wedge \varphi) = d * \beta = 0,$$

which implies that $d\alpha \in \mathcal{B}^2(M) = \{0\}$, and so

$$d\alpha = 0. \tag{48}$$

Also, we have

$$d * \alpha = -\frac{1}{4}d(\beta \wedge \varphi) = 0, \tag{49}$$

because $d\beta \wedge \varphi = d\varphi = 0$. Now, from (48) and (49) it follows that $\alpha \in \mathcal{H}^1(M)$. Moreover,

$$F^*(\alpha) = F(\alpha) = -\frac{1}{4} * (*(\beta \wedge \varphi) \wedge \varphi) = \beta,$$

that is, F^* is surjective. This completes the proof of (i). □

THEOREM 4.2. *Let M be a compact parallel G_2-manifold with fundamental 3-form φ. Then the connecting homomorphisms $H^3(d)$ and $H^4(d)$, of the exact sequence* (36), *vanish. Therefore, we have*

$$\begin{aligned} H^4(M) &\cong H^0(M) \oplus \hat{H}^4(\mathcal{B}(M)), \\ H^5(M) &\cong \hat{H}^5(\mathcal{B}(M)). \end{aligned} \tag{50}$$

Proof. First we see that the connecting homomorphism $H^3(d)$: $\check{H}^3(\mathcal{A}(M)) \longrightarrow \hat{H}^4(\mathcal{B}(M))$ is zero. Let $\beta \in \check{H}^3(\mathcal{A}(M))$. Since the map F^*: $\mathcal{H}^1(M) \longrightarrow \check{\mathcal{H}}^3(\mathcal{A}(M))$ given by (45) is an isomorphism, we have $\beta = *(\alpha \wedge \varphi)$, where α is a harmonic 1-form on M. Then α is parallel (with respect to the Levi-Civita connection of M) because the Ricci curvature of M is identically zero [Bo]. Hence $\alpha \wedge \varphi$ is also parallel. This implies that the differential 4-form $\alpha \wedge \varphi$ is harmonic. Thus we obtain

$$d\beta = d * (\alpha \wedge \varphi) = 0,$$

which implies that $H^3(d) = 0$.

On the other hand, let f be a differentiable function on M such that it satisfies $f * \varphi \in \check{\mathcal{H}}^4(\mathcal{A}(M))$. Then $df = 0$ and $H^4(d)(f * \varphi) = [d(f * \varphi)]_{\mathcal{B}} = 0$, because $d * \varphi = 0$. This proves that $H^4(d) = 0$. Now (50) follows from Proposition 3.6 and Theorem 4.1 (ii). □

Let us consider now the quotient

$$\frac{\hat{H}^3(\mathcal{B}(M))}{\check{H}^2(\mathcal{A}(M))/H^2(M)}. \tag{51}$$

We note that (51) is defined for any compact G_2-manifold M, as the cohomology groups $\check{H}^2(\mathcal{A}(M))$ and $\hat{H}^3(\mathcal{B}(M))$ are defined even if the manifold is not locally conformal calibrated. If M is locally conformal calibrated, then the exactness of (36) implies that

$$\frac{H^3(M)}{\dfrac{\hat{H}^3(\mathcal{B}(M))}{\check{H}^2(\mathcal{A}(M))/H^2(M)}} \cong \operatorname{Ker} H^3(d) \subseteq \check{H}^3(\mathcal{A}(M)), \tag{52}$$

and therefore (51) is of finite dimension. However, from Proposition 3.1 and Proposition 3.3, we know that the dimensions of the cohomology groups $\check{H}^2(\mathcal{A}(M))$ and

$\hat{H}^3(\mathcal{B}(M))$ are not necessarily finite. Moreover, if M is calibrated then (52) and Theorem 4.1 (i) imply that the dimension of (51) is $\geq b_3(M) - b_1(M)$, where $b_q(M)$ denotes the q-th Betti number of M.

COROLLARY 4.3. *Let M be a compact parallel G_2-manifold. Then*

$$H^3(M) \cong H^1(M) \oplus \frac{\hat{H}^3(\mathcal{B}(M))}{\check{H}^2(\mathcal{A}(M))/H^2(M)}. \tag{53}$$

Proof. From Theorem 4.1 (i) and Theorem 4.2 it follows that $\operatorname{Ker} H^3(d) = \check{H}^3(\mathcal{A}(M)) \cong H^1(M)$. Therefore, (53) follows from (52). □

Remark 4.4. Notice that for compact calibrated G_2-manifolds, (53) is satisfied if and only if $H^3(d) = 0$. In fact, this follows directly from (52) and Theorem 4.1 (i).

From Theorem 4.2 and Corollary 4.3 it follows that for any compact parallel G_2-manifold M, the dimensions of (51) and $\hat{H}^4(\mathcal{B}(M))$ are $b_3(M) - b_1(M)$ and $b_3(M) - b_0(M)$, respectively. In particular, for such a manifold M we get

$$b_3(M) \geq b_1(M) \qquad \text{and} \qquad b_3(M) \geq b_0(M),$$

which provides a proof of these topological conditions, different from the proof given in [Bo].

Now let us consider the long exact sequence (36). In Theorem 4.2 we have proved that the connecting homomorphisms $H^3(d)$ and $H^4(d)$ are zero for any compact parallel G_2-manifold. Next, we show an *example of a compact parallel G_2-manifold for which $H^2(d)$ is non-zero*:

Let $\mathbb{R}^7$ be the 7-dimensional Euclidean space $\mathbb{R}^7 = \{(x_0, \ldots, x_6) \mid x_i \in \mathbb{R}, 0 \leq i \leq 6\}$. A basis for the left invariant 1-forms on $\mathbb{R}^7$ is given by $\{dx_i; 0 \leq i \leq 6\}$. Now, we take the compact quotient $\Gamma\backslash\mathbb{R}^7$, where Γ is the uniform subgroup of $\mathbb{R}^7$ consisting of those elements whose coordinates are integers. Thus $\Gamma\backslash\mathbb{R}^7$ is a 7-dimensional torus $\mathbb{T}^7$; and the 1-forms dx_i $(0 \leq i \leq 6)$ all descend to 1-forms α_i $(0 \leq i \leq 6)$ on $\mathbb{T}^7$ such that

$$d\alpha_i = 0, \qquad 0 \leq i \leq 6.$$

Consider the functions $f_0\colon \mathbb{R}^7 \longrightarrow \mathbb{R}$ and $g_0\colon \mathbb{R}^7 \longrightarrow \mathbb{R}$ defined by

$$f_0(x) = \sin(2\pi x_0), \qquad g_0(x) = \cos(2\pi x_0)$$

for $x = (x_0, \ldots, x_6) \in \mathbb{R}^7$. One can check that $f_0(x+k) = f_0(x)$ and $g_0(x+k) = g_0(x)$ for $x \in \mathbb{R}^7$ and $k \in \Gamma$. Thus both functions f_0 and g_0 descend to functions f and g on $\mathbb{T}^7$, respectively, and they satisfy

$$df = 2\pi g\alpha_0, \qquad dg = -2\pi f\alpha_0. \tag{54}$$

Consider the metric $\langle\, ,\, \rangle$ on $\mathbb{T}^7$ given by

$$\langle\, ,\, \rangle = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2.$$

Define the 3-form φ on $\mathbb{T}^7$ by

$$\begin{aligned} \varphi = {} & \alpha_0 \wedge \alpha_1 \wedge \alpha_3 + \alpha_1 \wedge \alpha_2 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4 \wedge \alpha_6 \\ & + \alpha_0 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_5 \wedge \alpha_6 + \alpha_0 \wedge \alpha_2 \wedge \alpha_6. \end{aligned} \tag{55}$$

Then it is clear that $d\varphi = d * \varphi = 0$. Therefore $\mathbb{T}^7$ is a compact parallel G_2-manifold whose fundamental 3-form is the 3-form φ given by (55).

Now let us consider the 2-form β on $\mathbb{T}^7$ given by

$$\beta = f\alpha_1 \wedge \alpha_3. \tag{56}$$

Using (54), we find that $d\beta = 2\pi g\alpha_0 \wedge \alpha_1 \wedge \alpha_3$, that is, β is non closed. However, since $d\beta \wedge \varphi = 0$, we get $\check{d}_2(\beta) = 0$ and therefore β defines a non-zero cohomology class $[\beta]_{\mathcal{A}}$ in $\check{H}^2(\mathcal{A}(\mathbb{T}^7))$. Thus

$$H^2(d)([\beta]_{\mathcal{A}}) = [d\beta]_{\mathcal{B}} = 2\pi g\alpha_0 \wedge \alpha_1 \wedge \alpha_3 \neq 0,$$

that is, the connecting homomorphism $H^2(d)$ is non-zero. From Proposition 3.6 and Theorem 4.1 (i) it follows that

$$\begin{aligned} H^2(\mathbb{T}^7) &\ncong \check{H}^2(\mathcal{A}(\mathbb{T}^7)), \\ H^3(\mathbb{T}^7) &\ncong H^1(\mathbb{T}^7) \oplus \hat{H}^3(\mathcal{B}(\mathbb{T}^7)), \end{aligned}$$

for the compact parallel G_2-manifold $\mathbb{T}^7$.

5. A theorem of Nomizu type for the coeffective cohomology

In this section we prove that there exists a canonical isomorphism between the coeffective cohomology of a compact calibrated G_2-nilmanifold $\Gamma\backslash K$ and the coeffective cohomology of the Lie algebra $\mathfrak{K}$ of K. We also prove that this result holds for compact completely solvable calibrated G_2-manifolds.

Let M be a 7-dimensional compact nilmanifold; that is, $M = \Gamma\backslash K$, where K is a 7-dimensional connected, simply-connected and nilpotent Lie group, and Γ is a discrete subgroup of K such that the quotient space $\Gamma\backslash K$ is compact. The most immediate example of such a manifold is the torus $\mathbb{T}^7$. It is easy to see that each left invariant differential form on K descends to the quotient $\Gamma\backslash K$. For convenience, if μ is a left invariant differential form on K we also denote by μ the differential form induced on M.

Next, let us suppose that K is a G_2-manifold with left invariant metric $\langle\, ,\, \rangle$ and left invariant 2-fold vector cross product P. Then the metric and the vector product descend to a metric $\langle\ ,\ \rangle$ and a 2-fold vector cross product P on M, respectively. Let φ be the left invariant fundamental 3-form on K. Then M is a G_2-nilmanifold with fundamental form φ. Moreover, M is locally conformal calibrated (in particular, calibrated) if and only if K is locally conformal calibrated (in particular, calibrated).

Denote by $\mathfrak{K}$ the Lie algebra of K. Let

$$\cdots \longrightarrow \Lambda^{q-1}(\mathfrak{K}^*) \xrightarrow{d} \Lambda^q(\mathfrak{K}^*) \xrightarrow{d} \Lambda^{q+1}(\mathfrak{K}^*) \longrightarrow \cdots$$

be the Chevalley-Eilenberg complex, where $\Lambda^q(\mathfrak{K}^*)$ denotes the space of left invariant differential q-forms on K. The Chevalley-Eilenberg cohomology is defined by

$$H^q(\mathfrak{K}^*) = \frac{\operatorname{Ker}\{d\colon \Lambda^q(\mathfrak{K}^*) \longrightarrow \Lambda^{q+1}(\mathfrak{K}^*)\}}{\operatorname{Im}\{d\colon \Lambda^{q-1}(\mathfrak{K}^*) \longrightarrow \Lambda^q(\mathfrak{K}^*)\}}.$$

In 1954 Nomizu [N] proved the following theorem which reduces the computation of the de Rham cohomology of compact nilmanifolds to the calculation at the Lie algebra level:

THEOREM 5.1 ([N]). *Let $M = \Gamma\backslash K$ be a compact nilmanifold of dimension m and denote by $\tau_q\colon H^q(\mathfrak{K}^*) \longrightarrow H^q(\Gamma\backslash K)$, $0 \leq q \leq m$, the homomorphism of cohomology groups defined by*

$$\tau_q(\{\alpha\}) = [\alpha] \in H^q(\Gamma\backslash K)$$

for $\{\alpha\} \in H^q(\mathfrak{K}^*)$, where $\mathfrak{K}$ denotes the Lie algebra of K. Then τ_q is an isomorphism for $0 \leq q \leq m$.

Now we introduce the differential complexes

$$\begin{aligned} (57) \qquad 0 \longrightarrow \mathcal{B}^3(\mathfrak{K}^*) &= \Lambda^3_1(\mathfrak{K}^*) \oplus \Lambda^3_{27}(\mathfrak{K}^*) \xrightarrow{\hat{d}} \mathcal{B}^4(\mathfrak{K}^*) \\ &= \Lambda^4_7(\mathfrak{K}^*) \oplus \Lambda^4_{27}(\mathfrak{K}^*) \xrightarrow{\hat{d}} \Lambda^5(\mathfrak{K}^*) \xrightarrow{d} \Lambda^6(\mathfrak{K}^*) \\ &\xrightarrow{d} \Lambda^7(\mathfrak{K}^*) \longrightarrow 0 \end{aligned}$$

and

$$(58) \qquad 0 \longrightarrow \mathbb{R} \xrightarrow{d=0} \Lambda^1(\mathfrak{K}^*) \xrightarrow{d} \Lambda^2(\mathfrak{K}^*) \xrightarrow{\check{d}} \Lambda^3_7(\mathfrak{K}^*) \xrightarrow{\check{d}} \Lambda^4_1(\mathfrak{K}^*) \longrightarrow 0,$$

where the spaces $\Lambda^q_l(\mathfrak{K}^*)$ and the maps $\hat{d}$ and $\check{d}$ are defined by relations similar to (3), (9) and (12), respectively. Notice that the complexes (57) and (58) are differential subcomplexes of (9) and (13), respectively. We denote by $\hat{H}^*(\mathcal{B}(\mathfrak{K}^*))$ the cohomology of the complex (57), and by $\check{H}^*(\mathcal{A}(\mathfrak{K}^*))$ the cohomology of (58).

LEMMA 5.2. *Let $M = \Gamma\backslash K$ be a compact calibrated G_2-nilmanifold. Suppose that the fundamental 3-form φ on M stems from a left invariant fundamental 3-form* *on K. Denote by $\sigma_q\colon \check{H}^q(\mathcal{A}(\mathfrak{K}^*)) \longrightarrow \check{H}^q(\mathcal{A}(M))$, $q = 3, 4$, the homomorphism of cohomology groups defined by*

$$\sigma_q(\{\alpha\}_{\mathcal{A}}) = [\alpha]_{\mathcal{A}} \in \check{H}^q(\mathcal{A}(M))$$

for $\{\alpha\}_{\mathcal{A}} \in \check{H}^q(\mathcal{A}(\mathfrak{K}^*))$. Then σ_q is an isomorphism for $q = 3, 4$.

Proof. First we prove that the homomorphism $f_q\colon \check{H}^q(\mathcal{A}(M)) \longrightarrow H^{q+3}(M)$ defined by $f_q([\alpha]_{\mathcal{A}}) = [\alpha\wedge\varphi]$ for $[\alpha]_{\mathcal{A}} \in \check{H}^q(\mathcal{A}(M))$ is an isomorphism for $q = 3, 4$. In fact, taking into account that M is calibrated, this follows from the commutativity of all the squares in the diagram

$$\begin{array}{ccccccc} \Lambda^2(M) & \xrightarrow{\check{d}_2} & \Lambda^3_7(M) & \xrightarrow{\check{d}_3} & \Lambda^4_1(M) & \longrightarrow & 0 \\ \downarrow \wedge\varphi & & \downarrow \wedge\varphi & & \downarrow \wedge\varphi & & \\ \Lambda^5(M) & \xrightarrow{d} & \Lambda^6(M) & \xrightarrow{d} & \Lambda^7(M) & \longrightarrow & 0, \end{array} \tag{59}$$

and from the fact that the wedge product by φ makes the vertical arrows isomorphisms.

Moreover, if we consider the diagram (59) at the Lie algebra level it follows that the homomorphism $g_q\colon \check{H}^q(\mathcal{A}(\mathfrak{K}^*)) \longrightarrow H^{q+3}(\mathfrak{K}^*)$ given by $g_q(\{\alpha\}_{\mathcal{A}}) = \{\alpha \wedge \varphi\}$ for $\{\alpha\}_{\mathcal{A}} \in \check{H}^q(\mathcal{A}(\mathfrak{K}^*))$ is also an isomorphism for $q = 3, 4$.

Therefore, we can consider the diagram

$$\begin{array}{ccc} \check{H}^q(\mathcal{A}(\mathfrak{K}^*)) & \xrightarrow{\sigma_q} & \check{H}^q(\mathcal{A}(M)) \\ \downarrow f_q & & \downarrow g_q \\ H^{q+3}(\mathfrak{K}^*) & \xrightarrow{\tau_{q+3}} & H^{q+3}(M) \end{array}$$

for $q = 3, 4$. As this diagram is commutative and f_q, g_q and τ_{q+3} are isomorphisms for $q = 3, 4$ (see Theorem 5.1), we see that σ_q is an isomorphism for $q = 3, 4$. $\square$

In order to prove a theorem of Nomizu type for the G_2-coeffective cohomology, we need to consider for the Lie algebra $\mathfrak{K}$ the corresponding long exact sequence given by (36) for any locally conformal calibrated G_2-manifold. It will be convenient, for $\mathfrak{K}$, to change the notation of the homomorphisms of (36). We write $H^q(\widetilde{p})$, $H^q(\widetilde{d})$ and $H^q(\widetilde{i})$ instead of $H^q(p)$, $H^q(d)$ and $H^q(i)$, respectively. Then we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^2(\mathfrak{K}^*) & \xrightarrow{H^2(\widetilde{p})} & \check{H}^2(\mathcal{A}(\mathfrak{K}^*)) & \xrightarrow{H^2(\widetilde{d})} & \\ \hat{H}^3(\mathcal{B}(\mathfrak{K}^*)) & \xrightarrow{H^3(\widetilde{i})} & H^3(\mathfrak{K}^*) & \xrightarrow{H^3(\widetilde{p})} & \check{H}^3(\mathcal{A}(\mathfrak{K}^*)) & \xrightarrow{H^3(\widetilde{d})} & \\ \hat{H}^4(\mathcal{B}(\mathfrak{K}^*)) & \xrightarrow{H^4(\widetilde{i})} & H^4(\mathfrak{K}^*) & \xrightarrow{H^4(\widetilde{p})} & \check{H}^4(\mathcal{A}(\mathfrak{K}^*)) & \xrightarrow{H^4(\widetilde{d})} & \\ \hat{H}^5(\mathcal{B}(\mathfrak{K}^*)) & \xrightarrow{H^5(\widetilde{i})} & H^5(\mathfrak{K}^*) & \longrightarrow & 0. & & \end{array} \tag{60}$$

THEOREM 5.3. *Let $M = \Gamma\backslash K$ be a compact calibrated G_2-nilmanifold for which the fundamental 3-form φ stems from a left invariant fundamental 3-form on K. Denote by $\delta_q\colon \hat{H}^q(\mathcal{B}(\mathfrak{K}^*)) \longrightarrow \hat{H}^q(\mathcal{B}(M))$, $q = 4, 5$, the homomorphism of cohomology groups defined by*

$$\delta_q(\{\alpha\}_{\mathcal{B}}) = [\alpha]_{\mathcal{B}} \in \hat{H}^q(\mathcal{B}(M)),$$

for $\{\alpha\}_{\mathcal{B}} \in \hat{H}^q(\mathcal{B}(\mathfrak{K}^*))$, where $\mathfrak{K}$ denotes the Lie algebra of K. Then δ_q is an isomorphism for $q = 4, 5$.

Proof. Let us consider the diagram

$$\begin{array}{ccccccccc}
H^{q-1}(\mathfrak{K}^*) & \overset{H^{q-1}(\widetilde{p})}{\longrightarrow} & \check{H}^{q-1}(\mathcal{A}(\mathfrak{K}^*)) & \overset{H^{q-1}(\widetilde{d})}{\longrightarrow} & \hat{H}^q(\mathcal{B}(\mathfrak{K}^*)) & \overset{H^q(\widetilde{i})}{\longrightarrow} & H^q(\mathfrak{K}^*) & \overset{H^q(\widetilde{p})}{\longrightarrow} & \check{H}^q(\mathcal{A}(\mathfrak{K}^*)) \\
\downarrow{\tau_{q-1}} & & \downarrow{\sigma_{q-1}} & & \downarrow{\delta_q} & & \downarrow{\tau_q} & & \downarrow{\sigma_q} \\
H^{q-1}(M) & \overset{H^{q-1}(p)}{\longrightarrow} & \check{H}^{q-1}(\mathcal{A}(M)) & \overset{H^{q-1}(d)}{\longrightarrow} & \hat{H}^q(\mathcal{B}(M)) & \overset{H^q(i)}{\longrightarrow} & H^q(M) & \overset{H^q(p)}{\longrightarrow} & \check{H}^q(\mathcal{A}(M)),
\end{array}$$

where τ_{q-1}, τ_q, σ_{q-1} and σ_q are the canonical isomorphisms given in Theorem 5.1 and Lemma 5.2 for $q = 4, 5$. Notice that $\check{H}^5(\mathcal{A}(\mathfrak{K}^*)) = \check{H}^5(\mathcal{A}(M)) = \{0\}$ and the isomorphism σ_5 is zero.

It is easy to see that the homomorphism δ_q makes the squares commutative for $q = 4, 5$. Moreover, from (36) and (60) it follows that the two horizontal rows in the diagram are exact. Then the Five Lemma implies that δ_q is an isomorphism for $q = 4, 5$. □

COROLLARY 5.4. *Let $M = \Gamma\backslash K$ be a compact G_2-nilmanifold in the conditions of Theorem 5.3. Then there is a canonical isomorphism*

$$\frac{\hat{H}^3(\mathcal{B}(\mathfrak{K}^*))}{\check{H}^2(\mathcal{A}(\mathfrak{K}^*))/H^2(\mathfrak{K}^*)} \cong \frac{\hat{H}^3(\mathcal{B}(M))}{\check{H}^2(\mathcal{A}(M))/H^2(M)}.$$

Proof. From the exactness of the sequence (60) it follows that

$$\frac{H^3(\mathfrak{K}^*)}{\dfrac{\hat{H}^3(\mathcal{B}(\mathfrak{K}^*))}{\check{H}^2(\mathcal{A}(\mathfrak{K}^*))/H^2(\mathfrak{K}^*)}} \cong \operatorname{Ker} H^3(\widetilde{d})\,. \tag{61}$$

Now, taking into account Lemma 5.2 and Theorem 5.3 it follows that the diagram

$$\begin{array}{ccc}
\check{H}^3(\mathcal{A}(\mathfrak{K}^*)) & \overset{H^3(\widetilde{d})}{\longrightarrow} & \hat{H}^4(\mathcal{B}(\mathfrak{K}^*)) \\
\downarrow \sigma_3 & & \downarrow \delta_4 \\
\check{H}^3(\mathcal{A}(M)) & \overset{H^3(d)}{\longrightarrow} & \hat{H}^4(\mathcal{B}(M))
\end{array}$$

is commutative. Moreover, since σ_3 and δ_4 are isomorphisms we obtain a canonical isomorphism

$$\operatorname{Ker} H^3(\widetilde{d}) \cong \operatorname{Ker} H^3(d). \tag{62}$$

Finally, the result follows from (52), (61) and (62), taking into account that $H^3(\mathfrak{K}^*) \cong H^3(M)$ by Theorem 5.1. $\square$

Remark 5.5. The example exhibited at the end of Section 4 shows that a theorem of Nomizu type does not hold for the cohomology groups $\check{H}^2(\mathcal{A}(M))$ and $\hat{H}^3(\mathcal{B}(M))$ of arbitrary compact calibrated G_2-nilmanifolds M. In fact, it is easy to see that there are no left invariant representatives for the cohomology classes $[\beta]_{\mathcal{A}}$ and $[d\beta]_{\mathcal{B}}$, β being the 2-form on $\mathbb{T}^7$ given by (56). Therefore, for the torus $\mathbb{T}^7$ we have

$$\check{H}^2(\mathcal{A}(\mathfrak{K}^*)) \ncong \check{H}^2(\mathcal{A}(\mathbb{T}^7)) \quad \text{and} \quad \hat{H}^3(\mathcal{B}(\mathfrak{K}^*)) \ncong \hat{H}^3(\mathcal{B}(\mathbb{T}^7)).$$

Remark 5.6. Hattori has extended Theorem 5.1 for compact completely solvable manifolds (see [H]). Taking into account this result one can deduce that Lemma 5.2, Theorem 5.3 and Corollary 5.4 still hold for compact completely solvable calibrated G_2-manifolds.

6. Examples

In this section we exhibit two examples of compact calibrated (non-parallel) G_2-nilmanifolds. The first of them, given in [F1], was the first known example of a calibrated G_2-manifold in the compact case. We prove that Theorem 4.2 holds for this manifold. The second example is a compact calibrated G_2-nilmanifold for which Theorem 4.2 and Corollary 4.3 fail.

Next we prove some results about G_2-nilmanifolds, which we shall use later.

PROPOSITION 6.1. *Let $M = \Gamma\backslash K$ be a compact G_2-nilmanifold for which the fundamental 3-form φ stems from a left invariant fundamental 3-form on K. Then M is parallel if and only if M is the torus $\mathbb{T}^7$.*

Proof. As we have seen at the end of Section 4, the torus $\mathbb{T}^7$ is a compact parallel G_2-nilmanifold. Suppose that $M = \Gamma\backslash K$ is a compact parallel G_2-nilmanifold and denote by $\mathfrak{K}$ the Lie algebra of K. Since M is parallel, the Ricci curvature of M is identically zero [Bo]. Therefore, the Lie algebra $\mathfrak{K}$ must be abelian because otherwise there would exist a direction of strictly negative Ricci curvature and a direction of strictly positive Ricci curvature ([Wo], [M]). Therefore, since $\mathfrak{K}$ is abelian the nilmanifold $M = \Gamma\backslash K$ must be the torus $\mathbb{T}^7$. $\square$

Next we obtain a characterization of Theorem 4.2 for the particular case of nilmanifolds.

PROPOSITION 6.2. *Let $M = \Gamma\backslash K$ be a compact calibrated G_2-nilmanifold. Suppose that the fundamental 3-form φ on M arises from a left invariant fundamental 3-form on K. Then*

$$H^5(M) \cong \hat{H}^5(\mathcal{B}(M)) \tag{63}$$

if and only if

$$d*\varphi \in d(\Lambda^4_7(\mathfrak{K}^*) \oplus \Lambda^4_{27}(\mathfrak{K}^*)), \tag{64}$$

where $\mathfrak{K}$ denotes the Lie algebra of K. Moreover, if (63) *is satisfied then*

$$H^4(M) \cong H^0(M) \oplus \hat{H}^4(\mathcal{B}(M)) \tag{65}$$

if and only if

$$d(\Lambda^3_7(\mathfrak{K}^*)) \subset d(\Lambda^3_{27}(\mathfrak{K}^*)). \tag{66}$$

Proof. From Theorem 5.1 and Theorem 5.3 it follows that $H^5(M) \cong \hat{H}^5(\mathcal{B}(M))$ if and only if

$$H^5(\mathfrak{K}^*) = \frac{\{\alpha \in \Lambda^5(\mathfrak{K}^*) \mid d\alpha = 0\}}{d(\Lambda^4(\mathfrak{K}^*))} \cong \frac{\{\alpha \in \Lambda^5(\mathfrak{K}^*) \mid d\alpha = 0\}}{d(\mathcal{B}^4(\mathfrak{K}^*))} = \hat{H}^5(\mathcal{B}(\mathfrak{K}^*)).$$

Therefore, (63) is satisfied if and only if

$$d(\Lambda^4(\mathfrak{K}^*)) = d(\mathcal{B}^4(\mathfrak{K}^*)). \tag{67}$$

But $\Lambda^4(\mathfrak{K}^*) = \Lambda^4_1(\mathfrak{K}^*) \oplus \mathcal{B}^4(\mathfrak{K}^*)$, and so (67) is equivalent to $d(\Lambda^4_1(\mathfrak{K}^*)) \subset d(\mathcal{B}^4(\mathfrak{K}^*))$. Since $\Lambda^4_1(\mathfrak{K}^*)$ is generated by $*\varphi$, from the definition of $\mathcal{B}^4(\mathfrak{K}^*)$ we get the equivalence between (63) and (64).

Suppose now that (64) is satisfied. Using Theorem 5.1 and Theorem 5.3 again, it follows that $H^4(M) \cong H^0(M) \oplus \hat{H}^4(\mathcal{B}(M))$ if and only if

$$H^4(\mathfrak{K}^*) = \frac{Z^4(\mathfrak{K}^*)}{d(\Lambda^3(\mathfrak{K}^*))} \cong H^0(\mathfrak{K}^*) \oplus \frac{Z^4(\mathcal{B}(\mathfrak{K}^*))}{d(\mathcal{B}^3(\mathfrak{K}^*))} = H^0(\mathfrak{K}^*) \oplus \hat{H}^4(\mathcal{B}(\mathfrak{K}^*)),$$

where $Z^4(\mathfrak{K}^*) = \{\alpha \in \Lambda^4(\mathfrak{K}^*) \mid d\alpha = 0\}$ and $Z^4(\mathcal{B}(\mathfrak{K}^*)) = \{\alpha \in \mathcal{B}^4(\mathfrak{K}^*) \mid d\alpha = 0\}$. Since all these spaces are finite dimensional, (65) is satisfied if and only if

$$\dim Z^4(\mathfrak{K}^*) - \dim d(\Lambda^3(\mathfrak{K}^*)) = 1 + \dim Z^4(\mathcal{B}(\mathfrak{K}^*)) - \dim d(\mathcal{B}^3(\mathfrak{K}^*)). \tag{68}$$

From (64) we have $d*\varphi \in d(\mathcal{B}^4(\mathfrak{K}^*))$, which implies that there is $\gamma \in \mathcal{B}^4(\mathfrak{K}^*)$ such that $d(*\varphi - \gamma) = 0$. Since $\Lambda^4(\mathfrak{K}^*) = \Lambda^4_1(\mathfrak{K}^*) \oplus \mathcal{B}^4(\mathfrak{K}^*)$ and $\Lambda^4_1(\mathfrak{K}^*) = \langle *\varphi \rangle$, we get $Z^4(\mathfrak{K}^*) = \langle *\varphi - \gamma \rangle \oplus Z^4(\mathcal{B}(\mathfrak{K}^*))$. Therefore, $\dim Z^4(\mathfrak{K}^*) = 1 + \dim Z^4(\mathcal{B}(\mathfrak{K}^*))$. Using this equality, it follows that (68) is equivalent to

$$\dim d(\Lambda^3(\mathfrak{K}^*)) = \dim d(\mathcal{B}^3(\mathfrak{K}^*)).$$

Since $\mathcal{B}^3(\mathfrak{K}^*) \subset \Lambda^3(\mathfrak{K}^*)$, from the definition of the space $\mathcal{B}^3(\mathfrak{K}^*)$ we get

$$d(\Lambda^3_1(\mathfrak{K}^*) \oplus \Lambda^3_7(\mathfrak{K}^*) \oplus \Lambda^3_{27}(\mathfrak{K}^*)) = d(\Lambda^3_1(\mathfrak{K}^*) \oplus \Lambda^3_{27}(\mathfrak{K}^*)). \tag{69}$$

But $\Lambda^3_1(\mathfrak{K}^*)$ is generated by φ and, since M is calibrated, $d(\Lambda^3_1(\mathfrak{K}^*)) = \{0\}$. So (69) is reduced to

$$d(\Lambda^3_7(\mathfrak{K}^*) \oplus \Lambda^3_{27}(\mathfrak{K}^*)) = d(\Lambda^3_{27}(\mathfrak{K}^*)),$$

which is equivalent to (66). □

Remark 6.3. It follows from Remark 5.6, that Proposition 6.2 also holds for compact completely solvable calibrated G_2-manifolds.

6.1. *Example* 1.

Consider the 7-dimensional compact nilmanifold $M = \Gamma\backslash K$, where K is a simply-connected nilpotent Lie group defined by left invariant 1-forms $\{\alpha_1, \alpha_2, \beta, \gamma_1, \gamma_2, \eta_1, \eta_2\}$ such that

$$\begin{cases} d\alpha_1 = d\alpha_2 = d\beta = d\eta_1 = d\eta_2 = 0, \\ d\gamma_1 = -\alpha_1 \wedge \beta, \\ d\gamma_2 = -\alpha_2 \wedge \beta, \end{cases} \tag{70}$$

and Γ is a uniform subgroup of K. This manifold can be seen as $M = \Gamma(1,2)\backslash H(1,2) \times \mathbb{T}^2$, where $\mathbb{T}^2$ denotes the 2-dimensional torus, $H(1,2)$ is the generalized Heisenberg group which consists of all matrices of the form

$$\begin{pmatrix} 1 & 0 & x_1 & z_1 \\ 0 & 1 & x_2 & z_2 \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x_1, x_2, y, z_1, z_2 \in \mathbb{R}$, and $\Gamma(1,2)$ is the subgroup of $H(1,2)$ consisting of those matrices whose entries $\{x_1, x_2, y, z_1, z_2\}$ are integers (see [F1]).

THEOREM 6.4 ([F1]). *There exists a vector cross product on M such that the fundamental* 3*-form is closed. Therefore, M is a compact calibrated G_2-nilmanifold.*

Proof. The 3-form φ on M defined by

$$\begin{aligned} \varphi = {} & -\alpha_1 \wedge \gamma_1 \wedge \eta_2 + \alpha_2 \wedge \gamma_2 \wedge \eta_2 + \alpha_1 \wedge \gamma_2 \wedge \eta_1 + \alpha_2 \wedge \gamma_1 \wedge \eta_1 \\ & + \beta \wedge \gamma_1 \wedge \gamma_2 + \alpha_1 \wedge \alpha_2 \wedge \beta - \beta \wedge \eta_1 \wedge \eta_2 \end{aligned} \tag{71}$$

is closed. Consider the metric given by

$$\langle\, , \rangle = \alpha_1^2 + \alpha_2^2 + \beta^2 + \gamma_1^2 + \gamma_2^2 + \eta_1^2 + \eta_2^2.$$

Let $\{E_0, \ldots, E_6\}$ be the basis dual to $\{\alpha_1, \gamma_2, \eta_2, \eta_1, \alpha_2, \beta, \gamma_1\}$. Then a 2-fold vector cross product P on M is given by $P(E_i, E_j) = -P(E_j, E_i)$, and $P(E_i, E_{i+1}) = E_{i+3}$, $P(E_{i+3}, E_i) = E_{i+1}$, $P(E_{i+1}, E_{i+3}) = E_i$ $(i \in \mathbb{Z}_7)$. □

From Proposition 6.1 it follows that the compact calibrated G_2-nilmanifold M is non-parallel.

PROPOSITION 6.5 ([F1]). *The Betti numbers of M are as follows*:

$$b_1(M) = 5, \qquad b_2(M) = 13, \qquad b_3(M) = 21.$$

THEOREM 6.6. *The calibrated G_2-manifold M satisfies*

$$\begin{aligned} H^4(M) &\cong H^0(M) \oplus \hat{H}^4(\mathcal{B}(M)), \\ H^5(M) &\cong \hat{H}^5(\mathcal{B}(M)). \end{aligned}$$

Proof. From Proposition 6.2 it is sufficient to prove (64) and (66). From (70) and (71) it is easy to verify that

$$\begin{aligned} *\varphi &= \alpha_2 \wedge \beta \wedge \gamma_1 \wedge \eta_2 + \alpha_1 \wedge \beta \wedge \gamma_1 \wedge \eta_1 - \alpha_1 \wedge \alpha_2 \wedge \gamma_1 \wedge \gamma_2 \\ &\quad + \alpha_1 \wedge \beta \wedge \gamma_2 \wedge \eta_2 + \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2 + \alpha_1 \wedge \alpha_2 \wedge \eta_1 \wedge \eta_2 \\ &\quad - \alpha_2 \wedge \beta \wedge \gamma_2 \wedge \eta_1 \end{aligned}$$

and

$$\begin{aligned} d * \varphi &= d(\gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2) \\ &= -\alpha_1 \wedge \beta \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2 + \alpha_2 \wedge \beta \wedge \gamma_1 \wedge \eta_1 \wedge \eta_2 \\ &= d(\mu), \end{aligned}$$

where $\mu = \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2 - \alpha_2 \wedge \beta \wedge \gamma_1 \wedge \eta_2$. Using (71) it is easy to see that $\mu \wedge \varphi = 0$, that is, $\mu \in \Lambda^4_7(\mathfrak{K}^*) \oplus \Lambda^4_{27}(\mathfrak{K}^*)$. Therefore, (64) is satisfied or, equivalently, $H^5(M) \cong \hat{H}^5(\mathcal{B}(M))$.

To prove (66), we note that

$$\begin{aligned} \omega_1 &= \alpha_1 \wedge \gamma_2 \wedge \eta_2 + \alpha_1 \wedge \gamma_1 \wedge \eta_1 - \alpha_2 \wedge \gamma_2 \wedge \eta_1 + \alpha_2 \wedge \gamma_1 \wedge \eta_2, \\ \omega_2 &= -\alpha_1 \wedge \alpha_2 \wedge \gamma_2 + \alpha_1 \wedge \beta \wedge \eta_1 + \gamma_2 \wedge \eta_1 \wedge \eta_2 + \alpha_2 \wedge \beta \wedge \eta_2, \\ \omega_3 &= -\alpha_1 \wedge \beta \wedge \gamma_2 - \alpha_1 \wedge \alpha_2 \wedge \eta_1 - \gamma_1 \wedge \gamma_2 \wedge \eta_1 - \alpha_2 \wedge \beta \wedge \gamma_1, \\ \omega_4 &= -\alpha_1 \wedge \gamma_1 \wedge \gamma_2 + \alpha_1 \wedge \eta_1 \wedge \eta_2 + \beta \wedge \gamma_2 \wedge \eta_1 - \beta \wedge \gamma_1 \wedge \eta_2, \\ \omega_5 &= -\alpha_1 \wedge \alpha_2 \wedge \eta_2 + \alpha_1 \wedge \beta \wedge \gamma_1 - \gamma_1 \wedge \gamma_2 \wedge \eta_2 - \alpha_2 \wedge \beta \wedge \gamma_2, \\ \omega_6 &= -\alpha_1 \wedge \beta \wedge \eta_2 - \alpha_1 \wedge \alpha_2 \wedge \gamma_1 + \gamma_1 \wedge \eta_1 \wedge \eta_2 + \alpha_2 \wedge \beta \wedge \eta_1, \\ \omega_7 &= \beta \wedge \gamma_2 \wedge \eta_2 - \alpha_2 \wedge \gamma_1 \wedge \gamma_2 + \alpha_2 \wedge \eta_1 \wedge \eta_2 + \beta \wedge \gamma_1 \wedge \eta_1, \end{aligned}$$

form a basis of the space $\Lambda^3_7(\mathfrak{K}^*)$. The form ω_1 is closed and therefore, to prove (66) it is sufficient to prove that $d\omega_i \in d(\Lambda^3_{27}(\mathfrak{K}^*))$ for $2 \leq i \leq 7$. Let us consider

$$\begin{aligned} &\mu_2 = \alpha_1 \wedge \alpha_2 \wedge \gamma_2 + \gamma_2 \wedge \eta_1 \wedge \eta_2, && \mu_3 = \alpha_1 \wedge \beta \wedge \gamma_2 - \gamma_1 \wedge \gamma_2 \wedge \eta_1, \\ &\mu_4 = -\alpha_1 \wedge \gamma_1 \wedge \gamma_2 - \alpha_1 \wedge \eta_1 \wedge \eta_2, && \mu_5 = \alpha_1 \wedge \alpha_2 \wedge \eta_2 - \gamma_1 \wedge \gamma_2 \wedge \eta_2, \\ &\mu_6 = \alpha_1 \wedge \beta \wedge \eta_2 + \gamma_1 \wedge \eta_1 \wedge \eta_2, && \mu_7 = -\beta \wedge \gamma_2 \wedge \eta_2 - \alpha_2 \wedge \gamma_1 \wedge \gamma_2. \end{aligned}$$

A straightforward computation using (70) shows that $d(\omega_i) = d(\mu_i)$ and $\mu_i \wedge \varphi = \mu_i \wedge *\varphi = 0$ for $2 \leq i \leq 7$. Then it follows from (3) that $\mu_i \in \Lambda^3_{27}(\mathfrak{K}^*)$ and so (66) is satisfied. □

COROLLARY 6.7. *The calibrated G_2-manifold M satisfies*

$$H^3(M) \cong H^1(M) \oplus \frac{\hat{H}^3(\mathcal{B}(M))}{\check{H}^2(\mathcal{A}(M))/H^2(M)}.$$

Proof. Proposition 3.6 and Theorem 6.6 imply that the connecting homomorphism $H^3(d)$ is zero. Therefore, taking into account Remark 4.4, the result follows. □

6.2. *Example* 2.

Let K be the 7-dimensional connected, simply-connected and nilpotent Lie group defined by left invariant 1-forms $\{\alpha_0, \ldots, \alpha_6\}$ such that

$$(72) \qquad \begin{cases} d\alpha_0 = d\alpha_1 = d\alpha_2 = d\alpha_3 = 0, \\ d\alpha_4 = \alpha_0 \wedge \alpha_1 + \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_3, \\ d\alpha_5 = \alpha_0 \wedge \alpha_3 + \alpha_1 \wedge \alpha_3, \\ d\alpha_6 = -\alpha_0 \wedge \alpha_1 - \alpha_0 \wedge \alpha_3 + \alpha_3 \wedge \alpha_5. \end{cases}$$

Since the coefficients in the structure equations given by (72) are integers, a well-known result of Mal'čev [Ma] implies that K has a uniform subgroup Γ. Consider the compact nilmanifold $N = \Gamma \backslash K$.

THEOREM 6.8. *There exists a vector cross product on N for which the fundamental 3-form is closed. Therefore N is a compact calibrated (non-parallel) G_2-nilmanifold.*

Proof. Let φ be the 3-form on N defined by

$$(73) \qquad \begin{aligned} \varphi = {} & \alpha_0 \wedge \alpha_1 \wedge \alpha_3 + \alpha_1 \wedge \alpha_2 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4 \wedge \alpha_6 \\ & + \alpha_0 \wedge \alpha_4 \wedge \alpha_5 + \alpha_1 \wedge \alpha_5 \wedge \alpha_6 + \alpha_0 \wedge \alpha_2 \wedge \alpha_6. \end{aligned}$$

From (72) it is easy to verify that φ is closed.

Define a metric on N by

$$\langle \, , \, \rangle = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2.$$

Let $\{E_0, \ldots, E_6\}$ be the basis dual to $\{\alpha_0, \ldots, \alpha_6\}$. Then a 2-fold vector cross product P on N is given by $P(E_i, E_j) = -P(E_j, E_i)$, and $P(E_i, E_{i+1}) = E_{i+3}$, $P(E_{i+3}, E_i) = E_{i+1}$, $P(E_{i+1}, E_{i+3}) = E_i$ ($i \in \mathbb{Z}_7$). One can check that P satisfies the axioms for a 2-fold vector cross product and moreover, that the form φ given by (73) is the fundamental 3-form. Finally, from Proposition 6.1 it follows that $d * \varphi \neq 0$, that is, the calibrated G_2-nilmanifold N is not parallel. □

PROPOSITION 6.9. *The Betti numbers of N are as follows*:

$$b_1(N) = 4, \qquad b_2(N) = 8, \qquad b_3(N) = 13.$$

Proof. An easy computation, using Theorem 5.1, permits to obtain explicitly all the de Rham cohomology groups of N. $\square$

THEOREM 6.10. *For the calibrated G_2-manifold N we have*

$$\begin{aligned} H^4(N) &\ncong H^0(N) \oplus \hat{H}^4(\mathcal{B}(N)), \\ H^5(N) &\cong \hat{H}^5(\mathcal{B}(N)). \end{aligned}$$

Proof. First we prove that condition (64) in Proposition 6.2 is satisfied. It will be convenient to introduce an abbreviated notation for wedge products. We write $\alpha_{ij} = \alpha_i \wedge \alpha_j$, $\alpha_{ijk} = \alpha_i \wedge \alpha_j \wedge \alpha_k$, and so forth.

From (72) and (73) it is easy to verify that

$$*\varphi = -\alpha_{2456} + \alpha_{0356} - \alpha_{0146} + \alpha_{0125} + \alpha_{1236} - \alpha_{0234} - \alpha_{1345},$$

and

$$\begin{aligned} d*\varphi &= d(-\alpha_{2456} - \alpha_{0146}) \\ &= -\alpha_{01236} - \alpha_{01245} + \alpha_{01256} - \alpha_{01345} + \alpha_{02345} + \alpha_{02346} + \alpha_{12346} - \alpha_{12356} \\ &= d(\gamma), \end{aligned}$$

where $\gamma = -\alpha_{2456} - \alpha_{0146} + 2\alpha_{1345}$. Moreover, from (73) we have $\gamma \wedge \varphi = 0$, that is, $\gamma \in \Lambda^4_7(\mathfrak{K}^*) \oplus \Lambda^4_{27}(\mathfrak{K}^*)$. Therefore (64) is satisfied or, equivalently, $H^5(N) \cong \hat{H}^5(\mathcal{B}(N))$.

Next we show that $H^4(N) \ncong H^0(N) \oplus \hat{H}^4(\mathcal{B}(N))$ by proving that the condition (66) in Proposition 6.2 is not satisfied. To see this, it is sufficient to find a 3-form μ in $\Lambda^3_7(\mathfrak{K}^*)$ such that $d\mu \notin d(\Lambda^3_{27}(\mathfrak{K}^*))$.

Let us consider $\mu = \alpha_{024} + \alpha_{056} - \alpha_{126} - \alpha_{145}$. Using (73) it is easy to see that $\mu = *(\alpha_3 \wedge \varphi)$; therefore, from the description (3) it follows that $\mu \in \Lambda^3_7(\mathfrak{K}^*)$. Now, an easy computation using (72) shows that

$$(74) \qquad \begin{aligned} d(\alpha_{024}) &= -\alpha_{0123}, \\ d(\alpha_{056}) &= -\alpha_{0136}, \\ d(\alpha_{126}) &= -\alpha_{0123} + \alpha_{1235}, \\ d(\alpha_{145}) &= -\alpha_{0134} - \alpha_{1235}. \end{aligned}$$

Therefore $d\mu = \alpha_{0134} - \alpha_{0136}$. Moreover, a long but easy calculation shows that if η is a 3-form in $\Lambda^3_{27}(\mathfrak{K}^*)$ for which $d\mu = d\eta$ then η must be a linear combination of the forms

$$\eta_1 = \alpha_{024} - \alpha_{056}, \qquad \eta_2 = \alpha_{024} + \alpha_{126}, \qquad \eta_3 = \alpha_{024} + \alpha_{145}$$

and any other closed 3-form in $\Lambda^3_{27}(\mathfrak{K}^*)$. (Notice that $\eta_i \wedge \varphi = \eta_i \wedge *\varphi = 0$, that is, $\eta_i \in \Lambda^3_{27}(\mathfrak{K}^*)$ for $i = 1, 2, 3$.) So $d\mu \notin d(\Lambda^3_{27}(\mathfrak{K}^*))$ if and only if there do not exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ satisfying

$$d\mu = \lambda_1 d\eta_1 + \lambda_2 d\eta_2 + \lambda_3 d\eta_3. \tag{75}$$

From (74) we obtain

$$\begin{aligned}\lambda_1 d\eta_1 + \lambda_2 d\eta_2 + \lambda_3 d\eta_3 &= -(\lambda_1 + 2\lambda_2 + \lambda_3)\alpha_{0123} - \lambda_3\alpha_{0134} \\ &\quad + \lambda_1\alpha_{0136} + (\lambda_2 - \lambda_3)\alpha_{1235}.\end{aligned}$$

Therefore, since $d\mu = \alpha_{0134} - \alpha_{0136}$, it follows that (75) is equivalent to

$$\begin{cases}\lambda_1 + 2\lambda_2 + \lambda_3 = 0, \\ \lambda_2 - \lambda_3 = 0,\end{cases} \tag{76}$$

where $\lambda_1 = \lambda_3 = -1$. Since there does not exist λ_2 satisfying the equations (76) with $\lambda_1 = \lambda_3 = -1$, we get $d\mu \notin d(\Lambda^3_{27}(\mathfrak{K}^*))$. Therefore the condition (66) is not satisfied. □

COROLLARY 6.11. *For the calibrated G_2-manifold N we have*

$$H^3(N) \ncong H^1(N) \oplus \frac{\hat{H}^3(\mathcal{B}(N))}{\check{H}^2(\mathcal{A}(N))/H^2(N)}.$$

Proof. Theorem 6.10 and Proposition 3.6 imply that the connecting homomorphism $H^3(d)$ is non-zero, because $H^4(d)$ is identically zero. Therefore the result follows taking into account Remark 4.4. □

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