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RIMS OF CANTOR TREES

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ABSTRACT. This paper is concerned with trees constructed with geodesic arcs in hyperbolic space. We shall be interested in determining how close to geodesics (in hyperbolic space) are the geodesics paths (within the tree), and also in bounding the Hausdorff dimension of the so-called rim of the tree. These estimates and bounds are very useful in studying the Hausdorff dimension of some limit sets appearing in the theory of Fuchsian groups and related to the asymptotic behaviour of geodesics of Riemann surfaces.

For a compact set E in the unit circle $\partial\mathbf{D} := \{z\colon |z| = 1\}$ (or in $\mathbf{R}$), its Hausdorff dimension can be defined in terms of the content. For $\eta \in [0, 1]$ the η-content of E is defined by

$$\operatorname{Cont}_\eta(E) := \inf \sum_i r_i^\eta$$

where the infimum is taken over all coverings of E by balls of radius r_i in $\partial\mathbf{D}$ (or, respectively, by intervals of radius r_i in $\mathbf{R}$), and the Hausdorff dimension is defined by

$$\dim(E) := \sup\{\eta\colon \operatorname{Cont}_\eta(E) > 0\} = \inf\{\eta\colon \operatorname{Cont}_\eta(E) = 0\}.$$

For $\eta > \dim(E)$, $\operatorname{Cont}_\eta(E) = 0$, and for $\eta < \dim(E)$, $\operatorname{Cont}_\eta(E) > 0$.

To estimate Hausdorff dimensions from above one simply has to exhibit an η and (arbitrarily) small coverings (in terms of η). Usually the estimation from below is more complicated. The best technique is due to Frostman (e.g., see [Ca]): construct a probability measure ν with support on the set E and such that for all arcs J on $\partial\mathbf{D}$,

$$\nu(J) \le C|J|^\eta \tag{0.1}$$

for some constant $C > 0$, and exponent $\eta > 0$. From this inequality it follows that for all coverings by balls $\{J_s\}$ of the set E,

$$1 = \nu(E) \le \sum_s \nu(J_s) \le \sum_s C|J_s|^\eta$$

and therefore, $\operatorname{Cont}_\eta(E) \ge 1/C > 0$.

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For instance, consider the usual ternary Cantor set $\mathcal{C}$ in [0, 1]. This set can be described by

$$\mathcal{C} := \bigcap_{j=0}^{\infty} \bigcup_{I_j \in \mathcal{E}_j} I_j$$

where $\mathcal{E}_j$ denotes the family of the 2^j intervals of length $1/3^j$ constructed inductively in the following way: $\mathcal{E}_0 = \{I_0\}$ where I_0 is the interval [0, 1], and the family $\mathcal{E}_j$ is obtained from the family $\mathcal{E}_{j-1}$ by writing each interval $I_{j-1} \in \mathcal{E}_{j-1}$ as the disjoint union of three intervals of equal length and keeping the two extreme intervals. The Hausdorff dimension of $\mathcal{C}$ is $\log 2/\log 3$.

To obtain the upper bound for the Hausdorff dimension of $\mathcal{C}$ we may use the covering $\{I\colon\ I \in \mathcal{E}_j\}$ (with j fixed). Notice that

$$\sum_{I \in \mathcal{E}_j} |I|^\eta = 2^j \left(\frac{1}{3^j}\right)^\eta$$

and therefore for $\eta > \log 2/\log 3$ we have $\mathrm{Cont}_\eta(\mathcal{C}) = 0$. Consequently, $\dim(\mathcal{C}) \leq \log 2/\log 3$.

On the other hand, to obtain the lower bound we define a measure ν with support in $\mathcal{C}$. For each interval $I \in \mathcal{E}_j$ we define

$$\nu(I) := \frac{1}{2^j}$$

and for any set J,

$$\nu(J) := \inf \sum_{U \in \mathcal{U}} \nu(U)$$

where the infimum is taken over all the coverings $\mathcal{U}$ of J with arcs in $\{I\colon\ I \in \cup_j \mathcal{E}_j\}$.

This measure satisfies (0.1) for $\eta \leq \log 2/\log 3$, so $\mathrm{Cont}_\eta(E) > 0$, and hence $\dim(\mathcal{C}) \geq \log 2/\log 3$.

In this paper we shall be considering tree-structures formed with geodesic arcs in the hyperbolic space, and certain attached boundary-structures which we shall call Rims. Next we give a model example of such trees and rims, although too particular for applications. The general framework will be introduced in Section 2.

Let us consider the upper half plane $\mathbf{H}^2 := \{z\colon\ \Im z > 0\}$ endowed with the hyperbolic metric. We can associate to the usual ternary Cantor set $\mathcal{C}$ a tree $\mathcal{T}$ in $\mathbf{H}^2$ as follows:
For each interval $I_j := [a_j, b_j] \in \mathcal{E}_j$ let $G(I_j)$ denote the geodesic in $\mathbf{H}^2$ with endpoints a_j and b_j, and let $N(I_j)$ denote the point in $G(I_j)$ with largest imaginary part. We define the set of vertices of $\mathcal{T}$ by

$$\mathcal{V} := \left\{N(I)\colon\ I \in \bigcup_j \mathcal{E}_j\right\}.$$

The arcs in $\mathcal{T}$ will be the set of geodesic arcs from a vertex $N(I)$ to a vertex $N(I')$ whenever $I \in \mathcal{E}_j$, $I' \in \mathcal{E}_{j+1}$, and $I' \subset I$.

Then the usual Cantor set appears as the collection of the endpoints of all those piecewise geodesics in $\mathcal{T}$. We shall call the set of these endpoints the Rim of $\mathcal{T}$, which in this case is simply $\mathcal{C}$.

Let us notice that in the applications we will always start with the tree as given as opposed to this example where we start with the Rim as given and then construct a tree (see Section 2).

We are interested in estimating from below the Hausdorff dimension of the Rim from the geometric information of the tree: length of the arcs, angles between consecutive arcs, and branching numbers.

The main result is the following statement:

THEOREM. *If $\mathcal{T}$ is a Cantor tree of order δ, then the rim of $\mathcal{T}$,* $\mathrm{Rim}(\mathcal{T})$, *has non zero η-content for all $\eta < \delta$. In particular,*

$$\dim(\mathrm{Rim}(\mathcal{T})) \geq \text{order of } \mathcal{T}.$$

For a complete statement see Section 2.3.

The order of a tree is an exponent which measures, in a certain way, the distribution of the vertices of the tree. We refer to Section 2.2 for the proper definitions of Cantor tree and of order of a tree.

An interesting application of this theorem on trees appear in [FM] where we estimate the size of the set of escaping geodesics in a Riemann surface.

This paper is organized in two sections: the first contains some geometric results which will be useful in the proof of the theorem, and the second contains the precise definition of Cantor tree, its properties, and finally the proof of the theorem.

1. Some hyperbolic trigonometry

We need some information concerning the relations between angles, length, visual angles (harmonic measure), and shadows in the Poincaré disk model $\mathbf{D} := \{z\colon |z| < 1\}$. We collect this information in this section. We shall be using repeatedly the sinh and cosh rules of hyperbolic geometry as stated in [Be, p.148], and we will use $d(\,,\,)$ to denote hyperbolic distance. Along the way we shall introduce some convenient notation which we shall use later on in the paper.

1.1. *Angles and lengths.* Let E be a closed set on $\overline{\mathbf{D}}$, and $z \in \mathbf{D}\backslash E$. We will denote by $\omega(z, E)$ the *harmonic measure* from the point z of the set E in the component of $\overline{\mathbf{D}}\backslash E$ which contains z.

LEMMA 1.1.1. *Let G be a geodesic in $\mathbf{D}$, and $z \in \mathbf{D}\backslash G$. Let Ω denote the component of $\mathbf{D}\backslash G$ which contains z. If I denotes the arc $\partial\mathbf{D}\backslash\partial\Omega$, then*

$$\omega(z, G) = 2\,\omega(z, I)$$

Proof. We first observe that, by symmetry, if $z \in G$ then $\omega(z, I) = \frac{1}{2}$, and then we apply the maximum principle.

LEMMA 1.1.2. *Let $z \in \mathbf{D}$, and let G be a geodesic arc in $\mathbf{D}$. Then*

$$e^{d(z,G)} = \operatorname{cotan}\left(\frac{\pi}{4}\omega(z, G)\right).$$

Moreover, there exists $C > 1$ such that if $d(z, G) \geq 1$, then for all $u \in G$,

$$\frac{1}{C}\,\omega(z, G)\,\sin\theta_u \leq e^{-d(z,u)} \leq C\,\omega(z, G)\,\sin\theta_u$$

where θ_u denotes the acute angle at u between G and the geodesic through z and u.

Proof. Let Ω denote the component of $\mathbf{D}\backslash G$ which contains z, and let I denote the complementary arc $\partial\mathbf{D}\backslash\partial\Omega$.

Let $L := d(z, G)$. Moving to $\mathbf{H}^2$, and mapping z to $e^L\mathbf{i}$, and I to $[-1, 1]$, we can easily see that

$$\omega(z, I) = \frac{2}{\pi}\arctan e^{-L}.$$

Hence, using Lemma 1.1.1 we get $e^L = \operatorname{cotan}\left(\frac{\pi}{4}\omega(z, G)\right)$.

Now let $v \in G$ be chosen so that $d(z, v) = d(z, G) = L$. Without loss of generality we may assume that $z = 0$, and $v \in [0, 1)$.

Using the sinh rule in the triangle Δ with vertices 0, u, and v we get

$$\sinh d(0, u) = \frac{\sinh L}{\sin\theta_u}.$$

But since $L \geq 1$, we have $\sinh L \asymp e^L$ and $\sinh d(0, u) \asymp e^{d(0,u)}$. Moreover,

$$e^L = \operatorname{cotan}\left(\frac{\pi}{4}\omega(z, G)\right) \asymp \frac{1}{\omega(z, G)}.$$

Therefore, $e^{-d(0,u)} \asymp \omega(z, G)\sin\theta$.

The lemma above is stated in a conformally invariant way in terms of harmonic measure. We will use it to estimate the Lebesgue measure of boundary sets. The method is explained in the following elementary lemma, whose proof is omitted.

LEMMA 1.1.3. *For each $C > 1$ there exists $C' > 0$ such that if $z \in \mathbf{D}\setminus\{0\}$, and I denotes the arc in $\partial\mathbf{D}$ with center $z/|z|$ and radius $C(1-|z|)$, then for all closed sets $E \subset I$,*

$$\frac{1}{C'}\frac{|E|}{|I|} \leq \omega(z, E) \leq C'\frac{|E|}{|I|}.$$

We shall also need the following result whose proof is straightforward.

LEMMA 1.1.4. *Consider the two regions $\Omega_\pm$ of the unit disk $\mathbf{D}$:*

$$\Omega_\pm = \left\{z = re^{i\theta}\colon\ |\theta \pm \frac{\pi}{2}| < \frac{\pi}{4}\right\}.$$

Then for any pair of points z, w with $z \in \Omega_+$ and $w \in \Omega_-$ the geodesic arc γ from z to w satisfies

$$d(\gamma, 0) \leq \log\frac{1}{\tan(\pi/8)} = \operatorname{arg\,cosh}\frac{1}{\sin\pi/4}$$

1.2. *Angles and shadows.* Let ℓ and s be two geodesic arcs in $\mathbf{D}$, and let $\gamma\colon [a, b] \to \mathbf{D}$ and $\eta\colon [c, d] \to \mathbf{D}$ be parametrizations such that $\ell = \gamma([a, b])$, $s = \eta([c, d])$. If $\gamma(b) = \eta(c)$, then by the angle between ℓ and s we mean the angle from $\gamma'(b)$ to $\eta'(c)$. On the other hand, if $\gamma(a) = \eta(c)$, then by the angle between ℓ and s we mean the angle from $\gamma'(a)$ to $\eta'(c)$. Angles are given mod 2π, between $-\pi$ and π.

Given $z \in \mathbf{D}$, $\xi \in \partial\mathbf{D}$ and $\theta \in (0, \pi)$, we define the *θ-cone from z to ξ* as the set $Q_\theta(z, \xi)$ of points $\omega \in \overline{\mathbf{D}}$ such that the angle at z between the geodesic emanating from z and going through ω and the geodesic emanating from z with endpoint ξ is less than or equal to θ. We also define the *θ-shadow from z to ξ* as the set

$$S_\theta(z, \xi) := Q_\theta(z, \xi) \cap \partial\mathbf{D}.$$

If J is an arc in $\partial\mathbf{D}$ such that $J = S_\theta(z, \xi)$, then we will denote by $Q_J(z)$ the set $Q_\theta(z, \xi)$.

For points $z, u \in \mathbf{D}$ the symbol $[zu]$ denotes the oriented geodesic arc from z to u. By the final endpoint of the oriented geodesic G containing the oriented geodesic arc $[zu]$, we mean the point η of intersection between G and the unit circle, such that u separates z from η. The other intersection point is the beginning endpoint.

LEMMA 1.2.1. *Let $z, u, v \in \mathbf{D}$ be such that the absolute value β of the angle at u between the oriented geodesic arcs $[z\,u]$ and $[u\,v]$ satisfies*

$$\beta \leq \frac{\pi}{4}.$$

There is a constant $C > 1$ *such that if*

$$\Lambda := \min\{d(z,u), d(u,v)\} \geq C,$$

then the following hold:

(i) $d(z,v) \geq d(z,u) + d(u,v) - 1$.
(ii) *If* $\xi, \eta \in \partial\mathbf{D}$ *denote, respectively, the final endpoint of the geodesic from* z *through* u, *and the final endpoint of the geodesic from* u *through* v, *then for all* $0 < \alpha < \pi/2$,

$$\frac{\alpha}{C}\, e^{-d(z,v)} \leq \omega(z, S_\alpha(v,\eta)) \leq C\,\alpha\, e^{-d(z,v)}$$

and

$$\frac{1}{C}\, e^{d(u,v)} \leq \frac{\omega(z, S_\alpha(u,\xi))}{\omega(z, S_\alpha(v,\eta))} \leq C\, e^{d(u,v)}.$$

Proof. We shall denote by Δ the triangle with vertices z, u, v.

(i) It follows from the cosine rule, and the estimate $\sinh x \leq e^x/2 \leq \cosh x \leq e^x$ for $x \geq 0$, that

$$d(z,v) \geq d(z,u) + d(u,v) - \log \frac{4}{1 - \cos(\pi - \beta)}.$$

(ii) Without loss of generality we assume that $z = 0$. It follows from the sinh rule and (i) for large C that the internal angle at v in the triangle Δ is at most $\frac{\pi}{4}$. Therefore $S_\alpha(v,\eta)$ is contained in $S_{3\pi/2}(v,\mu)$ with $\mu = \frac{v}{|v|}$. And from Lemma 1.1.3 we obtain

$$\begin{aligned} (1.2.1) \qquad \omega(0, S_\alpha(v,\mu)) &= \frac{|S_\alpha(v,\mu)|}{2\pi} \asymp \omega(v, S_\alpha(v,\eta))\, |S_{3\pi/2}(v,\mu)| \\ &= 2\alpha\, \omega(0, S_{3\pi/2}(v,\mu)). \end{aligned}$$

Using (1.2.1) and Lemma 1.1.2 we get

$$(1.2.2) \qquad \omega(0, S_\alpha(v,\eta)) \asymp \alpha\, e^{-d(0,v)}$$

Finally, from (1.2.2) and Lemma 1.1.2 we obtain

$$\frac{\omega(0, S_\alpha(u,\xi))}{\omega(0, S_\alpha(v,\eta))} \asymp e^{d(u,v)}.$$

The following lemmas will give some properties of shadows and cones which we will need later.

LEMMA 1.2.2. *Let z, v, u_1, u_2 be four points in $\mathbf{D}$. For each j, $j = 1, 2$, we define Δ_j as the triangle with vertices z, v, u_j, we let γ_j be the interior angle of the triangle Δ_j at u_j, and we define $\xi_j \in \partial\mathbf{D}$ as the final endpoint of the geodesic from v through u_j.*

Assume that

$$\gamma_j \leq \pi/4, \ j = 1, 2$$

and

$$Q_{\pi/2}(u_1, \xi_1) \cap Q_{\pi/2}(u_2, \xi_2) = \emptyset$$

Then if α, respectively β, is the angle between the geodesic arcs from z, respectively v, to the points u_1 and u_2 we have

$$\alpha \geq C\beta e^{-d(z,v)}$$

with C a certain positive constant.

Proof. The geodesic from z through u_j ends at a point $\eta_j \in \partial\mathbf{D}$. Clearly, $\eta_j \in S_{\pi/4}(u_j, \xi_j)$. Let I, respectively J, be the arc in $\partial\mathbf{D}$ with extremes η_j's, respectively ξ_j's. Observe that

$$\begin{aligned} \omega(z, I) &= \alpha/2\pi \\ \omega(v, J) &= \beta/2\pi \end{aligned}$$

Using Lemma 1.1.3 one sees that

$$\omega(v, J) \asymp \omega(v, I).$$

The result follows upon applying Harnack's inequality to the harmonic function $\omega(\cdot, I)$.

LEMMA 1.2.3. *Let $z, u, v \in \mathbf{D}$, and let ℓ, s, denote the geodesics emanating from z and going through u and v, respectively. If $L := \min\{d(z, u), d(z, v)\}$ is such that*

$$e^L \geq \operatorname{cotan}\frac{\theta}{16}$$

where θ is the angle between ℓ and s at z, $0 \leq \theta \leq \pi$, then

$$Q_{\pi/2}(u, \xi) \cap Q_{\pi/2}(v, \eta) = \emptyset$$

where $\xi, \eta \in \partial\mathbf{D}$ denote respectively the final endpoints of ℓ and s.

Proof. Without loss of generality we may assume that $z = 0$, and $u \in [0, 1)$. Let us denote by I_ξ and I_η the arcs $S_{\pi/2}(u, \xi)$ and $S_{\pi/2}(v, \eta)$, respectively. It is enough to show that I_ξ, I_η are disjoint.

Notice that $I_\xi = S_\alpha(0, \xi)$ with $\alpha = 2\pi\omega(0, I_\xi)$, and $I_\eta = S_\beta(0, \eta)$ with $\beta = 2\pi\omega(0, I_\eta)$. Hence to conclude that $I_\xi \cap I_\eta = \emptyset$, it is enough to show that

$$\max\{\alpha, \beta\} = 2\pi \max\{\omega(0, I_\xi), \omega(0, I_\eta)\} < \frac{1}{2}\theta. \tag{1.2.3}$$

From Lemmas 1.1.1 and 1.1.2 we have

$$e^{-L} := \max\{e^{-d(0,u)}, e^{-d(0,v)}\} = \max\left\{\tan\left(\frac{\pi}{4}\omega(0, I_\xi)\right), \tan\left(\frac{\pi}{4}\omega(0, I_\eta)\right)\right\}.$$

Since $e^{-L} \leq \tan\frac{\theta}{16}$ we get (1.2.3).

LEMMA 1.2.4. *There is a constant C such that if $z, u \in \mathbf{D}$ with $d(z, u) \geq C$, and $0 < \alpha < \frac{3\pi}{4}$, then*

$$Q_\alpha(u, \xi) \subset Q_{\alpha/4}(z, \xi)$$

with $\xi \in \partial\mathbf{D}$ the final endpoint of the geodesic emanating from z and going through u.

Proof. Without loss of generality we may assume that $z = 0$, $\xi = 1$, and $0 < u < 1$. Let $E = S_\alpha(u, 1)$ and $I = S_\beta(u, 1)$, where $\beta = \frac{3\pi}{4}$. Then by Lemma 1.1.3 we have

$$|E| \asymp \omega(u, E)|I| \asymp \alpha e^{-d(0,u)}.$$

Therefore, $|E| \leq \frac{\alpha}{4}$, if C is large enough.

1.3. *Estimates on piecewise geodesics.* The following lemma will allow us to compare polygonal piecewise geodesic paths with geodesics.

LEMMA 1.3.1. *There is a constant Λ_0 such that if $\{z_n\}_{n=0}^\infty$ denotes a sequence of points in $\mathbf{D}$ such that*

$$\beta_n \leq \pi/4$$

where β_n is the absolute value of the angle at z_n between the oriented geodesic arcs $[z_{n-1}\, z_n]$ and $[z_n\, z_{n+1}]$, and such that

$$\Lambda := \inf\{d(z_n, z_{n+1})\} \geq \Lambda_0,$$

then the following conclusions hold:

(i) $d(z_0, z_n) \to \infty$ *and, moreover z_n converges in the Euclidean metric to a single point in $\partial\mathbf{D}$.*

(ii) *There is a constant $C > 0$ such that if γ denotes the geodesic going from z_0 to $\xi := \lim_{n\to\infty} z_n$ then*

$$\sup_n d(z_n, \gamma) < C.$$

(iii) *If φ_n denotes the maximum of the absolute values of the angles at z_0 and z_n between the geodesic from z_0 through z_n, and the polygonal piecewise geodesic path from z_0 to z_n, then*

$$\varphi_n \le e^{C-\Lambda}$$

with C a positive constant.

(iv) *Let $\eta_n \in \partial\mathbf{D}$ denote the final endpoint of the geodesic from z_0 through z_n. Then for $m \ge n$,*

$$Q_{\pi/2}(z_m, \eta_m) \subset Q_{\pi/2}(z_n, \eta_n).$$

Proof. Observe that from Lemma 1.2.1 (i) we have

$$d(z_n, z_0) \ge \left(\sum_{j=1}^{n} d(z_j, z_{j-1})\right) - n \ge n(\Lambda_0 - 1).$$

Thus $\lim_{n\to\infty} d(z_n, z_0) = \infty$, and therefore z_n escapes to the boundary of $\mathbf{D}$. This gives the first part of (i).

To show that (iii) holds, observe that z_0 and z_n play symmetric roles here. So we just have to check the estimate of the angle at z_0. The worst case (i.e., the largest φ_n) occurs when all the intermediate angles are as large as possible,

$$\beta_n \equiv \frac{\pi}{4},$$

and the lengths of the geodesic arcs involved are as small as possible,

$$d(z_{n-1}, z_n) \equiv \Lambda \ge \Lambda_0.$$

In that case all the points $\{z_n\}$ would lie on a single curve γ of constant geodesic curvature. For Λ_0 large enough, this constant geodesic curvature is smaller than 1, say $\cos\omega$. The curve γ is the portion inside the unit disk of a circular arc which intersects the unit circle with angle ω. For Λ_0 large enough the angle ω is at least $\pi/4$, say.

Now observe that for such a circular arc γ, if $0 \in \gamma$ and 1 is an endpoint of γ, say, and if $z = re^{i\theta} \in \gamma$ with $d(0, z) = \Lambda$ then the angle, φ, at 0, between the geodesic arc $[0\,z]$ and the geodesic from 0 to the boundary point 1 satisfies

$$|\varphi| \le C(1 - |z|) \le Ce^{-\Lambda}.$$

One deduces that our φ_n satisfies

$$\varphi_n \leq Ce^{-\Lambda}.$$

Define ξ_n as the final endpoint of the geodesic from z_{n-1} through z_n and recall that η_n is the final endpoint of the geodesic from z_0 through z_n.

Observe that since $\beta_n \leq \pi/4$, by Lemma 1.2.4, for Λ_0 large enough we have

$$Q_{\pi/4}(z_n, \xi_{n+1}) \subset Q_{\pi/2}(z_n, \xi_n) \subset Q_{\pi/4}(z_{n-1}, \xi_n). \tag{1.3.1}$$

Assume that Λ is so large that $e^{C-\Lambda} \leq \frac{\pi}{4}$. Then from (iii) we deduce that

$$Q_{\pi/4}(z_n, \xi_n) \subset Q_{\pi/2}(z_n, \eta_n). \tag{1.3.2}$$

Now

$$\operatorname{diam}(Q_{\pi/2}(z_n, \eta_n)) \leq C(z_0)e^{-d(z_n,z_0)} \leq C(z_0)e^{-(\Lambda_0-1)n}. \tag{1.3.3}$$

From (1.3.1), (1.3.2) and (1.3.3) we deduce the second half of (i).

Observe that from the above one obtains $\xi \in Q_{\frac{\pi}{4}}(z_n, \xi_n)$. From this it follows easily that the angle at z_n between the oriented geodesic arcs $[z_0\, z_n]$ and $[z_n\, \xi]$ is at most $\pi/2$. Finally, using Lemma 1.1.4, one deduces (ii).

It remains to verify that (iv) holds. And clearly by induction it is enough to consider the case $m = n+1$. Denote by φ the bound on the φ_n's: $\varphi = e^{C-\Lambda}$ and by β the bound on the β_n's ($\beta \leq \pi/4$). From part (iii) it is easy to see that

$$Q_{\pi/2}(z_{n+1}, \eta_{n+1}) \subset Q_{\varphi+(\pi/2)}(z_{n+1}, \xi_{n+1})$$

and

$$Q_{(\pi/2)-\varphi-\beta}(z_n, \xi_{n+1}) \subset Q_{\pi/2}(z_n, \eta_n). \tag{1.3.4}$$

Moreover, observe that by Lemma 1.2.4 we have

$$Q_{(\pi/2)+\varphi}(z_{n+1}, \xi_{n+1}) \subset Q_{(\pi/2)-\varphi-\beta}(z_n, \xi_{n+1}) \tag{1.3.5}$$

as long as φ satisfies the inequalities $\varphi + \pi/2 \leq 4\,(\pi/2 - \varphi - \beta)$. Combining (1.3.4) and (1.3.5) we conclude that (iv) holds.

Now consider two finite sequences $\{p_j\}_{j=0}^n$ and $\{q_j\}_{j=0}^n$ in $\mathbf{D}$, which satisfy the conditions of Lemma 1.3.1 and are such that for some m, $0 \leq m \leq n-1$, we have $p_j = q_j$, $j = 0, 1, \dots, m$. In particular, Λ_0 is a common lower bound of the lengths of the geodesic arcs between consecutive points of these sequences. Let α, respectively β, be the angle between the geodesic arcs $[p_0 p_n]$ and $[q_0 q_n]$, respectively $[p_m p_n]$ and $[q_m q_n]$. The next lemma gives us a useful inequality between the angles α and β under certain conditions.

LEMMA 1.3.2. *There are positive constants C_1, C_2 such that if Λ_0 is large enough, and*

$$M := \min_{j\geq m}\{\min\{d(p_j, p_{j+1}), d(q_j, q_{j+1})\}\} \geq \log \frac{C_1}{\theta}$$

where θ denotes the angle between the geodesic arcs $[p_m p_{m+1}]$ and $[q_m q_{m+1}]$, then

$$\alpha \geq C_2 \beta e^{-d(p_0, p_m)} = C_2 \beta e^{-d(q_0, q_m)}.$$

Proof. This lemma follows from applying Lemma 1.2.2 to $z = p_0$, $v = p_m = q_m$, $u_1 = p_n$, and $u_2 = q_n$. To apply Lemma 1.2.2 we need the following:

(i) $\gamma_j \leq \dfrac{\pi}{4}$ for $j = 1, 2$ where γ_1 (respectively γ_2) is the interior angle at p_n (resp. q_n) of the triangle with vertices p_0, $p_m = q_m$, and p_n (resp. q_n).

(ii) $Q_{\pi/2}(p_n, \xi_1) \cap Q_{\pi/2}(q_n, \xi_2) = \emptyset$ where ξ_1 (respectively ξ_2) denotes the endpoint of the geodesic from $p_m = q_m$ through p_n (resp. q_n).

From Lemma 1.3.1 (iii) we deduce that

$$\gamma_j \leq 2e^{C-\Lambda_0}$$

and taking Λ_0 large enough we have (i).

On the other hand, (ii) follows from Lemma 1.2.3 if we show that

$$e^L \geq \operatorname{cotan}\frac{\beta}{16} \tag{1.3.6}$$

where $L := \min\{d(p_m, p_n), d(q_m, q_n)\}$. But from Lemma 1.3.1 (iii) we have

$$\beta \geq \theta - 2e^{C-M}$$

and from Lemma 1.2.1 (i) we get

$$L \geq M + (n - m - 1)(\Lambda_0 - 1) \geq M$$

Hence, taking M large enough (depending on θ) we obtain (1.3.6).

2. Trees and Hausdorff dimension

2.1. *Trees.* In this paper by a *tree* $\mathcal{T}$ we shall mean a graph whose vertices, $\mathcal{V}$, are points in $\mathbf{D}$, whose edges, $\mathcal{A}$, are geodesic arcs connecting pair of vertices, and which as an abstract graph is a planar tree. By this we mean that it is connected, has no cycles, and any pair of open edges do not intersect. (An open edge is an edge with the two vertices that it connects removed.)

We fix a root vertex v_0 in $\mathcal{V}$ and classify the vertices according to their graph distance from v_0 into generations:

$$V_n = \{v \in \mathcal{V}\colon \text{graph-distance}(v, v_0) = n\}.$$

The vertices in V_n are called the vertices of the n^{th} generation. Of course, $V_0 = \{v_0\}$. The edges in $\mathcal{A}$ will be oriented away from the root, and we define A_n as the collection of those edges of $\mathcal{T}$ connecting a vertex in V_n with a vertex in V_{n+1}.

For any vertex $v \in V_n$ we will denote by $P(v)$ the *parent* of v, that is, the unique neighbor of v which belongs to V_{n-1}. Of course, v_0 has no parent. Also we will denote by $H(v)$ the set of *children* of v, that is, the set of neighbors of v in V_{n+1}.

We shall introduce now some definitions and notations that will be useful in the rest of this section. For a given tree $\mathcal{T}$ and for all $n \geq 1$, we use the following notation.

$L_n(\mathcal{T})$ is the infimum of the lengths of all the edges in A_n.

$\theta_n(\mathcal{T})$ is the infimum of the absolute values of the angles between any two edges in A_n, if they share a common vertex. If all the vertices in V_n have only one child, then $\theta_n(\mathcal{T}) := 2\pi$.

$b_n(\mathcal{T})$ is the supremum of the absolute values of the angles between consecutive edges one in A_{n-1} and the other in A_n.

$N_n(\mathcal{T}) + 1$ is the infimum of the branching number of vertices in V_n. Recall that the branching number of a vertex v is the number of neighbors of v.

We refer to Section 1 for details on angle measuring.

We may write

$$L_n(\mathcal{T}) = \inf\{\text{length}[P(v)v]\colon v \in V_{n+1}\},$$

$$\theta_n(\mathcal{T}) = \inf\{|\angle([vu_1], [vu_2])|\colon v \in V_n,\ u_1, u_2 \in H(v)\};$$

(If $\#H(v) = 1$ for all $v \in V_n$, then $\theta_n(\mathcal{T}) = 2\pi$.) And

$$b_n(\mathcal{T}) = \sup\{|\angle([P(v)v], [vu])|\colon v \in V_n,\ u \in H(v)\},$$

$$N_n(\mathcal{T}) = \inf_{v \in V_n} \#H(v).$$

We will use $\#U$ to denote the cardinality of the set U.

To avoid artificial technicalities we shall require from the outset that any trees under consideration satisfy

$$\text{(2.1.1)} \quad \beta(\mathcal{T}) = \sup_{n\geq 1} b_n(\mathcal{T}) < \pi/8 \qquad \text{and} \qquad \Lambda(\mathcal{T}) = \inf_{n\geq 1} L_n(\mathcal{T}) \geq \Lambda_0$$

where $\Lambda_0 > 1$ is the constant given in Lemmas 1.3.1 and 1.3.2.

The *branch* from $v_n \in V_n$ to $v_m \in V_m$ $(m > n)$ is the *unique* finite sequence of vertices $\{v_i\}_{i=n}^{m}$ such that v_{i+1} is a child of v_i; we will use $[v_n\, v_{n+1} \cdots v_m]$ to denote the polygonal piecewise geodesic path from v_n to v_m given by the union of the edges of the tree joining the vertices of $\{v_i\}_{i=n}^{m}$. An *infinite branch emanating from* $v_n \in V_n$ is an infinite sequence of vertices $\{v_i\}_{i=n}^{\infty}$ such that $v_{i+1} \in H(v_i)$.

Observe that Lemma 1.3.1 implies that if $\{v_i\}_i$ is an infinite branch, then $\{v_i\}$ converges in the Euclidean metric to a point in $\partial\mathbf{D}$; we will refer to this point as the *endpoint of the branch* $\{v_i\}_i$. The *rim of* $\mathcal{T}$, $\operatorname{Rim}(\mathcal{T})$, is the set of all the endpoints of all infinite branches of the tree $\mathcal{T}$.

For any $v \in \mathcal{V}$, we will use ξ_v to denote the final endpoint of the oriented geodesic emanating from $P(v)$ and going through v. Moreover we will define $\operatorname{Rim}(\mathcal{T})_n$ as the set

$$\{\xi_v \colon v \in V_n\}$$

Notice that if $\xi \in \operatorname{Rim}(\mathcal{T})$ then there exists a sequence $\{\xi_n\}$ with $\xi_n \in \operatorname{Rim}(\mathcal{T})_n$ of $\mathcal{T}$, such that $\xi_n \to \xi$.

We are interested in determining the size, i.e., the Hausdorff dimension of the rim of $\mathcal{T}$, in terms of the "growth" of $\mathcal{T}$.

The following lemma will allow us to describe a kind of tree $\mathcal{T}$ such that the rim of $\mathcal{T}$ can be written as a Cantor type set.

For simplicity we will use $Q_\theta(v)$ and $S_\theta(v)$ to denote the cone $Q_\theta(v, \xi_v)$ and the shadow $S_\theta(v, \xi_v)$ respectively. We refer to Section 1.2 for the general definitions of $Q_\theta(v, \xi)$ and $S_\theta(v, \xi)$.

From Lemmas 1.2.3 and 1.2.4 we get the following result.

LEMMA 2.1.1. *There exists $C > 0$ such that if $L_n(\mathcal{T}) \geq C + \log(16/\theta_n(\mathcal{T}))$, then*

(i) $S_{\pi/2}(u) \cap S_{\pi/2}(v) = \emptyset$ *for all* $u, v \in V_{n+1}$ *such that* $P(u) = P(v)$,
(ii) $Q_\alpha(v) \subset Q_{\alpha/2}(P(v))$, *for all* $v \in V \backslash \{v_0\}$ *and all* α *such that* $4\beta(\mathcal{T}) < \alpha < \pi/2$.

We remark that it follows from Lemma 2.1.1 that if $\xi \in \operatorname{Rim}(\mathcal{T})$, with $\xi := \lim_{n\to\infty} \xi_{v_n}$ and $v_n \in V_n$, then ξ is also determined as $\{\xi\} = \cap_{i=0}^{\infty} I_i$ with $I_{v_0} = \partial\mathbf{D}$ and I_{v_i} $(i > 0)$ the arc $S_\alpha(v_i)$ with $4\beta(\mathcal{T}) < \alpha < \pi/2$. Consequently, the rim of $\mathcal{T}$ can be described as the Cantor type set

$$\operatorname{Rim}(\mathcal{T}) := \bigcap_{n=0}^{\infty} \bigcup_{v \in V_n} I_v.$$

Observe that $\operatorname{Rim}(\mathcal{T})$, does not depend on α, of course. However its description above as a Cantor set, apparently depends on α, but in fact it does not. (All this assumes $4\beta < \alpha < \frac{\pi}{2}$.)

From Lemmas 1.1.3, 1.2.1, and 2.1.1 (ii) we have the next result.

LEMMA 2.1.2. *There is a constant $C > 1$ such that if $\Lambda(\mathcal{T}) \geq C$, then the following hold*:

(i) $d(v_0, v) \geq d(v_0, P(v)) + d(P(v), v) - C$ *for all* $v \in \mathcal{V}\backslash\{v_0\}$.

(ii) *If J is an arc such that $J \subset S_{\pi/2}(v)$ with $v \in \mathcal{V}$, then*

$$\frac{1}{C}\,|J|e^{d(v_0,v)} \leq \omega(v_0, J) \leq C\,|J|e^{d(v_0,v)}.$$

(iii) *If $v \in \mathcal{V}$ and $0 \leq \alpha \leq \pi/2$, then*

$$\frac{\alpha}{C}\,e^{-d(v_0,v)} \leq |S_\alpha(v)| \leq C\,\alpha\,e^{-d(v_0,v)}$$

and

$$\frac{1}{C}\,e^{-d(P(v),v)} \leq \frac{|S_\alpha(v)|}{|S_\alpha(P(v))|} \leq C\,e^{-d(P(v),v)}.$$

(iv) *For all $v \in \mathcal{V}\backslash\{v_0\}$, and $4\beta(\mathcal{T}) < \alpha < \pi/2$, the set $A := S_\alpha(v)\backslash S_{\alpha/2}(v)$ is a disjoint union of two arcs J_1, $J_2 \in \partial\mathbf{D}$ such that*

$$|J_i| \geq \frac{1}{C^2}\,|S_\alpha(v)| \qquad i = 1, 2.$$

The following corollary shall be frequently used.

COROLLARY 2.1.1. *There exists a positive constant C_0 such that if $\{L_n(\mathcal{T})\}$ is an increasing sequence, $e^{L_n(\mathcal{T})} \geq C_0/\theta_n(\mathcal{T})$, and $4\beta(\mathcal{T}) < \alpha < \pi/2$, then the conclusions of Lemmas* 1.3.1, 1.3.2, 2.1.1, *and* 2.1.2 *hold for any branch of $\mathcal{T}$.*

2.2. *Cantor trees.* We say that a tree $\mathcal{T}$ is a *Cantor tree* if, for some $s > 0$, the following hold.

(i) For all $\epsilon \in (0, 1)$ there exists $n(\epsilon) \in \mathbf{N}$ such that for all $v \in V_{n+1}$ with $n \geq n(\epsilon)$,

$$\text{length}([P(v)v]) \leq (1+\epsilon)L_n(\mathcal{T}),$$

i.e.,

$$\frac{\max \text{length}([P(v)v])}{\min \text{length}([P(v)v])} \longrightarrow 1$$

when $n \to \infty$. Observe that $\text{length}([P(v)v]) \geq L_n(\mathcal{T})$ for all $v \in V_{n+1}$.

(ii) The sequence $\{L_n(\mathcal{T})\}$ is monotonically increasing to infinity, and $e^{L_n(\mathcal{T})} \geq C_0/\theta_n(\mathcal{T})$, with C_0 the constant in Corollary 2.1.1. This implies that the conclusions of Corollary 2.1.1 hold.

(iii) There exists a sequence $\{\tau_n\}_{n=1}^{\infty}$ with $\tau_n \geq 0$ and $\lim_{n\to\infty} \tau_n/L_{n-1} = 0$ satisfying the following:
If J is an arc on $\partial\mathbf{D}$ such that $J \subset S_{\pi/2}(v)$, with $v \in V_n$, and moreover $|J| \geq \frac{1}{C_0} e^{-d(v_0,\tilde{u})}$ for some $\tilde{u} \in H(v)$, then

$$\frac{\#\{u \in H(v) \cap Q_J(v)\}}{N_n(\mathcal{T})} \leq e^{\tau_n} \left(\frac{|J|}{e^{-d(v_0,v)}}\right)^s .$$

We recall that $Q_J(v)$ denotes the cone with vertex v and shadow J; see Section 1.2.

The tree described in the introduction is a Cantor tree with $\epsilon = 0$, τ_n a constant, and $s < \log 2/\log 3$.

We define *the order of the Cantor tree $\mathcal{T}$* as the supremum of all s satisfying the above conditions. Observe that for each $s <$ order of $\mathcal{T}$, the conditions above are satisfied.

2.2. *Rate of growth of a Cantor tree.* For a Cantor tree as above we have the following two useful estimates:

$$N_n(\mathcal{T}) \geq M^s e^{-\tau_n} e^{sL_n(\mathcal{T})} \tag{2.2.1}$$

$$N_n(\mathcal{T}) \leq Ce^{(1+\epsilon)L_n(\mathcal{T})} \text{ for } n \geq n(\epsilon) \tag{2.2.2}$$

where M and C are absolute constants ($M = 2/(\pi C_0^2)$).

Notice that the inequalities above imply that the order of $\mathcal{T}$ is at most 1. Here we are using the fact that L_n *increases* to ∞ and that τ_n/L_{n-1} tends to 0.

To verify (2.2.1) observe that if we take $J := S_{\pi/2}(u)$ with $u \in H(v)$ ($v \in V_n$), then from Lemma 2.1.2 (iii) (which holds because of condition (ii) of the definition of Cantor tree), $|J| \geq 1/C_0\, e^{-d(v_0,u)}$, and therefore condition (iii) of the definition of a Cantor tree says that

$$\frac{1}{N_n(\mathcal{T})} \leq e^{\tau_n} \frac{|S_{\pi/2}(u)|^s}{e^{-s\,d(v_0,v)}}. \tag{2.2.3}$$

Besides, from Lemma 2.1.2, (iii), it follows that

$$\left(\frac{\pi}{2C_0^2}\right) e^{-d(u,v)} \leq \frac{|S_{\pi/2}(u)|}{e^{-d(v_0,v)}} \leq \frac{\pi C_0^2}{2}\, e^{-d(u,v)},$$

and by definition of $L_n(\mathcal{T})$,

$$\frac{|S_{\pi/2}(u)|}{e^{-d(v_0,v)}} \leq \frac{\pi C_0^2}{2} e^{-L_n(\mathcal{T})}.$$

Now, using (2.2.3) we obtain (2.2.1).

To verify (2.2.2) observe that for all $4\beta(\mathcal{T}) < \alpha < \pi/2$, from Lemma 2.1.1 we have

$$\bigcup_{u\in H(v)} S_\alpha(u) \subset S_\alpha(v)$$

and moreover, the union is a disjoint union. Then by comparing Lebesgue measures and using Lemma 2.1.2 again, and from the property (ii) of a Cantor tree, we obtain (2.2.2).

2.3. *Dimension of rims of Cantor trees.* Now we are ready to prove the theorem mentioned in the introduction which gives a relation between the size of the rim and the order of a Cantor tree, and which also tells us about distances and angles between the piecewise geodesics forming the tree.

THEOREM. *A Cantor tree $\mathcal{T}$ of order δ has the following properties*:

(i) *The rim of $\mathcal{T}$,* $\mathrm{Rim}(\mathcal{T})$, *has non zero η-content for all $\eta < \delta$. In particular,*

$$\dim(\mathrm{Rim}(\mathcal{T})) \geq \textit{order of } \mathcal{T}.$$

(ii) *All piecewise geodesic paths within the tree are quasigeodesics (in the hyperbolic metric) with a fixed constant; more precisely there exists a constant C (depending on all the constants appearing in the definition of the Cantor tree $\mathcal{T}$) such that if $\{v_j\}_{j=m}^n$ is a branch of $\mathcal{T}$, then*

$$\sup_{z\in[v_m v_{m+1}\dots v_n]} d(z,[v_m v_n]) < C.$$

In particular, if $\xi = \lim_{j\to\infty} v_j$ belongs to the rim of $\mathcal{T}$, then

$$\sup_{z\in[v_0 v_1\dots]} d(z,[v_0\xi]) < C.$$

(iii) *If $\{u_j\}_{j=m}^n$ and $\{v_j\}_{j=m}^n$ are two branches in $\mathcal{T}$ such that $u_j = v_j$ for $j = m, m+1, \dots, s$ with $m \leq s \leq n-1$, then*

$$e^{d(u_m,u_s)} = e^{d(v_m,v_s)} \geq C\,\frac{\beta}{\alpha}$$

where α (respectively β) denotes the interior angle at $u_m = v_m$ (resp. $u_s = v_s$) of the triangle with vertices u_n, v_n, and $u_m = v_m$ (resp. $u_s = v_s$), and C is a positive constant.

For more on the theory of quasigeodesics we refer to [GH].

Proof. Property (iii) follows directly from Lemma 1.3.2 which holds due to the second property of a Cantor tree. The Property (ii) is a consequence of Lemma 1.3.1 (ii) and the fact that if ℓ is a geodesic arc in $\mathbf{D}$, and C is a positive constant, then the region

$$\Omega = \{z \in \mathbf{D}\colon\ d(z, \ell) \leq C\}$$

is hyperbolically convex.

It remains only to prove (i). First, we fix $\eta < \delta$, and we define $\epsilon := (\delta - \eta)/(2\eta)$. Then we take $\eta_1 \in (\eta(1+\epsilon), \delta)$ such that properties (i)–(iii) of the Cantor tree $\mathcal{T}$ hold for $s = \eta_1$. We will denote by $\{\tau_n\}_{n=1}^{\infty}$ the sequence given by property (iii) (with $s = \eta_1$).

Let us fix α such that $4\beta(\mathcal{T}) < \alpha < \pi/2$. We recall that $\beta(\mathcal{T}) < \pi/8$, (see (2.1.1)).

We have already described the rim of $\mathcal{T}$, $\operatorname{Rim}(\mathcal{T})$, as the Cantor type set

$$\operatorname{Rim}(\mathcal{T}) := \bigcap_{n=0}^{\infty} \bigcup_{v \in V_n} I_v.$$

with $I_{v_0} = \partial\mathbf{D}$, and $I_{v_i} = S_\alpha(v_i)$ for $v_i \in V_n$, $n > 0$.

As we mentioned in the introduction to show that the set $\operatorname{Rim}(\mathcal{T})$ has non zero η-content it is enough to construct a *probability* measure ν with support on $\operatorname{Rim}(\mathcal{T})$ and such that for all arcs J on $\partial\mathbf{D}$,

$$\nu(J) \leq C\,|J|^{\eta} \tag{2.3.1}$$

for some positive constant $C > 0$.

First, we will define the measure $\nu(I_v)$ inductively, for each $v \in \mathcal{V}$, (from one generation to the next). First, $\nu(I_{v_0}) = 1$, and then for all $v \in V\backslash\{v_0\}$,

$$\nu(I_v) = \nu(I_v \cap \operatorname{Rim}(\mathcal{T})) := \frac{1}{\#H(P(v))}\, \nu(I_{P(v)}).$$

In other terms, the mass of the parent is equidistributed among its children. Next, for any set $J \subset \partial\mathbf{D}$,

$$\nu(J) := \inf \sum_{U \in \mathcal{U}} \nu(U),$$

where the infimum is taken over all the coverings $\mathcal{U}$ of J with arcs in $\{I_v\colon\ v \in \mathcal{V}\}$.

The set-function ν defined above is a measure (e.g., see [R]).

Now, we verify that (2.3.1) holds for $J = I_v$, $v \in \mathcal{V}$. From Lemma 2.1.2 and the property (i) of a Cantor tree, there exists $n_1 \in \mathbf{N}$ such that for all $v \in V_{n+1}$ with $n > n_1$,

$$\frac{1}{C_0}\, e^{-(1+\epsilon)L_n(\mathcal{T})} \leq \frac{|I_v|}{|I_{P(v)}|} \leq C_0\, e^{-L_n(\mathcal{T})}. \tag{2.3.2}$$

To simplify notation we will write L_n and N_n instead of $L_n(\mathcal{T})$ and $N_n(\mathcal{T})$, respectively.

Since $\lim_{n\to\infty} \tau_n/L_{n-1} = 0$, and $\{L_n\}$ is a sequence increasing to ∞, we can take $n_2 \in \mathbf{N}$ large enough so that for all $n \geq n_2$,

$$e^{(\eta_1-\eta(1+\epsilon))L_n} \geq \frac{C_0^\eta}{M^{\eta_1}} \left(\frac{\pi}{2}\right)^{\eta_1} e^{\tau_{n+1}} \tag{2.3.3}$$

with $M = \frac{2}{\pi C_0^2}$.

Let $k_0 = \max\{n_1, n_2\}$.

We will prove inductively that for all $v \in V_{k+1}$,

$$\nu(I_v) \leq C' \frac{|I_v|^\eta}{e^{\tau_{k+1}}} \tag{2.3.4}$$

with C' a positive constant. Notice that since $\tau_{k+1} \geq 0$, (2.3.4) would imply that (2.3.1) holds for I_v. But (2.3.4) is a stronger inequality, which we shall use later; see (2.3.9) and (2.3.10).

It is clear that there exists $C' > 0$ such that all I_v with $v \in \bigcup_{n=0}^{k_0} V_n$ satisfy (2.3.4).

Now consider the arc I_v with $v \in V_{k+1}$ and $k \geq k_0$. By definition, we have

$$\nu(I_v) = \frac{\nu(I_{P(v)})}{\#H(P(v))} \leq \frac{\nu(I_{P(v)})}{N_k}.$$

Using the inductive hypothesis, and (2.3.2), we get

$$\nu(I_v) \leq C' \frac{|I_{P(v)}|^\eta}{e^{\tau_k} N_k} \leq C' \frac{(C_0 e^{(1+\epsilon)L_k} |I_v|)^\eta}{e^{\tau_k} N_k}.$$

Moreover, (see (2.2.1)) we know that

$$\frac{1}{N_k} \leq \frac{1}{M^{\eta_1}} \frac{e^{\tau_k}}{e^{\eta_1 L_k}}.$$

Hence, we have

$$\nu(I_v) \leq C' \frac{C_0^\eta}{M^{\eta_1} e^{(\eta_1-\eta(1+\epsilon))L_k}} |I_v|^\eta.$$

And from (2.3.3) it follows that I_v satisfies (2.3.4).

Finally, we verify that (2.3.1) holds for an arbitrary arc $J \subset \partial\mathbf{D}$. Let us assume that $J \neq I_v$ for all $v \in V$, and $J \cap \mathrm{Rim}(\mathcal{T}) \neq \emptyset$ (otherwise $\nu(J) = 0$). If $J \cap \mathrm{Rim}(\mathcal{T}) \not\subset I_v$ for any $v \in V_{k_0}$, then $|J| \geq c$ for some positive constant c, and therefore

$$\nu(J) \leq 1 \leq \frac{1}{c^\eta} |J|^\eta.$$

So, we may assume that $J \subset I_v$, for some $v \in V_{k_0}$.

Let I_u be the smallest arc in the family $\{I_w\colon\ w \in \cup_{i\geq k_0} V_i\}$ containing the arc J, and denote by m the generation of the vertex u, i.e., $u \in V_m$.

We have

$$(2.3.5)\quad \nu(J) \leq \sum_{\substack{w\in H(u)\\ I_w\cap J\cap \mathrm{Rim}(\mathcal{T})\neq\emptyset}} \nu(I_w) \leq \#\{w\in H(u)\colon\ I_w \cap J \cap \mathrm{Rim}(\mathcal{T}) \neq \emptyset\}\,\frac{\nu(I_u)}{N_m}.$$

Let us fix I_w with $w \in H(u)$ such that $I_w \cap J \cap \mathrm{Rim}(\mathcal{T}) \neq \emptyset$. (Notice that there is at least one $w \in H(u)$ such that $I_w \cap J \cap \mathrm{Rim}(\mathcal{T}) \neq \emptyset$, because $J \cap \mathrm{Rim}(\mathcal{T}) \neq \emptyset$ and $J \subset S_{\alpha/2}(u)$).

From Lemma 2.1.1 (ii) we get $I_w \cap \mathrm{Rim}(\mathcal{T}) \subset S_{\alpha/2}(w)$, and from the definition of I_u we have $J \not\subset I_w$. Then, by Lemma 2.1.2 it follows that

$$(2.3.6)\qquad |J| \geq \frac{1}{C_0^2}|I_w| \geq \frac{\alpha}{C_0^3} e^{-d(v_0,w)}.$$

Using the fact that $S_\alpha(w) \subset S_{\alpha/2}(u)$ for all $w \in H(u)$ (this follows from Lemma 2.1.1), and (2.3.6), it is not difficult to see that there exists an arc J_1 in $\partial\mathbf{D}$ and a positive constant c depending only on C_0 such that

$$(2.3.7)\qquad J \subset J_1 \subset S_{\pi/2}(u) \quad\text{and}\quad \frac{1}{C_0}e^{-d(v_0,w)} \leq |J_1| \leq (1+c)|J|,$$

and such that

$$(2.3.8)\qquad \{w \in H(u)\colon\ I_w \cap J \cap \mathrm{Rim}(\mathcal{T}) \neq \emptyset\} \subset \{w \in H(u) \cap Q_{J_1}(u)\}.$$

From property (iii) of a Cantor tree (with $s = \eta_1$) for J_1 and (2.3.7) we have

$$\frac{\#\{w \in H(u) \cap Q_{J_1}(u)\}}{N_m} \leq e^{\tau_m}\frac{|J_1|^{\eta_1}}{e^{-\eta_1 d(v_0,u)}} \leq (1+c)^{\eta_1} e^{\tau_m}\frac{|J|^{\eta_1}}{e^{-\eta_1 d(v_0,u)}},$$

and therefore using (2.3.5) and (2.3.8) we get

$$(2.3.9)\qquad \nu(J) \leq (1+c)^{\eta_1} e^{\tau_m}\frac{|J|^{\eta_1}}{e^{-\eta_1 d(v_0,u)}}\,\nu(I_v).$$

From (2.3.9), (2.3.4), and Lemma 2.1.2, it follows that

$$(2.3.10)\qquad \nu(J) \leq C'\frac{|J|^{\eta_1}}{e^{-(\eta_1-\eta)d(v_0,u)}}.$$

Since $J \subset I_u$, by Lemma 2.1.2 we have

$$(2.3.11)\qquad |J| \leq C_0\alpha\, e^{-d(v_0,u)}.$$

Finally, since $\eta_1 - \eta > 0$ from (2.3.10) and (2.3.11), we get (2.3.1).

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