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THE EXCEPTIONAL SET IN THE FOUR PRIME SQUARES PROBLEM

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ABSTRACT. In this paper we prove that, with at most $O(N^{13/15+\varepsilon})$ exceptions, all positive even integers $n \leq N$ with $n \equiv 4(\text{mod } 24)$ can be written as sums of four squares of primes.

1. Introduction

In 1938, Hua [10] proved that each large integer congruent to $5(\text{mod } 24)$ can be written as a sum of five squares of primes. In view of this result and Lagrange's theorem of four squares, it seems reasonable to conjecture that each large integer $n \equiv 4(\text{mod } 24)$ is a sum of four squares of primes,

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2. \tag{1.1}$$

However, a result of this strength seems out of reach at present. The purpose of this paper is to establish the following approximation to this conjecture.

THEOREM 1. *Let* $N \geq 2$, *and let* $E(N)$ *be the number of positive integers* $\equiv 4(\text{mod } 24)$, *not exceeding* N, *which cannot be written in the form in* (1.1). *Then for any* $\theta > 13/15$ *we have*

$$E(N) \ll N^{\theta}. \tag{1.2}$$

The first result in this direction is due to Hua [10], who proved that $E(N) \ll N \log^{-A} N$ for some absolute constant $A > 0$. Later Schwarz [21] showed that A can be taken arbitrarily.

There are other approximations to the above mentioned conjecture, and our Theorem 1 can be compared with them. Greaves [8] gave a lower bound for the number of representations of an integer as a sum of two squares of integers and two squares of primes. Later Shields [22], Plaksin [18], and Kovalchik [13] obtained, among other things, an asymptotic formula in this problem. Recently Brüdern and Fouvry [2] proved that every large $n \equiv 4(\text{mod } 24)$ is the sum of four squares of integers with each of their prime factors greater than $n^{1/68.86}$.

We prove our Theorem 1 by the circle method. Here the main difficulty arises in treating the enlarged major arcs. The idea of the proof will be explained in §2.

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Notation. As usual, $\varphi(n)$, $\mu(n)$, and $\Lambda(n)$ stand for the function of Euler, Möbius, and von Mangoldt respectively, $d(n)$ is the divisor function, and $d_\nu(n)$ is the generalized divisor function which is defined as the number of representations of n as products of ν positive integers. We use $\chi \bmod q$ and $\chi^0 \bmod q$ to denote a Dirichlet character and the principal character modulo q, and $L(s,\chi)$ is the Dirichlet L-function. For integers $a, b, \ldots$ we denote by $[a, b, \ldots]$ their least common multiple. N is a large integer, and $L = \log N$. And $r \sim R$ means $R < r \leq 2R$. If there is no ambiguity, we express $\frac{a}{b} + \theta$ as $a/b + \theta$ or $\theta + a/b$. The same convention will be applied for quotients. The letter ε denotes a positive constant which is arbitrarily small.

2. Outline of the method

In order to apply the circle method, we set

$$P = N^{2/15-\varepsilon}, \qquad Q = N/(PL^{14}). \tag{2.1}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1 + 1/Q]$ may be written in the form

$$\alpha = a/q + \lambda, \qquad |\lambda| \leq 1/(qQ) \tag{2.2}$$

for some integers a, q with $1 \leq a \leq q \leq Q$ and $(a, q) = 1$. We denote by $\mathcal{M}(a, q)$ the set of α satisfying (2.2), and define the major arcs $\mathcal{M}$ and the minor arcs $C(\mathcal{M})$ as follows:

$$\mathcal{M} = \bigcup_{q \leq P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{M}(a, q), \qquad C(\mathcal{M}) = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \backslash \mathcal{M}. \tag{2.3}$$

It follows from $2P \leq Q$ that the major arcs $\mathcal{M}(a, q)$ are mutually disjoint. Our Theorem 1 is a consequence of the following:

THEOREM 2. *Let $\mathcal{M}$ be as in* (2.3) *with P determined by* (2.1). *And let*

$$T(\alpha) = \sum_{p^2 \leq N} (\log p) e(p^2 \alpha). \tag{2.4}$$

Then for $2 \leq n \leq N$, we have

$$\int_{\mathcal{M}} T^4(\alpha) e(-n\alpha) d\alpha = \frac{\pi^2}{16} \mathfrak{S}(n) n + O\left(\frac{N}{\log N}\right). \tag{2.5}$$

Here $\mathfrak{S}(n)$ is defined in (3.3), *and satisfies $\mathfrak{S}(n) \gg 1$ for $n \equiv 4 \pmod{24}$.*

In the following proof for our Theorem 1, we only need this theorem for $N/2 < n \leq N$, but here we consider a much wider range $2 \leq n \leq N$ for general interest.

Note that the theorem only gives an O-result if n is much smaller than N. However it is useful in a later paper [24] even in its weak form.

We can readily derive Theorem 1 from Theorem 2.

Proof of Theorem 1. Let N be a sufficiently large integer, and $N/2 < n \le N$ with $n \equiv 4 (\mathrm{mod}\, 24)$. Let

$$r(n) = \sum_{n=p_1^2+\cdots+p_4^2} (\log p_1)\cdots(\log p_4).$$

Then we have

$$r(n) = \int_0^1 T^4(\alpha)e(-n\alpha)d\alpha = \int_{\mathcal{M}} + \int_{C(\mathcal{M})}, \tag{2.6}$$

where $\mathcal{M}$, $C(\mathcal{M})$, and $T(\alpha)$ are as in (2.3) and (2.4).

To estimate the contribution from the minor arcs, one notes that each $\alpha \in C(\mathcal{M})$ can be written as (2.2) for some $P < q \le Q$ and $1 \le a \le q$ with $(q, a) = 1$. We now apply Theorem 2 of Ghosh [7], which states that, for $\alpha \in C(\mathcal{M})$,

$$T(\alpha) \ll N^{1/2+\varepsilon}(P^{-1} + N^{-1/4} + QN^{-1})^{1/4} \ll N^{1/2-1/30+2\varepsilon}. \tag{2.7}$$

Also, we easily derive the following mean-value estimate for $T(\alpha)$:

$$\int_0^1 |T(\alpha)|^4 d\alpha \ll L^4 \sum_{\substack{m_1^2+m_2^2=m_3^2+m_4^2 \\ m_j^2 \le N}} 1 \ll N^{1+\varepsilon}. \tag{2.8}$$

It therefore follows from Bessel's inequality, (2.7), and (2.8) that

$$\begin{aligned}\sum_{N/2<n\le N} \left|\int_{C(\mathcal{M})}\right|^2 &\ll \int_{C(\mathcal{M})} |T(\alpha)|^8 d\alpha \\ &\ll \left\{\max_{\alpha\in C(\mathcal{M})} |T(\alpha)|^4\right\} \int_0^1 |T(\alpha)|^4 d\alpha \ll N^{3-2/15+9\varepsilon}.\end{aligned}$$

Therefore, for all $N/2 < n \le N$ with $n \equiv 4 (\mathrm{mod}\, 24)$, except for a subset $\mathfrak{E}(N)$ of cardinality $O(N^{13/15+11\varepsilon})$, we have

$$\left|\int_{C(\mathcal{M})}\right| \ll N^{1-\varepsilon}. \tag{2.9}$$

The contribution from the major arcs can be handled by Theorem 2. We conclude from Theorem 2, (2.6), and (2.9) that for all $N/2 < n \le N$ with $n \equiv 4 (\mathrm{mod}\, 24)$ and $n \notin \mathfrak{E}(N)$,

$$r(n) = \frac{\pi^2}{16}\mathfrak{S}(n)n + O\left(\frac{n}{\log n}\right).$$

From this and the fact that $\mathfrak{S}(n) \gg 1$, Theorem 1 clearly follows.

Now it only remains to prove Theorem 2, which takes up the rest of the paper. One easily sees from (2.1) that our major arcs is quite large. In contrast to the previous works [14], [16], [6] which treat the enlarged major arcs by the Deuring-Heilbronn phenomenon, we prove Theorem 2 by a different approach, which has recently been used by Bauer, Liu and Zhan [3]. This approach reveals that in the context of this paper, the possible existence of Siegel zero does not have special influence, hence the Deuring-Heilbronn phenomenon can be avoided. The key point of this approach is that there are four prime variables in our problem (while there are only two in Linnik [14] and Gallagher [6]), and we can take advantage of this by saving the factor $r_0^{-1+\varepsilon}$ in Lemma 3.1 below. With this saving, our enlarged major arcs can be treated by the large sieve inequality, Gallagher's lemma, and classical results on the distribution of zeros of L-functions (see Lemmas 3.3–3.6). Our novelties in this paper described above not only give better results (note that Theorem 2 holds with $P = N^{2/15-\varepsilon}$), but also lead us to a technically simpler proof.

3. Preliminaries

For χ mod q, define

$$C(\chi, a) = \sum_{h=1}^{q} \bar{\chi}(h) e\left(\frac{ah^2}{q}\right), \qquad C(q, a) = C(\chi^0, a). \tag{3.1}$$

If $\chi_1, \ldots, \chi_4$ are characters mod q, then we write

$$B(n, q, \chi_1, \ldots, \chi_4) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) C(\chi_1, a) \cdots C(\chi_4, a),$$
$$B(n, q) = B(n, q, \chi^0, \ldots, \chi^0), \tag{3.2}$$

and

$$A(n, q) = \frac{B(n, q)}{\varphi^4(q)}, \qquad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q). \tag{3.3}$$

This $\mathfrak{S}(n)$ is the singular series in Theorem 2.

In the following sections, we will need the following results.

LEMMA 3.1. *Let χ_j mod r_j with $j = 1, \ldots, 4$ be primitive characters, $r_0 = [r_1, \ldots, r_4]$, and χ^0 the principal character* mod q. *Then*

$$\sum_{\substack{q \le x \\ r_0 | q}} \frac{1}{\varphi^4(q)} |B(n, q, \chi_1\chi^0, \ldots, \chi_4\chi^0)| \ll r_0^{-1+\varepsilon} \log^{17} x.$$

Proof. This is Lemma 4.4 in [15].

LEMMA 3.2. (i) *We have*

$$\sum_{q>x} |A(n,q)| \ll x^{-1+\varepsilon} d(n). \tag{3.4}$$

Thus the singular series $\mathfrak{S}(n)$ *is absolutely convergent.*

(ii) *For* $n \equiv 4 \pmod{24}$, *one has*

$$c_1 < \mathfrak{S}(n) \ll (\log\log n)^{11}$$

with some absolute constant $c_1 > 0$; *while for* $n \not\equiv 4 \pmod{24}$, *one has* $\mathfrak{S}(n) = 0$.

Proof. Part (i) is (4.12) of [15] and Part (ii) is Proposition 4.3 of [15].

LEMMA 3.3. *Let* $P \geq 2$ *and* $T \geq 2$, *and* $k = 0$ *or* 1. *Then we have*

$$\sum_{q \leq P} \sum_{\chi \bmod q}{}^{*} \int_{-T}^{T} \left| L^{(k)}\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \ll P^2 T \log^{4(k+1)}(P^2 T).$$

Here and in the sequel, the sum $\sum^*$ *is over all primitive characters.*

LEMMA 3.4. *Let* $P \geq 2$, $T \geq 2$, *and* a_m *with* $m = 1, 2, \dots$ *be a sequence of complex numbers. Then we have*

$$\sum_{q \leq P} \sum_{\chi \bmod q}{}^{*} \int_{-T}^{T} \left| \sum_{m=M_0}^{M_0+M} \frac{a_m \chi(m)}{m^{it}} \right|^2 dt \ll \sum_{m=M_0}^{M_0+M} (P^2 T + m)|a_m|^2.$$

LEMMA 3.5. *For* $T \geq 2$, *let* $N^*(\alpha, q, T)$ *denote the number of zeros of all the L-functions* $L(s, \chi)$ *with primitive characters* $\chi \bmod q$ *in the region* $\operatorname{Re} s \geq \alpha$, $|\operatorname{Im} s| \leq T$. *Then*

$$N^*(\alpha, q, T) \ll (qT)^{12(1-\alpha)/5} \log^c(qT)$$

where $c > 0$ *is an absolute constant.*

LEMMA 3.6. *Let* $T \geq 2$. *There is an absolute constant* $c_2 > 0$ *such that* $\prod_{\chi \bmod q} L(s, \chi)$ *is zero-free in the region*

$$\operatorname{Re} s \geq 1 - c_2/\max\{\log q, \log^{4/5} T\}, \qquad |\operatorname{Im} s| \leq T,$$

except for the possible Siegel zero.

Lemmas 3.3–3.6 are well-known results in number theory. For the proofs of Lemmas 3.3–3.5, see for example pp. 640 and 642, 634, and 669 in Pan-Pan [17]. For Lemma 3.3, see also Bombieri [1], and for a slightly weak form of Lemma 3.5 which suffices for our purposes, see Huxley [12]. For the proof of Lemma 3.6, see Satz VIII.6.2 in Prachar [19].

4. An explicit expression

Let $M = NL^{-12}$, and

$$S(\alpha) = \sum_{M<p^2\le N} (\log p)e(p^2\alpha).$$

It is convenient to establish the asymptotic formula

$$\int_{\mathcal{M}} S^4(\alpha)e(-n\alpha)d\alpha = \frac{\pi^2}{16}\mathfrak{S}(n)n + O\left(\frac{N}{\log N}\right), \tag{4.1}$$

and then in §6 we derive Theorem 2 (i.e., (2.5)) from (4.1). The purpose of this section is to establish an explicit expression for the left-hand side of (4.1) (see Lemma 4.1 below). And in §§5 and 6 we shall estimate this explicit expression to obtain (4.1). Define

$$V(\lambda) = \sum_{M<m^2\le N} e(m^2\lambda),$$

$$W(\chi,\lambda) = \sum_{M<p^2\le N} (\log p)\chi(p)e(p^2\lambda) - \delta_\chi \sum_{M<m^2\le N} e(m^2\lambda), \tag{4.2}$$

where $\delta_\chi = 1$ or 0 according as χ is principal or not. Also, define

$$J = \sum_{r\le P} r^{-1/4+\varepsilon} \sum_{\chi \bmod r}{}^{*} \max_{|\lambda|\le 1/(rQ)} |W(\chi,\lambda)|,$$

and

$$K = \sum_{r\le P} r^{-1/4+\varepsilon} \sum_{\chi \bmod r}{}^{*} \left(\int_{-1/(rQ)}^{1/(rQ)} |W(\chi,\lambda)|^2 d\lambda\right)^{1/2}.$$

Now we state the main result of this section.

LEMMA 4.1. *Let n, $\mathcal{M}$ be as in Theorem 2. Then*

$$\int_{\mathcal{M}} S^4(\alpha)e(-n\alpha)d\alpha = \frac{\pi^2}{16}\mathfrak{S}(n)n + O\{(J^2K^2 + J^2K + J^2 + N^{1/2}J)L^{23}\} + O(NL^{-1}),$$

where $\mathfrak{S}(n)$ is the singular series defined as in (3.3).

Proof. Introducing Dirichlet characters, we can rewrite the exponential sum $S(\alpha)$ (see for example [4], §26, (2)) as

$$S\left(\frac{a}{q}+\lambda\right) = \frac{C(q,a)}{\varphi(q)}V(\lambda) + \frac{1}{\varphi(q)}\sum_{\chi \bmod q} C(\chi,a)W(\chi,\lambda). \tag{4.3}$$

Thus,

$$\int_{\mathcal{M}} S^4(\alpha)e(-n\alpha)d\alpha = I_0 + 4I_1 + 6I_2 + 4I_3 + I_4, \tag{4.4}$$

where

$$I_j = \sum_{q\le P} \frac{1}{\varphi^4(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} C^{4-j}(q,a)e\left(-\frac{an}{q}\right) \int_{-1/(qQ)}^{1/(qQ)} V^{4-j}(\lambda) \left\{ \sum_{\chi \bmod q} C(\chi,a)W(\chi,\lambda) \right\}^j e(-n\lambda)d\lambda.$$

We will prove that I_0 gives the main term, and I_1, I_2, I_3, I_4 the error term.

We begin with I_4, the most complicated one. Reducing the characters in I_4 into primitive characters, we have

$$\begin{aligned} |I_4| &= \left| \sum_{q\le P} \frac{1}{\varphi^4(q)} \sum_{\chi_1 \bmod q} \cdots \right. \\ &\qquad \left. \sum_{\chi_4 \bmod q} B(n,q,\chi_1,\dots,\chi_4) \int_{-1/(qQ)}^{1/(qQ)} W(\chi_1,\lambda)\cdots W(\chi_4,\lambda)e(-n\lambda)d\lambda \right| \\ &\le \sum_{r_1\le P}\cdots\sum_{r_4\le P} \sum_{\chi_1 \bmod r_1}{}^{*}\cdots \sum_{\chi_4 \bmod r_4}{}^{*} \sum_{\substack{q\le P\\ r_0|q}} \frac{|B(n,q,\chi_1\chi^0,\dots,\chi_4\chi^0)|}{\varphi^4(q)} \\ &\qquad \times \int_{-1/(qQ)}^{1/(qQ)} |W(\chi_1\chi^0,\lambda)|\cdots|W(\chi_4\chi^0,\lambda)|d\lambda, \end{aligned}$$

where χ^0 is the principal character modulo q and $r_0 = [r_1, \dots, r_4]$. For $q \le P$ and $M < p^2 \le N$, we have $(q, p) = 1$. Using this and (4.2), we have $W(\chi_j\chi^0, \lambda) = W(\chi_j, \lambda)$ for the primitive characters χ_j above. Using this and Lemma 3.1, we obtain

$$\begin{aligned} |I_4| &\le \sum_{r_1\le P}\cdots\sum_{r_4\le P} \sum_{\chi_1 \bmod r_1}{}^{*}\cdots \sum_{\chi_4 \bmod r_4}{}^{*} \int_{-1/(r_0Q)}^{1/(r_0Q)} |W(\chi_1,\lambda)|\cdots|W(\chi_4,\lambda)|d\lambda \\ &\qquad \times \sum_{\substack{q\le P\\ r_0|q}} \frac{|B(n,q,\chi_1\chi^0,\dots,\chi_4\chi^0)|}{\varphi^4(q)} \\ &\ll L^{17} \sum_{r_1\le P}\cdots\sum_{r_4\le P} r_0^{-1+\varepsilon} \sum_{\chi_1 \bmod r_1}{}^{*}\cdots \sum_{\chi_4 \bmod r_4}{}^{*} \int_{-1/(r_0Q)}^{1/(r_0Q)} |W(\chi_1,\lambda)|\cdots|W(\chi_4,\lambda)|d\lambda. \end{aligned}$$

If we apply the inequality

$$r_0^{-1+\varepsilon} \le r_1^{-1/4+\varepsilon} r_2^{-1/4+\varepsilon} r_3^{-1/4+\varepsilon} r_4^{-1/4+\varepsilon} \tag{4.5}$$

to the above quantity and use Cauchy's inequality, then we get

$$\begin{aligned}|I_4| &\ll L^{17}\left\{\sum_{r_1\le P} r_1^{-1/4+\varepsilon} \sum_{\chi_1 \bmod r_1}{}^{*} \max_{|\lambda|\le 1/(r_1Q)} |W(\chi_1,\lambda)|\right\}\\ &\quad\times\left\{\sum_{r_2\le P} r_2^{-1/4+\varepsilon} \sum_{\chi_2 \bmod r_2}{}^{*} \max_{|\lambda|\le 1/(r_2Q)} |W(\chi_2,\lambda)|\right\}\\ &\quad\times\left\{\sum_{r_3\le P} r_3^{-1/4+\varepsilon} \sum_{\chi_3 \bmod r_3}{}^{*} \left(\int_{-1/(r_3Q)}^{1/(r_3Q)} |W(\chi_3,\lambda)|^2 d\lambda\right)^{1/2}\right\}\\ &\quad\times\left\{\sum_{r_4\le P} r_4^{-1/4+\varepsilon} \sum_{\chi_4 \bmod r_4}{}^{*} \left(\int_{-1/(r_4Q)}^{1/(r_4Q)} |W(\chi_4,\lambda)|^2 d\lambda\right)^{1/2}\right\}\\ &= J^2K^2L^{17}.\end{aligned} \tag{4.6}$$

Similarly, we can bound I_3, I_2, and I_1 in terms of J and K, to get

$$\begin{aligned}|I_3|+|I_2|+|I_1| \ll L^{17}\Bigg\{&J^2K\left(\int_{-1/Q}^{1/Q}|V(\lambda)|^2 d\lambda\right)^{1/2} + J^2\int_{-1/Q}^{1/Q}|V(\lambda)|^2 d\lambda\\ &+ J\left(\int_{-1/Q}^{1/Q}|V(\lambda)|^2 d\lambda\right)\max_{|\lambda|\le 1/Q}|V(\lambda)|\Bigg\}.\end{aligned} \tag{4.7}$$

By Lemma 7.11 of [11],

$$V(\lambda) = \int_{M^{1/2}}^{N^{1/2}} e(\lambda u^2)du + O(1) = \frac{1}{2}\sum_{M<m\le N} m^{-1/2}e(m\lambda) + O(1). \tag{4.8}$$

Using this and the elementary estimate

$$\sum_{M<m\le N} m^{-1/2}e(m\lambda) \ll \min(N^{1/2}, M^{-1/2}|\lambda|^{-1}), \tag{4.9}$$

one easily gets

$$\max_{|\lambda|\le 1/Q}|V(\lambda)| \ll N^{1/2},$$

$$\int_{-1/Q}^{1/Q}|V(\lambda)|^2 d\lambda \ll \int_0^{1/\sqrt{MN}} N d\lambda + \int_{1/\sqrt{MN}}^{\infty} M^{-1}\lambda^{-2} d\lambda \ll L^6.$$

It thus follows from (4.6) and (4.7) that

$$|I_4|+|I_3|+|I_2|+|I_1| \ll \{J^2K^2 + J^2K + J^2 + N^{1/2}J\}L^{23}. \tag{4.10}$$

It remains to compute I_0. Substituting (4.8) into I_0, we have

$$I_0 = \frac{1}{16}\sum_{q\le P}\frac{B(n,q)}{\varphi^4(q)}\int_{-1/(qQ)}^{1/(qQ)}\left\{\sum_{M<m\le N} m^{-1/2}e(m\lambda)\right\}^4 e(-n\lambda)d\lambda$$
$$+ O\left\{\sum_{q\le P}\frac{|B(n,q)|}{\varphi^4(q)}\int_{-1/(qQ)}^{1/(qQ)}\left|\sum_{M<m\le N} m^{-1/2}e(m\lambda)\right|^3 d\lambda\right\}. \quad (4.11)$$

By (4.9) and Lemma 3.1 with $r_0 = 1$, the O-term in (4.11) can be estimated as

$$\ll \sum_{q\le P}\frac{|B(n,q)|}{\varphi^4(q)}\left\{\int_0^{1/\sqrt{MN}} N^{3/2}d\lambda + \int_{1/\sqrt{MN}}^{\infty} M^{-3/2}|\lambda|^{-3}d\lambda\right\} \ll N^{1/2}L^{23}.$$

Now we extend the integral in the main term of (4.11) to $[-1/2, 1/2]$; by a similar argument we see that the resulting error is

$$\ll L^{17}\int_{1/(PQ)}^{1/2} M^{-2}|\lambda|^{-4}d\lambda \ll M^{-2}(PQ)^3L^{17} \ll NL^{-1},$$

where we have used (2.1). Thus the main term of (4.11) becomes

$$\frac{1}{16}\sum_{q\le P}\frac{B(n,q)}{\varphi^4(q)}\sum_{\substack{M<m_1,\dots,m_4\le N\\ m_1+\dots+m_4=n}}(m_1\cdots m_4)^{-1/2} + O(NL^{-1}).$$

By (3.4), the first sum above is $\mathfrak{S}(n) + O(L^{-1})$. The second sum can be calculated as

$$\sum_{\substack{1\le m_1,\dots,m_4\le N\\ m_1+\dots+m_4=n}}(m_1\cdots m_4)^{-1/2} + O(M^{1/2}N^{1/2}) = \frac{\Gamma^4(1/2)}{\Gamma(2)}n\{1+O(n^{-1/2})\} + O(NL^{-6})$$
$$= \pi^2 n + O(NL^{-6}),$$

on appealing to Lemmas 7.17 and 7.18 of Hua [11]. Now by Lemma 3.2 (ii), (4.11) becomes

$$I_0 = \frac{\pi^2}{16}\mathfrak{S}(n)n + O(NL^{-1}). \quad (4.12)$$

Lemma 4.1 now follows from (4.4), (4.10), and (4.12).

5. Estimation of J

We have

$$J \ll L\max_{R\le P} J_R$$

where J_R is defined similarly to J except that the sum is over $r \sim R$. The estimation of J_R falls naturally into two cases according as R is small or large. For $R > L^B$, where B is some positive constant, one appeals to contour integration, mean-value estimates for the Dirichlet L-functions or their derivatives, the large sieve inequality, and Heath-Brown's identity. While for $R \le L^B$, one uses the classical zero-density estimates and zero-free region for the Dirichlet L-functions.

We first establish the following result for large R. In Lemma 5.5 we shall consider small R.

LEMMA 5.1. *Let $A > 0$ be arbitrary. Then there exists a constant $B = B(A) > 0$ such that when $L^B < R \le P$,*

$$J_R \ll N^{1/2}L^{-A},$$

where the implied constant depends at most on A.

To prove this result, it suffices to show that

$$\sum_{r\sim R}\ \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda|\le 1/(rQ)} |W(\chi,\lambda)| \ll R^{1/4-\varepsilon}N^{1/2}L^{-A} \tag{5.1}$$

for $L^B < R \le P$ and arbitrary $A > 0$. Let

$$\hat{W}(\chi,\lambda) = \sum_{M<m^2\le N} (\Lambda(m)\chi(m) - \delta_\chi)e(m^2\lambda). \tag{5.2}$$

Then

$$W(\chi,\lambda) - \hat{W}(\chi,\lambda) = -\sum_{j\ge 2}\ \sum_{M<p^{2j}\le N} (\log p)\chi(p)e(p^{2j}\lambda) \ll N^{1/4}. \tag{5.3}$$

Thus (5.1) is a consequence of the estimate

$$\sum_{r\sim R}\ \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda|\le 1/(rQ)} |\hat{W}(\chi,\lambda)| \ll R^{1/4-\varepsilon}N^{1/2}L^{-A}, \tag{5.4}$$

where $R \le P$ and $A > 0$ is arbitrary.

Let $M^{1/2} < u \le N^{1/2}$, and let $M_1, \dots, M_{10}$ be positive integers such that

$$2^{-10}M^{1/2} \le M_1 \cdots M_{10} < u, \qquad \text{and} \qquad 2M_6, \dots, 2M_{10} \le u^{1/5}. \tag{5.5}$$

For $j = 1, \dots, 10$ let

$$a_j(m) = \begin{cases} \log m & \text{if } \ j = 1, \\ 1 & \text{if } \ j = 2, 3, 4, 5, \\ \mu(m) & \text{if } \ j = 6, 7, 8, 9, 10. \end{cases}$$

We define the following functions of a complex variable s:

$$f_j(s) = f_j(s,\chi) = \sum_{m\sim M_j} \frac{a_j(m)\chi(m)}{m^s}, \qquad F(s) = F(s,\chi) = f_1(s)\cdots f_{10}(s).$$

Now we recall Heath-Brown's identity (see Lemma 1 in [9]) for $k = 5$, which states that

$$\frac{\zeta'}{\zeta}(s) = \sum_{j=1}^{5} \binom{5}{j} (-1)^{j-1} \zeta'(s) \zeta^{j-1}(s) G^j(s) + \frac{\zeta'}{\zeta}(s)(1 - \zeta(s)G(s))^5,$$

where $\zeta(s)$ is the Riemann zeta-function, and $G(s) = \sum_{m \le u^{1/5}} \mu(m) m^{-s}$. We choose $k = 5$ because the identity with $k \le 4$ will give weaker results, and when $k \ge 6$ it produces the same estimate as the case $k = 5$. Equating coefficients of the Dirichlet series on both sides provides an identity for $-\Lambda(m)$. Also, for $m \le u$ the coefficient of m^{-s} in

$$-\frac{\zeta'}{\zeta}(s)(1 - \zeta(s)G(s))^5$$

is zero. Thus,

$$\Lambda(m) = \sum_{j=1}^{5} \binom{5}{j} (-1)^{j-1} \sum_{\substack{m_1 \cdots m_{2j} = m \\ m_{j+1}, \ldots, m_{2j} \le u}} (\log m_1) \mu(m_{j+1}) \cdots \mu(m_{2j}).$$

Applying this identity to the sum

$$\sum_{M^{1/2} < m \le u} \Lambda(m) \chi(m), \tag{5.6}$$

one finds that (5.6) is a linear combination of $O(L^{10})$ terms, each of which is of the form

$$\sigma(u; \mathbf{M}) = \sum_{\substack{m_1 \sim M_1 \\ M^{1/2} < m_1 \cdots m_{10} \le u}} \cdots \sum_{m_{10} \sim M_{10}} a_1(m_1) \chi(m_1) \cdots a_{10}(m_{10}) \chi(m_{10})$$

where $\mathbf{M}$ denotes the vector $(M_1, M_2, \ldots, M_{10})$. By using Perron's summation formula (see for example, Lemma 3.12 in [23] or Theorem 2, p.98 in [17]) and then shifting the contour to the left, the above $\sigma(u; \mathbf{M})$ is

$$\frac{1}{2\pi i} \int_{1+1/L-iT}^{1+1/L+iT} F(s, \chi) \frac{u^s - M^{s/2}}{s} ds + O\left(\frac{N^{1/2} L^2}{T}\right)$$
$$= \frac{1}{2\pi i} \left\{ \int_{1+1/L-iT}^{1/2-iT} + \int_{1/2-iT}^{1/2+iT} + \int_{1/2+iT}^{1+1/L+iT} \right\} + O\left(\frac{N^{1/2} L^2}{T}\right),$$

where T is a parameter satisfying $2 \le T \le N^{1/2}$. The integral on the two horizontal segments above can be easily estimated as

$$\ll \max_{1/2 \le \sigma \le 1+1/L} |F(\sigma \pm iT, \chi)| \frac{u^\sigma}{T} \ll \max_{1/2 \le \sigma \le 1+1/L} N^{(1-\sigma)/2} L \frac{u^\sigma}{T} \ll \frac{N^{1/2} L}{T}$$

on using the trivial estimate

$$\begin{aligned} F(\sigma \pm iT, \chi) &\ll |f_1(\sigma \pm iT, \chi)| \cdots |f_{10}(\sigma \pm iT, \chi)| \\ &\ll (M_1^{1-\sigma} L) M_2^{1-\sigma} \cdots M_{10}^{1-\sigma} \ll N^{(1-\sigma)/2} L. \end{aligned}$$

Thus,

$$\sigma(u; \mathbf{M}) = \frac{1}{2\pi} \int_{-T}^{T} F\left(\frac{1}{2} + it, \chi\right) \frac{u^{\frac{1}{2}+it} - M^{\frac{1}{2}(\frac{1}{2}+it)}}{\frac{1}{2} + it} dt + O\left(\frac{N^{1/2} L^2}{T}\right).$$

Since $R > L^B$ (so $\chi \neq \chi^0$), in (5.2) we have

$$\hat{W}(\chi, \lambda) = \sum_{M < m^2 \leq N} \Lambda(m) \chi(m) e(m^2 \lambda) = \int_{M^{1/2}}^{N^{1/2}} e(u^2 \lambda) d\left\{ \sum_{M^{1/2} < m \leq u} \Lambda(m) \chi(m) \right\},$$

and consequently $\hat{W}(\chi, \lambda)$ is a linear combination $O(L^{10})$ terms, each of which is of the form

$$\begin{aligned} &\int_{M^{1/2}}^{N^{1/2}} e(u^2 \lambda) d\sigma(u; \mathbf{M}) \\ &= \frac{1}{2\pi} \int_{-T}^{T} F\left(\frac{1}{2} + it, \chi\right) \int_{M^{1/2}}^{N^{1/2}} u^{-1/2+it} e(u^2 \lambda) du\, dt + O\left(\frac{N^{1/2} L^2}{T} (1 + |\lambda| N)\right). \end{aligned}$$

By taking $T = N^{1/2}$ and changing variables in the inner integral, we deduce from the above formulae that

$$\begin{aligned} |\hat{W}(\chi, \lambda)| \ll L^{10} \max_{\mathbf{M}} \Bigg| \int_{-T}^{T} F\left(\frac{1}{2} + it, \chi\right) \\ \int_{M}^{N} v^{-3/4} e\left(\frac{t}{4\pi} \log v + \lambda v\right) dv\, dt \Bigg| + N^{2/15} L^{12}, \end{aligned} \tag{5.7}$$

where the maximum is taken over all $\mathbf{M} = (M_1, M_2, \ldots, M_{10})$. Since

$$\frac{d}{dv}\left(\frac{t}{4\pi} \log v + \lambda v\right) = \frac{t}{4\pi v} + \lambda, \qquad \frac{d^2}{dv^2}\left(\frac{t}{4\pi} \log v + \lambda v\right) = -\frac{t}{4\pi v^2},$$

by Lemmas 4.4 and 4.3 in [23], the inner integral in (5.7) can be estimated as

$$\begin{aligned} &\ll M^{-3/4} \min\left\{ \frac{N}{(|t| + 1)^{1/2}}, \frac{N}{\min_{M < v \leq N} |t + 4\pi \lambda v|} \right\} \\ &\ll \begin{cases} N^{1/4} L^9 / (|t| + 1)^{1/2} & \text{if } |t| \leq T_0, \\ N^{1/4} L^9 / |t| & \text{if } T_0 < |t| \leq T, \end{cases} \end{aligned} \tag{5.8}$$

where $T_0 = 8\pi N/(RQ)$. Here the choice of T_0 is to ensure that $|t + 4\pi\lambda v| > |t|/2$ whenever $|t| > T_0$; in fact,

$$|t + 4\pi\lambda v| \geq |t| - 4\pi|v|/(rQ) > |t|/2 + T_0/2 - 4\pi N/(RQ) \geq |t|/2.$$

Therefore it follows from (5.7) and (5.8) that the lemma (more precisely, the $\ll$ in (5.4)) is a consequence of the following two estimates: For $0 < T_1 \leq T_0$, we have

$$\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt \ll R^{1/4-\varepsilon}N^{1/4}(T_1+1)^{1/2}L^{-A}, \tag{5.9}$$

while for $T_0 < T_2 \leq T$, we have

$$\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_{T_2}^{2T_2}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt \ll R^{1/4-\varepsilon}N^{1/4}T_2L^{-A}. \tag{5.10}$$

Both (5.9) and (5.10) are deduced from the following bound.

LEMMA 5.2. *Let $F(s,\chi)$ be defined as above. Then for any $R \geq 1$ and $T_3 > 0$,*

$$\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_{T_3}^{2T_3}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt \ll (R^2T_3 + RT_3^{1/2}N^{3/20} + N^{1/4})L^c. \tag{5.11}$$

Now we can complete the proof of Lemma 5.1.

Proof of Lemma 5.1. By taking $T_3 = T_1$ in Lemma 5.2, the left-hand side of (5.9) is now

$$\ll (R^2T_1 + RT_1^{1/2}N^{3/20} + N^{1/4})L^c \ll R^{1/4-\varepsilon}N^{1/4}(T_1+1)^{1/2}L^{-A},$$

provided that $L^B < R \leq P = N^{2/15-\varepsilon}$ with a sufficiently large B. Here $L^B < R$ guarantees that $N^{1/4}L^c$ is dominated by the quantity on the right-hand side. This establishes (5.9). Similarly we can prove (5.10) by taking $T_3 = T_2$ in Lemma 5.2. Lemma 5.1 now follows.

It remains to prove Lemma 5.2, which follows from the following two propositions.

PROPOSITION 5.3. *If there exist M_i and M_j with $1 \leq i < j \leq 5$ such that $M_iM_j > N^{1/5}$, then* (5.11) *is true.*

Proof. Without loss of generality, we may suppose that $i = 1$, and $j = 2$. Using Perron's summation formula and then shifting the path of integration to the left as before, we get

$$\begin{aligned} f_1\left(\frac{1}{2}+it,\chi\right) &= \frac{1}{2\pi i}\int_{1/2+1/L-iN}^{1/2+1/L+iN} L'\left(\frac{1}{2}+it+w,\chi\right)\frac{(2M_1)^w - M_1^w}{w}dw + O(L^2) \\ &= \frac{1}{2\pi i}\left\{\int_{1/2+1/L-iN}^{-iN} + \int_{-iN}^{iN} + \int_{iN}^{1/2+1/L+iN}\right\} + O(L^2). \end{aligned}$$

Here one notes that the function $\frac{(2M_1)^w - M_1^w}{w}$ has a removable singularity at $w = 0$. Thus, on the above vertical segment from $-iN$ to iN, we have

$$\frac{(2M_1)^w - M_1^w}{w} \ll \frac{1}{1+|v|}$$

where $w = u + iv$. Using the well-known bounds (see for example [17], p. 271, Exercise 6 and p. 264, (13))

$$L'(\sigma + it, \chi) \ll \begin{cases} r^{(1-\sigma)/2}|t|^{1-\sigma}\log^2(r|t|) & \text{for } \ 0 < \sigma < 1, |t| \geq 2, \\ \log^2(r|t|) & \text{for } \ \sigma \geq 1, |t| \geq 2, \end{cases}$$

the contribution from the horizontal segments can be estimated as

$$\begin{aligned} &\ll \max_{0 \leq u \leq 1/2+1/L} r^{(1-(1/2+u))/2}(N+|t|)^{1-(1/2+u)}\log^2(r(N+|t|))\frac{M_1^u}{N} \\ &\ll L^2 \max_{0\leq u\leq 1/2+1/L} r^{1/4-u/2}N^{-1/2-u}M_1^u \ll L^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} f_1\left(\frac{1}{2}+it,\chi\right) &\ll \int_{-N}^{N}\left|L'\left(\frac{1}{2}+it+iv,\chi\right)\right|\frac{dv}{1+|v|} + L^2 \\ &\ll L\left\{\int_{-N}^{N}\left|L'\left(\frac{1}{2}+it+iv,\chi\right)\right|^4\frac{dv}{1+|v|}\right\}^{1/4} + L^2 \end{aligned}$$

by Hölder's inequality. Thus,

$$\begin{aligned} &\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_{T_3}^{2T_3}\left|f_1\left(\frac{1}{2}+it,\chi\right)\right|^4 dt \\ &\quad \ll L^4\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_{T_3}^{2T_3}dt\left\{\int_{|v|\leq 6T_3} + \int_{6T_3\leq|v|\leq N}\right\}\left|L'\left(\frac{1}{2}+it+iv,\chi\right)\right|^4\frac{dv}{1+|v|} \\ &\quad + R^2T_3L^8 =: \Sigma_1 + \Sigma_2 + R^2T_3L^8, \end{aligned}$$

where Σ_1 and Σ_2 denote the contributions from the two integrals within the braces respectively. Clearly,

$$\begin{aligned} \Sigma_1 &= L^4\int_{|v|\leq 6T_3}\frac{dv}{1+|v|}\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_{T_3+v}^{2T_3+v}\left|L'\left(\frac{1}{2}+iw,\chi\right)\right|^4 dw \\ &\ll L^4\int_{|v|\leq 6T_3}\frac{dv}{1+|v|}\sum_{r\sim R}\sum_{\chi \bmod r}{}^{*}\int_{-8T_3}^{8T_3}\left|L'\left(\frac{1}{2}+iw,\chi\right)\right|^4 dw \ll R^2T_3L^{13} \end{aligned}$$

on using Lemma 3.3 in the last step. To bound Σ_2, one first changes the order of integration to get

$$\Sigma_2 = L^4 \int_{T_3}^{2T_3} dt \sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \int_{6T_3\le|w-t|\le N} \left|L'\left(\frac{1}{2}+iw,\chi\right)\right|^4 \frac{dw}{1+|w-t|}.$$

Now $6T_3 \le |w-t| \le N$ implies that either $6T_3+t \le w \le N+t$ or $-N+t \le w \le -6T_3+t$. So, since $T_3 \le t \le 2T_3$, one deduces that in either case $|w-t| - |w|/2 \ge |w|/2 - |t| \ge 0$, and this shows that $|w-t| \ge |w|/2$. Consequently, by Lemma 3.3,

$$\Sigma_2 \ll L^5 \int_{T_3}^{2T_3} dt \max_{4T_3\le x\le N+2T_3} \frac{1}{x} \sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \int_x^{2x} \left|L'\left(\frac{1}{2}+iw,\chi\right)\right|^4 dw \ll R^2T_3L^{13}.$$

Collecting the above estimates for Σ_1 and Σ_2, one obtains

$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_3}^{2T_3} \left|f_1\left(\frac{1}{2}+it,\chi\right)\right|^4 dt \ll R^2T_3L^{13}. \tag{5.12}$$

Arguing similarly, we also have

$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_3}^{2T_3} \left|f_2\left(\frac{1}{2}+it,\chi\right)\right|^4 dt \ll R^2T_3L^{13}. \tag{5.13}$$

Since

$$\prod_{j=3}^{10} f_j\left(\frac{1}{2}+it,\chi\right) = \sum_{M_3\cdots M_{10}<m\le 2^8 M_3\cdots M_{10}} \frac{b(m)\chi(m)}{m^{1/2+it}}$$

with $b(m) \le d_8(m)$, by Lemma 3.4 one has

$$\begin{aligned}\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_3}^{2T_3} &\left|\prod_{j=3}^{10} f_j\left(\frac{1}{2}+it,\chi\right)\right|^2 dt \\ &\ll \sum_{M_3\cdots M_{10}<m\le 2^8 M_3\cdots M_{10}} \frac{(R^2T_3+m)d_8^2(m)}{m} \\ &\ll (R^2T_3+M_3\cdots M_{10})L^c \ll \left\{R^2T_3 + \frac{N^{1/2}}{M_1M_2}\right\} L^c. \end{aligned} \tag{5.14}$$

One thus concludes from Hölder's inequality, (5.12), (5.13), and (5.14) that

$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_3}^{2T_3} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt$$

$$\ll \left\{ \sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{T_3}^{2T_3} \left| f_1\left(\frac{1}{2}+it,\chi\right)\right|^4 dt \right\}^{1/4}$$
$$\times \left\{ \sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{T_3}^{2T_3} \left| f_2\left(\frac{1}{2}+it,\chi\right)\right|^4 dt \right\}^{1/4}$$
$$\times \left\{ \sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{T_3}^{2T_3} \left| \prod_{j=3}^{10} f_j\left(\frac{1}{2}+it,\chi\right)\right|^2 dt \right\}^{1/2}$$
$$\ll (R^2T_3)^{1/2} \left\{ R^2T_3 + \frac{N^{1/2}}{M_1M_2} \right\}^{1/2} L^c \ll (R^2T_3 + RT_3^{1/2}N^{3/20})L^c,$$

since $M_1M_2 > N^{1/5}$. This proves Proposition 5.3.

PROPOSITION 5.4. *If there is a partition* $\{J_1, J_2\}$ *of the set* $\{1, \dots, 10\}$ *such that*

$$\prod_{j\in J_1} M_j + \prod_{j\in J_2} M_j \ll N^{3/10},$$

then (5.11) *is true.*

Proof. For $\nu = 1, 2$ define

$$F_\nu(s,\chi) := \prod_{j\in J_\nu} f_j(s,\chi) = \sum_{n\ll N_\nu} \frac{b_\nu(n)\chi(n)}{n^s},$$

where $N_\nu = \prod_{j\in J_\nu} M_j$ and $b_\nu(n) \ll Ld_{10}(n)$. Applying Lemma 3.4 we see that

$$\sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{T_3}^{2T_3} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt$$
$$\ll \left\{ \sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{T_3}^{2T_3} \left| F_1\left(\frac{1}{2}+it,\chi\right)\right|^2 dt \right\}^{1/2}$$
$$\times \left\{ \sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{T_3}^{2T_3} \left| F_2\left(\frac{1}{2}+it,\chi\right)\right|^2 dt \right\}^{1/2}$$
$$\ll \left\{ R^2T_3 + \sum_{n\ll N_1} |b_1(n)|^2 \right\}^{1/2} \left\{ R^2T_3 + \sum_{n\ll N_2} |b_2(n)|^2 \right\}^{1/2} L^c$$
$$\ll (R^2T_3 + N_1)^{1/2}(R^2T_3 + N_2)^{1/2}L^c$$
$$\ll (R^2T_3 + RT_3^{1/2}N^{3/20} + N^{1/4})L^c, \tag{5.15}$$

since $N_1 + N_2 \ll N^{3/10}$ and $N_1N_2 \ll N^{1/2}$. This proves Proposition 5.4.

Proof of Lemma 5.2. In view of Proposition 5.3, we may assume that $M_iM_j \le N^{1/5}$ for all i, j satisfying $1 \le i < j \le 5$. It follows that there is at most one M_j with $1 \le j \le 5$ such that $M_j > N^{1/10}$. Without loss of generality, we can suppose this exceptional M_j is M_1, so for $j = 2, 3, 4, 5$ we have $M_j \le N^{1/10}$. From this and the assumption that $M_6, \dots, M_{10} \le N^{1/10}$, we deduce that $M_j \le N^{1/10}$ holds for $j = 2, 3, \dots, 10$.

Although M_1 may exceed $N^{1/10}$, it is bounded from above by the inequality $M_1M_2 \le N^{1/5}$. From this and the assumption $M^{1/2} \ll M_1 \cdots M_{10} \ll N^{1/2}$, we see that there is an integer l with $2 \le l \le 8$, such that

$$M_1 \cdots M_l \le N^{1/5}, \qquad \text{but} \qquad M_1 \cdots M_{l+1} > N^{1/5}.$$

Take $N_1 = M_1 \cdots M_{l+1}$ and $N_2 = M_{l+2} \cdots M_{10}$. Then we have

$$N^{1/5} \ll N_1 \ll N^{1/5}M_{l+1} \ll N^{1/5}N^{1/10} \ll N^{3/10} \quad \text{and} \quad N_2 \ll N^{1/2}/N_1 \ll N^{3/10}.$$

Thus we have $N_1 + N_2 \ll N^{3/10}$, i.e., the assumption of Proposition 5.4 is satisfied. Lemma 5.2 now follows from Proposition 5.4.

Now we treat the case $R \le L^B$.

LEMMA 5.5. *Let $A > 0$ and $B > 0$ be arbitrary. Then for $R \le L^B$, we have*

$$J_R \ll N^{1/2}L^{-A},$$

where the implied constant depends at most on B.

Proof. We use the explicit formula (see [4], p.109 and 120, or [17], p.313)

$$\sum_{m \le u} \Lambda(m)\chi(m) = \delta_\chi u - \sum_{|\gamma| \le T} \frac{u^\rho}{\rho} + O\left\{\left(\frac{u}{T} + 1\right)\log^2(quT)\right\} \tag{5.16}$$

where $\rho = \beta + i\gamma$ is a non-trivial zero of the function $L(s, \chi)$, and $2 \le T \le u$ is a parameter. Taking $T = N^{1/6}$ in (5.16), and then inserting it into $\hat{W}(\chi, \lambda)$, it follows from the fact that $M^{1/2} < u \le N^{1/2}$, $M = NL^{-12}$, and (2.1) that

$$\begin{aligned} \hat{W}(\chi, \lambda) &= \int_{M^{1/2}}^{N^{1/2}} e(u^2\lambda)d\left\{\sum_{n \le u}(\Lambda(m)\chi(m) - \delta_\chi)\right\} \\ &= \int_{M^{1/2}}^{N^{1/2}} e(u^2\lambda) \sum_{|\gamma| \le N^{1/6}} u^{\rho-1}du + O\{N^{1/3}(1 + |\lambda|N)L^2\} \\ &\ll N^{1/2}L^3 \sum_{|\gamma| \le N^{1/6}} N^{(\beta-1)/2} + O(N^{7/15}). \end{aligned}$$

Now let $\eta(T) = c_2 \log^{-4/5} T$. By Lemma 3.6, $\prod_{\chi \bmod q} L(s, \chi)$ is zero-free in the region $\sigma \ge 1 - \eta(T)$, $|t| \le T$ except for the possible Siegel zero. But by Siegel's theorem (see, for example, [4], §21) the Siegel zero does not exist in the present situation, since $r \le L^B$. Thus, by Lemma 3.5,

$$\sum_{|\gamma|\le N^{1/6}} N^{(\beta-1)/2} \ll L^c \int_0^{1-\eta(N^{1/6})} (N^{1/6})^{12(1-\alpha)/5} N^{(\alpha-1)/2} d\alpha$$
$$\ll L^c N^{-\eta(N^{1/6})/10} \ll \exp(-c_3 L^{1/5}).$$

Consequently,

$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda|\le 1/(rQ)} |\hat{W}(\chi,\lambda)| \ll N^{1/2} L^{-A}, \tag{5.17}$$

where $R \le P$, and $A > 0$ is arbitrary. Lemma 5.5 now follows from (5.17), (5.2), and (5.3).

6. Estimation of K

In this section, we estimate K by establishing the following Lemma 6.1. We remark that in proving Lemma 6.1 we need not distinguish the two cases $R > L^B$ and $R \le L^B$ as in Lemmas 5.1 and 5.5, since we need not save a factor L^{-A} on the right-hand side of (6.1).

LEMMA 6.1. *We have*

$$K \ll L^c \tag{6.1}$$

where $c > 0$ is some absolute constant.

Proof. By the definition of K and (5.3), we have

$$K \ll L \max_{R\le P} \sum_{r\sim R} r^{-1/4+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |W(\chi,\lambda)|^2 d\lambda \right)^{1/2}$$
$$\ll L \max_{R\le P} \sum_{r\sim R} r^{-1/4+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |\hat{W}(\chi,\lambda)|^2 d\lambda \right)^{1/2} + 1.$$

Thus to establish (6.1), it suffices to show that

$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |\hat{W}(\chi,\lambda)|^2 d\lambda \right)^{1/2} \ll R^{1/4-\varepsilon} L^c \tag{6.2}$$

for $R \le P$ and some $c > 0$.

By Gallagher's lemma (see [5], Lemma 1), we have

$$\int_{-1/(rQ)}^{1/(rQ)} |\hat{W}(\chi,\lambda)|^2 d\lambda \ll \left(\frac{1}{RQ}\right)^2 \int_{-\infty}^{\infty} \left| \sum_{\substack{v<m^2\le v+rQ \\ M<m^2\le N}} (\Lambda(m)\chi(m) - \delta_\chi) \right|^2 dv$$

$$\ll \left(\frac{1}{RQ}\right)^2 \int_{M-rQ}^{N} \left| \sum_{\substack{v<m^2\le v+rQ \\ M<m^2\le N}} (\Lambda(m)\chi(m) - \delta_\chi) \right|^2 dv. \quad (6.3)$$

Let $X = \max(v, M)$ and $Y = \min(v + rQ, N)$. Then the sum in (6.3) can be written as

$$\sum_{X<m^2\le Y} (\Lambda(m)\chi(m) - \delta_\chi). \quad (6.4)$$

Using Heath-Brown's identity to this sum, and applying Perron's formula as before, we see that (6.4) is a linear combination of $O(L^{10})$ terms, each of which has the form

$$\sigma(u; \mathbf{M}) := \frac{1}{2\pi} \int_{-T}^{T} F\left(\frac{1}{2} + it, \chi\right) \frac{Y^{\frac{1}{2}(\frac{1}{2}+it)} - X^{\frac{1}{2}(\frac{1}{2}+it)}}{\frac{1}{2} + it} dt + O\left(\frac{N^{1/2}L^2}{T}\right),$$

where $\mathbf{M}$, $F(s, \chi)$ are as in §5, and T is a parameter satisfying $2 \le T \le N^{1/2}$. One easily sees that

$$\frac{Y^{\frac{1}{2}(\frac{1}{2}+it)} - X^{\frac{1}{2}(\frac{1}{2}+it)}}{\frac{1}{2} + it} = \frac{1}{2} \int_X^Y u^{-3/4+it/2} du = \frac{1}{2} \int_X^Y u^{-3/4} e\left(\frac{t}{4\pi} \log u\right) du.$$

The integral can be easily estimated as

$$\ll Y^{1/4} - X^{1/4} \ll (v + rQ)^{1/4} - v^{1/4} \ll v^{1/4}\{(1 + rQ/v)^{1/4} - 1\}.$$

Since v satisfies $M - rQ \le v \le N$, and $rQ \le 2RQ \le 2PQ = 2NL^{-14} = 2ML^{-2}$, the above quantity is $\ll v^{-3/4}RQ \ll M^{-3/4}RQ$. On the other hand, one has trivially

$$\frac{Y^{\frac{1}{2}(\frac{1}{2}+it)} - X^{\frac{1}{2}(\frac{1}{2}+it)}}{\frac{1}{2} + it} \ll \frac{Y^{1/4}}{|t|} \ll \frac{N^{1/4}}{|t|}.$$

Collecting the two upper bounds, we get

$$\frac{Y^{\frac{1}{2}(\frac{1}{2}+it)} - X^{\frac{1}{2}(\frac{1}{2}+it)}}{\frac{1}{2} + it} \ll \min\left(M^{-3/4}RQ, \frac{N^{1/4}}{|t|}\right) \ll L^9 \min\left(\frac{RQ}{N^{3/4}}, \frac{N^{1/4}}{|t|}\right).$$

Taking

$$T = N^{1/2}, \qquad T_0 = N/(QR),$$

we see that

$$\sigma(u;\mathbf{M}) \ll \frac{RQL^9}{N^{3/4}} \int_{|t|\le T_0} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt + N^{1/4}L^9 \int_{T_0<|t|\le T} \left|F\left(\frac{1}{2}+it,\chi\right)\right| \frac{dt}{|t|} + O(L^2).$$

And consequently (6.3) becomes

$$\int_{-1/(rQ)}^{1/(rQ)} |\hat{W}(\chi,\lambda)|^2 d\lambda \ll N^{-1/2}L^{38} \max_{\mathbf{M}} \left(\int_{|t|\le T_0} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt\right)^2 + \frac{N^{3/2}L^{38}}{(QR)^2} \max_{\mathbf{M}} \left(\int_{T_0<|t|\le T} \left|F\left(\frac{1}{2}+it,\chi\right)\right| \frac{dt}{|t|}\right)^2 + \frac{NL^{24}}{(QR)^2}.$$

Now the left-hand side of (6.2) is

$$\ll N^{-1/4}L^{19} \max_{\mathbf{M}} \sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{|t|\le T_0} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt + \frac{N^{3/4}L^{19}}{RQ} \max_{\mathbf{M}} \sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{T_0<|t|\le T} \left|F\left(\frac{1}{2}+it,\chi\right)\right| \frac{dt}{|t|} + \frac{N^{1/2}RL^{12}}{Q}.$$

Thus, to prove (6.2) it suffices to show that the estimate

$$\sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{T_1}^{2T_1} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll R^{1/4-\varepsilon}N^{1/4}L^c \tag{6.5}$$

holds for $R \le P$ and $0 < T_1 \le T_0$, and

$$\sum_{r\sim R} \sum_{\chi \bmod r}{}^{*} \int_{T_2}^{2T_2} \left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll R^{1/4-\varepsilon}(RQ)N^{-3/4}T_2L^c \tag{6.6}$$

holds for $R \le P$ and $T_0 < T_2 \le T$.

The estimates (6.5) and (6.6) follows from Lemma 5.2. The proof of Lemma 6.1 is completed.

Proof of Theorem 2. By Lemmas 4.1, 5.1, 5.5, and 6.1, we get (4.1). In view of Lemma 3.2, it remains only to derive (2.5) from (4.1).

Applying the inequality $|a^4 - b^4| \le |a-b|\{|a|+|b|\}^3$, we get

$$\int_{\mathcal{M}} \{T^4(\alpha) - S^4(\alpha)\} e(-n\alpha) d\alpha \ll \int_0^1 |T(\alpha) - S(\alpha)||T(\alpha)|^3 d\alpha + \int_0^1 |T(\alpha) - S(\alpha)||S(\alpha)|^3 d\alpha. \tag{6.7}$$

By Hölder's inequality, the last integral above is

$$\ll \left(\int_0^1 |T(\alpha) - S(\alpha)|^4 d\alpha\right)^{1/4} \left(\int_0^1 |S(\alpha)|^4 d\alpha\right)^{3/4} =: H_1^{1/4} H_2^{3/4}, \qquad \text{say.} \tag{6.8}$$

Here H_2 does not exceed $\log^4 N$ times the number of solutions of

$$p_1^2 + p_2^2 = p_3^2 + p_4^2, \qquad p_j \leq N^{1/2}. \tag{6.9}$$

By [20], Satz 3 the number of solutions of (6.9) with $p_1 p_2 \neq p_3 p_4$ is $O(NL^{-3})$. Also by the prime number theorem, (6.9) has approximately $8N \log^{-2} N$ trivial solutions, namely those satisfying $p_1 p_2 = p_3 p_4$. Therefore,

$$H_2 \leq 8(1 + \varepsilon) N \log^2 N \ll NL^2. \tag{6.10}$$

The integral H_1 is less than $\log^4 N$ times the number of solutions of (6.9) with $p_j \leq N^{1/2}$ replaced by $p_j \leq M^{1/2}$, and consequently $H_1 \ll ML^2$ by a similar argument. Putting these upper bounds into (6.8) and using $M = NL^{-12}$, one sees that the last integral in (6.7) is $\ll NL^{-1}$. The same estimate also holds for the next-to-last integral in (6.7), and hence the quantity in (6.7) is bounded by NL^{-1}. The desired result (2.5) now follows from (6.7) and (4.1). Theorem 2 is proved.

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References

[1] E. Bombieri, *Le grand crible dans la theorie analytique des nombres*, Asterisque, No.18, Soc. Math. France, Paris, 1974.
[2] J. Brüdern and E. Fouvry, *Lagrange's four squares theorem with almost prime variables*, J. Reine Angew. Math. **454**(1994), 59–96.
[3] C. Bauer, M. C. Liu and T. Zhan, *On sums of three prime squares*, J. Number Theory, to appear.
[4] H. Davenport, *Multiplicative number theory*, 2nd ed., Springer, Berlin 1980.
[5] P. X. Gallagher, *A large sieve density estimate near* $\sigma = 1$, Invent. Math. **11**(1970), 329–339.
[6] P. X. Gallagher, *Primes and powers of* 2, Invent. Math. **29**(1975), 125–142.
[7] A. Ghosh, *The distribution of* αp^2 *modulo* 1, Proc. London Math. Soc. (Ser. 3) **42**(1981), 252–269.
[8] G. Greaves, *On the representation of a number in the form* $x^2 + y^2 + p^2 + q^2$ *where p and q are odd primes*, Acta Arith. **29**(1976), 257–274.
[9] D. R. Heath-Brown, *Prime numbers in short intervals and a generalized Vaughan's identity*, Canad. J. Math. **34**(1982), 1365–1377.
[10] L. K. Hua, *Some results in the additive prime number theory*, Quart. J. Math. (Oxford) **9**(1938), 68–80.
[11] ———, *Additive theory of prime numbers* (in Chinese), Science Press, Beijing 1957; English translation, Amer. Math. Soc., Providence, Rhode Island 1965.

[12] M. N. Huxley, *Large values of Dirichlet polynomials (III)*, Acta Arith. **26**(1974/75), 435–444.

[13] F. B. Kovalchik, *Some analogies of the Hardy-Littlewood equation*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov **116**(1982), 86–95, 163.

[14] Yu. V. Linnik, *Addition of prime numbers and powers of one and the same number*, Mat. Sb. (N.S.) **32**(1953), 3–60.

[15] J. Y. Liu, M. C. Liu, and T. Zhan, *Squares of primes and powers of two*, Monat. Math., **128**(1999), 283–313.

[16] M. C. Liu and K. M. Tsang, "Small prime solutions of linear equations" in *Théorie des Nombres*, J. M. De Koninck and C. Levesque (éds.), Walter de Gruyter, 1989, 395–624.

[17] C. D. Pan, and C. B. Pan, *Fundamentals of analytic number theory* (in Chinese), Science Press, Beijing 1991.

[18] V. A. Plaksin, *An asymptotic formula for the number of solutions of a nonlinear equation for prime numbers*, Math. USSR Izv. **18**(1982), 275–348.

[19] K. Prachar, *Primzahlverteilung*, Springer, Berlin 1957.

[20] G. J. Rieger, *Über die Summe aus einem Quadrat und einem Primzahlquadrat*, J. Reine Angew. Math. **251**(1968), 89–100.

[21] W. Schwarz, *Zur Darstellung von Zahlen durch Summen von Primzahlpotenzen II*, J. Riene Angew. Math. **206**(1961), 78–112.

[22] P. Shields, *Some applications of sieve methods in number theory*, Thesis, University of Wales, 1979.

[23] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., University Press, Oxford 1986.

[24] J. Y. Liu and M. C. Liu, *Representation of even integers as sums of primes and powers of* 2, J. Number Theory, to appear.

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