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ARENS REGULARITY AND WEAK SEQUENTIAL COMPLETENESS FOR QUOTIENTS OF THE FOURIER ALGEBRA

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ABSTRACT. This is a study of Arens regularity in the context of quotients of the Fourier algebra on a non-discrete locally compact abelian group (or compact group).

(1) If a compact set E of G is of bounded synthesis and is the support of a pseudofunction, then $A(E)$ is weakly sequentially complete. (This implies that every point of E is a Day point.)

(2) If a compact set E supports a synthesizable pseudofunction, then $A(E)$ has Day points. (The existence of a Day point implies that $A(E)$ is not Arens regular.)

We use be L^2-methods of proof which do not have obvious extensions to the case of $A_p(E)$.

Related results, context (historical and mathematical), and open questions are given.

1. Introduction

This introduction first gives a summary of the sections of the paper and then states the definitions which we shall need, as well as providing some background for the results. Further background will be found also in the later sections.

For related results of the author, see [6] (every lca group has countable subsets E such that $A(E)$ is Arens regular in every bounded multiplication while neither $\tilde{A}(E)$ nor $A(E)^{**}$ is Arens regular), [7] ($A_p(E)$ is not regular if E supports a synthesizable pseudofunction), and [8] ($A(E+F)$ is not regular if E, F are perfect compact).

1.1. *Survey of Arens regularity.* Section 2 is a brief discussion of (non) Arens regularity in the context of group algebras. Readers familiar with the subject may skip it. In that section we show (in particular) that if E has a tenting sequence that is also a Sidon sequence, then $A(E)$ is not Arens regular. (The result is stated in terms of general Banach algebras, and, in that form, the result may be new.)

1.2. *General lemmas.* In Section 3 we give technical lemmas that are used in later sections and most usefully grouped together.

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1.3. *Discussion of results.*

Result 1 *of Abstract.* This result is proved in Section 4, first for abelian groups and then for non-abelian compact groups. The result may not be new, as it was stated, in a stronger form (with no proof and no hypotheses on synthesizability) by Meyer in 1970 [19, 6.2.10]. Unfortunately, even Result 1 is false without some sort of synthesizability assumption. A discussion appears in Remarks 4.1.4 (i).

The proof for the non-abelian case uses a simple lemma concerning operators on Hilbert space to estimate the norm of a sum of pseudofunctions. That lemma appears to be new.

Result 1 greatly extends (for $p = 2$) the result of [10, p. 131], which shows that certain subspaces of $A_p(G)$ are weakly sequentially complete. The result here for quotient spaces appears to be new, and somewhat surprising, in spite of Remarks 4.1.4. For the sake of completeness we include, as Proposition 4.1.1, the result from [11] which says that weak sequential completeness for $A_p(E)$ implies the existence of Day points. The conclusion about Day points was inspired by results of Granirer [11]. However, the results of this paper (e.g., 5.1.3–5.1.4) do not include Granirer's results, since his results apply to symmetric sets E, not all of which support pseudofunctions, while our results apply to (some) sets supporting pseudofunctions.

Result 2 *of Abstract.* In Section 5 of this paper, we show how one can obtain the result of Granirer [11] for sets of multiplicity, i.e., the set supports a synthesizable pseudomeasure whose Fourier-Stieltjes transform "vanishes at infinity". (The relevant definitions and some lemmas are given in Section 4.) We first prove the abelian case. It is, again, unclear if the abelian case is new (see above).

Since ultrathin sets never support pseudofunctions, Result 2 is in a different direction from Granirer's. Also, Result 2 is very much an "L^2" result, and the proof does not have evident generalization to the case of $A_p(E)$ for $p \neq 2$. See Section 6 where we give evidence for the difficulty of such a generalization.

The proof for compact non-abelian groups is given at the end of the section. It, like the abelian proof, is still an "L^2" result. See [7], where a weaker version of Result 2 is obtained for quotients of the $A_p(G)$ algebras of Herz.

We end with a list of open questions.

1.4. *Notation and definitions.*

Definition 1.4.1. A Banach algebra A is *Arens regular* if the two Arens multiplications on its second dual A^{**} coincide.

$L^1(G)$ is Arens non-regular, for every infinite locally compact Hausdorff group. (Due to Young [26] for the non-abelian case and Civin and Yood [3] for the abelian case.) On the other hand, C^*-algebras are Arens regular [3]. See Section 2 for more on Arens regularity and non regularity.

Let G be a locally compact group, and, for abelian G, let Γ denote the dual group of G. $A(G)$ is the Fourier algebra of G, and for a compact subset E of G, $A(E)$ denotes the set of restrictions of elements of $A(G)$ to E, each restriction being given its quotient norm. Elements of $A(G)^*$ are called *pseudomeasures* and the collection of them is denoted $PM(G)$. An important subspace of $PM((G)$ is the space of *pseudofunctions* $PF(G)$: a *pseudofunction* is an element of $PM(G)$ whose Fourier transform vanishes at infinity on Γ (in the case of abelian groups) or (in the non-abelian case and true for the abelian case as well) a pseudofunction is an element of $PM(G)$ that is in the norm closure of $L^1(G)$. (Of course $A(G) \simeq L^1(\Gamma)$ and $A(G)^* \simeq L^\infty(\Gamma)$, and so it makes sense to talk about "vanishing at infinity".) Pseudofunctions are important in questions of multiplicity and spectral synthesis; e.g., see, [9], [17], [16] and their references. A pseudomeasure S is called *synthesizable* (or *admits synthesis*) if it is the weak* limit of measures concentrated on the support of S; e.g., see [9, p. 69ff] for background and major results on spectral synthesis and non-synthesis in abelian groups. The pseudomeasures (resp. pseudofunctions) supported on a closed $E \subset G$ will be denoted by $PM(E)$ (resp. $PF(E)$).

1.5. *Tenting sequences and Day points.*

Definition 1.5.1. Let $M > 0$, E a closed subset of the locally compact group G, and $a \in E$. A *tenting-M sequence at a* is a sequence $\{f_n\} \subset A(E)$ such that:

(1.5.1) $\|f_n\|_{A(E)} \leq M$ for all $n \geq 1$;

(1.5.2) $f_n(a) = 1$ for all $n \geq 1$;

(1.5.3) For every neighborhood U of a, there is $N \geq 1$ with $\operatorname{Supp} f_n \subset U$ for $n \geq N$.

Remarks. (i) Tenting sequences are marginally more general than *familles moyanntes* of Lust-Piquard [18].

(ii) Any weak* $(\sigma(A(G)^*, A(G)^{**}))$ accumulation point of a 1-tenting sequence on $A(G)$ is an invariant mean on $A(G)^*$ [12], [18]. See also Remark 2.5.1.

(iii) Nothing is lost if (1.5.3) is replaced with the following:

(1.5.3 bis) For every open neighborhood U of a, $\lim_{n\to\infty} \|\mathbf{1}_{G\setminus U} f_n\|_{A(E\cap(G\setminus U))} = 0$.

To adapt the proofs that appear in this paper to ((1.5.3 bis)), just replace $\{f_n\}$ with $\{g_n f_n\}$ where the g_n are one in a neighborhood of a and have norms tending to 1. With the correct choices of g_n, one has $\|g_n f_n - f_n\|_{A(E} \to 0$, which is enough for everything that follows.

(iv) Lust-Piquard [18, p. 192] showed (for sets in metrizable abelian groups—the statement is slightly different in the more general abelian case) that if every tenting sequence in $A(E)$ is weak (that is, $\sigma(A(E)^*, A(E))$ Cauchy, then every element $\mu \in A(E)^*$ is totally ergodic, that is, $\delta_x * \mu$ is ergodic for every $x \in G$. In the same paper, Lust-Piquard [18, p. 212] shows that $A(E)^{**}$ has a commutative Arens multiplication (and therefore $A(E)$ is Arens regular) if and only if $A(E)^* \subset WAP$, that is, $\mu \in A(E)^*$ implies $\{\rho\mu: \rho \in \Gamma\}$ is weakly (that is, $\sigma(A(G)^{**}, A(G)^*)$) compact. That result seems to be a special case of an earlier result of N. J. Young: a locally convex semi-topological algebra is Arens regular if and only if every continuous linear functional on the algebra is weakly almost periodic [28, Lemma 2] (see also [27]).

(v) Granirer [11] has shown that if the subset E of an locally compact group contains the translate of an ultrathin set (e.g., see, [9, p. 333ff and p. 88 (7)]), then $A(E)$ is not Arens regular (because $A(E)$ contains a tenting Sidon sequence, i.e., there exist Day points; see also Corollary 5.1.3 and Corollary 5.1.4 below).

Definition 1.5.2. A bounded sequence $\{f_n\} \subset A$ (where A is a Banach space) is a *Sidon sequence* if there exists $\delta > 0$ such that for each integer $N > 0$ and complex numbers $c_1, \dots, c_N$, $\| \sum_{j=0}^N c_j f_j \|_A \geq \delta \sum_{j=0}^N |c_j|$. The Sidon sequence is a *strong Sidon sequence* if for each $\delta < 1$ there exists J such that $\| \sum_{j=J}^N c_j f_j \|_A \geq \delta \sum_{j=J}^N |c_j| \liminf_{j \geq J} \|f_j\|_A$ holds for all integers $N \geq J$ and all complex numbers $c_J, \dots, c_N$.

Sidon sequences are useful in constructing copies of ℓ^1 inside A (particularly $A(E)$), and hence, copies of ℓ^∞ in the dual (or quotients of the dual) of A (or $A(E)$), with all the complexity that ℓ^∞ implies. It is perhaps worth pointing out that a Sidon sequence $\{f_n\}$ cannot converge weakly. Indeed, the closed subspace spanned by $\{f_n\}$ is isomorphic to one-sided ℓ^1, $\{f_n\}$ corresponds to the usual basis of ℓ^1, and, obviously, the usual basis of ℓ^1 cannot converge weakly. Strong Sidon sequences give almost isometric copies of ℓ^1.

Definition 1.5.3. Let E be a compact subset of the locally compact group G. If there is a tenting-M sequence in $A(E)$ at $a \in E$ which contains a Sidon subsequence, we say that a is a *Day-M* point. Day-1 points will be called simply *Day points*. The Day points in E will be denoted by $D(E)$. If there is a tenting-1 sequence at $a \in E$ which contains a strong Sidon subsequence, we call a a strong Day point, and we denote by $D_s(E)$ the set of strong Day points for $A(E)$.

Day points were first defined in Granirer [11]. Strong Day points allow for more precise estimates than Day points, and imply that $A(E)^*$ has quotients which are arbitrarily close (in Banach-Mazur distance) to ℓ^∞.

Definition 1.5.4. A Banach space is a *Sidon space* if every subsequence has either a norm-convergent subsequence or a Sidon subsequence.

Sidon spaces were first defined by Meyer [19].

2. Arens regularity and tenting sequences

2.1. *Overview.* In this section, we give a criterion for Arens regularity and then show that various algebras are not regular. The results of this section are not new.

The following result from [21, 4.2] is well-known, but it will be useful to have a formal statement.

2.2. *Criterion for Arens regularity.*

THEOREM 2.2.1. *Let A be a Banach algebra. The following are equivalent.*

(1) *A is Arens regular.*
(2) *For all $\{g_n\}$ and $\{h_m\}$ bounded sequences in A and $S \in A^*$, the existence of the two limits*

$$\lim_{n\to\infty}\lim_{m\to\infty}\langle S, g_n h_m\rangle \text{ and } \lim_{m\to\infty}\lim_{n\to\infty}\langle S, g_n h_m\rangle$$

implies their equality:

$$\lim_{n\to\infty}\lim_{m\to\infty}\langle S, g_n h_m\rangle = \lim_{m\to\infty}\lim_{n\to\infty}\langle S, g_n h_m\rangle. \tag{2.2.1}$$

Remarks 2.2.2. It will be useful to have some proofs of the occurance of non-regularity (these are, of course, not new). The first argument reappears, e.g., at (2.2.2). The second result will be used repeatedly.

(i) One-sided ℓ^1 is not Arens regular (with the usual multiplication). Indeed, let $X = \{10^{2m+1} + 10^{2n}\colon m > n\}$, and let $S = \mathbf{1}_X$. We let $g_n = \delta_{10^{2n}}$, $n \geq 1$ and $h_m = \delta_{10^{2m+1}}$, $m \geq 1$. Then

$$\lim_{n\to\infty}\lim_{m\to\infty}\langle S, g_n h_m\rangle = 1$$

and

$$\lim_{m\to\infty}\lim_{n\to\infty}\langle S, g_n h_m\rangle = 0,$$

so one-sided ℓ^1 is not Arens regular.

(ii) One sided ℓ^1 is not Arens regular, where $\mathbb{N}$ is given maximum multiplication. (See [22, p. 106], where the weakly almost periodic functions on this algebra are identified with $c_o + \mathbb{C}\mathbf{1}$.) Indeed, let $S = \mathbf{1}_X$, where $X = \{2m+1\colon m \geq 1\}$. Let $g_m = \delta_{2m+1}$ and $h_n = \delta_{2n}$, for $n, m \geq 1$. Then

$$\lim_{m\to\infty}\lim_{n\to\infty}\langle S, g_m h_n\rangle = \lim_{m\to\infty}\lim_{n\to\infty}\langle S, \delta_{2n}\rangle = 0, \tag{2.2.3}$$

but

$$\lim_{n\to\infty}\lim_{m\to\infty}\langle S, g_m h_n\rangle = \lim_{n\to\infty}\lim_{m\to\infty}\langle S, \delta_{2m+1}\rangle = 1. \tag{2.2.2}$$

The conclusion now follows from Theorem 2.2.1.

2.3. *Point of spectral synthesis.*

Definition 2.3.1. Let A be a commutative Banach algebra with maximal ideal space ΔA. Let $a \in \Delta A$. If every element $f \in A$ with $\hat{f}(a) = 0$ can be approximated in A-norm by elements that vanish in a (Gelfand topology) neighborhood of a, then we say that a is a *point of spectral synthesis* for A.

2.4. *Day points imply non regularity.*

THEOREM 2.4.1. *Suppose A is a commutative Banach algebra with maximal ideal space ΔA, and $a \in \Delta A$ is both a Day-M point for some $0 < M < \infty$ and a point of spectral synthesis. Then A is not Arens regular.*

LEMMA 2.4.2. *Let $\{f_n\}$ be a Sidon sequence in a Banach space Y. Let $X \subset \mathbb{N}$. Then there exists $\mu \in Y^*$ such that $n \mapsto < \mu, f_n >$ is the function $\mathbf{1}_X$.*

Proof of Lemma 2.4.2. Left to the reader. □

Proof of Theorem 2.4.1. We begin with some simple observations. First, by replacing each f_n with a function f_n' with $\|f_n - f_n'\| < \lambda < \frac{\delta}{4}$, we may assume that $f_n \equiv 1$ in a neighborhood of a and that $\{f_n\}$ is still Sidon. Here $\lambda > 0$ and δ is from the definition of Sidon. [That follows from (i) the assumption that singletons are points of spectral synthesis, and $f_n(a) = 1$ for all n, which allow the approximation, and (ii) an examination of the estimate defining Sidon sequence, which uses the approximation (the latter point is from [19, Prop. 1, p. 243]).] Second, by passing to a subsequence, we may assume that

$$f_n f_m = f_n \text{ if } n > m. \tag{2.4.1}$$

[That follows from the preceding step. We do not care what f_n^2 equals! This puts us essentially in the situation of Remarks 2.2.2 (ii), which we apply below.]

Let $\mu \in A(E)^*$ be such that

$$\langle \mu, f_k \rangle = \begin{cases} 1 \text{ if } k = 2m + 1 \text{ for some } m > 0, \\ 0 \text{ otherwise.} \end{cases} \tag{2.4.2}$$

Such a μ exists by Lemma 2.4.2.

Let

$$g_m = f_{2m+1} \quad \text{and} \quad h_n = f_{2n}. \tag{2.4.3}$$

Now we may apply (the argument of) Remarks 2.2.2 (ii). □

Remark 2.4.3. The requirement that $f(a) = 1$ is necessary. For example, one can have a pup-tenting Sidon sequence (see Problem 25) $\{f_n\}$ such that $\|f_n f_m\| < 2^{-min(n,m)}$ for all n, m. This makes the preceding proof fail.

2.5. *Day points and TIMs.*

Remark 2.5.1. (i) Let x be a Day point for $A(E)$. Then there are at least c distinct elements in $\mathcal{F}(x, E) = \{f \in A(E)^{**}: fg = f \text{ for all } g \in A(E) \text{ with } g(a) = 1\}$. The proof is immediate using ultrafiltres. Compare with Problem 21, where Sidonicity is not assumed.

(ii) Each element of $\mathcal{F}(x, E)$ gives rise to a "translation invariant mean" on $A(G)^*$. See [11].

3. Lemmas and background for Results 1–2

This section contains technical lemmas which are for the most part used in more than one section below, and which, in any case, are conveniently grouped together. Most proofs are written to include the non-abelian case (hence we use $\hat{G}$ to denote the dual group (abelian case) or the dual object (non-abelian case). We put the notation and definitions which we use for the non-abelian case at the end of this section, in subsection 3.5. See [15, Section 34ff] for additional details.

3.1. *Reduction to the compact metrizable case.* We assume that G is either an infinite compact group (not necessarily abelian) or an abelian non-discrete, non-compact group.

We begin with the reduction to the metrizable case, assuming compactness if G is not abelian. We suppose that we have a sequence $f_n \in A(E)$. Then the f_n collectively involve only a countable number of Fourier coefficients (whether we are in the abelian case or not), and those together determine a closed normal subgroup H of G such that G/H is metrizable. It is easy to see that the image πE (in G/H) of E is of bounded synthesis if E is, and that pseudofunctions on E are carried to pseudofunctions on πE, with synthesizability preserved. If there is a weak limit of f_n, then there is a weak limit for πf_n (in the obvious abuse of notation). If the f_n are a tenting sequence, then πf_n is a tenting sequence for πE and if πf_n has a Sidon subsequence then, a fortiori, f_n has a Sidon subsequence.

Now we give the reduction to the compact abelian case, assuming that G is locally compact, abelian and non-discrete. Under these conditions, G has an open subgroup of the form $\mathbb{R}^n \times H$, where $n \geq 1$ and H has a compact open subgroup C. The compact set E (assumed in Sections 4–5) is contained in a set of the form $W \times (B + C)$, where $W \subset \mathbb{R}^n$ is compact and B is a finite subset of H. We now choose an image of $\mathbb{Z}^n$ in $\mathbb{R}^n$ such that the mapping $\pi_R\colon R^n \to \mathbb{R}^n/\mathbb{Z}^n$ has $A(E)$ and $A(\pi_R E)$ isomorphic with Banach space distance at most (say) 3. [This is standard: we can assume that W is a symmetric $n-$cube centered at $\mathbf{0}$ and choose Z_n so that $(W + W + W) \cap Z^n) = \{\mathbf{0}\}$.] This reduces to the case that G has a compact open subgroup.

The finite set B generates a finitely generated subgroup L of G, obviously. We may assume that the torsion subgroup F of L is already contained in B, so $L = F + \mathbb{Z}^m$

for some $m \geq 0$. If $m = 0$, we are done. Otherwise we chose a subgroup of the form $t\mathbb{Z}^m$ for some large integer t. Then as before, we have a projection of $G \to G/(t\mathbb{Z}^m)$ which carries E to an isomorphic copy, preserving all the pseudomeasure properties. We omit the remaining details.

3.2. *From pseudomeasures to (pseudo)measures.* We begin with several lemmas. We first note that because $A(G)$ is a commutative Banach algebra, $A(G)^*$ is an $A(G)$ module. Thus, fS makes sense for $f \in A(G)$, $S \in A(G)^*$.

LEMMA 3.2.1. *Let S be a non-zero pseudomeasure on the metrizable compact group G. Let $a \in \operatorname{Supp} S$. Then there exists a sequence h_n of compactly supported elements of $A(G)$ such that* (1.5.3) *holds (with h_n in place of f_n), $\|h_n S\|_{PM} = 1$ for all n, $\lim_n \langle h_n, S\rangle = \lim_n \langle 1, h_n S\rangle = 1$, and $\lim_n h_n S = \delta_a$ weak* in $PM(G)$.*

Proof of Lemma 3.2.1. We use the fact that a singleton (e.g., $\{a\}$) is a set of spectral synthesis. For example, see [9, A.3] for a proof in the case of $G = T$. The general abelian case follows from [24, 7.2.4]. For the non-abelian case, see, e.g., [15, 34.46(d)] or [20, 19.19].

We also use the fact that for each $a \in G$ and every neighborhood V of a, there exists $f \in A(G)$ with $\|f\|_{A(G)} = 1$, $f(a) = 1$, and $f = 0$ outside V:e.g., just apply [15, 34.21].

We may assume that $a = 0$. Choose a neighborhood base $\{U_n \colon n \geq 1\}$ of compact sets at 0 with $U_{n+1} \subset U_n$ for $n \geq 1$.

Because $0 \in \operatorname{Supp} S$, for each $n \geq 1$ there exists $g_n \in A(G)$ such that $\operatorname{Supp} g_n \subset U_n$ and $\langle g_n, S\rangle \neq 0$. Thus, $g_n S \neq 0$, so there exists $g_n' \in A(G)$ such that

$$\left|\langle g_n', g_n S\rangle - \|g_n S\|\right| < 2^{-n}\|g_n S\| \text{ and } \|g_n'\| = 1.$$

Let $h_n = \|g_n S\|^{-1} g_n' g_n$ for $n \geq 1$. Then

$$\|h_n S\| \leq 1 \text{ for } n \geq 1,$$

and

$$\lim_n \langle h_n S, 1\rangle = \lim_n \langle h_n, S\rangle = \lim_n \|g_n S\|^{-1}\langle g_n S, g_n'\rangle = 1.$$

Furthermore, because $\{0\}$ is a set of spectral synthesis, $\lim_n \langle f, h_n S\rangle = f(0)$. [Indeed, for any $\epsilon > 0$, we can write $f = f(0)k + m$ where $k = 1$ in a neighborhood of 0 and $\|m\|_{A(G)} < \epsilon$. Then $\lim_n \langle k, h_n S\rangle = 1$, so $|\lim_n \langle f, h_n S\rangle - f(0)| \leq \sup_n |\langle m, h_n S\rangle| \leq \|m\| = \epsilon$.]

Hence, $(h_n S)\hat{} \to 1$ pointwise on the dual group. It follows that $\lim_n h_n S$ exists weak* and equals δ_0. □

Remark. We need to be careful about "limits" of the form $\lim_n f_n S$ ($f \in A(G)$, $S \in PM(G)$). In some circumstances there can be multiple (or even c!) accumulation points. See [9, p. 394] for more on this point and for a reference.

LEMMA 3.2.2. *Let S be a non-zero pseudomeasure on the metrizable compact group G. Let $f \in A(G)$.*

(1) *If $S \in PF(G)$, then $fS \in PF(G)$.*
(2) *If S is synthesizable, then fS is synthesizable.*
(3) *When fS is synthesizable, $\langle fS, g\rangle = \langle fS, g'\rangle$ whenever $g = g'$ on the support of fS.*

Proof of Lemma 3.2.2. We first prove this in the case that G is abelian.

(1) By definition $S \in PF(G)$ if and only if $\hat{S} \in C_o(\Gamma)$. Since $\hat{f} \in L^1(\Gamma)$, $\hat{f} * \hat{S} \in C_o(\hat{G})$, so $f\hat{S} \in PF(G)$.
(2) Since S is of synthesis, there is a net $\{\mu_\alpha\}$ of measures in the support of S that converges weak* to S in $A(G)^*$. Of course, each $f\mu_\alpha$ is concentrated on the support of fS, so $f\mu_\alpha \to fS$ weak*. Hence, fS is synthesizable.
(3) If $g = g'$ on E, then

$$\langle g, fS\rangle = \lim_\alpha \langle g, f\mu_\alpha\rangle = \lim_\alpha \langle g', f\mu_\alpha\rangle = \langle g', fS\rangle,$$

by the definition of synthesis and the fact that the $f\mu_\alpha$ are measures.

Here is the proof for the non-abelian case.

(1) By definition, $S \in PF(G) \subset L^\infty(\hat{G})$ if and only if S is a norm limit of elements that have finite support in $\hat{G}$. Since $\hat{f} \in L^1(\hat{G})$, $\hat{f} * \hat{S} \in C_o(\hat{G})$.
(2) Since S is of synthesis, there is a net $\{\mu_\alpha\}$ measures in the support of S that converges weak* to S in $A(G)^*$. Of course, each $f\mu_\alpha$ is concentrated on the support of fS, so $f\mu_\alpha \to fS$ weak*. Hence, fS is synthesizable.
(3) If $g = g'$ on E, then

$$\langle g, fS\rangle = \lim_\alpha \langle g, f\mu_\alpha\rangle = \lim_\alpha \langle g', f\mu_\alpha\rangle = \langle g', fS\rangle,$$

by the definition of synthesis and the fact that the $f\mu_\alpha$ are measures. □

The following lemma is stated for both the abelian case and the non-abelian case, though we will need a non-abelian version (Lemma 3.2.5, given below).

LEMMA 3.2.3. *Let E be a compact subset of the compact group G. Suppose that E is the support of a pseudofunction T and that μ is a measure supported on E. Then for every $\epsilon > 0$ and finite subset $F \subset \hat{G}$, there exists a pseudofunction S on E such that*

$$|\hat{S} - \hat{\mu}| < \epsilon \text{ on } F \tag{3.2.1}$$

and

$$\|S\|_{PM} \leq (1+\epsilon)\|\|\hat{\mu}\|_\infty. \tag{3.2.2}$$

COROLLARY 3.2.4. *Let G be a compact metrizable group, and E be a closed subset of G such that*

(1) *E is a set of bounded spectral synthesis; and*
(2) *E is the support of a nonzero pseudofunction.*

Then every pseudomeasure on E is the weak limit of a (pseudomeasure norm) bounded sequence of pseudofunctions supported on E.*

Proof of Corollary 3.2.4. Approximate the pseudomeasure boundedly by measures and apply Lemma 3.2.3 to each approximant. □

Proof of Lemma 3.2.3. We first assume that μ is discrete and has finite support, say $x_1, \dots, x_m$.

The obvious method is to use Lemma 3.2.1 (applied at each x_j) to obtain a pseudofunction which is weakly close to μ. This will give us (3.2.1) but (3.2.2) might fail if, for example, the transforms of the approximants to the non-negative masses decreased less rapidly at infinity than the approximants to the positive masses. To avoid that difficulty, we do the approximation in stages.

Let N be any integer greater than $1+2\|\mu\|_{M(G)}/\|\hat{\mu}\|_\infty$. Let $\nu = \frac{1}{N}\mu$. Let $F_1 = F$. Let T_1 be a pseudofunction (which exists by application of Lemma 3.2.1 to each of the point masses) such that $|\hat{T}_1 - \hat{\nu}| < \epsilon/2N$ on F_1 and $\|T_1\|_{PM} \le \|\nu\|_{M(G)}$. Let $F_2 = F_1 \cup \{\lambda\colon |\hat{T}_1| > \epsilon/2N\}$. Since T_1 is a pseudofunction, F_2 is finite.

Inductively, for $2 \le \ell \le N$, we find $F_2 \subseteq \cdots \subseteq F_N$ and $T_2, \dots, T_N$ such that

$$|\hat{T}_\ell - \hat{\nu}| < \epsilon/2N \text{ on } F_\ell$$

and

$$\|T_\ell\|_{PM} \le \|\nu\|_{M(G)}.$$

Straightforward calculations now show that $S = \sum_1^N T_\ell$ has the required properties.

That completes the proof in the case that μ is a finitely supported discrete measure. In the general case we replace μ at each stage with a discrete measure. We use a Riemann sum argument, the terms in the sum depending on the finite set of characters F_n. This produces at each stage a (probably new) discrete measure, which is approximated itself (using the first part of the proof). A smaller error term is needed to allow for the final summation: the resulting pseudofunctions are summed, as in the discrete case.

Here is how we begin. As before, we let N be any integer greater than $1 + 2\|\mu\|_{M(G)}/\|\mu\|_{PM}$. Let $\nu = \frac{1}{N}\mu$. Let $F_1 = F$. Let ν' be a discrete measure with $\|\nu'\|_{M(G)} = \|\nu\|_{M(G)}$ and $|\hat{\nu}' - \hat{\nu}| < \epsilon/2N$ on F_1. Use the discrete case of the lemma to find a pseudofunction T_1 with $\|T_1\|_{PM} \le \|\nu\|_{M(G)}$ and $|\hat{T}_1 - \hat{\nu}| < \epsilon/2N$ on F_1 (if

$\hat{T}_1$ is sufficiently close to $\hat{\nu}'$, then $\hat{T}_1$ will be close to $\hat{\nu}$). Let $F_2 = \{\lambda\colon |\hat{T}_1| > \epsilon/2N\}$. We now proceed as above. We omit further details.

Non-abelian case. The proof is identical to the abelian case, except that we replace estimates of the form $|\hat{S}(\lambda)| < \epsilon/2N$ with estimates of the form $\|\hat{S}(\sigma)\|_{\mathcal{B}(\mathcal{H}_\sigma)} < \epsilon/2N$, and similarly when the inequality is the reverse. The set F_2 is defined by $F_2 = \{\sigma\colon \|\hat{T}_1(\sigma)\|_{\mathcal{B}(\mathcal{H}_\sigma)} > \epsilon/2N\}$. We omit further details. □

We now observe that the convolution $S * T$ of two pseudomeasures is well-defined by duality theory. In the case of abelian G, the convolution $S * T$ corresponds to the pointwise multiplication $\hat{S}\hat{T}$ of the transforms on the dual group. In the non-abelian case, we have a similar representation. With that observation, we can state a non-abelian version of Lemma 3.2.3:

LEMMA 3.2.5. *Let E be a compact subset of the compact group G. Suppose that E is the support of a pseudofunction T, that W is a pseudofunction supported on G, and that μ is a measure supported on E. Then for every $\epsilon > 0$ there exists a pseudofunction S on E such that*

$$\|S * W - \mu * W\|_{PM} < \epsilon, \quad \|W * S - W * \mu\|_{PM} < \epsilon \tag{3.2.3}$$

and

$$\|S\|_{PM} \leq (1+\epsilon)\|\|\hat{\mu}\|_{PM}. \tag{3.2.4}$$

Proof of Lemma 3.2.5. The proof is similar to that of Lemma 3.2.3. We indicate the differences between the two proofs.

Since $W \in PF(G)$, a 2ϵ-argument (as in the proof of Lemma 3.2.3) shows that we may assume that W is in fact a trigonometric polynomial. But then the requirements of (3.2.3) are met by ensuring that S is sufficiently close (weak*) to μ. Now the proof of Lemma 3.2.3 applies directly, taking into account the definition of the norm of $\ell^\infty(\hat{G})$. □

3.3. *Convolution by a pseudofunction is a compact operator.*

LEMMA 3.3.1. *Let W be a pseudofunction on the compact group G. Suppose that S_α is a bounded net of pseudomeasures on G convergine weak* to S in $PM(G)$.Then $S_\alpha * W$ converges in pseudomeasure norm to $S * W$ and $W * S_\alpha$ converges in norm to $W * S$. In other words, the mappings $S \mapsto S * W$ and $S \mapsto W * S$ are compact from $PM(G) \to PM(G)$.*

There is less to this than meets the eye: in the abelian case it says that if $g \in c_o$, then the mapping $f \mapsto fg$ from $\ell^\infty \to c_o$ is compact.

Proof of Lemma 3.3.1. We prove the assertion for $S * W$. Indeed, since W is a pseudofunction, we may approximate W in PM-norm by a sequence $\{W_n\}$ of trigonometric polynomials (times Haar measure on G). Of course, $S_\alpha * W_n$ are continuous funtions—even trigonometric polynomials—and $S_\alpha * W$ thus converges in uniform norm (!) [look at the Fourier coefficients, abelian or not] to the trigonometric polynomial $S * W_n$ on G. Now a 3ϵ-argument completes the proof. □

3.4. *Sums of pseudofunctions in a von Neumann algebra.* We now state and prove the simple (but apparently new) lemma which we use to sum pseudofunctions on non-abelian groups. It is more particular to the non-abelian case, and is the reason we need the formulation of (3.2.3). This lemma is used in application to operators on $L^2(G)$, that is, to $A(G)$.

LEMMA 3.4.1. *Let $\mathfrak{A}$ be a von Neumann algebra and $A, B \in \mathfrak{A}$. Suppose that $\|A\| = 1 \geq \|B\| = b$. For every $\epsilon > 0$ there exists $\delta > 0$ such that $\|AB^*\| < \delta$ and $\|A^*B\| < \delta$ imply $\|A + B\| < 1 + \epsilon$.*

Proof of Lemma 3.4.1. We use the fact that for a von Neumann algebra, the norm is the spectral radius; therefore, to estimate the norm of an operator D, it suffices to estimate the norm of powers of DD^*.

Thus, $\|A + B\|$ equals the square root of $\|(A + B)(A^* + B^*)\|$, and that (by multiplying) equals the square root of the spectral radius of $AA^* + AB^* + BA^* + BB^*$. We expand $(AA^* + AB^* + BA^* + BB^*)^n$ into its 4^n terms (remembering that we cannot assume commutativity): each term consists of words of length $2n$. We use the triangle inequality to replace the norm of the sum with the sum of the norms.

We note that $\|AB^*\| = \|BA^*\|$ and $\|A^*B\| = \|B^*A\|$. Then (with a slight rearrangement of the usual binomial order of terms), we have

$$\begin{aligned}((A + B)(A + B)^*)^2 = {} & AA^*AA^* + BB^*BB^* \\ & + AA^*BB^* + BB^*AA^* + AA^*AB^* + AA^*BA^* \\ & + AB^*AA^* + AB^*AB^* + AB^*BA^* + AB^*BB^* \\ & + BA^*AA^* + BA^*AB^* + BA^*BA^* + BA^*BB^* \\ & + BB^*AB^* + BB^*BA^*\end{aligned}$$

In the above expression, we note that every term except the first two (the "bad" ones) have within them one or more juxtapositions of the forms AB^*, BA^* A^*B, or B^*A. Furthermore, the product of any two different "bad" terms (in either order) is not a "bad" term. It follows that

$$\|A + B\|^{4n} = \|((A + B)(A + B)^*)^{2n}\| \leq 2 + (16^n - 2)\delta.$$

Let n be such that $3^{1/4n} < 1+\epsilon$, and let $\delta = 16^{-n}$. Then $\|A+B\| \le \|((A+B)(A+B)^*)^{2n}\|^{1/4n} \le 3^{1/4n} < 1+\epsilon$. □

From Lemma 3.4.1 we can conclude that if S, T are two pseudofunctions on G with $\|ST^*\|_{A(G)^*}$ and $\|S^*T\|_{A(G)^*}$ small, then $\|S+T\|_{A(G)^*}$ is not much larger than the maximum of the norms of S, T.

3.5. *Background for non-abelian compact groups.* We give the non-abelian group facts and notation used in this paper.

Let G be a compact group, and let $A(G) = L^2(G) * L^2(G)$ be the usual Fourier algebra [5]. We let $\hat{G}$ denote the dual object, that is, $\hat{G}$ is a maximal set of inequivalent unitary representations $U_\sigma\colon G \to \mathfrak{B}(H_\sigma)$ on the Hilbert space H_σ. Here $\mathfrak{B}(H)$ denotes the space of all bounded linear operators on the Hilbert space H. Let $\mathfrak{P} = \Pi_{\sigma\in\hat{G}}\mathfrak{B}(H_\sigma)$. Let $L^\infty(\hat{G})$ be the set of all elements $T = (T_\sigma)_{\sigma\in\hat{G}})$ such that $\sup\|T_\sigma\|_{B(H_\sigma)} < \infty$; then $L^\infty(\hat{G})$ is (isomorphic to) the dual space of $A(G)$; e.g., see, [15, 34.19 and 34.46]. As in the abelian case, some of the elements of $L^\infty(\hat{G})$ come from measures on G: each regular bounded Borel measure μ on G gives rise to an element of $L^\infty(\hat{G})$ via the "Fourier-Stieltjes" transform [15, 34.1-2 and 34.24] $\hat{\mu}(\sigma) \in \mathfrak{B}(H_\sigma)$ where

$$\text{(3.5.1)} \qquad \langle\hat{\mu}(\sigma)\eta, \zeta\rangle = \int \langle U_\sigma(x)\eta, \zeta\rangle d\mu(x) \text{ for all } \eta, \zeta \in H_\sigma \text{ and all } \sigma.$$

The finite linear combinations of the functions $x \mapsto \langle U_\sigma(x)\eta, \zeta\rangle$ $(\eta, \zeta \in H_\sigma)$ are called "trigonometric polynomials" [15, 27.7].

Definition 3.5.1. An element $S \in A(G)^*$ is in $A(G)^*_o$ if S is the norm-limit of elements such that $\{\sigma\colon T_\sigma \neq 0\}$ is finite. We'll call such S *pseudofunctions* for short and denote the set of all pseudofunctions by $PF(G)$. The set of pseudofunctions supported on a compact set E will be denoted by $PF(E)$.

Each pseudofunction translates norm-continuously, that is, the mapping $x \mapsto \delta_x * S$ is continuous from G to $A(G)^*$, where the latter space is given its dual space norm topology. Here $\delta_x * S$ is the element of $A(G)^*$ defined by $f \mapsto \langle\delta_{-x} * f, S\rangle$.

The *support* of elements of $A(G)^*$ is determined analogously to the abelian case; see [15, 34.46(b)]. An element $S \in A(G)^* = L^\infty(\hat{G})$ is *synthesizable* if S is the weak* limit of measures concentrated on the support of S (the limit being taken in the weak* topology). A closed subset $E \subset G$ is of *synthesis* if every pseudomeasure supported on E is synthesizable. Singletons in amenable groups are sets of spectral synthesis. See, e.g., [15, 34.46(d)] or [20, 19.19] for a proof of that fact.

4. Weak sequential completeness and Arens non-regularity of $A(E)$

4.1. *Results and general remarks.* In this section we are inspired by a result of Granirer [11, proof of Theorem 3] to give a stronger result (though subject to a slightly stronger hypothesis) than Theorem 5.1.1, namely, Theorem 4.1.3. We begin with an abstract version of [11, proof of Theorem 3].

PROPOSITION 4.1.1 [11, Proof of Theorem 3]. *Suppose that A is a commutative, regular Banach algebra with maximal ideal space ΔA, $E \subset \Delta$ is closed, $a \in E$ is not an isolated point in E and that $A(E)$ is weakly sequentially complete. Then every tenting sequence at a in $A(E)$ has a Sidon subsequence.*

Proof of Proposition 4.1.1. This proof is lightly adapted from [11, Proof of Theorem 3]. Suppose that $a \in E$ and that f_n is a tenting sequence in $A(E)$ at a. By a well-known theorem of Rosenthal, [23, p. 808], either the tenting sequence has a Sidon subsequence or a weakly convergent subsequence. We show that "weakly convergent subsequence" is not possible.

Indeed if f_{N_j} were weakly convergent, then because $A(E)$ is weakly sequentially complete, there exists $f \in A(E)$ with $f_{n_j} \to f$ weakly. But because f_{n_j} is a tenting sequence,

$$f_{n_j}(x) \to 0 \tag{4.1.1}$$

for all $x \in E\backslash\{a\}$ and $f_{n_j}(a) \to 1$. It follows that f is not continuous, an absurdity. □

Ülger [25, Theorem 3.3] shows that a weakly sequentially complete Banach algebra (commutative or not) with no unit and with a bounded approximate identity is not Arens regular. Proposition 4.1.1 suggests a slightly stronger result and an easier proof, but only in the commutative case, as follows.

COROLLARY 4.1.2. *Let A be a commutative Banach algebra with no identity but with a bounded approximate identity h_i. Suppose that A is weakly sequentially complete and that the elements with compact support are dense. Then ∞ is a Day point for the unitized $A + \mathbb{C}\mathbf{1}$. In particular, A is not Arens regular.*

Sketch of proof. Indeed with $E = \Delta A \cup \{\infty\}$ and $a = \infty$, the proof of Proposition 4.1.1 applies, since $\mathbf{1} - h_i$ satisfies (4.1.1) at ∞. (We pass to a separable quotient of $A + \mathbb{C}\mathbf{1}$ if necessary.)

Hence $A + \mathbb{C}\mathbf{1}$ is not Arens regular. Since adjoining an identity does not affect regularity, A is not regular (see also the proof of Arens regularity of "blocked algebras" in [6]). □

Here is the main result of this section (Result 1 of the abstract).

THEOREM 4.1.3. *Let G be a non-discrete locally compact abelian group or a non-discrete compact group, and E be a closed subset of G such that*

(1) *E is a set of bounded spectral synthesis; and*
(2) *E is the support of a nonzero pseudofunction.*

Then $A(E)$ is weakly sequentially complete.

Remark 4.1.4. (i) Is the abelian group version of this result new? Y. Meyer [19, 6.2.10] states an even stronger conclusion for the line, but with neither the first hypothesis on E, nor the hypothesis of spectral synthsis on the pseudomeasure whose support is E. However, if E is a Helson set supporting a non-zero pseudofunction, then $A(E) = C(E)$, for which the conclusion of Theorem 4.1.3 is false. Since such Helson sets exist, by a famous result of Körner (see [9, 4.6.4]), an additional hypothesis such as "synthesizable" is needed. However, [19] provides no details of the argument. Thus, it is unclear if either the result or proof here are new.

(ii) Granirer and Cowling [10, p. 131] show that if the compact set E has non-empty interior, then the set of elements of the Herz algebra $A_p(G)$ supported on E is weakly sequentially complete. This is then used to show that if $aEb \cap H$ has non-empty (relative) interior in the infinite closed subgroup $H \subset G$, then E has Day points for $A_p(E)$. It is that argument which inspired us.

(iii) If $F = E_1 + E_2$, where the E_j are disjoint compact subsets of a Kronecker or K_p-set, then $A(F)$ is not weakly sequentially complete unless F is finite. [Look at sequences of the form $1 \otimes f_j$, where the f_j are tents in $C(E_2)$ and $g_j \otimes 1$ (g_j tents in $C(E_1)$).] Hence, we can have non-regularity without weak sequential completeness.

(iv) Hypothesis (1) can be weakened slightly to "bounded approximation by pseudofunctions", and hypothesis (2) weakened to "non-zero synthesizable pseudofunction". It would be interesting to know if such a weakening is meaningful, that is, if there are sets not of (bounded?) synthesis for which bounded synthesis held for synthesizable pseudomeasures.

COROLLARY 4.1.5. *Let G be a compact group or a locally compact abelian group. Let E be a compact subset of G such that for elements $a, b \in G$ and a closed infinite subgroup H of G, $aEb \cap H$ has a closed subset which is of bounded synthesis (for H) and which supports a pseudofunction (for H). Then E has Day points.*

Proof of Corollary 4.1.5. As per [11, proof of Theorem 3]. □

4.2. *Proof of Theorem* 4.1.3*—abelian case.* We may assume that the group is compact and metrizable, by subsection 3.1.

Let $\{f_n\}$ be a sequence in $A(E)$ that converges weakly. We first reduce the proof to a matter of showing that (a suitably modified version of) the sequence $f_n \to 0$ weakly.

We may assume that $\|f_n\|_{A(E)} \leq 1$. Let $\{g_n\} \subset A(G)$ be such that $g_{n|E} = f_n$ and $\|g_n\|_{A(G)} \leq 2\|f_n\|_{A(E)}$ for all n.

Since $\hat{G}$ is countable and discrete, we may assume that $\hat{g}_n \to g$ pointwise on $\hat{G}$. Of course $g \in L^1(\hat{G})$ with $\|g\|_{L^1(\hat{G})} \leq 2$. Let $h = \hat{g}$. Consider

$$k_n = f_n - h_{|E}, \; n \geq 1.$$

It will now suffice to show that $k_n \to 0$ weakly in $A(E)$.

So suppose that $k_n \not\to 0$ weakly. Then there exists $S \in A(E)^*$ such that

$$\langle k_n, S\rangle \to 1. \tag{4.2.1}$$

We shall show that (4.2.1) implies that k_n has a Sidon subsequence (which cannot not converge weakly—see the paragraph following the definition of Sidon sequence), giving a contradiction. Thus, (4.2.1) cannot occur, and the theorem will be proved.

To this point we have not used the hypotheses on E. We now do so. First, because E is of spectral synthesis, the quantities $\langle k_n, W\rangle$ do not depend on the coset representatives we use for k_n, whatever be the $W \in PM(E)$. With this in mind, we choose $g_n - h$ as our representatives. [We will eventually have a "gliding hump" argument ([2]), since $\hat{g}_n - \hat{h} = \hat{g}_n - g$ converges pointwise to 0 on $\hat{G}$. Intuitively, we think of finding finite sets $F_n \subset \hat{G}$ where "most of" $\hat{k}_n$ lives. In fact, that will be implicit, because we need to keep our weak approximations in mind.]

The idea is to replace S with a sum $\sum c_\ell S_\ell$ of pseudofunctions, so that $\hat{S}_\ell$ agrees with $\hat{S}$ as far as k_ℓ is concerned, and $\hat{S}_\ell$ is small on the other k_m's and $\{c_\ell\}$ is a bounded sequence of complex numbers. Then the Sidonicity of k_ℓ will follow. We shall need to pass—in this proof at two places—to a subsequence of the k_m's, so that "subsequence of" k_m will become implicit. We may assume that $\|S\|_{PM} = 1$.

We use induction. Let $n(1) = 1$. We apply the property of "bounded spectral synthesis" to find a finite regular Borel measure μ_1 on E such that

$$|\langle k_{n(1)}, \mu_1\rangle - \langle k_1, S\rangle| < \frac{1}{10}$$

and

$$\|\mu_1\|_{PM} \leq C\|S\|_{PM},$$

where C is a fixed constant. Now use Lemma 3.2.3 to find a pseudofunction T_1 on E such that

$$|\langle k_{n(1)}, \mu_1\rangle - \langle k_{n(1)}, T_1\rangle| < \frac{1}{10} \tag{4.2.2}$$

and

$$\|T_1\|_{PM} \leq C\left(1 + \frac{1}{10}\right). \tag{4.2.3}$$

Thus,

$$|\langle k_{n(1)}, S\rangle - \langle k_{n(1)}, T_1\rangle| < \frac{2}{10}. \tag{4.2.4}$$

Assume that $m \geq 1$, that pseudofunctions $T_1, \dots, T_m$ supported on E, and that integers $n(1) < n(2) < \cdots < n(m)$ have been found so that

$$\left|\hat{T}_\ell(\lambda) - \hat{S}(\lambda)\right| < \frac{1}{10^\ell} \text{ if } \sum_{1\leq k<\ell} |\hat{T}_k(\lambda)| > \frac{1}{10^\ell}, \quad 1 \leq \ell \leq m, \tag{4.2.5}$$

$$|\langle k_{n(\ell)}, S\rangle - \langle k_\ell, T_\ell\rangle| < \frac{2}{10^\ell}, \quad 1 \leq \ell \leq m, \tag{4.2.6}$$

$$|\langle k_{n(j)}, T_\ell\rangle| < \frac{1}{10^j}, \quad 1 \leq \ell < j \leq m, \tag{4.2.7}$$

$$\|T_\ell\|_{PM} < C\left(1 + \frac{1}{10^\ell}\right), \quad 1 \leq \ell \leq m. \tag{4.2.8}$$

The vacuous sum in (4.2.5) for $\ell = 1$ means that we ignore the condition (4.2.5) for that ℓ. To obtain (4.2.7), we passed to a subsequence (that was the first time—see next paragraph).

We now choose $N > n(m)$ such that $|\langle k_n, T_\ell\rangle| < \frac{1}{10^{m+1}}$ for $1 \leq \ell \leq m$, and $n \geq N$. That uses the two facts (i) the T_ℓ are pseudofunctions and (ii) the $\hat{k}_n$ go to zero pointwise on $\hat{G}$, as well as the fact that the values $\langle k_j, T_\ell\rangle$ are independent of the particular coset representatives chosen for the restrictions.

Let $n(m+1) = N$. Now chose a finite measure μ (again using bounded synthesis) which satisfies

$$\left|\hat{\mu}(\lambda) - \hat{S}(\lambda)\right| \frac{1}{10^{m+1}} \tag{4.2.9}$$

for all λ with

$$\sum_{1\leq k\leq m} \left|\hat{T}_k(\lambda)\right| > \frac{1}{10^{m+1}} \tag{4.2.10}$$

and which also satisfies

$$|\langle k_{n(m+1)}, \mu\rangle - \langle k_1, S\rangle| < \frac{1}{10^{m+1}}. \tag{4.2.11}$$

Replace μ by a pseudofunction (using Lemma 3.2.3) T_{m+1} such that

$$\left|\langle k_{n(m+1)}, T_{m+1}\rangle - \langle k_{n(m+1)}, \mu\rangle\right| < \frac{1}{10^{m+1}}, \tag{4.2.12}$$

and

$$\left|\hat{S}(\lambda) - \hat{T}_{m+1}(\lambda)\right| < \frac{1}{10^{m+1}} \tag{4.2.13a}$$

whenever

$$\sum_{1 \le k \le m} \left|\hat{T}_k(\lambda)\right| > \frac{1}{10^{m+1}}. \tag{4.2.13b}$$

That completes the induction.

We now replace $\{k_{n(m)}\}$ with its subsequence $\{k_{n(2m)}\}$ and consider the sums

$$W = \sum_{\ell} c_\ell (T_{m+1} - T_m). \tag{4.2.14}$$

Here $\{c_\ell\}$ is any bounded sequence of complex numbers. Straightforward calculations will show that the sum in (4.2.14) converges weak* in $PM(E)$ (with norm the order of $C \sum |c_\ell|$) and that $|\langle k_{2m}, W\rangle - c_m| \to 0$, which is enough to prove that k_{2m} is a Sidon sequence. □

4.3. *Proof of Theorem* 4.1.3—*non-abelian case.* We may assume that the group is metrizable, by subsection 3.1.

We indicate the changes that need to be made to allow for the (possible) non-commutativity of G, the changes basically involve the summing of the pseudofunctions.

We proceed, without change from the abelian case, to obtain (4.2.4), and all that preceeds (4.2.4). Note that we have advanced the formula numbers to have agreement (in the third component) with the abelian proof. Assume that $m \ge 1$, that pseudofunctions $T_1, \dots, T_m$ supported on E, and that integers $n(1) < n(2) < \cdots < n(m)$ have been found so that

$$\left\|\hat{T}_\ell(\sigma) - \hat{S}(\sigma)\right\|_{H_\sigma} < \frac{1}{10^\ell} \text{ if } \sum_{1 \le k < \ell} \left\|\hat{T}_k(\sigma)\right\|_{H_\sigma} > \frac{1}{10^\ell}, \quad 1 \le \ell \le m, \tag{4.3.5}$$

$$|\langle k_{n(\ell)}, S\rangle - \langle k_\ell, T_\ell\rangle| < \frac{2}{10^\ell}, \quad 1 \le \ell \le m, \tag{4.3.6}$$

$$|\langle k_{n(j)}, T_\ell\rangle| < \frac{1}{10^j}, \quad 1 \le \ell < j \le m, \tag{4.3.7}$$

$$\|T_\ell\|_{PM} < C\left(1 + \frac{1}{10^\ell}\right), \quad 1 \le \ell \le m. \tag{4.3.8}$$

As in the abelian case, the vacuous sum in (4.3.5) for $\ell = 1$ means that we ignore the condition (4.3.5) for that ℓ. To obtain (4.3.7), we passed to a subsequence (that was the first time—see next paragraph).

We now choose $N > n(m)$ such that $|\langle k_n, T_\ell \rangle| < \frac{1}{10^{m+1}}$ for $1 \leq \ell \leq m$, and $n \geq N$. That uses the two facts (i) the T_ℓ are pseudofunctions and (ii) the $\hat{k}_n$ go to zero pointwise on $\hat{G}$, as well as the fact that the values $\langle k_j, T_\ell \rangle$ are independent of the particular coset representatives chosen for the restrictions.

Let $n(m+1) = N$. Now chose a finite measure μ (again using bounded synthesis) which satisfies

$$\|\hat{\mu}(\sigma) - \hat{S}(\sigma)\|_{H_\sigma} < \frac{1}{10^{m+1}} \tag{4.3.9}$$

for all representations σ with

$$\sum_{1 \leq k \leq m} \|\hat{T}_k(\sigma)\|_{H_\sigma} > \frac{1}{10^{m+1}} \tag{4.3.10}$$

and which also satisfies

$$|\langle k_{n(m+1)}, \mu \rangle - \langle k_1, S \rangle| < \frac{1}{10^{m+1}}. \tag{4.3.11}$$

Replace μ by a pseudofunction (using Lemma 3.2.3) T_{m+1} such that

$$|\langle k_{n(m+1)}, T_{m+1} \rangle - \langle k_{n(m+1)}, \mu \rangle| < \frac{1}{10^{m+1}}, \tag{4.3.12}$$

and

$$\left\| \hat{S}(\sigma) - \hat{T}_{m+1}(\sigma) \right\|_{H_\sigma} < \frac{1}{10^{m+1}} \tag{4.3.13}$$

for all σ with

$$\sum_{1 \leq k \leq m} \left\| \hat{T}_k(\sigma) \right\|_{H_\sigma} > \frac{1}{10^{m+1}}. \tag{4.3.14}$$

That completes the induction.

We now replace $\{k_{n(m)}\}$ with its subsequence $\{k_{n(2m)}\}$ and consider the sums

$$W = \sum_{\ell} c_\ell \left(T_{m+1} - T_m \right). \tag{4.3.15}$$

Here $\{c_\ell\}$ is any bounded sequence of complex numbers. Straightforward calculations will show that the sum in (4.3.15) converges weak* in $PM(E)$ (wth norm the order of $C \sum |c_\ell|$) and that $|\langle k_{2m}, W \rangle - c_m| \to 0$, which is enough to prove that k_{2m} is a Sidon sequence. □

5. $PF(E)$ and Arens regularity of $A(E)$

5.1. *Overview.* We prove Result 2 of the abstract. We also discuss its history.

THEOREM 5.1.1. *Let G be a non-discrete locally compact abelian group or a non-discrete compact group. Let E be a closed subset of G. Suppose that E supports a non-zero synthesizable pseudofunction S, $a \in \operatorname{Supp} S$ and $M \geq 1$. Then every tenting-M sequence at a has a Sidon subsequence. Furthermore, a is a strong Day point.*

Remarks 5.1.2. (i) The restrictions of compact and metrizable are not necessary, but they are convenient in the proof. The reduction to the general case is routine and left to the interested reader.

(ii) Theorem 5.1.1 does show that the second alternative in the definition of Sidon space (that is, every tenting sequence has a Sidon subsequence) may arise for all sets supporting a synthesizable pseudofunction, which is certainly evidence for the assertion of [19, 6.2.10].

(iii) a must be a non-isolated point of E, since if there is an isolated point in the support of a pseudomeasure on a non-discrete group, then the pseudomeasure cannot be a pseudofunction.

COROLLARY 5.1.3. *Let G be a locally non-discrete compact abelian group or a non-discrete compact group. Let E be a closed subset of G and H a closed non-discrete infinite subgroup of G and $a, b \in G$. If $aEb \cap H$ supports a non-zero synthesizable pseudofunction S for H, then $D(E) \neq \phi$. In fact $D(E) \supset \bigcup\{a^{-1} \operatorname{Supp} Sb^{-1}\colon S \in PF(aEb \cap H)\}$.*

Proof of Corollary 5.1.3. Let $\{f_n\} \subset A(E)$ be a tenting sequence at $x \in aEb \cap H$. Then the restrictions of the f_n to $aEb \cap H$ form a tenting sequence in $A(aEb \cap H)$. By the theorem, there is a Sidon subseqence of the restrictions. It is obvious that the unrestricted subsequence is also a Sidon sequence. □

COROLLARY 5.1.4. *Let G be a non-discretelocally compact abelian group or a non-discrete compact group. Let E be a compact subset of G and H a closed non-discrete infinite subgroup of G and $a, b \in G$. If $aEb \cap H$ supports a non-zero synthesizable pseudofunction S for H, then $A(E)$ is not Arens regular.*

Proof of Corollary 5.1.4. This is immediate from Theorem 5.1.1 and Corollary 5.1.3. □

5.2. *Proof of Theorem* 5.1.1—*abelian case.* We may assume that the group is compact and metrizable, by subsection 3.1.

The reader will note that, in what follows, if the estimates are done with smaller quantities (replacing the constant "3", for example, at each induction step with "$1+\epsilon_j$", where $\epsilon_j \to 0^+$), one will obtain the assertion about strong Day points. We omit those details.

In any case, the proof uses Lemma 3.2.1 to produce two sequences of pseudofunctions ν_j, τ_j and elements g_j, h_j of $A(G)$ having the properties that for $1 \le j \le J < \infty$,

$$\nu_j = g_j S,\ \tau_j = h_j S,\ a \notin \operatorname{Supp}(\nu_j + \tau_j), \tag{5.2.1a}$$

$$\|\nu_j\|_{PM} = \|\tau_j\|_{PM} = 1, \tag{5.2.1b}$$

and

$$\left\| \sum_1^J |\hat{\nu}_j - \hat{\tau}_j| \right\|_\infty \le 3 - 8^{-J}. \tag{5.2.2}$$

The supremum is taken over the dual group Γ of G. Also, $g_j, h_j \in A(G)$ are such that ν_j, τ_j are weak* close to unit point masses (which can be done by Lemma 3.2.1).

We will also find a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that for $k > j \ge 1$

$$\operatorname{Supp} f_{n_k} \cap \operatorname{Supp}(\nu_j + \tau_j) = \phi = \operatorname{Supp} f_{n_j} \cap \operatorname{Supp} \tau_j; \tag{5.2.3}$$

while for $k > j$

$$|\langle f_{n_j}, \nu_k - \tau_k \rangle| < 8^{-k \cdot j}; \tag{5.2.4}$$

and, for all j,

$$\left|1 - \langle f_{n_j} \nu_j \rangle\right| < 8^{-j^2}. \tag{5.2.5}$$

It then follows that any finite sum $g = \sum_1^N c_k f_{n_k}$ has norm that can be estimated as follows. First,

$$\left| \left\langle g, \sum_1^N \frac{\bar{c}_k}{|c_k|} (\nu_k - \tau_k) \right\rangle \right| \ge \sum_1^N |c_k| \left(1 - \frac{8}{49}\right), \tag{5.2.6}$$

by iterating (5.2.4)–(5.2.5). (If $c_k = 0$, we define $\frac{\bar{c}_k}{|c_k|}$ to be 0.) Since

$$\left\| \sum_1^N \frac{\bar{c}_j}{|c_j|} (\hat{\nu}_j - \hat{\tau}_j) \right\|_\infty \le 3, \tag{5.2.7}$$

$$\|g\|_{A(E)} \ge \frac{1 - \frac{8}{49}}{3} \sum_1^N |c_j|. \tag{5.2.8}$$

Now apply (5.2.8) to conclude that f_{n_j} is a Sidon sequence with $d \ge \frac{1-\frac{8}{49}}{3}$.

[Note: the quantity $\frac{8}{49}$ appearing in (5.2.6) and (5.2.8) comes from calculating the error $\sum_{j,k} 8^{-k\cdot j} = \sum_k \frac{8^{-k}}{1-8^{-k}} \le \sum_k \frac{8}{7} 8^{-k} = \frac{8}{7}\frac{8^{-1}}{1-8^{-1}} = \frac{8}{7}\cdot\frac{1}{8}\cdot\frac{8}{7} = \frac{8}{49}$.]

It remains to show how (5.2.1)–(5.2.5) can be achieved.

We begin (here we use the metrizability) by chosing a neighborhood basis $\{U_n\}$ at a consisting of open relatively compact sets such that

$$\bar{U}_{n+1} \subset U_n \text{ for } n \ge 1. \tag{5.2.9}$$

We may assume (by passing to a subsequence) that

$$\operatorname{Supp} f_n \subset U_n \text{ for } n \ge 1. \tag{5.2.10}$$

We begin with some preliminary remarks concerning (5.2.2)–(5.2.5).

Ad (5.2.2. If we did not have to worry about (5.2.2)–(5.2.5), we could achieve (5.2.2) by using only the facts that S is a pseudofunction supported on E, and $\nu_j = g_j S$, $\tau_j = h_j S$, so that if $1 \le K$, and ν_j, τ_j $(1 \le j \le K)$ are such that (5.2.2) holds for $1 \le J \le K$, then there is a finite (a. k. a. compact) subset $L \subset \Gamma$ such that

$$\sum_1^J |\hat{\nu}_j - \hat{\tau}_j| < 16^{-J} \tag{5.2.11}$$

outside L, and, so for one of the compact neighborhoods U_r of a, it is true that if ν_{K+1}, τ_{K+1} are *any* probability measures supported in U_r, then $|\hat{\nu}_{K+1} - \hat{\tau}_{K+1}| < 16^{-K}$ on L. We replace "any probability measure" with pseudofunctions using Lemma 3.2.2, to ensure that ν_j and τ_j are pseudofunctions which satisfy the (5.2.11); that will follow from taking pointwise limits on L (a consequence of the weak* convergence). Now (5.2.2) follows for $J = K + 1$.

Ad (5.2.3). This is easily obtained by using (1.5.3), since we ensure that the (closed) supports of the ν_j and τ_j exclude a.

Ad (5.2.4). For any J, the functions f_{n_j}, $1 \le j \le J$ are a fixed finite collection. Their continuity will tell us that (5.2.4) will follow if ν_{J+1}, τ_{J+1} are point masses sufficiently close to a. We now use Lemma 3.2.1 (applied to those two point masses) to replace those two point masses with pseudomeasures.

Ad (5.2.5). We will achieve this by ensuring that the support of τ_j misses the support of f_{n_j} (see the remark above concerning (5.2.3)), and also ensuring that ν_j is sufficiently close (weak*) to a unit point mass near a (see the preceding paragraph).

We now begin with τ_1: we choose τ_1 to be any pseudofunction of the form $g_1 S$ whose support *excludes* a and which has pseudomeasure norm 1.

We now choose f_{n_1} such that $\operatorname{Supp} f_{n_1} \cap \operatorname{Supp} \tau_1 = \phi$, which is possible by (1.5.1)–(1.5.3). Now use (1.5.2) and a weak* approximation of a point mass (using the fact that $f_{n_1} \in A(E)$) to choose a pseudofunction ν_1 of the form $h_1 S$ such that $a \notin \operatorname{Supp} \nu_1$, (5.2.5) holds for $j = 1$ and $\|\nu_1\| = 1$.

We have now satisfied (5.2.1)–(5.2.5) ((5.2.4) is vacuous here) for $j = 1$. That begins the induction.

We now suppose that $K \geq 1$ and that we have found pseudofunctions ν_j, τ_j such that (5.2.1)–(5.2.5) hold for $1 \leq J \leq K$.

Choose a neighborhood U of a such that U does not intersect the supports of any of the ν_j, τ_j, $1 \leq j \leq K$. That is possible by (5.2.1). If necessary, make U smaller, so that (5.3.2) holds for $J = K+1$ for all pairs ν'_{K+1}, τ'_{K+1} of probability measures supported in U. We, of course, immediately replace those probability measures by weak* approximants of the form hS as above. See *Ad* (5.3.2) for how to do this. If necessary, make U smaller still so that (5.2.4) holds for all pairs ν_{K+1}, τ_{K+} of probablity measures supported in U. See *Ad* (5.2.4) for how to do this.

Now use (1.5.3) to choose n_{K+1} so large that $U \cap \operatorname{Supp} \mu$ is not contained in $\operatorname{Supp} f_{n_{K+1}}$. With this, we can choose $\tau_{K+1} = g_j S$ to be a pseudofunction supported in $U \setminus \operatorname{Supp} f_{n_{K+1}}$ that is sufficiently close (weak*) to a probability measure (e.g., a unit point mass) supported in that set.

Finally, choose $\nu_{K+1} = h_j S$ such that $a \notin \operatorname{Supp} \nu_{K+1}$ but (5.2.5) holds for $j = K+1$. See *Ad* (5.2.5) for how to do this.

That completes the induction and the proof. □

COROLLARY 5.2.1. *Let E be a subset of the locally compact abelian group G. Suppose that H is a closed non-discrete subgroup of G and that E contains the translate, $K + x$,of a relatively open subset K of H. Then E contains Day-*1 *points.*

Proof. The Haar measure of H, restricted to a compact subset of K, is a measure (even) whose Fourier-Stieltjes transform vanishes at infinity on $\hat{H}$. Now apply Corollary 5.1.3. □

Remark 5.2.2. Compare with [11, Cor. 10, p. 417], for example.

5.3. *Proof of Theorem* 5.1.1*—nonabelian case.* We give the Theorem 5.1.1 in the non-abelian case. Here, G will denote a compact (hence, amenable) group. We may assume that the group is metrizable, by subsection 3.1.

The proof closely follows the lines of the proof of Theorem 5.1.1, but using Lemma 3.4.1 to give the conclusion that we can sum appropriate elements of $PF(G)$ with the norm being bounded. We note that if x, y are sufficiently close to each other, then all eight norm differences

$$\left\|(\delta_{\pm x} - \delta_{\pm y}) * S\right\|_{A(G)^*} \text{ and } \left\|S * (\delta_{\pm x} - \delta_{\pm y})\right\|_{A(G)^*} \tag{5.3.1}$$

are small (by a 2ϵ-argument using the fact that S is a pseudofunction , i.e., norm limit of elements of $L^1(G)$), and (5.3.1) also holds if the point masses are replaced with elements of $A(G)^*$ with supports close to each other and norm 1, where we again use the fact that S is a pseudofunction to obtain the conclusion. (The fact that S is a pseudofunction is used by approximating S in PM-norm by an element of $L^1(G)$, using continuity of translation, from both sides, in L^1, and a 3ϵ-argument.)

Use Lemma 3.4.1 (as indicated above) to produce two sequences of pseudofunctions ν_j, τ_j and elements g_j, h_j of $(A(G)$ having the properties that for $1 \le j \le J$,

(5.3.2a) $$\nu_j = g_j S,\ \tau_j = h_j S,\ \ a \notin \operatorname{Supp}(\nu_j + \tau_j),$$

(5.3.2b) $$\|\nu_j\|_{A(G)^*} = \|\tau_j\|_{A(G)^*} = 1,$$

(5.3.3) $$\sum_1^J \|\nu_j - \tau_j\|_{A(G)^*} \le 3 - 8^{-J}.$$

We are using the notation of 5.2 to make the parallels clearer. Also, $g_j, h_j \in A(G)$ are such that ν_j, τ_j are weak* close to unit point masses (which can be done by Lemma 3.2.1).

We will also find a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that for $k > j \ge 1$,

(5.3.4) $$\operatorname{Supp} f_{n_k} \cap \operatorname{Supp}(\nu_j + \tau_j) = \phi = \operatorname{Supp} f_{n_j} \cap \operatorname{Supp} \tau_j;$$

while for $k > j$,

(5.3.5) $$\left|\langle f_{n_j} \nu_k - \tau_k\rangle\right| < 8^{-k\cdot j};$$

and, for all j,

(5.3.6) $$\left|1 - \langle f_{n_j} \nu_j\rangle\right| < 8^{-j^2}.$$

Then any finite sum $g = \sum_1^N c_j f_{n_j}$ has norm that can be estimated as follows. First,

(5.3.7) $$\left|\langle g, \sum \frac{\bar{c}_j}{|c_j|}(\nu_j - \tau_j)\rangle\right| \ge \sum_1^N |c_j| \left(1 - \frac{8}{49}\right),$$

by iterating (5.3.5)–(5.3.6). Since

(5.3.8) $$\left\|\sum \frac{\bar{c}_j}{|c_j|}(\nu_j - \tau_j)\right\|_{A(G)^*} \le 3,$$

we have

(5.3.9) $$\|g\|_{A(E)} \ge \frac{1 - \frac{8}{49}}{3} \sum_1^N |c_j|.$$

Now apply (5.3.3) to conclude that f_{n_j} is a Sidon sequence with $d \ge \frac{1-\frac{8}{49}}{3}$.

It remains to show how (5.3.2)–(5.3.6) can be achieved.

We begin by choosing a neighborhood basis $\{U_n\}$ at a consisting of open relatively compact sets such that

(5.3.10) $$\bar{U}_{n+1} \subset U_n \text{ for } n \ge 1.$$

We may assume (by passing to a subsequence) that

$$\text{(5.3.11)} \qquad \operatorname{Supp} f_n \subset U_n \text{ for } n \geq 1.$$

We begin with some preliminary remarks concerning (5.3.3)–(5.3.6). Many details are identical to those in the corresponding parts of Section 5.

Ad (5.3.3). If we did not have to worry about (5.3.4)–(5.3.6), we could achieve (5.3.3) by using only the facts that S is a pseudomeasure, and $\nu_j = g_j S$, $\tau_j = h_j S$ have norm 1, so that if $1 \leq K$, and ν_j, τ_j $(1 \leq j \leq K)$ are such that (5.3.3) holds for $1 \leq J \leq K$, then there is a finite subset (because the dual object $\hat{G}$ is discrete) $L \subset \hat{G}$ such that $\sum_1^J \|\nu_j - \tau_j\|_{\mathfrak{B}(H_\rho)} < 16^{-J}$ for $\rho \notin L$, and so for one of the compact neighborhoods U_r of a, it is true that if ν_{K+1}, τ_{K+1} are *any* probability measures supported in U_r, then

$$\text{(5.3.12)} \qquad \|\nu_{K+1} - \tau_{K+1}\|_{\mathfrak{B}(H_\tau)} < 16^{-K}/d_\tau$$

for all $\tau \in L$. Now replace the probability measures ν_{K+1}, τ_{K+1} with pseudomeasures of the forms $\nu_{K+1} = g_{K+1}S$, $\tau_{K+1} = h_{K+1}S$ still (by weak* approximation) satisfying (5.3.12). Now (5.3.3) follows for $J = K + 1$.

Ad (5.3.4). No change from the discussion *Ad* (5.2.3): this is easily obtained by using (1.5.3), since we ensure that the (closed) supports of the ν_j and τ_j exclude a.

Ad (5.3.5). No change from the discussion *Ad* (5.2.4).

Ad (5.3.6). We will achieve this by ensuring that the support of τ_j misses the support of f_{n_j} (see the remark above concerning (5.3.4)), and also ensuring that the support of ν_j is sufficiently close to a (see the preceding paragraph).

We begin with τ_1. We choose $\tau_1 = g_1 S$ to be any pseudomeasure of that form, of norm 1, whose support *excludes* a. We now choose f_{n_1} such that $\operatorname{Supp} f_{n_1} \cap \operatorname{Supp} \tau_1 = \phi$, which is possible by (1.5.3). Now use (1.5.2) and the continuity of f_{n_1} to choose a first an $x \neq a$ and then a pseudomeasure $\nu_1 = h_1 S$ weak* close to x such that $a \notin \operatorname{Supp} \nu_1$ and (5.3.6) hold for $j = 1$.

We have now satisfied (5.3.2)–(5.3.6) (item (5.3.5) is vacuous here) for $j = 1$. This begins the induction.

We now suppose that $K \geq 1$ and that we have found pseudomeasures $\nu_j = g_j S$, $\tau_j = h_j S$ such that (5.3.2)–(5.3.6) hold for $1 \leq J \leq K$.

Choose a neighborhood U of a such that U does not intersect the supports of any of the pseudomeasures ν_j, τ_j, $1 \leq j \leq K$. If necessary, make U smaller, so that (5.3.2) holds for $J = K + 1$ for all pairs ν_{K+1}, τ_{K+1} of probability measures supported in U. See *Ad* (5.3.2) for how to do this. If necessary, make U smaller still so that (5.3.3) holds for all pairs ν_{K+1}, τ_{K+} of probability measures supported in U. Then take weak* approximants of the form hS. See *Ad* (5.3.5) for how to do this.

Now use (1.5.3) to choose n_{K+1} so large that $U \cap \operatorname{Supp} \mu$ is not contained in $\operatorname{Supp} f_{n_{K+1}}$. With this, we can choose $\tau_{K+1} = g_j S$ to be any norm 1 pseudomeasure supported in $U \setminus \operatorname{Supp} f_{n_{K+1}}$.

Finally, choose $\nu_{K+1} = h_j S$ such that $a \notin \operatorname{Supp} \nu_{K+1}$ but (5.3.6) holds for $j = K+1$. See *Ad* (5.3.6) for how to do this.

That completes the induction and the proof. □

6. No A_p-analogs of Theorem 4.1.3 and Theorem 5.1.1?

6.1. *Discussion.* For $1 < p < \infty$, $A_p(G)$ denotes the algebras of Herz [13, 14]: $f \in A_p(G)$ iff there exist $g_i \in L^p(G)$ and $h_j \in L^q(G)$ with $f = \sum g_j * h_j$ and $\sum \|g_j\|_p \|h_j\|_q < \infty$. The norm if f is the infimum of the sums $\sum \|g_j\|_p \|h_j\|_q$ subject to $f = \sum g_j * h_j$. When G is amenable, the set $A_p(G)$ is in fact a regular Banach algebra. Of course, $A_2(G) = A(G)$, the usual Fourier algebra.

In this section we give evidence that Theorem 4.1.3 and Theorem 5.1.1 may not generalize to the A_p-spaces. The difficulty involves the extension of the lemma about von Neumann algebras, Lemma 3.4.1.

An opposite extreme from a von Neumann algebra might be $L^1(\mathbb{T})$. For that latter space, we have the statement that follows about its elements, which is also a statement about multipliers on $L^1(\mathbb{T})$.

PROPOSITION 6.1.1. *There exist $f, g \in L^1(\mathbb{T})$ such that $\|f\|_1 = \|g\|_1 = 1$, $f * g = 0$, and $\|f + g\|_1 = 2$.*

Proof. We let $f_1 = 10(\mathbf{1}_{|[0,\frac{1}{10}]} * \mathbf{1}_{|[-\frac{1}{10},0]})$. Then f_1 has L^1-norm 1. We let $f(x) = f_1(2x)$. Then f has L^1-norm 1, is supported on $[-\frac{1}{5}, \frac{1}{5}] \cup [\pi - \frac{1}{5}, \pi + \frac{1}{5}]$ and has Fourier transform supported on the even integers. Let $g(x) = e^{\frac{ix}{2\pi}} f(x + \frac{\pi}{4})$. Then g also has L^1-norm 1, is supported on $[\frac{\pi}{4} - \frac{1}{5}, \frac{\pi}{4} + \frac{1}{5}] \cup [\frac{3\pi}{4} - \frac{1}{5}, \frac{3\pi}{4} + \frac{1}{5}]$, and has Fourier transform supported on the odd integers. Thus $f * g = 0$, and $\|f + g\| = 2$. □

6.2. *Difficulties for $A_p(E)$.* The following observation suggests that it will be difficult to generalize Theorem 5.1.1 to the case of A_p-algebras.

COROLLARY 6.2.1. *There exist $1 < p < 2$ and functions $f, g \in L_1(\mathbb{T})$ whose norms $\|\cdot\|_{M_p}$ as multipliers on $L^p(\mathbb{T})$ satisfy $\|f\|_{M_p} \leq 1$, $\|g\|_{M_p} \leq 1$, $f * g = 0$, but $\|f + g\| \geq 3/2$.*

Proof. Let f, g be the functions given in the proof of Proposition 6.1.1. Since $f, g \in L^1(\mathbb{T})$, it follows that $f\|_{M_p} \leq 1$ and $\|g\|_{M_p} \leq 1$. But $\|\hat{f}\|_\infty = \|\hat{g}\|_\infty = 1$, so $\|f\|_{M_p} \geq \|f\|_{M_2} = \|\hat{f}\|_\infty = 1$ and $\|g\|_{M_p} \geq \|g\|_{M_2} = \|\hat{g}\|_\infty = 1$. Thus $\|f\|_{M_p} = \|g\|_{M_p} = 1$. And, by approximation, if p is sufficiently close to 1, then the norm of $f + g$ as a multiplier on $L^p(\mathbb{T})$ is close to its L^1-norm of 2, that is $\|f + g\|_{M_p} \geq \frac{3}{2}$: let h be a trigonometric polynomial such that $\|h\|_1 = 1$ and

$\|h*(f+g)\|_1 > \frac{3}{2}$. Let $G(p) = \|h*(f+g)\|_p/\|h\|_p > \frac{3}{2}$. Then G is continuous in p $(0 < p < \infty)$. Since $G(1) > \frac{3}{2}$, $G(p) > \frac{3}{2}$ for some $p > 1$. Hence, for that p, $\|f+g\|_{M_p} > \frac{3}{2}$. □

7. Problems

General questions.

1. If $A(E)^{**}$ is Arens regular, is E Helson (i.e., $A(E) = C(E)$). Or: if $A(E)$ and $A(E)^{**}$, $A(E)^{****}, \ldots$ are all Arens regular, is $A(E) = C(E)$? Suggested by H. H. Dales [4]
2. For which non-Helson subsets E of a locally compact abelian group is $A(E)$ Arens regular and $A(E)^{**}$ not Arens regular?
3. Can $A(E)^{**}$ be Arens regular for non-Helson E?
4. If E is analytic, must $A(E)^{**}$ be non-Arens regular?
5. If $A(E)$ and $A(F)$ are Arens regular, is $A(E \cup F)$ Arens regular? [4]

Questions about A_p.

6. Under what conditions on E (and $1 < p < 2$?) can we say that $A_p(E)$ is not Arens regular.
7. Does there exist E such that $A_p(E)$ is weakly sequentially complete (or Arens regular) for some p and not others? Do (must?) the "good" p form an interval?
8. If $p \neq 2$, give conditions on E (and p?) so that $A_p(E)$ contains a tenting Sidon sequence.
9. If $p \neq 2$, can $A_p(\mathbb{T})$ contain a tenting sequence that is strong Sidon? In other words, does $\mathbb{T}$ contain strong Day points for $A_p(\mathbb{T})$?
10. If $1 < p < 2$, and $E \subset \mathbb{T}$, can $\tilde{A}_p(E)$ contain an element f of norm one such that $\{f^n\}$ is a Sidon sequence? What if $E = \mathbb{T}$?
11. Is M_p Arens regular? What about PF_p?

WSC and bounded synthesis.

12. Let S^2 denote the unit sphere in $\mathbb{R}^3$. Is $A(S^2)$ weakly sequentially complete? [Surface area measure on S^2 is a pseudofunction, so every point in S^2 is a Day point for $A(S^2)$, but S^2 is not a set of spectral synthesis [24].]
13. Find a compact set $E \subset G$ (for some compact abelian group G, ideally $\mathbb{T}$) such that E is not a Helson set, and $A(E)$ fails the conclusion of Theorem 4.1.3. Alternatively, show that no such E exists. See Remarks 4.1.4.
14. Does bounded synthesis hold for the synthesizable pseudomeasures on S^2? If so, the proof of Theorem 4.1.3 shows that $A(S^2)$ is wsc. More generally, when can bounded synthesis hold for the synthesizable pseudomeasures on a set of non-spectral synthesis? [A Helson set of non-synthesis is one such example. Are there others?]
15. Suppose E is the closure of an open set. Is $A_p(E)$ weakly sequentially complete for some $1 < p < \infty$ (all p)? [Note that $A_p(E)$ is not quite the same as $\{f \in A_p(G) : \operatorname{Supp} f \subset E\}$, which is wsc. See [10, p. 131].]

16. Suppose E is the support of a synthesizable pseudofunction and that E has empty interior. Is $A_p(E)$ weakly sequentially complete for some $1 < p < \infty$ (all p)?

17. Under what conditions does $A(E)$ non Arens regular imply $A(E)$ weakly sequentially complete? [See Remarks 4.1.4(iv).]

18. Does E countable and compact imply that $A_p(E)$ is not weakly sequentially complete?

19. Does there exist E such that $A(E)$ (or $A_p(E)$ for some, all? p) is neither weakly sequentially complete nor Arens regular?

20. If E is of spectral synthesis and is the support of a pseudofunction, must E be of bounded synthesis? Must $A(E)$ be wsc?

Translation-invariant means.

21. Can a tenting sequence have just 2 weak accumulation points? Or is it the case that the set of accumulation points of a tenting sequence is either a singleton or infinite (uncountable?)? See Remark 2.5.1.

22. If $A(E)$ is not Arens regular, must $A(E)^{**}$ contain c translation-invariant means?

Tenting sequences.

23. If $A(E)$ is not Arens regular, must $A(E)$ contain a Sidon tenting sequence?

24. Are tenting-M sequences needed, that is, if E has a tenting-M sequence at $a \in E$, is there a tenting-1 sequence in $A(E)$ at a?

25. Consider a generalization of tenting sequences: if for some $\delta > 0$ we have (instead of (1.2))

$$(1.2') \qquad \|f_n\|_{A(E)} \geq \delta \text{ for all } n \geq 1, \text{ and } \lim_n f_n(0) = 0.$$

we say $\{f_n\}$ is a *pup tenting-M* sequence. (The other requirements are unchanged.) What conditions are needed for $A(E)$ to be non Arens regular if $A(E)$ has a pup tenting sequence which is Sidon? Meyer and Granirer [11], [19] show that if $E \subset \mathbb{R}$ is ultrathin, then every pup tenting sequence in $A(E)$ has a Sidon subsequence.

26. If E has a pup-tenting sequence at $a \in E$, is there a tenting sequence in $A(E)$ at a?

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