FUNCTION TOPOLOGIES ON ABELIAN GROUPS

BY
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If P is a direct product (i.e., complete direct sum) of copies of the ring Z of rational integers, then P may be viewed as a topological abelian group in the Cartesian product topology corresponding to the discrete topology on Z. The structure of P as topological group has recently been studied by R. J. Nunke [8], who proved the following interesting result: If P has countably many factors and A is a closed subgroup of P, then A is likewise isomorphic to a direct product of copies of Z. If P has uncountably many factors, then there are closed subgroups of P which are not direct products.

We shall investigate this situation from a somewhat different point of view which is standard in the theory of topological vector spaces. Our starting point is the trivial observation that $P = \mathrm{Hom}_Z(F, Z)$, where F is a free abelian group. The product topology on P is simply the weak topology induced by F relative to the discrete topology on Z, F being viewed in the obvious way as a set of functions from P to Z. This is a special case of the situation which arises if we give to an arbitrary abelian group A the weak topology induced by a subgroup B of $\mathrm{Hom}_Z(A, Z)$, again relative to the discrete topology on Z. In this paper we study these topologies and their implications concerning the structure of certain groups.

In Section 1 we derive simple criteria for density and continuity in the afore-mentioned topologies which are analogous to well-known facts concerning topological vector spaces (see e.g. [4, Chapter 4]). The main theorem of Section 2 characterizes up to isomorphism the abelian groups which can arise as closed subgroups of direct products of copies of Z. Of course, this theorem contains the afore-mentioned result of Nunke as a special case. In Section 3 we turn to the direct product P of countably many copies of Z. After giving a complete description of closed subgroups of P, we prove the following analogue of a result of Mackey on topological vector spaces [4, p. 58]: If F_1 and F_2 are pure dense subgroups of P of countable rank, then there exists an automorphism of P which maps F_1 onto F_2. In Section 4 we apply our results to construct a pure subgroup A of P which is not free, but whose localization at any prime p is a free module over the corresponding discrete valuation ring[1] R_p. We show that A provides a negative answer to a question of J. Rotman concerning subgroups of P [10, p. 252].

Our exposition will be phrased for principal ideal domains. Hence through-

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[1] R_p is the subring of Q consisting of all elements of Q which, when reduced to lowest terms, have denominators relatively prime to p. We refer the reader to [6] for a discussion of the process of localization of a module over a commutative ring R with respect to a prime ideal in R.

out the paper R will denote a principal ideal domain with quotient field Q. All R-modules considered will be unitary. If A is an R-module, tA will represent the torsion submodule of A. We shall write $R^{\omega}A = \bigcap aA$, where a traces the nonzero elements of R. The rank of A is defined to be the Q-dimension of the vector space[2] $Q \otimes A$. If p is a prime in R, then A_p will denote the localization of A at p, which is a module over the discrete valuation ring R_p.

We shall adhere to the notation and terminology of [2]. In particular, if A is an R-module we shall write $A^* = \text{Hom}(A, R)$; A^* is called the *dual* of A. If $x \in A$ and $y \in A^*$, we shall denote the value of y on x by $\langle x, y\rangle$ or $\langle y, x\rangle$. If B is another module and $f : A \to B$ is a homomorphism, then the induced homomorphism $f^* : B^* \to A^*$ is called the *adjoint* of f. The natural homomorphism of A into A^{**} will be denoted by i_A. Following Bass [1], we shall call A *torsionless* if i_A is a monomorphism (such a module is called an e.h. module by Nunke [8]). If A is torsionless, we shall often tacitly identify A with its image in A^{**}. We shall have use for the following facts [2]: Every dual is torsionless, and every torsionless module is $\aleph_1$-free (i.e., every submodule of countable rank is free).

A will be called *reflexive* if i_A is an isomorphism, and *locally free* if every pure submodule of A of finite rank is a free direct summand of A. It was proved in Proposition 2.2 of [2] that A is locally free if and only if i_A is a monomorphism of A onto a pure submodule of A^{**}.

For further discussion of the above concepts, we refer the reader to [1], [2], [8], or [9].

Finally, we proclaim once and for all that the only topology on R which we shall consider is the discrete topology.

1. The weak topology

Throughout this section we shall deal with the following pair of objects: an R-module A and a submodule B of A^*.

DEFINITION 1.1. The B-topology on A is defined to be the weak topology induced on A by B. That is, it is the coarsest topology on A with the property that every element of B is a continuous function from A to R.

One special case of this definition warrants special mention. If A is an R-module, then $i_A(A) \subseteq A^{**}$, and so we obtain, via Definition 1.1, the $i_A(A)$-topology on A. By abuse of language, this topology will often be referred to as the A-topology on A^*.

We first summarize without proof the relevant formal properties of the weak topology.

[2] $Q \otimes A$ means $Q \otimes_R A$. Throughout this paper we shall omit the superfluous subscripts on the functors Hom, Ext, etc., which refer to the coefficient ring.

PROPOSITION 1.2. *The following conditions hold for any R-module A and submodule B of* A^*:

(1) *A is a topological R-module in the B-topology.*

(2) *If* $y_1, \cdots, y_n \in B$, *then* $U = \bigcap_{i=1}^n \ker(y_i)$ *is an open neighborhood of zero in the B-topology. Furthermore, the collection of all subsets of A of this form is a neighborhood basis of zero for this topology.*

(3) *If C is a subset of A, then* $x \in \bar{C}$ *if and only if, for any* $y_1, \cdots, y_n \in B$, *there exists* $x' \in C$ *such that* $\langle x', y_i \rangle = \langle x, y_i \rangle$ *for* $i \leqq n$.

(4) *The B-topology on A is Hausdorff if and only if, for any* $x \neq 0$ *in A, there exists* $y \in B$ *such that* $\langle x, y \rangle \neq 0$.

(5) *If F is a free R-module of rank* α, *then* $P = F^*$ *is a direct product of* α *copies of R. The F-topology on P is simply the product topology.*

(6) *Let* $A_1 \subseteq A$, *and let j be the inclusion mapping. If* $B_1 = j^*(B) \subseteq A_1^*$, *then the* B_1-*topology on* A_1 *is simply the relative topology induced on* A_1 *by the B-topology on A.*

The last assertion follows from the fact that, if $y \in B$, then $j^*(y)$ is the restriction of y to A_1.

In order to undertake a discussion of density and continuity in the topology described above, we shall need an important lemma.

LEMMA 1.3. *Let A be an R-module, and let* $y \neq 0$ *generate a pure submodule of* A^*. *Then there exists* $x \in A$ *such that* $\langle x, y \rangle = 1$.

Proof. Let $I = \langle A, y \rangle = \{\langle x, y \rangle \mid x \in A\}$. I is an ideal in R, and so $I = Ra$ for some $a \in R$, since R is a principal ideal domain. Since $y \neq 0$, $I \neq 0$, and so $a \neq 0$. Define a mapping $g : A \rightarrow R$ by $g(x) = a^{-1}\langle x, y \rangle$; then it is easy to see that there exists $z \in A^*$ such that $g(x) = \langle x, z \rangle$ for all $x \in A$, and $az = y$. Since y generates a pure submodule of A^*, it follows that a is a unit in R, in which case $I = R$, and there exists $x \in A$ such that $\langle x, y \rangle = 1$, completing the proof.

The following theorem provides an analogue for abelian groups of a well-known result on topological vector spaces [4, Theorem 1, p. 69].

THEOREM 1.4. *Let A be an R-module, and B a submodule of* A^*. *Then the following conditions hold for any submodule* A_1 *of A*:

(a) *B is pure in* A^*, *and* A_1 *is dense in A in the B-topology.*

(b) *B is pure in* A^*; *and if* $x \in A$ *and* $y_1, \cdots, y_n \in B$, *then there exists* $x' \in A_1$ *such that* $\langle x', y_i \rangle = \langle x, y_i \rangle$ *for all* $i \leqq n$.

(c) *If* $y \in B$ *generates a pure submodule of B, then there exists* $x \in A_1$ *such that* $\langle x, y \rangle = 1$.

Proof. (a) $\Leftrightarrow$ (b). The equivalence of (a) and (b) follows easily from condition (2) of Proposition 1.2.

(b) $\Rightarrow$ (c). Assume (b) holds, and let y generate a pure submodule of B. Since B is pure in A^*, we see that Ry is also a pure submodule of A^*, and so by Lemma 1.3 there exists $x \in A$ such that $\langle x, y\rangle = 1$. We may then apply (b) to obtain $x' \in A_1$ such that $\langle x', y\rangle = 1$. Condition (c) then follows upon replacing x by x'.

(c) $\Rightarrow$ (b). We show first that B is pure in A^*. Observe that, since A^* is locally free [2, Proposition 2.1], every pure submodule of A^* of rank one is cyclic. Let y generate such a submodule, and set $B \cap Ry = Ry'$; then $y' = ay$ for some $a \in R$. Ry' is a pure submodule of B, and so, if $y' \neq 0$, we obtain from (c) that there exists $x' \in A_1$ such that $\langle x', y'\rangle = 1$. Then $a\langle x', y\rangle = \langle x', ay\rangle = \langle x', y'\rangle = 1$; i.e., a is a unit in R, and $Ry = Ry' \subseteq B$. We have shown that every pure submodule of A^* of rank one is either contained in B or has trivial intersection with B, from which it follows that B is pure in A^*.

Now let $y_1, \cdots, y_n \in B$, and let F be the smallest pure submodule of B containing $y_1, \cdots, y_n$ (i.e., the intersection of all pure submodules of B containing $y_1, \cdots, y_n$; since A^* is torsion-free, this is again a pure submodule of B). By a previous remark, A^* is $\aleph_1$-free, and so F is a free module of finite rank. Let $j : F \rightarrow A^*$ be the inclusion mapping, and let $j^* : A^{**} \rightarrow F^*$ be the adjoint of j. Set $f = j^* i_A$; then $f : A \rightarrow F^*$, and we see that, if $x \in A$ and $y \in F$, then $\langle f(x), y\rangle = \langle x, y\rangle$.

Since R is a principal ideal domain and F^* is free of finite rank, there exists a basis $v_1, \cdots, v_r$ of F^* and $x_1, \cdots, x_r \in A_1$ such that $f(x_1), \cdots, f(x_r)$ generate $f(A_1)$, and $f(x_i) = a_i v_i$ for some $a_1, \cdots, a_r \in R$. Let $u_1, \cdots, u_r$ be a basis of F dual to the basis $v_1, \cdots, v_r$ of F^*; i.e., $\langle v_i, u_j\rangle = \delta_{ij}$ for $i, j \leqq r$. Then clearly each u_i generates a pure submodule of F, and hence a pure submodule of B, since F is pure in B. Therefore, by hypothesis, there exists $x_i' \in A_1$ such that $\langle x_i', u_i\rangle = 1$. Let $f(x_i') = b_{i1} f(x_1) + \cdots + b_{ir} f(x_r)$; then we obtain from a routine computation that $1 = \langle x_i', u_i\rangle = a_i b_{ii}$. Hence each a_i is a unit in R, from which it follows that $f(A_1) = F^*$.

Now, if $x \in A$, we have from the preceding discussion that there exists $x' \in A_1$ such that $f(x') = f(x)$. Hence, since $y_1, \cdots, y_n \in F$, it follows that $\langle x', y_i\rangle = \langle f(x'), y_i\rangle = \langle f(x), y_i\rangle = \langle x, y_i\rangle$ for $i \leqq n$. This establishes (b) and completes the proof of the theorem.

One special case of the preceding theorem is worthy of mention.

COROLLARY 1.5. *If A is an R-module, then $i_A(A)$ is dense in A^{**} in the A^*-topology. That is, if $y_1, \cdots, y_n \in A^*$ and $z \in A^{**}$, then there exists $x \in A$ such that $\langle x, y_i\rangle = \langle z, y_i\rangle$ for all $i \leqq n$.*

Proof. We have from Proposition 2.1 of [2] that A^* is locally free, and hence, by an earlier remark, i_{A^*} is a monomorphism of A^* onto a pure submodule of A^{***}. (This follows also from Theorem 1.4 of [5].) The A^*-topology on A^{**} arises, of course, from the identification of A^* with its image in A^{***}.

Furthermore, we obtain from Lemma 1.3 that, if y generates a pure submodule of A^*, then there exists $x \in A$ such that $\langle i_A(x), y\rangle = \langle x, y\rangle = 1$. Our assertions then follow immediately from Theorem 1.4, and the proof is complete.

If A is a topological R-module, we shall denote by $\mathrm{CHom}(A, R)$ the submodule of A^* consisting of all continuous homomorphisms from A to R. We state without proof the following easy consequence of Definition 1.1.

LEMMA 1.6. *Let A be a topological R-module. Let C be an R-module endowed with the D-topology, where D is a submodule of C^*. Then a homomorphism $f : A \to C$ is continuous if and only if $f^*(D) \subseteq \mathrm{CHom}(A, R)$.*

Suppose now that A is given the B-topology, where B is a submodule of A^*. The relationship between B and $\mathrm{CHom}(A, R)$ is then easily described.

THEOREM 1.7. *Let A be an R-module with the B-topology, where B is a submodule of A^*, and let B_1 be the intersection of all pure submodules of A^* which contain B. Then $B_1 = \mathrm{CHom}(A, R)$, and the B_1-topology on A coincides with the B-topology.*

Proof. Let $y \in B_1$. Since B_1/B is a torsion module, there exists $a \neq 0$ in R such that $ay = y' \in B$. Then $\ker(y) = \ker(y')$ is open in A in the B-topology, since R is discrete. Hence, for the same reason, y is B-continuous. It then follows that $B_1 \subseteq \mathrm{CHom}(A, R)$.

On the other hand, if $y \in \mathrm{CHom}(A, R)$, then since R is discrete, $\ker(y)$ is B-open in A. Hence, by condition (2) of Proposition 1.2, there exist $y_1, \cdots, y_n \in B$ such that $K = \bigcap_{i=1}^n \ker(y_i) \subseteq \ker(y)$. If, for some $k \leqq n$, y_k is linearly dependent on $y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_n$, then

$$\ker(y_k) \supseteq \textstyle\bigcap_{i \neq k} \ker(y_i),$$

in which case $\ker(y_k)$ can be omitted from the intersection. Thus we may assume that $y_1, \cdots, y_n$ are linearly independent. Let F be a free R-module with basis $e_1, \cdots, e_n$, and define a homomorphism $f : A \to F$ by $f(x) = \langle x, y_1\rangle e_1 + \cdots + \langle x, y_n\rangle e_n$ for $x \in A$. Then clearly $K = \ker(f)$. Setting $C = \mathrm{Im}(f)$, we obtain the exact sequence

$$0 \to K \xrightarrow{\ j\ } A \xrightarrow{\ g\ } C \to 0,$$

where j is the inclusion mapping and g differs from f only by the obvious contraction of the range. This gives rise to the exact sequence

$$0 \to C^* \xrightarrow{\ g^*\ } A^* \xrightarrow{\ j^*\ } K^*.$$

Since $K \subseteq \ker(y_i)$ for $i \leqq n$ and $K \subseteq \ker(y)$, we have that $j^*(y_1) = \cdots = j^*(y_n) = j^*(y) = 0$, and so there exist $y_1', \cdots, y_n', y' \in C^*$ such that $g^*(y_i') = y_i$ and $g^*(y') = y$. But since $\mathrm{rank}(C) \leqq n$, we have that also $\mathrm{rank}(C^*) \leqq n$, and so $y_1', \cdots, y_n', y'$ are linearly dependent, in which case $y_1, \cdots, y_n, y$ are likewise linearly dependent. Since $y_1, \cdots, y_n$ are linearly independent, it follows that $ay = a_1 y_1 + \cdots + a_n y_n \in B$ for some $a, a_1, \cdots, a_n \in R$, where $a \neq 0$. Then $y \in B_1$. Thus $\mathrm{CHom}(A, R) = B_1$, completing the proof that $\mathrm{CHom}(A, R) = B_1$.

To say that the B_1-topology on A coincides with the B-topology is to say that the identity mapping on A is continuous from either topology to the other. That this is the case follows immediately from Lemma 1.6 and the fact that $B_1 = \mathrm{CHom}(A, R)$. This completes the proof of the theorem.

The following result is related to a theorem on continuous mappings of topological vector spaces [4, Theorem 1, p. 72].

THEOREM 1.8. *Let A and C be R-modules, and B, D submodules of A^*, C^*, respectively. Endow A with the B-topology, and C with the D-topology. Then a homomorphism $f : A \to C$ is continuous if and only if $f^*(D)/B \cap f^*(D)$ is a torsion module.*

Proof. By Lemma 1.6, f is continuous if and only if $f^*(D) \subseteq \mathrm{CHom}(A, R)$. But it follows immediately from Theorem 1.7 that this is true if and only if $f^*(D)/B \cap f^*(D)$ is a torsion module. This completes the proof.

2. Closed submodules

Before providing a description of closed submodules of direct products of copies of R, we must prove some preliminary results.

LEMMA 2.1. *Let A, B be R-modules, and $f : A \to B$ a homomorphism. If* $\mathrm{coker}(f)$ *is a torsion module, then $f^* : B^* \to A^*$ is a closed mapping from the B-topology on B^* to the A-topology on A^*. Furthermore, if R is not a field, then* $R^{\omega}(\mathrm{coker}(f^*)) = 0$.

Proof. Let C be a B-closed subset of B^*, and let y be in the A-closure of $f^*(C)$ in A^*. We shall define a homomorphism $g : B \to Q$ in the following way: If $x \in B$, then since $\mathrm{coker}(f)$ is a torsion module, there exist $a \neq 0$ in R and $x' \in A$ such that $ax = f(x')$, in which case we set $g(x) = a^{-1}\langle x', y\rangle$. Suppose that also $bx = f(x'')$ for some $x'' \in A$ and $b \neq 0$ in R. Then, since y is in the A-closure of $f^*(C)$, we obtain from condition (3) of Proposition 1.2 that there exists $z \in C$ such that $\langle x', y\rangle = \langle x', f^*(z)\rangle = \langle f(x'), z\rangle$, and $\langle x'', y\rangle = \langle x'', f^*(z)\rangle = \langle f(x''), z\rangle$. But $bf(x') = abx = af(x'')$, and so $b\langle x', y\rangle = \langle bf(x'), z\rangle = \langle af(x''), z\rangle = a\langle x'', y\rangle$, in which case $a^{-1}\langle x', y\rangle = b^{-1}\langle x'', y\rangle$, since $ab \neq 0$. It then follows that g is well defined. Furthermore, since $z \in C \subseteq B^*$, $\langle x, z\rangle$ is a well-defined element of R, and so $\langle x', y\rangle = \langle f(x'), z\rangle = \langle ax, z\rangle = a\langle x, z\rangle \in aR$, from which it follows that $g(x) \in R$. Hence $g(B) \subseteq R$. We omit the trivial verification that g is a homomorphism of B into R; i.e., there exists unique $v \in B^*$ such that $g(x) = \langle x, v\rangle$ for all $x \in B$. If $x \in A$, then it follows immediately from the definition of v that $\langle x, f^*(v)\rangle = \langle f(x), v\rangle = \langle x, y\rangle$. Hence $f^*(v) = y$.

Now let $x_1, \cdots, x_n \in B$. Then, since $\mathrm{coker}(f)$ is a torsion module, there exists $a \neq 0$ such that $ax_i = f(x_i')$ for $x_i' \in A$, $i \leqq n$. Since y is in the A-

closure of $f^*(C)$ in A^*, we have from condition (3) of Proposition (1.2) that there exists $z \in C$ such that $\langle x_i', f^*(z)\rangle = \langle x_i', y\rangle$ for all $i \leqq n$. Hence $a\langle x_i, z\rangle = \langle f(x_i), z\rangle = \langle x_i', f^*(z)\rangle = \langle x_i', y\rangle = a\langle x_i, v\rangle$, and so $\langle x_i, z\rangle = \langle x_i, v\rangle$ for $i \leqq n$. Applying condition (3) of Proposition 1.2 once again, we obtain that $v \in \bar{C} = C$, and so $y = f^*(v) \in f^*(C)$. It then follows that $f^*(C)$ is A-closed in A^*, completing the proof that f^* is a closed mapping.

Assume now that R is not a field. We shall present a short homological proof of the assertion that $R^\omega(\text{coker}(f^*)) = 0$, although there exists an easy but slightly longer direct proof which avoids the use of homological methods. Since coker(f) is a torsion module, $(\text{coker}(f))^* = 0$, and so the exact sequences

$$0 \to \ker(f) \to A \to \text{Im}(f) \to 0, \qquad 0 \to \text{Im}(f) \to B \to \text{coker}(f) \to 0$$

give rise to the exact sequences

$$0 \to (\text{Im}(f))^* \to A^* \to (\ker(f))^*,$$

$$0 \to B^* \to (\text{Im}(f))^* \to \text{Ext}\{\text{coker}(f), R\}.$$

But $(\ker(f))^*$ is torsionless, and so since R is not a field it follows easily that $R^\omega\{(\ker(f))^*\} = 0$; furthermore, since coker(f) is a torsion module, $R^\omega\{\text{Ext}[\text{coker}(f), R]\} = 0$ by Lemma 7.5 of [7, p. 236]. Since f^* is simply the composition $B^* \to (\text{Im}(f))^* \to A^*$, it follows easily that coker(f^*) is an extension of a submodule of Ext$\{\text{coker}(f), R\}$ by a submodule of $(\ker(f))^*$, and so $R^\omega\{\text{coker}(f^*)\} = 0$. This completes the proof of the lemma.

The following theorem, a special case of which is closely related to results of Bass [1, (4.4), p. 477] and Nunke [8, Theorem 8] contains most of the information which we have been able to gather concerning closed submodules. It says essentially that a closed submodule of a dual is likewise a dual.

THEOREM 2.2. *Let B be an R-module, and A a submodule of B^*. Then there exist a pure submodule C of A^* and a homomorphism $g : B \to C$ such that*

(a) *C is locally free, and* rank$(C) \leqq$ rank(B).

(b) *g^* is a continuous closed monomorphism of C^* into B^* (where C^* is given the C-topology, and B^* the B-topology).*

(c) *If $j : A \to B^*$ and $\mu : C \to A^*$ are the inclusion mappings, then $\sigma = \mu^* i_A$ is a monomorphism of A into C^* such that $g^*\sigma = j$. If C^* is identified with its image in B^*, then $A \subseteq C^*$, and C^* is the B-closure of A in B^*.*

(d) *$R^\omega(B^*/C^*) = 0$ if R is not a field.*

Proof. Let $f : B \to A^*$ be the composition of

$$i_B : B \to B^{**} \quad \text{and} \quad j^* : B^{**} \to A^*.$$

Let C be the smallest pure submodule of A^* containing Im(f), and $g : B \to C$ the homomorphism which differs from f only by contraction of the range. Since C is a pure submodule of A^*, it follows from Proposition 2.1 of [2] that C is locally free. Also, coker$(g) = C/\text{Im}(f)$ is a torsion module, and so we see immediately that rank$(C) \leqq$ rank(B). Thus (a) holds.

Now g is the composition of the epimorphism $B \to \operatorname{Im}(f)$ and the monomorphism $\operatorname{Im}(f) \to C$, the latter being the inclusion mapping and having a torsion cokernel. Since $\operatorname{Hom}(\cdot, R)$ is left exact, and since the dual of a torsion module is trivial, we see easily that $g^* : C^* \to B^*$ is a monomorphism. Since $\operatorname{coker}(g)$ is a torsion module, we may apply Theorem 1.8 and Lemma 2.1 to conclude that g^* is a continuous closed mapping of C^* into B^*, where C^* is given the C-topology and B^* the B-topology. This establishes (b).

Let $\mu : C \to A^*$ be the inclusion mapping. Then the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{i_B} & B^{**} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle j^*} \\ C & \xrightarrow{\mu} & A^* \end{array}$$

gives rise to the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i_A} & A^{**} & \xrightarrow{\mu^*} & C^* \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j^{**}} & & \downarrow{\scriptstyle g^*} \\ B^* & \xrightarrow{i_{B^*}} & B^{***} & \xrightarrow{(i_B)^*} & B^*. \end{array}$$

But we have from Theorem 1.4 of [5] that $(i_B)^* i_{B^*}$ is the identity mapping on B^*. Hence, setting $\sigma = \mu^* i_A$, we obtain the commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & C^* \\ & {\scriptstyle j}\searrow \quad \swarrow{\scriptstyle g^*} & \\ & B^*. & \end{array}$$

σ must be a monomorphism, since j is. Therefore, if C^* is identified with its image in B^*, then $A \subseteq C^*$. Note that, since g^* is a continuous, closed monomorphism, C^* is identified with a B-closed submodule of B^* in such a way that the C-topology on C^* coincides with the relative topology induced by the B-topology on B^*.

Now let y generate a pure submodule of C; then, since C is pure in A^*, Ry is likewise a pure submodule of A^*. Thus, by Lemma 1.3, there exists $x \in A$ such that $\langle x, y \rangle = 1$. We may then apply Theorem 1.4 to conclude that A is C-dense in C^*, and hence B-dense in C^*, since the two topologies coincide on C^*. Since C^* is a B-closed submodule of B^*, it follows that C^* is the B-closure of A in B^*. This establishes (c).

Finally, let us assume that R is not a field. Then, since $\operatorname{coker}(g)$ is a torsion module, and (according to the identification introduced above)

$A \subseteq C^* = \operatorname{Im}(g^*)$, it follows from Lemma 2.1 that $R^\omega(B^*/C^*) = 0$. This establishes (d) and completes the proof of the theorem.

One sees easily from Corollary 1.7 and condition (6) of Proposition 1.2 that the module C which appears in the theorem is simply $\operatorname{CHom}(A, R)$. This suggests the following alternate method of proof of the theorem: Define $C = \operatorname{CHom}(A, R)$, and let $g : B \to C$ be the restriction to B of the homomorphism of $\operatorname{CHom}(B, R)$ into C induced by the inclusion mapping $j : A \to B^*$. However, the proof presented above is somewhat shorter.

If A and B are topological R-modules, we shall call a mapping $j : A \to B$ an *isometry* if it is both an isomorphism and a homeomorphism. In this case A and B will be called *isometric*. We can now easily derive the principal result of this section, which characterizes up to isometry the closed submodules of arbitrary direct products of copies of R.

THEOREM 2.3. *Let α be a cardinal number, and P the direct product of α copies of R. View P as a topological R-module with the product topology. Then*

(a) *If A is a closed submodule of P, then there exist a locally free R-module C of rank less than or equal to α, and an isometry $\sigma : A \approx C^*$ (where C^* is given the C-topology, and A the relative topology induced by the product topology on P).*

(b) *If C is a locally free R-module of rank α, then there exists a continuous closed monomorphism $\tau : C^* \to P$ (C^* having the C-topology). Thus C^* is isometric to a closed submodule of P.*

Proof. (a) We have observed in condition (5) of Proposition 1.2 that $P = F^*$, where F is a free R-module of rank α, and the product topology on P coincides with the F-topology. Since A is F-closed in P, (a) then follows easily from Theorem 2.2 upon setting $B = F$.

(b) Since C is torsion-free and has rank α, we see easily that there exists a free submodule F of C of rank α such that C/F is a torsion module. Then $(C/F)^* = 0$; hence, if $j : F \to C$ is the inclusion mapping, then $j^* : C^* \to F^*$ is a monomorphism. But, by condition (5) of Proposition 1.2, F^* is isometric to P, where F^* has the F-topology. Furthermore, since $\operatorname{coker}(j) = C/F$ is a torsion module, we may apply Theorem 1.8 and Lemma 2.1 to conclude that j^* is a continuous closed monomorphism of C^* into P. Hence (b) holds, and the proof is complete.

It is an easy consequence of the results of Section 2 of [2] that any dual is the dual of some locally free module. Hence, by Theorem 2.3, any dual is isometric to a closed submodule of a direct product of copies of R.

Now let R be a field, and P a direct product of copies of R. If A is a subspace of P, then it is known that a necessary and sufficient condition that A be closed in P in the product topology is that A be the "annihilator" of some subspace B of F, where $P = F^*$ [4, Theorem 1, p. 72]. That is,

$$A = \{y \in P \mid \langle x, y\rangle = 0 \text{ for all } x \in B\}.$$

We shall present an example to show that this simple characterization of closed submodules cannot be extended to arbitrary principal ideal domains, even for the special case in which A is a pure submodule of P.

Let R be a principal ideal domain which is not a field, and let Q be the quotient field of R. Select a free R-module G and a submodule F of G so that $G/F \approx Q/R$. If $j : F \to G$ is the inclusion mapping, then since Q/R is a torsion module, the exact sequence

$$0 \to F \xrightarrow{j} G \to Q/R \to 0$$

gives rise to the exact cohomology sequence

$$0 \to G^* \xrightarrow{j^*} F^* \to \mathrm{Ext}(Q/R, R) \to 0.$$

Set $P = F^*$, $A = G^*$, and give to P the F-topology (i.e., the product topology) and to A the G-topology. Since coker(j) is a torsion module, we obtain from Lemma 2.1 and Theorem 1.8 that j^* is a continuous closed monomorphism, and therefore defines an isometry of A onto a closed submodule of P. We shall identify A with its image in P.

Furthermore, since $\mathrm{Hom}(Q/R, Q/R) \approx \prod_p \bar{R}_p$ (where p traces all primes in R, and $\bar{R}_p$ is the completion of the valuation ring R_p), it follows from a routine computation that $\mathrm{coker}(j^*) \approx \mathrm{Ext}(Q/R, R) \approx \prod_p R_p$, which is torsion-free. Hence A is a pure submodule of P.

Now let x be any nonzero element of F. Since G is free, it is clear that there exists $y \in G^* = A$ such that $\langle x, y \rangle \neq 0$. It then follows immediately that A cannot be the annihilator (in the sense described above) of any submodule of F.

It is true, however, that any such annihilator is a closed submodule of P. The proof is routine and will be omitted.

3. Submodules of the direct product of countably many copies of R

We now investigate more closely the situation described in Theorem 2.3 for the special case in which $\alpha = \aleph_0$. Part (a) of the result we obtain is (for the special case in which R is the ring of rational integers) simply the theorem of Nunke mentioned in the introduction [8, Theorem 3].

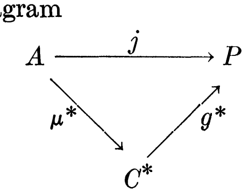
THEOREM 3.1. *Let P be the direct product of a countably infinite family of copies of R, viewed as a topological R-module in the product topology. Let A be a submodule of P. Then*

(a) *If A is a closed submodule of P, then A is isometric (in the relative topology) to the direct product of a countable family of copies of R (in the product topology). If R is not a field, then P/A is reduced.*

(b) *If R has infinitely many primes, A is isomorphic to a direct product of countably many copies of R, and P/A is reduced, then A is closed in P, and the relative topology on A coincides with the product topology.*

Proof. (a) We have from Theorem 2.3 that there exist a locally free module C of countable rank and an isometry $\sigma : A \approx C^*$. C, being locally free, is then $\aleph_1$-free, as was observed in the introduction; hence, since C has countable rank, it follows that C must be free. Thus, by (5) of Proposition 1.2, A is a direct product of a countable family of copies of R, and the product topology on A coincides with the C-topology on C^*, which, since σ is an isometry, coincides with the relative topology on A. That P/A is reduced if R is not a field follows from Theorem 2.2(d).

(b) By (5) of Proposition 1.2, $P = F^*$, where F is free of countably infinite rank and the product topology on P is the same as the F-topology. If R has infinitely many primes, then it follows from an argument similar to that of [3, Theorem 47.3] that $P^* = F^{**} = F$; i.e., F is reflexive (see also [8]). If A is a direct product of countably many copies of R, then for the same reason $A^* = F_0$ is free of countable rank, and $A^{**} = A$. (Throughout this discussion we shall identify reflexive modules with their double duals.) By Theorem 2.2, there exist a pure submodule C of $A^* = F_0$ and a homomorphism $g : F \to C$ such that $g^* : C^* \to P$ defines an isometry of C^* with the closure of A in P. In this case i_A is the identity mapping, and so, by Theorem 2.2 again, the diagram



is commutative, where $j : A \to P$ and $\mu : C \to F_0$ are the inclusion mappings. But, since F_0 and C are free, the exact sequence

$$0 \to C \xrightarrow{\mu} F_0 \to F_0/C \to 0$$

gives rise to the exact cohomology sequence

$$0 \to (F_0/C)^* \to A \xrightarrow{\mu^*} C^* \to \operatorname{Ext}(F_0/C, R) \to 0.$$

Since μ^* is a monomorphism, $(F_0/C)^* = 0$. Since C is a pure submodule of F_0 and the latter is free, we see that F_0/C is torsion-free, in which case coker $(\mu^*) \approx \operatorname{Ext}(F_0/C, R)$ is divisible. (See [7, Theorem 4.5, p. 230]; there is also an elementary direct proof of this fact.) But since P/A is reduced, it follows easily from the diagram above that $\operatorname{Ext}(F_0/C, R) = 0$. We may then apply Theorem 8.5 of [7, p. 240] to conclude that $F_0/C = 0$; i.e., μ is an isomorphism. It then follows from Theorem 1.8 and Lemma 2.1 that $\mu^* : A \approx C^*$ is an isometry between A (with the F_0-topology, i.e., the product topology) and C^* (with the C-topology). (b) then follows immediately from the diagram above and the properties of μ^* and g^* already discussed. This completes the proof of the theorem.

We end this section with a theorem on dense submodules of products which strongly resembles, in both statement and proof, a well-known result of Mackey on topological vector spaces [4, p. 58].

THEOREM 3.2. *Let F be a free R-module of countably infinite rank, and set $P = F^*$. Let F_0 be a pure submodule of P of countable rank which is dense in P with respect to the F-topology. Then F_0 is free, and there exist a basis $x_1, x_2, \cdots$ of F and a basis $y_1, y_2, \cdots$ of F_0 such that $\langle x_i, y_j \rangle = \delta_{ij}$.*

Proof. Since P is a direct product of copies of R and $F_0 \subseteq P$, F_0 is torsionless and hence $\aleph_1$-free, by a previous remark. Since F_0 has countable rank, it follows that F_0 is free. Set $P_0 = F_0^*$. Let $j : F_0 \to P$ be the inclusion mapping, and define $g : F \to P_0$ to be the composition of $i_A : F \to P^*$ with $j^* : P^* \to P_0$. If $x \in F$ and $y \in F_0$, then $\langle g(x), y \rangle = \langle j^* i_A(x), y \rangle = \langle i_A(x), j(y) \rangle = \langle x, j(y) \rangle = \langle x, y \rangle$. Let $x \neq 0$ be in F; then, since F is free, it is easy to see that there exists $y \in P$ such that $\langle x, y \rangle \neq 0$. Since F_0 is F-dense in P, there exists $y' \in F_0$ such that $\langle g(x), y' \rangle = \langle x, y' \rangle = \langle x, y \rangle \neq 0$, in which case $g(x) \neq 0$. Thus g is a monomorphism. In the following discussion we shall identify F with its image in P_0.

Now let y generate a pure submodule of F_0; then, since F_0 is pure in P, Ry is also a pure submodule of P. Hence, by Lemma 1.3, there exists $x \in F$ such that $\langle x, y \rangle = 1$. We may then apply Theorem 1.4 to conclude that F is F_0-dense in P_0.

We should remark at this point that F and F_0, being free, are identified in the usual way with pure submodules of F^{**} and F_0^{**}, respectively (see [2, Proposition 2.2]). Since $F^{**} = P^*$ and $F_0^{**} = P_0^*$, these identifications give rise to the F-topology on P and the F_0-topology on P_0.

We shall construct $\{x_n\}$ and $\{y_n\}$ inductively. Let $u_1, u_2, \cdots$ be any basis of F, and $v_1, v_2, \cdots$ any basis of F_0. Since x_1 generates a pure submodule of F, and F_0 is F-dense in P, we obtain from Theorem 1.4 that there exists $y_1 \in F_0$ such that $\langle u_1, y_1 \rangle = 1$. Set $x_1 = u_1$. Proceeding by induction on n, we assume now that $x_1, \cdots, x_{n-1}$ and $y_1, \cdots, y_{n-1}$ have been constructed so that the conditions of the theorem are satisfied.

If n is odd, let u_{i_n} be the first u_i which is not contained in the submodule F_n of F generated by $x_1, \cdots, x_{n-1}$. Since $\langle x_i, y_j \rangle = \delta_{ij}$ for $i, j < n$, we see that F_n is a pure submodule of F, and so F/F_n is free. Let $u + F_n$ generate the pure submodule of F/F_n of rank one which contains $u_{i_n} + F_n$; this submodule is a direct summand of F/F_n. Therefore there exists $y_n' \in P$ such that $\langle u, y_n' \rangle = 1$ and $\langle x_i, y_n' \rangle = 0$ for $i < n$. Since F_0 is F-dense in P, we may then apply Theorem 1.4 to conclude that there exists $y_n \in F_0$ such that $\langle u, y_n \rangle = \langle u, y_n' \rangle = 1$ and $\langle x_i, y_n \rangle = \langle x_i, y_n' \rangle = 0$ for $i < n$. Set $x_n = u - \langle u, y_1 \rangle x_1 - \cdots - \langle u, y_{n-1} \rangle x_{n-1}$; then $\langle x_n, y_n \rangle = \langle u, y_n \rangle = 1$ and $\langle x_n, y_i \rangle = 0$ for $i < n$. Clearly u_{i_n} is contained in the submodule of F generated by $x_1, \cdots, x_n$.

If n is even, let v_{i_n} be the first v_i which is not contained in the submodule of F_0 generated by $y_1, \cdots, y_{n-1}$. Since F is F_0-dense in P_0, it follows from an argument similar to that of the preceding paragraph that there exist $x_n \in F$ and $y_n \in F_0$ such that $\langle x_i, y_j \rangle = \delta_{ij}$ for $i, j \leqq n$ and v_{i_n} is contained in the submodule of F_0 generated by $y_1, \cdots, y_n$. This completes the construction of the sequences $\{x_n\} \subseteq F$ and $\{y_n\} \subseteq F_0$ such that $\langle x_i, y_j \rangle = \delta_{ij}$ for all i, j. It is clear from the nature of their construction that $\{x_n\}$ is a basis of F and $\{y_n\}$ is a basis of F_0, and so we are done.

COROLLARY 3.3. *Let F be a free R-module of countably infinite rank, and set $P = F^*$. Let F_1 and F_2 be pure free submodules of P of countable rank which are dense in P with respect to the F-topology. Then there exists an automorphism σ of F such that $\sigma^*(F_1) = F_2$, where σ^* is the automorphism of P which is the adjoint of σ.*

Proof. We have from Theorem 3.2 that there exist bases $\{x_n^{(1)}\}$ and $\{x_n^{(2)}\}$ of F, a basis $\{y_n^{(1)}\}$ of F_1, and a basis $\{y_n^{(2)}\}$ of F_2 such that

$$\langle x_i^{(1)}, y_j^{(1)} \rangle = \delta_{ij} = \langle x_i^{(2)}, y_j^{(2)} \rangle.$$

Define σ by the condition $\sigma(x_n^{(2)}) = x_n^{(1)}$; then σ is an automorphism of F, and $\sigma^*(y_n^{(1)}) = y_n^{(2)}$ for all n. Hence $\sigma^*(F_1) = F_2$, completing the proof.

4. An example

Throughout this section R will be a principal ideal domain with an infinite number of primes, and P will be the direct product of a countably infinite family of copies of R. As usual, we may view P as the dual of a free R-module F of countably infinite rank.

This section was motivated in part by the following considerations. We shall say that a torsion-free R-module A satisfies the Density Condition if it possesses the following property [10, p. 249]: If B is any pure submodule of A, and B_1/B is the maximal divisible submodule of A/B, then $\text{rank}(B_1) = \text{rank}(B)$. Rotman [10, p. 252] has in essence raised the question: If A is a pure submodule of P satisfying the Density Condition, then is A free? We shall provide a negative answer to this question by constructing a pure nonfree submodule A of P possessing the following property which is easily seen to be stronger than the Density Condition: If p is any prime in R, then A_p is a free module over the discrete valuation ring R_p.

In the construction of A, we shall have use for a certain type of torsion-free R-module of rank one. Let $\Pi = \{p_1, p_2, \cdots\}$ be an infinite sequence of distinct primes in R, and define

$$I = I(\Pi) = \bigcup_{n=1}^{\infty} Rp_1^{-1} \cdots p_n^{-1}.$$

$I(\Pi)$ is easily seen to be a submodule of Q (and therefore a torsion-free R-module of rank one), and $I_p \approx R_p$ as R_p-module for any prime p in R. However, I is not isomorphic to R.

We shall build A by an inductive process, the essential step of which is provided by the following lemma.

LEMMA 4.1. *Let F_1 be a pure free submodule of P of countable rank which is dense in P in the F-topology. Let $I = I(\Pi)$, where $\Pi = \{p_1, p_2, \cdots\}$ is an infinite sequence of distinct primes in R. Then there exists a pure free submodule F_2 of P such that $F_1 \subseteq F_2$ and $F_2/F_1 \approx I$. F_2 is also dense in P in the F-topology.*

Proof. By Theorem 3.2, there exist bases $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$ of F and F_1, respectively, such that $\langle x_i, y_j \rangle = \delta_{ij}$. Define F_2 to be the subset of P consisting of all $y \in P$ satisfying the following condition: There exist $a \in R$ and an integer $k \geqq 1$ (both depending on y) such that $\langle x_n, y \rangle = ap_{k+1} \cdots p_n$ for large n. In addition, define a mapping $f : F_2 \to I$ by $f(y) = ap_1^{-1} \cdots p_k^{-1}$ (where y is as just described). We shall omit the routine verification that F_2 is a submodule of P containing F_1, and f is an epimorphism of F_2 onto I with kernel F_1. Of course, F_2 is F-dense in P, since F_1 is.

We now show that F_2 is a pure submodule of P. Let $y \in P$, and assume $py \in F_2$, where p is a prime in R. Then there exist $a \in R$ and $k \geqq 1$ such that $p\langle x_n, y \rangle = \langle x_n, py \rangle = ap_{k+1} \cdots p_n$ for large n. Either p/a or $p = p_i$ for some $i > k$. In the first case, we may write $a = pa'$ and let y' be the unique element of P such that $\langle x_n, y' \rangle = a'p_{k-1} \cdots p_n$ for all $n \geqq 1$; then $y' \in F_2$. Set $z = p(y - y')$; then $\langle x_n, z \rangle = 0$ for large n, from which we obtain by a standard argument that $z \in F_1$. Since F_1 is, by hypothesis, a pure submodule of P, we then have that $y - y' \in F_2$, and so $y \in F_2$.

Suppose, on the other hand, that $p = p_i$ for some $i > k$. Let y'' be the unique element of P such that $\langle x_n, y'' \rangle = 0$ for $n \leqq i$ and $\langle x_n, y'' \rangle = p_{i+1} \cdots p_n$ for $n > i$; then $y'' \in F_2$. If $b = p_1 p_2 \cdots p_{i-1}$, then $p(y - by'') \in F_1$, and so $y - by''$ is in F_1, since F_1 is pure in P. Then $y \in F_2$. We have shown that, if $py \in F_2$, where $y \in P$ and p is any prime in R, then $y \in F_2$. From this fact it follows easily that F_2 is pure in P, completing the proof of the lemma.

We now select an infinite sequence $\Pi = \{p_1, p_2, \cdots\}$ of distinct primes in R, and set $I = I(\Pi)$. For each ordinal α less than the first uncountable ordinal Ω, we shall construct a pure submodule $F^{(\alpha)}$ of P such that the following conditions are satisfied: (i) $F^{(\alpha)}$ is free of countable rank, and is dense in P in the F-topology, (ii) $F^{(\alpha)} \subseteq F^{(\alpha+1)}$, and $F^{(\alpha+1)}/F^{(\alpha)} \approx I$. Let $F^{(1)}$ be any pure free submodule of P of countable rank which is F-dense in P. If all $F^{(\beta)}$ for $\beta < \alpha$ have been constructed, set $F^{(\alpha)} = \bigcup_{\beta<\alpha} F^{(\beta)}$ if α is a limit ordinal; otherwise, if $\alpha = \beta + 1$, obtain $F^{\alpha)}$ via Lemma 4.1. Finally, set $A = \bigcup_{\alpha<\Omega} F^{(\alpha)}$.

THEOREM 4.2. *The R-module A just constructed possesses the following properties*:

(a) *A is a pure submodule of P, and is hence locally free.*

(b) *A is not free*; *in fact, if the Continuum Hypothesis holds and R is countable, then* $\operatorname{Ext}(A, R) \neq 0$.

(c) *If p is any prime in R, then A_p is a free module over the discrete valuation ring R_p.*

(d) *A satisfies the Density Condition.*

(e) $\mathrm{Ext}(A, C) = 0$ *for any primary R-module C.*

Proof. (a) Let $x \in P$, and suppose $ax \in A$, where $a \neq 0$ is in R. Then $ax \in F^{(\alpha)}$ for some α. Since $F^{(\alpha)}$ is a pure submodule of P and $a \neq 0$, we obtain that $x \in F^{(\alpha)}$, and so $x \in A$. It then follows that A is pure in P. That A is locally free then follows from Proposition 2.1 of [2] and the fact that $P = F^*$.

(b) We first show that, if $y \in A^*$ and $\langle F^{(1)}, y \rangle = 0$, then $y = 0$. Suppose on the contrary that $y \neq 0$, and select the least α such that $\langle F^{(\alpha)}, y \rangle \neq 0$. Then $\alpha > 1$, and it is clear that α cannot be a limit ordinal; hence $\alpha = \beta + 1$ for some $\beta < \Omega$. $\langle F^{(\beta)}, y \rangle = 0$, and so y gives rise to a nontrivial homomorphism z of $F^{(\alpha)}/F^{(\beta)}$ into R. But $F^{(\alpha)}/F^{(\beta)} \approx I$ has rank one, and so we see that z is a monomorphism of I into R. Since R is a principal ideal domain, it follows that $I \approx R$, a contradiction. Thus $y = 0$.

Assume now that A is free. Since A has uncountable rank and $F^{(1)}$ has countable rank, we see that $A = A_1 \oplus A_2$, where $F^{(1)} \subseteq A_1$ and $A_2 \neq 0$. Since A_2 is free, there exists a nontrivial homomorphism y of A_2 into R which may be extended to A by setting $y = 0$ on A_1. Then $y \neq 0$ but $\langle F^{(1)}, y \rangle = 0$, contradicting the preceding paragraph. It then follows that A is not free.

Now, setting $B = F^{(1)}$, we obtain easily from our previous remarks that the mapping $i_{A/B} : A/B \to (A/B)^{**}$ is trivial. Suppose that R is countable and $\mathrm{Ext}(A, R) = 0$. Then, assuming the Continuum Hypothesis and keeping in mind the fact that $\ker(i_{A/B}) = A/B$, we could apply Theorem 4.6 of [2] to conclude that $\mathrm{rank}(A) = \mathrm{rank}(B) = \aleph_0$, a contradiction. Hence[3] $\mathrm{Ext}(A, R) \neq 0$.

(c) Since localization preserves exact sequences, we have that

$$F_p^{(\alpha)} \subseteq F_p^{(\alpha+1)} \quad \text{and} \quad F_p^{(\alpha+1)}/F_p^{(\alpha)} \approx I_p \approx R_p$$

for any prime p in R, from which it follows that $F_p^{(\alpha)}$ is a direct summand of $F_p^{(\alpha+1)}$. Also, if α is a limit ordinal, $F_p^{(\alpha)} = \bigcup_{\beta<\alpha} F_p^{(\beta)}$. Of course, each $F_p^{(\alpha)}$ is a free R_p-module. Let $y_1, y_2, \cdots$ be a basis of $F_p^{(1)}$, and write $F_p^{(\alpha+1)} = F_p^{(\alpha)} \oplus R_p x_\alpha$, where $x_\alpha \in F_p^{(\alpha+1)}$; then it follows easily from a standard argument that the subset of A_p consisting of all y_n together with all x_α is a free R_p-basis of A_p. Thus A_p is a free R_p-module for any prime p in R.

(d) We shall show that A possesses the following property, which is stronger than the Density Condition: If $B \subseteq A$, and $B(p)/B$ is the maximal p-divisible submodule of A/B (where p is any prime in R), then

$$\mathrm{rank}(B(p)) = \mathrm{rank}(B).$$

[3] One may also deduce this fact from a special case of Theorem 4.6 of [2] which is much easier to prove. However, this proof also uses the Continuum Hypothesis.

For, since A_p is a free R_p-module, we may write $A_p = A_1 \oplus A_2$, where both summands are R_p-free, $B \subseteq A_1$, and $\text{rank}(A_1) = \text{rank}(B)$. Let $g : A \rightarrow A_2$ be the R-homomorphism which is the composition of the inclusion mapping of A into A_p with the projection of A_p onto A_2. Since A_2 is R_p-free, it is p-reduced both as R_p-module and as R-module. Since $g(B) = 0$, we obtain that $g(B(p)) = 0$, and so $B(p) \subseteq A_1$. Hence $\text{rank}(B(p)) = \text{rank}(A_1) = \text{rank}(B)$, completing the proof.

(e) Let p be a prime in R, and C a p-primary R-module. Then $C_p = C$; hence, if $0 \rightarrow C \rightarrow E \rightarrow A \rightarrow 0$ is an exact sequence of R-modules, we obtain by localization the exact sequence $0 \rightarrow C \rightarrow E_p \rightarrow A_p \rightarrow 0$ of R_p-modules. Since A_p is R_p-free, the second exact sequence splits, and this leads immediately to a splitting of the original sequence. We may then apply the discussion of the introduction to [7] to conclude that $\text{Ext}(A, C) = 0$. Thus (e) holds, and the proof of the theorem is complete.

We do not know whether $\text{Ext}(A, C) = 0$ for any torsion module C.

One can, with somewhat greater effort, use the method of this section to construct a pure submodule A of P which, in addition to satisfying the conditions of Theorem 4.2, is such that A^* has countable rank if the Continuum Hypothesis holds. In this case A satisfies the following stronger form of Theorem 4.2(b): A possesses no free direct summands of infinite rank.

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