CHARACTERIZATION OF THE THREE CLASSICAL PLANE GEOMETRIES

BY

HELMUT SALZMANN

The projective forms of the three classical plane geometries (the euclidean, elliptic, and hyperbolic plane) are topological planes, i.e., the set of points and the set of lines are topological spaces (the topology being uniquely determined by the order relation in the plane), and the operations of joining and intersecting are continuous (in both variables). A (not necessarily arguesian) topological projective plane is called "flat", if the point set P is a locally compact topological space of the topological dimension 2. In a flat plane $\mathbf{P} = (P, \mathfrak{L})$ both the set P of points and the set $\mathfrak{L}$ of lines are homeomorphic to the point set P_2 of the real projective plane [10, Satz P]. The dual $(\mathfrak{L}, P)$ of the flat plane $\mathbf{P}$ is, therefore, likewise a flat plane. Each quadrangle of such a flat plane generates an everywhere dense subplane [9]. Furthermore all collineations are continuous [9], and these two facts imply that the only collineation leaving any quadrangle fixed is the identity. It turns out that for the group $\Gamma = \Gamma_{\mathbf{P}}$ of all collineations of a flat plane $\mathbf{P}$ the topologies of uniform convergence and of convergence on an arbitrary fixed quadrangle coincide with the topology of pointwise convergence. In this "natural" topology the group Γ becomes a Lie group of dimension at most 8; see [11, Sätze 3.1–3.4 and 4.1]. If the flat plane $\mathbf{P}$ is not arguesian, then any nonsimple connected closed subgroup of $\Gamma_{\mathbf{P}}$ has a fixed point and, by duality, a fixed line [13, Satz 1.3].

The flat planes with a collineation group of dimension not less than 3 have been determined completely [12]–[14]: The only planes with a collineation group of dimension ≥ 4 are the Moulton planes [7], [5, §23]. There are three classes of flat planes with 3-dimensional collineation group, each class containing continuously many different planes. The planes of the first class generalize the Moulton planes and have a collineation group leaving exactly two points and two lines fixed. The second class consists of planes with one and only one flag (incident point-line-pair) fixed by all collineations. The planes of the third class have a simple collineation group isomorphic to the group Ω of proper (= even) hyperbolic motions, i.e., the commutator subgroup $\Omega_3(R, f)$ of the orthogonal group with respect to a symmetric bilinear form f of index 1 over the reals [3]. If the group Ω operates as a group of collineations of a flat plane $\mathbf{P} = (P, \mathfrak{L})$, possibly as a subgroup of the full collineation group Γ, then Ω leaves no point and no line fixed, and there are three domains of transitivity in the point set P, namely a simple closed curve K, its interior H, and its exterior. The points of H and the line segments

---

$L \cap H$ with $L \in \mathfrak{L}$ form an incidence structure $\mathbf{H}$ isomorphic to the ordinary hyperbolic plane, but the projective extension $\mathbf{P}$ of $\mathbf{H}$ need not be arguesian. For a detailed description of these "not necessarily arguesian hyperbolic projective planes" see [12, Sätze 14, 17, and 18]. The plane $\mathbf{P}$ is arguesian and therefore isomorphic to the real projective plane if and only if $\Omega < \Gamma$ [12, Satz 15].

Using these results we shall now characterize the real euclidean, elliptic, and hyperbolic plane in a similar fashion:

THEOREM. *A pair* $(\mathbf{P}, \Delta)$ *consisting of a flat plane* $\mathbf{P} = (P, \mathfrak{L})$ *and a "motion group"* Δ *satisfying the following conditions:*

   (a) Δ *is a 3-dimensional connected closed subgroup of the collineation group* $\Gamma_{\mathbf{P}}$ *in the natural topology,*

   (b) Δ *is not simply connected,*

   (c) Δ *leaves no point of* $\mathbf{P}$ *fixed,*

is a classical plane geometry; more precisely,

   (1)   *If* Δ *is not simple, then* Δ *fixes a line of* $\mathbf{P}$, *and* $(\mathbf{P}, \Delta)$ *is the euclidean plane,* Δ *being the group of its proper motions.*

   (2)   *If* Δ *is simple and compact, then* Δ *is flag-transitive on* $\mathbf{P}$, *and* $(\mathbf{P}, \Delta)$ *is the elliptic plane.*

   (3)   *If* Δ *is simple and not compact, then* $(\mathbf{P}, \Delta)$ *is a not necessarily arguesian hyperbolic projective plane, i.e.,* Δ *fixes a simple closed curve* K *in* P, *the interior* $\mathbf{H}$ *of* K *is the real hyperbolic plane, and* Δ *is its group of proper motions.* $\mathbf{P}$ *is arguesian if and only if* Δ *has index 2 in its normalizer* Θ *in* $\Gamma_{\mathbf{P}}$, *and* Θ *is then the full group of hyperbolic motions.*

A related characterization of the euclidean and hyperbolic plane has been given by Hilbert [5, Anhang IV] in his solution of the plane case of the Riemann-Helmholtz-Lie problem; see also Süss [15].

For the proof of the theorem we need the following

LEMMA. *A connected group* Φ *of collineations of a flat plane* $\mathbf{P}$ *either fixes a point of* $\mathbf{P}$ *or has trivial center* $Z(\Phi) = 1$.

Proof. Let $\zeta \neq 1$ be an element of the center $Z(\Phi)$. Because of the continuity of all collineations, ζ is a homeomorphism of the point set P. Therefore ζ has a fixed point p; see e.g. [4, p. 373]. The transitivity domain p^{Φ} is a connected subset of P consisting only of fixed points of ζ, for Φ is a connected topological transformation group of P and $p^{\alpha \zeta} = p^{\zeta \alpha} = p^{\alpha}$ for all $\alpha \in \Phi$. A connected set containing three points not on a line contains also a quadrangle. Consequently p^{Φ} is contained in a line L, since ζ cannot leave any quadrangle fixed. Either $p^{\Phi} = p$ is a fixed point of Φ, or the line L is uniquely determined and x^{Φ} is a connected subset of L for each $x \in L$. A homogeneous connected proper subset of L is a point or an open interval. Hence either there exists a point $x \in L$ with $x^{\Phi} = x$, or $p^{\Phi} = L$. But if Φ is

transitive on L, then ζ is a perspectivity with axis L and some center a, and $\zeta^\Phi = \zeta$ implies $a^\Phi = a$.

Proof of the theorem. Let Δ be a group satisfying conditions (a) and (c). Then Δ, being a closed subgroup of $\Gamma_\mathbf{P}$, is a 3-dimensional Lie group. By (c) and the lemma, $Z(\Delta) = 1$. The commutator subgroup Δ' of Δ is a connected normal subgroup $\neq 1$. By using Lie algebra methods it is easy to see that either Δ' is commutative, or $\Delta' = \Delta$; see [6, pp. 11–14]. In the latter case the Lie algebra of Δ is simple; hence every proper normal subgroup of Δ is discrete and contained in the center $Z(\Delta) = 1$, and Δ is isomorphic to one of the 3-dimensional simple groups SO_3 or Ω, both of which are not simply connected.

(1) If the motion group Δ is not simple, it has, therefore, a commutative, connected normal subgroup $\Delta' \neq 1$. By the lemma, Δ' fixes a point p of $\mathbf{P}$. Since Δ leaves no point fixed and $p^{\alpha\Delta'} = p^{\Delta'\alpha} = p^\alpha$ for each $\alpha \epsilon \Delta$, an argument completely analogous to the proof of the lemma shows that p^Δ is contained in a unique line L fixed by Δ, that Δ is transitive on L, and Δ' is a group of perspectivities with axis L. The group Δ' cannot contain any homology $\eta \neq 1$; for if $1 \neq \eta \epsilon \Delta'$ and $a^\eta = a \notin L$, then there is an element $\delta \epsilon \Delta$ with $a^\delta \neq a$ and $\eta^{\eta^\delta} \neq \eta$, which contradicts the fact that Δ' is a commutative normal subgroup of Δ. Hence Δ' is a group of translations with axis L. Let Σ be a one-parameter subgroup of Δ'. As Σ contains locally cyclic everywhere dense subgroups, e.g. the rationals, all the elements of Σ have the same center $a \epsilon L$, and Σ is the transitive translation group with center a; see [14, Hilfssatz 3]. The transitivity of Δ on L implies now that the translation group $\mathbf{T} = \Delta'$ is transitive on the affine plane $\mathbf{P} - L$, and $\mathbf{P}$ is arguesian by [8, §12]. The stability subgroup Δ_o of Δ fixing some point $o \notin L$ is then a complement of $\mathbf{T}$ in Δ. Since $\mathbf{T}$ is 2-dimensional, the group Δ_o is by (a) a connected 1-dimensional Lie group that is transitive on L. In the real plane there are continuously many nonisomorphic 3-dimensional connected collineation groups containing the group $\mathbf{T}$ of translations with axis L as a normal subgroup such that a complement of $\mathbf{T}$ is transitive on L, namely the groups

$$\Delta(r) = \left\{ \begin{pmatrix} 1 & x & y \\ & p_t & q_t \\ & -q_t & p_t \end{pmatrix}; t, x, y \epsilon R \right\} \quad \text{with} \quad \begin{cases} p_t = r^t \cos t \\ q_t = r^t \sin t \end{cases}.$$

All of these groups except $\Delta(1)$ are simply connected. Thus condition (b) which had not been assumed before, implies that $\Delta \cong \Delta(1)$ is the group of orientation-preserving euclidean motions.

(2) If the motion group Δ is simple and compact, it is isomorphic to the rotation group SO_3, i.e., the group of elliptic motions. Let Σ be a 1-dimensional connected subgroup of Δ. Then $\Sigma \cong SO_2$, and Σ contains one and

only one involution σ. Since σ cannot leave any quadrangle fixed, it is a reflection with some axis W and center $a \notin W$, and σ is uniquely determined by a and W; see [11, Folgerung 2.4]. By a theorem of Baer [2, p. 103, Lemma 1], the product of two reflections with the same axis and different centers is a translation. Because every translation of a flat plane generates an infinite discrete group of collineations, the compact group Δ cannot contain more than one reflection with given axis or given center. The orbits of Σ form a fibration of the point set $D = \{x \in P; a \neq x \notin W\}$, and any open half-line with end-point a is a cross-section for this fibration; see [12, Lemma 6]. Hence the transitivity domain a^Δ is either the whole plane P or the point a or an open or closed disk, because the connected set a^Δ is a union of orbits of Σ. If $a^\Delta = a$, then $\sigma^\Delta = \sigma$, which contradicts the simplicity of Δ. Since a^Δ is a compact homogeneous subset of P, it can be neither an open nor a closed disk. Therefore Δ is transitive on the point set P and dually also on the set of lines. Thus every point p and every line L of **P** is the center or the axis of a unique reflection $\alpha(p)$, respectively $\beta(L)$, of Δ, and $p \in L$ if and only if $\alpha(p)\beta(L)$ is an involution. This proves that **P** is isomorphic to the real elliptic plane; see Bachmann [1, §§16, 17]; also [11, Satz 4.3].

(3)   If the motion group Δ is simple and not compact, it is isomorphic to the group $\Omega \cong PSL_2(R)$. As mentioned already in the introduction, this case has been discussed completely in [12].

References

1. F. BACHMANN, *Aufbau der Geometrie aus dem Spiegelungsbegriff*, Berlin-Göttingen-Heidelberg, Springer, 1959.
2. R. BAER, *The fundamental theorems of elementary geometry. An axiomatic analysis*, Trans. Amer. Math. Soc., vol. 56 (1944), pp. 94–129.
3. J. DIEUDONNÉ, *La géométrie des groupes classiques*, Berlin-Göttingen-Heidelberg, Springer, 1955.
4. G. FEIGL, *Fixpunktsätze für spezielle n-dimensionale Mannigfaltigkeiten*, Math. Ann., Bd. 98 (1928), pp. 355–398.
5. D. HILBERT, *Grundlagen der Geometrie*, 9. Aufl., Stuttgart, 1962; Anhang IV = Math. Ann., Bd. 56 (1902), pp. 381–422.
6. N. JACOBSON, *Lie algebras*, New York, Interscience Publishers, 1962.
7. F. R. MOULTON, *A simple non-desarguesian plane geometry*, Trans. Amer. Math. Soc., vol. 3 (1902), pp. 192–195.
8. H. SALZMANN, *Topologische projektive Ebenen*, Math. Zeitschrift, Bd. 67 (1957), pp. 436–466.
9. ———, *Homomorphismen topologischer projektiver Ebenen*, Arch. Math., Bd. 10 (1959), pp. 51–55.
10. ———, *Topologische Struktur zweidimensionaler projektiver Ebenen*, Math. Zeitschrift, Bd. 71 (1959), pp. 408–413.
11. ———, *Kompakte zweidimensionale projektive Ebenen*, Math. Ann., Bd. 145 (1962), pp. 401–428.
12. ———, *Kompakte Ebenen mit einfacher Kollineationsgruppe*, Arch. Math., Bd. 13 (1962), pp. 98–109.
13. ———, *Zur Klassifikation topologischer Ebenen*, Math. Ann., Bd. 150 (1963), pp. 226–241.

14. ———, *Zur Klassifikation topologischer Ebenen II*, Abh. Math. Sem. Univ. Hamburg, to appear.
15. W. Süss, *Beiträge zur gruppentheoretischen Begründung der Geometrie*, Tôhoku Math. J., vol. 26 (1926), pp. 365–385.

UNIVERSITÄT
  FRANKFURT AM MAIN, GERMANY