A DECOMPOSITION THEOREM FOR E^4

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In [3, Theorem 8], it was shown that if G is an upper semicontinuous decomposition of E^3 into continua each lying in a horizontal plane but not separating that plane, then the decomposition space associated with G is E^3. The proof of this result depended on the theorem in that paper to the effect that if f is a regular mapping (see definition below) of a complete metric space X onto a finite-dimensional, locally compact, separable and contractible metric space Y and each inverse under f is homeomorphic to a 2-sphere, M, then X is homeomorphic to $M \times Y$, f corresponding to the projection map of $M \times Y$ onto Y. This result on regular mappings has been extended to the case where M is a 3-sphere [5]. It is now possible to prove

THEOREM 1. *If π is a fixed hyperplane of E^4, and G is an upper semicontinuous decomposition of E^4 into continua such that*

(1) *each element of G lies in a hyperplane parallel to π,*

(2) *if the element g of G lies in the hyperplane π' parallel to π, then $\pi' - g$ is homeomorphic to the complement in π' of a point, and*

(3) *for each hyperplane π' parallel to π, the decomposition space associated with the subcollection of G consisting of those elements that lie in π' is E^3,*

then the decomposition space associated with G is E^4.

Remarks. Condition (3) is required because there is no theorem for E^3 analogous to Moore's theorem for E^2 [8] to the effect that the decomposition space associated with an upper semicontinuous decomposition G of E^2 is E^2 provided that each of the elements of G is a continuum that does not separate E^2. That no such theorem exists for E^3 follows from an example of Bing [2] modified by Fort [4] to obtain a decomposition of E^3 into points and polygonal arcs whose decomposition space is not even a manifold. However, Bing [1] and McAuley [7] have established some conditions on the elements that imply that the decomposition space is E^3. For instance, the space is E^3 if each element of the decomposition is a point or a convex body (Bing) or if each element is a point or a straight line interval and there exists a countable collection Z of straight line intervals such that each interval in the decomposition is parallel to some element of Z (McAuley).

That condition (2) is required is demonstrated by the example to be found following the proof of Theorem 1.

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Proof of Theorem 1. Let Y be the decomposition space associated with G and q the associated mapping of E^4 onto Y, i.e., g is an element of G if and only if there is an element y of Y such that $q^{-1}(y) = g$. Let C_1 and C_2 be concentric 3-spheres in E^4, K the set of points between C_1 and C_2, and L the common part of K and a ray terminating at the common center of C_1 and C_2. For each point y of L, denote by S_y the sphere concentric with C_1 that contains y. There is a homeomorphism t of E^4 onto $K - L$ that carries each hyperplane parallel to π onto some set $S_y - y$. The collection H such that h belongs to it if and only if either $h = t(g)$ for some element g of G or h is a point of L is an upper semicontinuous decomposition of K. Let X be the decomposition space associated with H and f the associated proper mapping of K into X (inverses of compact sets are compact). Let g denote the mapping of X onto L such that $gf(S_y) = y$ for each point y of L. Note that, by hypothesis, each inverse under g is a 3-sphere. It will be proved that X is homeomorphic to $L \times S^3$ by first proving that g is h-2-regular.

DEFINITION. The proper mapping f of a metric space X onto a metric space Y is said to be *homotopy n-regular* (h-n-regular) provided that if $\varepsilon > 0$, $y \in Y$, and $x \in f^{-1}(y)$, then there exists a $\delta > 0$ such that each mapping of a k-sphere, $k \leqq n$, into $f^{-1}(y') \cap S(x, \delta)$, $y' \in Y$, is homotopic to 0 in $f^{-1}(y') \cap S(x, \varepsilon)$, where $S(x, \varepsilon)$ denotes, as usual, the open ε-neighborhood of x.

It is proved first that g is 0-regular. Let ε denote a positive number and let p denote a point of $g^{-1}(y)$, for some point y of L, where $p = f(h)$ for some h of H. There is a positive number d, such that if x is in the d-neighborhood, V_d, of $f^{-1}(p)$ in K, then $\rho(p, f(x)) < \varepsilon$. (The letter ρ will be used consistently to denote the metric in X.) But $f^{-1}(p)$ is connected; hence $V_d \cap S_y$ is arcwise connected. (V_d is a union of open, spherical 4-cells of radius d, each intersecting $f^{-1}(p)$ and meeting S_y in an open 3-cell.) Thus there is a positive number $e < d$ such that if a, b are in $S_x \cap V_e$, where x is any point of L, then there is an arc from a to b in $S_x \cap V_d$. There is a positive number δ such that if $\rho(p, q) < \delta$, then $f^{-1}(q) \in V_e$. Thus, if q and q' are points of X in $f(S_x)$ such that $\rho(p, q) < \delta$ and $\rho(p, q') < \delta$, then there is an arc ab in $S_x \cap V_d$ from a point a of $f^{-1}(q)$ to a point b of $f^{-1}(q')$, where $a \cup b \subset V_e$. It follows from the definition of d that $f(ab)$ is in the common part of $f(S_x) = g^{-1}(x)$ and the ε-neighborhood of p in X. Since f is continuous, $f(ab)$ contains an arc with endpoints q and q' ($= f(a)$ and $f(b)$ respectively). Thus g is h-0-regular.

To see that g is h-1-regular, let p, h, and ε be as above, and consider a 4-cell Z in K such that h ($= f^{-1}(p)$) is a subset of the interior of Z, Z meets each S_y in a 3-cell or not at all, and $\rho(f(x), p) < \varepsilon$ for each x in Z. The existence of Z is a consequence of condition (2) in the statement of the theorem. There is a positive number δ such that if $p' \in X$ and $\rho(p, p') < \delta$, then $f^{-1}(p') \in Z$. Suppose that ϕ is a mapping of the 1-sphere S^1 (bounding the disc R^2) into $g^{-1}(y') \cap S(p, \delta)$ (the $S(p, \delta)$ being a δ-neighborhood in X). For each x in S^1, let T_x be an open spherical neighborhood of $\phi(x)$ in $g^{-1}(y') \cap f(Z)$. A finite subcollection, $T_{x_1}, T_{x_2}, \cdots, T_{x_n}$ covers $\phi(S^1)$. Since ϕ is ε-homotopic to a piecewise linear homeomorphism for each ε, it may be assumed that ϕ is a piecewise linear homeomorphism. Furthermore, it may be assumed that $x_1, x_2, \cdots, x_n$ lie in that order on S^1, and that there are points $c_1, c_2, \cdots, c_n$ on S^1 such that for each i, $T_{x_i} \cap \phi(S^1)$ is connected, c_i lies between x_i and x_{i+1} (addition of subscripts being taken mod n), $\phi(c_i) \in T_{x_i} \cap T_{x_{i+1}}$, and $T(x_i) \cap T(x_{i+1})$ is connected. The set $f^{-1}(T_{x_i})$ is open and connected. Thus there are points $a_1, a_2, \cdots, a_n$ of $S_{y'}$ such that for each i, $a_i \in f^{-1}(T_{x_i}) \cap f^{-1}(T_{x_{i+1}})$ and there is an arc $a_{i-1}a_i$ in $f^{-1}(T_{x_i}) \cap S_{y'}$. Let $b_1, b_2, \cdots, b_n$ denote points in that order on S^1 and let α denote a mapping of S^1 into $f^{-1}(\bigcup T_{x_i})$ carrying each arc $b_{i-1}b_i$ homeomorphically onto $a_{i-1}a_i$. The mapping α can, since $f^{-1}(\bigcup T_{x_i}) \subset Z \cap S_y'$, be extended to a mapping α^* of the 2-cell R^2 into $Z \cap S_{y'}$. Then $f\alpha^*$ is a mapping of R^2 into $g^{-1}(y') \cap f(Z)$. Consider $f\alpha(b_{i-1})$, $f\alpha(b_i)$, $\phi(c_{i-1})$, and $\phi(c_i)$. For each i, there is an arc t_{i-1} in $T_{x_{i-1}} \cap T_{x_i}$ with endpoints $f\alpha(b_{i-1})$ and $\phi(c_{i-1})$. The set $t_{i-1} \cup t_i \cup f\alpha(b_{i-1}b_i) \cup \phi(c_{i-1}c_i)$ is a closed curve that is contractible in T_{x_i}. If these n contractions are fitted to $f\alpha^*$, an extension of ϕ to a mapping of R^2 into $g^{-1}(y') \cap f(Z)$, which lies in $S(p, \varepsilon)$, is obtained. Thus g is h-1-regular.

If ϕ maps S^2, the 2-sphere, into $g^{-1}(y') \cap S(p, \delta)$ and is not homotopic to 0 in $g^{-1}(y') \cap S(p, \varepsilon)$, then the Sphere Theorem (Papakyriakopoulous [9] and Whitehead [10]) is used to obtain a nonsingular 2-sphere in $g^{-1}(y') \cap S(p, \delta)$ that is not contractible in $g^{-1}(y') \cap S(p, \varepsilon)$. An argument similar to that above could now be used to prove that g is h-2-regular. However, it follows from [5] and [6, Theorem 6.1] that since g is h-1-regular, it is h-2-regular.

It now follows from the remarks in the opening paragraph that X is homeomorphic to $L \times S^3$ and thus, from the construction, that Y is homeomorphic to $K - L$ and, consequently, to E^4.

Example. Let T^* be a torus bounding the solid torus V^* and let g^* be a core of V^* (i.e., g^* is a simple closed curve in int V^* and V^* is a union of two 3-cells meeting in two disjoint discs such that each disc meets g^* in a point and each 3-cell meets g^* in an unknotted arc). Let h_1^* be a latitudinal simple closed curve on T^* that together with g^* bounds an annulus A^* in V^* that meets T^* only in h_1^*. Let h_2^* be a meridian simple closed curve on T^* bounding a disc D^* in V^* that meets g^* in a point, T^* in h_2^*, A^* in an arc, and h_1^* in a point. Each of these sets should be polyhedral with respect to some triangulation of V^*. Denote $h_1^* \cup h_2^*$ by h^*. There is a homeomorphism ϕ of $T^* \times [0, 1)$ (note the half-open interval) onto $V^* - g^*$ such that $\phi(x, 0) = x$, $\phi(h_1^*, t) \subset A^*$, $\phi(h_2^*, t) \subset D^*$, and that can be extended to a mapping ϕ^* of $T^* \times [0, 1]$ onto V^* such that $\phi^*(T^*, 1) = g^*$, $\phi^* \mid h_1^* \times 1$ is a homeomorphism, and $\phi^*(h_2^*, 1)$ is a point. In particular, considering T^* as $h_1^* \times h_2^*$, ϕ^* carries each $h_1^* \times x \times 1$ homeomorphically onto g^* and each $y \times h_2^* \times 1$ onto a point. Let H^* be the decomposition of V^* whose elements are g^*, each $\phi(h^*, t)$ for $0 \leqq t < 1$, and the remaining points of V^*. Then H^* is an upper semicontinuous decomposition of V^* and the associated decomposition space is a 3-cell. (Note that the decomposition of T^* whose elements are h^* and the points of $T^* - h^*$ has a 2-sphere as its associated decomposition space.)

Let V^{**} be a copy of V^* bounded by T^{**} and H^{**} the decomposition of V^{**} corresponding to H^*. Sew V^* and V^{**} together along their boundaries, sewing h_1^* to h_2^{**} and h_1^{**} to h_2^*. In this way a 3-sphere, S', is obtained with a decomposition H' whose decomposition space is also a 3-sphere (the two 3-cells, H^* and H^{**}, are sewed together along their boundaries to yield H'). If a degenerate element of H' is removed from S', a decomposition H of E^3 is obtained whose decomposition space is E^3 but each of whose nondegenerate elements has a complement in E^3 that is not simply connected.[2]

Now consider E^4 as $E^3 \times E^1$ and let G be a decomposition of E^4 whose elements are the points of $E^4 - (E^3 \times 0)$ and the continua $h \times 0$ for h in H. Suppose that the decomposition space associated with G is E^4. It will be proved that this assumption leads to a contradiction. Let f be the mapping of E^4 onto E^4 associated with G, i.e., the point inverses under f are the elements of G. The subset K of E^4 consisting of those points whose inverses under f are nondegenerate is an arc.

Let U_1 be a regular neighborhood in E^4 of a figure-eight element g of G such that U_1 contains neither of the simple closed curve elements of G but contains each element of G that it intersects. (I.e., g is a strong deformation retract of U_1.) The set U_1 may be considered as the union of two sets each of which is the topological product of a circle and an open 3-cell and whose intersection is an open 4-cell. Then $U_1^* = f(U_1)$ is an open neighborhood of $f(g)$ in E^4. Let V be a neighborhood of g such that $\bar{V} \subset U_1$, $f^{-1}(f(V)) = V$, $f(V)$ is an open 4-cell, and $f(\bar{V})$ is a 4-cell. Let U_2 be a regular neighborhood of g, as above, such that $f^{-1}(f(U_2)) = U_2$ and $U_2 \subset V$.

There is a simple closed curve C in $f(U_2) - (K \cap f(U_2))$ that fails to bound (homologically mod the integers) in $f(U_1) - (f(U_1) \cap K)$. (The curve $f^{-1}(C)$ may be constructed by looping around the common part of U_2 and some $E^3 \times t$, $t \neq 0$, which is possible by the construction of U_2.) However, $f(V)$ is a 4-cell and K is an arc, so it follows from the Alexander duality theorem that C does bound in $f(V) - (K \cap f(V))$. This contradiction implies that the decomposition space associated with G is not E^4, and thus that condition (2) may not be completely removed from the hypotheses of Theorem 1.

In fact, going back to the 3-dimensional case of Theorem 1, we can state the following.

[2] This example has also been described by Bing. See page 6 of *Topology of 3-manifolds*, M. K. Fort, Jr., editor, Englewood Cliffs, N. J., Prentice-Hall, 1962.

THEOREM 2. *If G is an upper semicontinuous decomposition of E^3 into continua each of which lies in a horizontal plane, then in order that the decomposition space associated with G be E^3 it is necessary that no element of G separate the horizontal plane in which it lies.*

Proof. Suppose that the decomposition space is E^3 and denote by f the mapping of E^3 onto itself whose point inverses are the elements of G. If an element g of G separates the horizontal plane π, it follows from the theorem of R. L. Moore [8] on decompositions of the plane that either (1) $f(\pi)$ is the union of an open disc and certain 2-spheres no one of which intersects the disc in more than one point, or (2) $f(\pi)$ contains an arc each noncut-point of which is an interior point of the arc relative to $f(\pi)$. If (2) holds, then an arc locally separates E^3; if (1) holds, then $f(\pi)$ separates E^3 into more than two components. Each of these situations is an obvious contradiction. Thus g fails to separate π.

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