THE EXISTENCE AND APPLICATIONS OF ANTICOMMUTATIVE COCHAIN ALGEBRAS[1]

BY

DONALD W. KAHN

The algebra of differential forms on a smooth manifold forms an anticommutative cochain algebra which has been studied for many years. As is shown by the theorems of Borel and Chevalley (see Chapter 2), such algebras have direct application in topology. Other applications, such as to symmetric spaces, as well as to topics directly related to differential geometry, are now well-known.

The purpose of this paper is to construct an anticommutative cochain algebra, over the real numbers, for any countable simplicial complex. This algebra has the expected properties with respect to maps, and the derived algebra is isomorphic to the real, singular cohomology algebra. In the second chapter, I use this algebra to give an extension of Borel's theorem.

It is easy to see that anticommutative cochains show the existence of new secondary cohomology operations, which are similar to the triple product of Massey. These operations will be the subject of a subsequent paper.

I would like to thank W. S. Massey and N. Stein for their help and encouragement in connection with this work.

I. THE COCHAIN ALGEBRA

The purpose of this chapter is to describe the construction of an anticommutative cochain algebra with real coefficients for a countable simplicial complex. After a preliminary algebraic section, the construction is described in Section 2. In Section 3, I show that the derived algebra of this cochain algebra is isomorphic to the singular cohomology algebra of the space of the complex.

1. Algebraic preliminaries

DEFINITION 1.1. A *real DG-algebra* will mean a graded, associative algebra over the real numbers, endowed with a unit, 1, and a differential, d, of degree 1.

We denote the degree of an element by a superscript.

In order to study systems of real DG-algebras, we introduce some notions which, in the case of groups, are discussed in [7]. Let M be a directed set. Then, there is the notion of an inverse family of real DG-algebras over M, in which we require that the projection maps be degree-preserving algebra

Received January 17, 1962.

[1] This work was partially supported by a National Science Foundation Predoctoral Fellowship, and by a contract with the Office of Naval Research.

homomorphisms, which commute with the differential. Given any set of real DG-algebras, in particular an inverse family, one may define their direct product, with the obvious coordinatewise multiplication and differentiation. The inverse limit may then be defined as a subalgebra of the direct product, and clearly has the structure of a real DG-algebra.

I shall use the following notations:

$\{\mathcal{G}_\alpha ; \pi^\beta_\alpha\}$	is the inverse family of real DG-algebras, $\mathcal{G}_\alpha$, with projection maps, π^β_α, defined whenever $\alpha < \beta$.
$\text{inv lim}_\alpha \mathcal{G}_\alpha = \mathcal{G}_\infty$	is the inverse limit algebra.
$d_\alpha, \quad d_\infty$	are the differentials on $\mathcal{G}_\alpha$ and $\mathcal{G}_\infty$, respectively.
$d^n_\alpha, \quad d^n_\infty$	shall mean $d_\alpha \mid \mathcal{G}^n_\alpha$ and $d_\infty \mid \mathcal{G}^n_\infty$, respectively.

$$H^n_\alpha = \text{Ker } d^n_\alpha / \text{Im } d^{n-1}_\alpha, \qquad H^n_\infty = \text{Ker } d^n_\infty / \text{Im } d^{n-1}_\infty,$$

$$H_\alpha = \textstyle\sum_n H^n_\alpha, \qquad H_\infty = \textstyle\sum_n H^n_\infty.$$

The following lemma is immediate.

LEMMA 1.1. *Let $\{\mathcal{G}_\alpha ; \pi^\beta_\alpha\}$ be an inverse system of real DG-algebras, over a directed set M. Then, the collection of derived groups and induced maps, $\{H_\alpha, (\pi^\beta_\alpha)^*\}$, is an inverse system of real, graded algebras, with multiplication induced from the multiplication in the $\mathcal{G}_\alpha$.*

In standard fashion, we define the algebra $\sum_n \text{inv lim}_\alpha H^n_\alpha$. For what follows in Sections 2 and 3, we must consider the relation between $\sum_n \text{inv lim}_\alpha H^n_\alpha$ and H_∞. If all of the algebras which we have mentioned above were finite-dimensional vector spaces, then Theorem 6.1 of VIII in [7] would imply that these two were isomorphic. Unfortunately, we cannot restrict attention to the case of finite-dimensional vector spaces. The following lemmas and theorem discuss the relation between these two algebras.

LEMMA 1.2. *Let $\{x_\alpha\}$ stand for a class in H_α, represented by x_α in $\mathcal{G}_\alpha$. Write an element x in $\mathcal{G}_\infty$ by (x_α), indicating that the α^{th} coordinate is x_α. Define a map*

$$F : H_\infty \to \textstyle\sum_n \text{inv lim}_\alpha H^n_\alpha \quad \text{by} \quad F\{(x_\alpha)\} = (\{x_\alpha\}),$$

where F acting on an element is written as F followed by that element. Then F is a natural (algebra) monomorphism.

Proof. F is clearly well-defined because if $\pi^\beta_\alpha x_\beta = x_\alpha$, then $(\pi^\beta_\alpha)^*\{x_\beta\} = \{x_\alpha\}$. F is obviously an algebra homomorphism, and is easily seen to have zero kernel.

For completeness, I shall show that F is natural.[2] Let $\bar{\mathcal{G}}_\alpha$, $\bar{H}_\alpha$, etc. be another such system. Suppose that we are given, for each α, an algebra homomorphism $f_\alpha : \mathcal{G}_\alpha \to \bar{\mathcal{G}}_\alpha$ which preserves all structures. Then $f = (f_\alpha)$

[2] In the remainder of the paper, such verifications will be left to the reader.

induces a map $\bar{f}: \text{inv} \lim_\alpha H^n_\alpha \to \text{inv} \lim_\alpha \bar{H}^n_\alpha$ and a map $f_*: H^n_\infty \to \bar{H}^n_\infty$, both of which clearly extend to algebra homomorphisms of the direct sums.

That F is natural means that the following diagram is commutative:

$$\begin{array}{ccc} H_\infty & \xrightarrow{f_*} & \bar{H}_\infty \\ \downarrow F & & \downarrow F \\ \sum_n \text{inv} \lim_\alpha H^n_\alpha & \xrightarrow{\bar{f}} & \sum_n \text{inv} \lim_\alpha \bar{H}^n_\alpha . \end{array}$$

But this is immediate, since $Ff_*\{x\} = Ff_*\{(x_\alpha)\} = F\{(\bar{x}_\alpha)\} = (\{\bar{x}_\alpha\})$; $\bar{f}F\{(x_\alpha)\} = \bar{f}(\{x_\alpha\}) = (\{\bar{x}_\alpha\})$, also. This completes the proof of the lemma.

Remark. I do not know whether F is onto, in general. I shall now give conditions on the directed set M, and on the maps π^β_α which will insure that F is onto. These conditions will be sufficient for the applications in later sections.

LEMMA 1.3. *Let M be a directed set which is countable and (hence) has a cofinal sequence. Consider an inverse family of real DG-algebras over M, and assume that all the projection maps π^β_α are onto. Suppose further that M has a least element. Then the map F of the previous lemma is onto.*

Proof. We must show that F is onto. That is, if $(y_\alpha) \in \text{inv} \lim_\alpha H^n_\alpha$, y_α being a cohomology class of H^n_α, we must find, for each α, an element x_α, $x_\alpha \in \text{Ker}\, d^n_\alpha$, such that $\{x_\alpha\} = y_\alpha$ and if $\alpha < \beta$, then $\pi^\beta_\alpha x_\beta = x_\alpha$. We proceed to construct (x_α) satisfying these conditions.

Let $\mu \in M$ be the least element. Write the cofinal sequence

$$\zeta_1 < \zeta_2 < \cdots .$$

We may clearly extend this sequence to a maximal cofinal sequence, which we shall write as $\mu < \mu_1 < \mu_2 < \cdots$.

The proof has two parts. First, I shall show how to construct the elements x_{μ_i}, corresponding to the maximal cofinal sequence. Choose $x_\mu \in y_\mu$, i.e., x_μ is a representative for y_μ. Now, $(\pi^{\mu_1}_\mu)^* y_{\mu_1} = y_\mu$. Hence, there are u_{μ_1} and u_μ, such that $\{u_{\mu_1}\} = y_{\mu_1}$, $\{u_\mu\} = y_\mu$, and $\pi^{\mu_1}_\mu u_{\mu_1} = u_\mu$. Choose w so that $x_\mu = u_\mu + dw$. Because $\pi^{\mu_1}_\mu$ is onto, we may pick w', $\pi^{\mu_1}_\mu w' = w$. Now, $\pi^{\mu_1}_\mu(u_{\mu_1} + dw') = u_\mu + \pi^{\mu_1}_\mu dw' = u_\mu + dw = x_\mu$. Hence, we may define $x_{\mu_1} = u_{\mu_1} + dw'$, and the relation is satisfied, i.e., $\pi^{\mu_1}_\mu x_{\mu_1} = x_\mu$.

As the maximal cofinal sequence is countable, one may use this process to define x_{μ_i} by induction, so that the two requirements are met.

Secondly, I show how the above construction permits us to define x_α, for each $\alpha \in M$. Suppose that $\alpha \in M$, $\alpha \neq \mu, \mu_i$, for any i. Let k be the smallest integer such that $\alpha < \mu_k$. x_{μ_k} has been defined. Define $x_\alpha = \pi^{\mu_k}_\alpha x_{\mu_k}$. This defines x_α for each $\alpha \in M$. Clearly, $x_\alpha \in \text{Ker}\, d^n_\alpha$ and $\{x_\alpha\} = y_\alpha$.

To prove that if $\alpha < \beta$, then $\pi_\alpha^\beta x_\beta = x_\alpha$, I distinguish four simple cases: Assume in each of these cases that $\alpha < \beta$.

1. α and β both belong to the cofinal sequence. This case is obvious.

2. β belongs to the cofinal sequence, α does not. There is a γ such that $\alpha < \gamma < \beta$ (possibly $\gamma = \beta$), where γ belongs to the sequence and by definition $x_\alpha = \pi_\alpha^\gamma x_\gamma$. But then $x_\gamma = \pi_\gamma^\beta x_\beta$. $x_\alpha = \pi_\alpha^\gamma x_\gamma = \pi_\alpha^\gamma \pi_\gamma^\beta x_\beta = \pi_\alpha^\beta x_\beta$.

3. α belongs to the sequence, β does not. x_β is defined in terms of x_γ, where γ belongs to the cofinal sequence. Then $\pi_\alpha^\beta x_\beta = \pi_\alpha^\beta \pi_\beta^\gamma x_\gamma = \pi_\alpha^\gamma x_\gamma = x_\alpha$.

4. Neither α nor β belongs to the cofinal sequence. Suppose that the definitions are $x_\alpha = \pi_\alpha^{\alpha'} x_{\alpha'}$, $x_\beta = \pi_\beta^{\beta'} x_{\beta'}$, where α' and β' belong to the sequence, $\alpha' < \beta'$. Then $x_\alpha = \pi_\alpha^{\alpha'} x_{\alpha'} = \pi_\alpha^{\alpha'} \pi_{\alpha'}^{\beta'} x_{\beta'} = \pi_\alpha^{\beta'} x_{\beta'} = \pi_\alpha^\beta \pi_\beta^{\beta'} x_{\beta'} = \pi_\alpha^\beta x_\beta$. This completes the proof of the lemma.

THEOREM 1.1. *Let M be a directed set which is countable, and (hence) contains a cofinal sequence. Let $\{\mathcal{G}_\alpha ; \pi_\alpha^\beta\}$ be an inverse system of real DG-algebras over M, such that each π_α^β is surjective. Then*

$$F : H_\infty \to \textstyle\sum_n \operatorname{inv} \lim_\alpha H_\alpha^n$$

is an algebra isomorphism which is natural, as explained in Lemma 1.2.

Proof. We define two subsets of M as follows: Let ζ_1 be in the cofinal sequence; set $M^+ = \{\gamma \mid \gamma > \zeta_1\}$ and $M^- = M - M^+$. Notice that $M = M^+ \cup M^-$. We must define x_α for each $\alpha \in M$, satisfying conditions as in the above lemma. But by that lemma, we may do this for each element of M^+ If $\gamma \in M^-$, choose $\beta \in M^+$ so that $\gamma < \beta$. (Recall that M is a directed set.) Define $x_\gamma = \pi_\gamma^\beta x_\beta$. To verify that the definition is good, suppose $\beta' \in M^+$, $\gamma < \beta'$. Choose $\delta \in M^+$ so that $\beta < \delta$, $\beta' < \delta$. Then we have $\pi_\beta^\delta x_\delta = x_\beta$; $\pi_{\beta'}^\delta x_\delta = x_{\beta'}$; $\pi_\gamma^\beta x_\beta = x_\gamma$, by definition. But then $x_\gamma = \pi_\gamma^\beta \pi_\beta^\delta x_\delta = \pi_\gamma^\delta x_\delta = \pi_\gamma^{\beta'} \pi_{\beta'}^\delta x_\delta = \pi_\gamma^{\beta'} x_{\beta'}$. Hence, the definition is good. There are four consistency relations to be verified, but these are similar to those of the above lemma, and are left to the reader.

Remark. This completes the algebraic preliminaries. Theorem 1.1 may be generalized to noncountable sets and other coefficient groups, but such considerations are not necessary for this paper.

2. An anticommutative cochain algebra

Let K be a countable simplicial complex, M the directed set of all finite subcomplexes, partially ordered by inclusion. We verify immediately that M has a cofinal sequence, for we make such a sequence by adding on the simplices one at a time. The first stage in the construction is the construction of an anticommutative cochain algebra on a finite subcomplex of K.

LEMMA 2.1. *In the language of* [3, I, *Section* 3] *a finite complex has a fine anticommutative R-cover, i.e., an anticommutative cochain algebra whose derived algebra is the "correct" cohomology algebra.*

Proof. Imbed the complex in a Euclidean space. Consider the cover of differential forms on the Euclidean space. The section of this cover over the complex is the desired cover. (Recall that the elements whose supports do not meet the complex are factored out.)

Our goal is to extend this procedure to countable simplicial complexes. As we shall see, we can then compute the real cohomology rings of certain fibre spaces.

THEOREM 2.1. *Let K be a countable simplicial complex, whose vertices have been ordered. Let M be the directed set of finite subcomplexes, ordered by inclusion. Fix, once and for all, an increasing sequence of Euclidean spaces*

$$E^1 \subset E^2 \subset \cdots \subset E^n \subset \cdots .$$

Then we may choose a family of imbeddings of the subcomplexes $\phi_\alpha : \alpha \to E^{n_\alpha}$ for each $\alpha \in M$ such that if $\alpha < \beta$, then

$$\begin{array}{ccc} \alpha & \subset & \beta \\ {\scriptstyle \phi_\alpha}\downarrow & & \downarrow{\scriptstyle \phi_\beta} \\ E^{n_\alpha} & \subset & E^{n_\beta} \end{array}$$

is a commutative diagram.

Proof. We imbed the increasing family of Euclidean spaces in Hilbert space, H,

$$E^1 \subset E^2 \subset \cdots \subset H$$

so that the n basis vectors in E^n go into the first n basis vectors in H, say

$$(1, 0, 0, \cdots), \quad (0, 1, 0, 0, \cdots), \quad \cdots, \quad \overbrace{(0, \cdots, 0,}^{(n-1)} 1, 0, \cdots)$$

in the representation l^2.

Let $\alpha \in M$. Let the vertices of α, in the ordering of K, be $v_1, \cdots, v_m$. We map the vertices of α into the basis vectors of H which correspond to them in the natural order. If two vertices of α are joined by a 1-simplex, map the 1-simplex linearly into the line joining the two corresponding basis vectors of H. Proceeding inductively in this way, we define a map

$$\phi_\alpha : \alpha \to E^{n_\alpha}.$$

Notice that ϕ_α preserves order, i.e., if $i < j$, then $\phi_\alpha(v_i)$ precedes $\phi_\alpha(v_j)$ in the natural order of the basis vectors in H. We define n_α as the smallest n such that $\phi_\alpha(\alpha)$ is contained in E^{n_α}. n_α is then a nondecreasing function of α, in the sense that if $\alpha < \beta$, then $n_\alpha \leqq n_\beta$.

I now describe an anticommutative cochain algebra for the complex K. By the above lemma, we associate with each $\alpha \in M$ an anticommutative cochain algebra, $\mathcal{G}_\alpha$, with respect to the imbedding $\phi_\alpha : \alpha \to E^{n_\alpha}$. Suppose $\alpha < \beta$. I shall define a map π_α^β. We have a commutative diagram

$$\begin{array}{ccc} \alpha & \subset & \beta \\ {\scriptstyle \phi_\alpha}\downarrow & & \downarrow{\scriptstyle \phi_\beta} \\ E^{n_\alpha} & \subset & E^{n_\beta}. \end{array}$$

Let c^n be a cochain on β, i.e., $c^n \in \mathcal{G}_\beta^n$. c^n comes from a form on E^{n_β}, say $\bar{c}^n$. The inclusion $E^{n_\alpha} \subset E^{n_\beta}$ induces a map on forms $i^* : \mathfrak{D}(E^{n_\beta}) \to \mathfrak{D}(E^{n_\alpha})$. The imbedding $\phi_\alpha : \alpha \to E^{n_\alpha}$ then gives an element of $\mathcal{G}_\alpha^n$ corresponding to $i^*(\bar{c}^n)$. This defines a map π_α^β.

It is clear that $\pi_\alpha^\beta : \mathcal{G}_\beta \to \mathcal{G}_\alpha$ is well-defined, and that if $\alpha < \beta < \gamma$, then $\pi_\alpha^\gamma = \pi_\alpha^\beta \pi_\beta^\gamma$, for the inclusions $E^{n_\alpha} \subset E^{n_\beta} \subset E^{n_\gamma}$ are transitive.

We must show that π_α^β is onto. However, this is immediate, since if $E^{n_\alpha} \subset E^{n_\beta}$, a form on E^{n_α} may clearly be extended to E^{n_β}.

DEFINITION 2.1. If K is a countable simplicial complex whose vertices have been ordered, and if M is the directed set of finite subcomplexes, then the limit algebra

$$\mathcal{G}_\infty = \text{inv lim}_{\alpha \in M} \mathcal{G}_\alpha$$

is the anticommutative cochain algebra for K, written $\mathcal{G}_\infty(K)$.

Remark. The above procedure depended on ordering the vertices of K. The reader can easily verify that $\mathcal{G}_\infty(K)$ does not depend on the ordering, up to an isomorphism. Hence, we shall not always specify the order in the following.

Recall that by Theorem 1.1, $H_\infty \cong \sum_n \text{inv lim}_\alpha H_\alpha^n$. This isomorphism will be used in the next section, where we shall study $H_\infty = \mathcal{H}(\mathcal{G}_\infty)$.

3. Cohomology of simplicial complexes

We make the following additional conventions:

A. K is a countable complex, with the weak topology.

B. $H_\Delta^n(K)$ is the n^{th} simplicial cohomology group of K with real coefficients.[3]

C. $H^n(K)$ is the n^{th} singular cohomology group of K with real coefficients.

D. $H^*(K) = \sum_n H^n(K)$, as an algebra with the cup-product.

The following lemma is an exercise in [7].

LEMMA 3.1. *Let $\{K_\alpha\}$, $\alpha \in M$, be the set of finite subcomplexes of K. Let $(\pi_\alpha^\beta)^* : H_\Delta^q(K_\beta) \to H_\Delta^q(K_\alpha)$ be induced by the inclusion $K_\alpha \subset K_\beta$, when $\alpha < \beta$. Then, for each q, $\{H_\Delta^q(K_\alpha); (\pi_\alpha^\beta)^*\}$ forms an inverse system of groups,*

[3] We use cohomology based on infinite cochains, dual to homology based on finite chains.

and

$$\text{inv lim}_\alpha\, H^q_\Delta(K_\alpha) \cong H^q_\Delta(K).$$

Proof. For any K_α (or K), denote by $Z^q(K_\alpha)$ (or $Z^q(K)$) and $B^q(K_\alpha)$ (or $B^q(K)$) the simplicial cocycles and coboundaries of K_α (or K), coefficients in the reals. We have an inverse family of exact sequences

$$0 \to B^q(K_\alpha) \to Z^q(K_\alpha) \to H^q_\Delta(K_\alpha) \to 0.$$

By [7, p. 228] it follows that

$$0 \to \text{inv lim}_\alpha\, B^q(K_\alpha) \to \text{inv lim}_\alpha\, Z^q(K_\alpha) \to \text{inv lim}_\alpha\, H^q_\Delta(K_\alpha) \to 0$$

is exact.

Since $\text{inv lim}_\alpha\, Z^q(K_\alpha) = Z^q(K)$ and $\text{inv lim}_\alpha\, B^q(K_\alpha) = B^q(K)$, we have the assertion.

Remark. This equivalence is easily seen to be natural in the category of simplicial complexes and simplicial maps.

LEMMA 3.2. *For any q, we have the following isomorphisms:*

$$\beta^q_{*\alpha} : H^q(K_\alpha) \xrightarrow{\cong} H^q_\Delta(K_\alpha); \qquad \phi^q_\alpha : \mathcal{H}^q(\mathcal{G}_\alpha) \xrightarrow{\cong} H^q(K_\alpha).$$

Proof. For the map $\beta^q_{*\alpha}$, see [7, p. 200]. For the right-hand map, see [2, exposé III]. We shall use $\mathcal{H}^q$ to mean q^{th} derived group, i.e., the derived group in dim q.

LEMMA 3.3. *There is an isomorphism*

$$\Phi : H_\infty \to H^*(K).$$

(*The map will be defined in the proof.*)

Proof. Φ is to be the composition of three isomorphisms.[4]

$$H_\infty \xrightarrow{F} \text{inv lim}_\alpha\, H_\alpha \xrightarrow{\phi_\infty} \text{inv lim}_\alpha\, H^*(K_\alpha) \xrightarrow{G} H^*(K).$$

F is as in Section 1. ϕ_∞ is defined by the collection ϕ_α, with ϕ_α as in the previous lemma. G is obtained as follows:

$$\text{inv lim}_\alpha\, H^*(K_\alpha) \xrightarrow{\cong} \text{inv lim}_\alpha\, H^*_\Delta(K_\alpha)$$

by the collection $(\beta_{*\alpha})$. By Lemma 3.1, $\text{inv lim}_\alpha\, H^*_\Delta(K_\alpha) \xrightarrow{\cong} H^*_\Delta(K)$. These two maps, followed by β_*^{-1}, give the map G. Everything here is an isomorphism.

We must now study multiplicative properties. F is a multiplicative map.

LEMMA 3.4. $G\cdot\phi_\infty : \text{inv lim}_\alpha\, H_\alpha \to H^*(K)$ *is a multiplicative map.*

[4] H^* means $\sum_n H^n$; H^*_Δ means $\sum_n H^n_\Delta$.

Proof. Consider the family of commutative diagrams:

$$\begin{array}{ccc} \text{inv lim}_\alpha H_\alpha & \xrightarrow{G\cdot\phi_\infty} & H^*(K) \\ \text{proj}\downarrow & & \downarrow i^* \\ H_\alpha & \xrightarrow{\phi_\alpha} & H^*(K_\alpha). \end{array}$$

We know that all the maps, except possibly $G\cdot\phi_\infty$, are multiplicative. Passing to the limit, we get the following commutative diagram:

$$\begin{array}{ccc} & & H^*(K) \\ & \overset{G\cdot\phi_\infty}{\nearrow} & \\ \text{inv lim}_\alpha H_\alpha & & \Big\downarrow (i^*) \\ & \underset{(\phi_\alpha)}{\searrow} & \\ & & \text{inv lim}_\alpha H^*(K_\alpha). \end{array}$$

$G\cdot\phi_\infty$ and (ϕ_α) are additive isomorphisms, so (i^*) is also. But i^* and (ϕ_α) are multiplicative. Therefore, $G\cdot\phi_\infty$ is a multiplicative isomorphism.

We collect these facts in the following theorem.

THEOREM 3.1. *If K is a countable simplicial complex, and $\mathcal{G}_\infty(K)$ is the cochain algebra constructed above, then there is an algebra isomorphism*

$$\Phi : H_\infty(K) \xrightarrow{\cong} H^*(K),$$

where

$$H_\infty(K) = \mathcal{H}(\mathcal{G}_\infty(K)).$$

For completeness, we briefly discuss the question of induced maps. First consider the case of two finite complexes, and a simplicial map $f : S \to T$. Suppose that S and T are given fine anticommutative R-covers, say $\mathcal{S}$ and $\mathfrak{T}$. Then, using the Leray theory of intersections of complexes (see [2]) we may define an induced map

$$\bar{f} : \mathfrak{T} \to f^{-1}(\mathfrak{T}) \circ \mathcal{S}.$$

However, in the case where $\mathcal{S}$ and $\mathfrak{T}$ are defined in terms of imbeddings $\phi_S : S \to E^{n_s}$ and $\phi_T : T \to E^{n_t}$, respectively, we may proceed in the following direct manner: Extend f to $\bar{f} : E^{n_s} \to E^{n_t}$ in a linear way. $\bar{f}$ defines a map $\bar{f}^*$ on forms. One easily checks that $\bar{f}^*$ gives a well-defined map

$$f^* : \mathfrak{T} \to \mathcal{S}.$$

Now, let K and L be countable simplicial complexes, and let $f : K \xrightarrow{\text{onto}} L$.

Denote, as usual, the anticommutative cochain algebras by $\mathcal{G}_\infty(K)$ and $\mathcal{G}_\infty(L)$. If α is a finite subcomplex of K, $f(\alpha)$ is a finite subcomplex of L. As f is onto, $\{f(\alpha)\}$ consists of all finite subcomplexes of L. We have defined maps

$$f^* : \mathcal{G}_{f(\alpha)}(L) \to \mathcal{G}_\alpha(K).$$

These are clearly natural with respect to inclusions, and define a map

$$\bar{f}^* : \text{inv lim}_\alpha\, \mathcal{G}_{f(\alpha)}(L) \to \text{inv lim}_\alpha\, \mathcal{G}_\alpha(K).$$

But, as $f(\alpha)$ ranges through all finite subcomplexes of L, this is just a map

$$f^\# : \text{inv lim}_\alpha\, \mathcal{G}_\alpha(L) \to \text{inv lim}_\alpha\, \mathcal{G}_\alpha(K).$$

DEFINITION 3.1. $f^\# : \mathcal{G}_\infty(L) - \mathcal{G}_\infty(K)$ is called the map induced by f.

The induced maps are clearly transitive, i.e., if

$$K_1 \xrightarrow[f_1]{\text{onto}} K_2 \xrightarrow[f_2]{\text{onto}} K_3\,,$$

then $(f_2 \cdot f_1)^\# = f_1^\# \cdot f_2^\#$.

THEOREM 3.2. *The map on cohomology which is induced by $f^\#$ coincides, under identification via the map Φ, with the map induced by f on singular cohomology.*

Proof. It is a question of proving the commutativity of the outer rectangle of the following diagram:

$$\begin{array}{ccccccc}
H_\infty(K) & \xrightarrow{F} & \text{inv lim}_\alpha\, H_\alpha(K) & \xrightarrow{\phi_\infty} & \text{inv lim}_\alpha\, H^*(K_\alpha) & \xrightarrow{G} & H^*(K) \\
\uparrow{\scriptstyle (f^\#)^*} & & \uparrow{\scriptstyle (f_\alpha)^*} & & \uparrow{\scriptstyle (f_\alpha^*)} & & \uparrow{\scriptstyle f^*} \\
H_\infty(L) & \xrightarrow{F} & \text{inv lim}_\alpha\, H_\alpha(L) & \xrightarrow{\phi_\infty} & \text{inv lim}_\alpha\, H^*(L_\alpha) & \xrightarrow{G} & H^*(L).
\end{array}$$

But the left and right squares are clearly commutative. As the assertion is true for finite subcomplexes, the middle square is commutative, also.

Now, let K_γ be a finite subcomplex of K, and let $\mathcal{G}_\infty(K_\gamma)$ be the anticommutative cochain algebra on K_γ, i.e., $\mathcal{G}_\gamma$. Consider the inclusion

$$i : K_\gamma \to K.$$

Then, we define an induced map

$$i^\# : \mathcal{G}_\infty(K) \to \mathcal{G}_\infty(K_\gamma)$$

by projection to the γ^{th} coordinate.

THEOREM 3.3. *The map $i^{\#*} : H_\infty(K) \to H_\gamma(K)$ corresponds, under identification via Φ, to the map induced on singular cohomology, $i^* : H^*(K) \to H^*(K_\gamma)$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} H_\infty(K) & \xrightarrow{F} & \text{inv lim}_\alpha H_\alpha(K) & \xrightarrow{\phi_\infty} & \text{inv lim}_\alpha H^*(K_\alpha) & \xrightarrow{G} & H^*(K) \\ & \searrow^{i^{\#*}} & \downarrow k & & \downarrow j & \swarrow^{i^*} & \\ & & H_\gamma(K) & \xrightarrow{\phi_\gamma} & H^*(K_\gamma) & & \end{array}$$

where k, j are projections to the γ^{th} coordinate. It is easily verified that each of the subdiagrams is commutative. As $\Phi = G \cdot \phi_\infty \cdot F$ and ϕ_γ are isomorphisms, the theorem follows.

Remark 1. This construction of a cochain algebra is canonical for ordered complexes. We have defined induced maps only for onto maps and inclusions of finite subcomplexes. The definition of induced maps in general, which will turn the construction into a functor, is left to the reader.

Remark 2. The construction of Section 1 did not depend on the coefficient domain being the real numbers. However, Section 2 depended on the real numbers in a strong way. It is very likely that there are analogous results for fields such as the complex numbers, etc. However, A. Borel has shown [2] that when the coefficients are a field of characteristic $p \neq 0$, there does not exist a construction of anticommutative cochain algebras, on the category of countable simplicial complexes, which gives the correct cohomology. This fact, which may be inferred, of course, from the existence and nontriviality of the Steenrod operations, does not seem to be mentioned explicitly anywhere else in the literature.

Remark 3. Allendoerfer and Eells [1] have given a generalization of de Rham's theorem to integral cohomology. They deal with classes of forms with singularities. Multiplication is not defined for the forms.

Remark 4. Eells [6] has given a description of forms on ∞-dimensional manifolds, modeled on a Banach space. It seems to me that there is a reasonable chance of constructing anticommutative cochains on countable simplicial complexes, by imbedding them in such manifolds.

II. Some Theorems on Fibre Spaces

In this chapter, I show how the previous construction may be used to extend a theorem of Borel [3, Théorème 24.1]. The generalization is not strong, but it is plausible that these techniques may be developed into a general theory. The first section (Section 4) contains a summary of the techniques and a statement of Borel's theorem. Section 5 contains our main theorem, while Section 6 is devoted to specific applications.

Real cohomology is to be understood. We denote Alexander-Spanier theory by $\tilde{H}^*(\ )$, singular theory by $H^*(\ )$. If X is a countable, connected simplicial complex, we write the anticommutative cochain algebra of the previous chapter as $\mathcal{G}(X)$, its derived algebra as $\mathcal{H}(\mathcal{G}(X))$. The spaces which we consider will be nice enough so that $\tilde{H}^*$ and H^* agree. The general reference for these questions is [5].

4. Borel's result

Consider a locally trivial fibre space $(E, F, B; p)$. Assume, temporarily, that E is a compact, finite-dimensional, separable metric space, which is connected and locally connected. Assume that F is connected, and that $\tilde{H}^*(F)$ is an exterior algebra on a vector space spanned by odd-dimensional transgressive elements. Assume that in the Leray spectral sequence of this fibration,

$$E_2 \cong \tilde{H}^*(B) \otimes \tilde{H}^*(F).$$

Under these assumptions, E and B both have fine, anticommutative R-covers (= couvertures), say $\mathcal{E}$ and $\mathcal{B}$. This permits the construction of a fine, anticommutative R-cover $\mathcal{C} = p^{-1}(\mathcal{B}) \circ \mathcal{E}$, and a map $\bar{p} : \mathcal{B} \to \mathcal{C}$. Let $x_1, \cdots, x_n$ be the transgressive generators of $\tilde{H}^*(F)$. Choose cochains of transgression $c_1, \cdots, c_n \in \mathcal{C}$, and representatives of the transgression $b_1, \cdots, b_n \in \mathcal{B}$, so that if $i : F \to E$ denotes the inclusion, we have

$$\{i^*(c_i)\} = x_i ; \qquad \bar{p}b_i = dc_i ; \qquad \text{all } i, \quad 1 \leqq i \leqq n.$$

DEFINITION 4.1. Set $\mathcal{L} = \mathcal{B} \otimes \tilde{H}^*(F)$. It is an anticommutative, graded, cochain algebra, for the total degree. Define a differential d, on $\mathcal{L}$, by setting

$$d(b \otimes 1) = db \otimes 1, \qquad d(1 \otimes x_i) = b_i \otimes 1$$

and extending to $\mathcal{L}$ in the obvious manner.

DEFINITION 4.2. Define a linear map

$$\bar{\lambda} : \mathcal{B} \otimes \{x_1, \cdots, x_n\} \to \mathcal{C}$$

by setting

$$\bar{\lambda}(b \otimes 1) = \bar{p}b, \qquad \bar{\lambda}(1 \otimes x_i) = c_i .$$

$\bar{\lambda}$ is a multiplicative homomorphism on $\mathcal{B} \otimes 1$. Since $\tilde{H}^*(F)$ is free, and $\mathcal{C}$ is anticommutative, $\bar{\lambda}$ extends uniquely to a multiplicative homomorphism

$$\lambda : \mathcal{L} \to \mathcal{C}.$$

It is immediate that $\lambda d = d\lambda$. Borel's result is then

THEOREM 4.1. *Under the above assumptions, λ induces an algebra isomorphism*

$$\lambda^* : \mathcal{H}(\mathcal{L}) \xrightarrow{\cong} \tilde{H}^*(E),$$

where $\mathcal{H}(\mathcal{L})$ is the derived algebra of $\mathcal{L}$. Furthermore, λ^* has certain naturality properties with respect to i and p.

I shall sketch those parts of the proof which are needed below. A filtration is defined on $\mathcal{L}$ by setting

$$\mathcal{L}^p = \sum_{i \geqq p} \mathcal{B}^i \otimes \tilde{H}^*(F).$$

Borel shows that in the resulting spectral sequence, say $\{\bar{E}_r, \bar{d}_r\}$, one has $\bar{E}_1 \cong \mathcal{B} \otimes \tilde{H}^*(F)$; $\bar{d}_1$ is partial differentiation with respect to $\mathcal{B}$.

$$\bar{E}_2 \cong \tilde{H}^*(B) \otimes \tilde{H}^*(F).$$

$\bar{E}_\infty$ is the graded algebra of $\mathcal{H}(\mathcal{L})$, with respect to the obvious filtration. He then shows that λ induces a homomorphism of $\{\bar{E}_r\}$ into $\{E_r\}$, the latter being the Leray sequence of the fibre map. Then he shows

$$\lambda^* : \bar{E}_2 \xrightarrow{\cong} E_2 .$$

Hence, it follows that

$$\lambda^* : \mathcal{H}(\mathcal{L}) \xrightarrow{\cong} \tilde{H}^*(E).$$

5. Main theorems

We now make the following general assumptions:

(1) $(E, F, B; p)$ is a locally trivial fibre space. E is 1-connected. We take F to be a finite complex. (Actually, the results will hold if F is a compact, connected, finite-dimensional, separable metric space which is HLC.)

(2) $H^*(F)$ is an exterior algebra on a vector space spanned by odd-dimensional transgressive elements.

(3) B is a 1-connected, countable simplicial complex. We suppose that there is a family of finite subcomplexes, B_i, beginning with a base point, such that

(a) $b_0 \subset B_1 \subset B_2 \subset \cdots \subset \bigcup_i B_i = B$,

(b) the inclusion map $i_n : B_n \to B$ induces isomorphisms

$$i_n^* : \pi_i(B_n) \xrightarrow{\cong} \pi_i(B), \qquad \text{for} \quad i < n.$$

Examples will be given in Section 6.

Denote the anticommutative cochain algebra on B by $\mathcal{G}(B)$, that on B_n (which occurs directly in the construction of Chapter I) by $\mathcal{G}(B_n)$. Then, we have a map

$$i_n^\# : \mathcal{G}(B) \to \mathcal{G}(B_n)$$

(see Theorem 3.3). It follows from (3) above, and Whitehead's theorem [12] that there is an isomorphism

$$i_{n*} : H_i(B_n ; R) \to H_i(B; R), \qquad i < n, \quad n \geqq 2.$$

As the coefficients are the real numbers, we conclude that

$$(i_n^\#)^* : \mathcal{H}(\mathcal{G}(B)) \to \mathcal{H}(\mathcal{G}(B_n))$$

is an isomorphism in dimensions $< n$.

Now, by (2) we may write

$$H^*(F) = \Lambda[u_1] \otimes \cdots \otimes \Lambda[u_n]$$

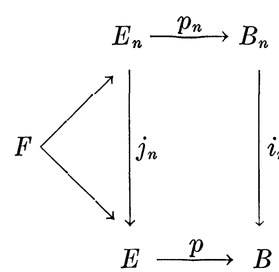
where dim u_i is odd, and each u_i is transgressive. If dim $u_i = 2p_i - 1$, then dim $\tau(u_i) = 2p_i$. We write

$$\tau(u_i) = v_i \in H^*(B).$$

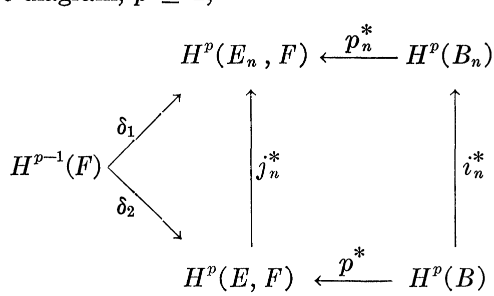
DEFINITION 5.1. Let $E_n = p^{-1}(B_n)$; $p_n = p \mid E_n$. Write the resulting fibre space as $(E_n, F, B_n; p_n)$. Denote the transgression in this fibre space by τ_n.

LEMMA 5.1. *In $(E_n, F, B_n; p_n)$, we have, for each i, $1 \leqq i \leqq n$,*
(a) *u_i is transgressive,*
(b) $\tau_n(u_i) = i_n^*(v_i)$.

Proof. Consider the commutative diagram



where the vertical arrows are inclusions. Taking the fibre to be the fibre over b_0, the inclusion map is a map of pairs $(E_n, F) \to (E, F)$. We then get a commutative diagram, $p \geqq 2$,



where δ_1 and δ_2 are the respective differentials of the pairs (E_n, F) and (E, F), and where we identify $H^p(B)$ and $H^p(B, b_0)$, etc. for $p > 0$.

If $\tau(u_i) = v_i \in H^p(B)$, then there is $z_i \in H^p(E, F)$ so that $p^* v_i = z_i$, $\delta_2 u_i = z_i$. From the diagram, $p_n^* i_n^* v_i = j_n^* p^* v_i = j_n^* z_i$; $\delta_1 u_i = j_n^* \delta_2 u_i = j_n^* z_i$. Hence if we set $j_n^* z_i = \bar{z}_i$; $i_n^* v_i = \bar{v}_i$, we have the relations $\delta_1 u_i = \bar{z}_i$; $p_n^* \bar{v}_i = \bar{z}_i$, as desired.

We shall need to specify the transgression on the cochain level. By the results of Chapter I, identify $H^*(B)$ and $\mathcal{H}(\mathcal{G}(B))$. Let v_i denote $\tau(u_i)$ or the corresponding element in $\mathcal{H}(\mathcal{G}(B))$. For each i, choose $b_i \in \mathcal{G}(B)$ so that $b_i \in v_i \in \mathcal{H}(\mathcal{G}(B))$. We put

$$b_i^m = i_m^\#(b_i) \in \mathcal{G}(B_m).$$

It follows from the above lemma that b_i^m represents $\tau_m(u_i)$.

DEFINITION 5.2. We put

$$\mathcal{L} = \mathcal{G}(B) \otimes H^*(F), \qquad \mathcal{L}_n = \mathcal{G}(B_n) \otimes H^*(F).$$

These algebras are graded by the total degree, and are clearly anticommutative. Differentials are defined by specifying

$$d\,(b \otimes 1) = db \otimes 1, \quad b \in \mathcal{G}(B), \qquad d_n(b \otimes 1) = db \otimes 1, \quad b \in \mathcal{G}(B_n),$$
$$d(1 \otimes u_i) = b_i \otimes 1, \qquad d_n(1 \otimes u_i) = b_i^n \otimes 1,$$

and extending to $\mathcal{L}$ and $\mathcal{L}_n$ in the obvious way.

LEMMA 5.2. *The map $f_n : \mathcal{L} \to \mathcal{L}_n$, defined by*

$$f_n = (i_n^\# \otimes \mathrm{Id}) : \mathcal{G}(B) \otimes H^*(F) \to \mathcal{G}(B_n) \otimes H^*(F)$$

commutes with the differentials, and induces a (multiplicative) homomorphism

$$f_n^* : \mathcal{H}(\mathcal{L}) \to \mathcal{H}(\mathcal{L}_n).$$

Proof. Since f_n is clearly multiplicative, in view of the definitions of the differentials, we need only check $d_n f_n = f_n d$ on $b \otimes u_i$. However,

$$d_n f_n(b \otimes u_i) = d_n(i_n^\# b \otimes u_i) = i_n^\# db \otimes u_i + (-1)^{\deg b}(b \cdot b_i^n) \otimes 1,$$
$$f_n d(b \otimes u_i) = f_n(db \otimes u_i + (-1)^{\deg b}(b \cdot b_i) \otimes 1)$$
$$= i_n^\# db \otimes u_i + (-1)^{\deg b} i_n^\#(b \cdot b_i^n) \otimes 1.$$

In order to study the maps f_n^* we introduce two spectral sequences. Following Borel [3] we define the following two filtrations:

$$\mathcal{L}_n^i = \textstyle\sum_{p \geqq i} \mathcal{G}^p(B_n) \otimes H^*(F), \qquad \mathcal{L}^i = \textstyle\sum_{p \geqq i} \mathcal{G}^p(B) \otimes H^*(F).$$

Denote the resulting spectral sequences respectively by $\{{}_n\bar{E}_r\}$ and $\{\bar{E}_r\}$. Then, as in [3, p. 184] we have the following.

LEMMA 5.3.

$${}_n\bar{E}_1 \cong \mathcal{G}(B_n) \otimes H^*(F), \qquad \bar{E}_1 \cong \mathcal{G}(B) \otimes H^*(F),$$
$${}_n\bar{E}_2 \cong \mathcal{H}(\mathcal{G}(B_n)) \otimes H^*(F), \qquad \bar{E}_2 \cong \mathcal{H}(\mathcal{G}(B)) \otimes H^*(F).$$

In both cases, d_1 is partial differentiation with respect to the first factor. ${}_n\bar{E}_\infty$ and $\bar{E}_\infty$ are the obvious graded rings associated with $\mathcal{H}(\mathcal{G}(B_n) \otimes H^*(F))$ and $\mathcal{H}(\mathcal{G}(B) \otimes H^*(F))$.

LEMMA 5.4. *Let s be the smallest integer so that $H^i(F) = 0$, $i > s$. Let $n > s$. Then $f_n^* : \mathcal{H}(\mathcal{L}) \to \mathcal{H}(\mathcal{L}_n)$ is a (multiplicative) isomorphism in dimensions $< n - s$.*

Proof. Clearly, f_n preserves filtrations, and induces a homomorphism

$$_2f_n^* : \bar{E}_2 \to {}_n\bar{E}_2 .$$

For $p < n$, this map is an isomorphism on $\bar{E}_2^{p,0}$. Then by [3, p. 130], f_n^* is an isomorphism in dimensions $< n - s$.

THEOREM 5.1. *Let $(E, F, B; p)$ be a fibre space satisfying the three conditions given above. Suppose $H^*(E)$ is finitely generated as an algebra (i.e., is a quotient algebra of a tensor product of a polynomial algebra on finitely-many even-dimensional generators and an exterior algebra on finitely-many odd-dimensional generators). Then*

$$H^*(E) \cong \mathcal{H}(\mathcal{G}(B) \otimes H^*(F))$$

as algebras.

Proof. In light of the first assumption, E_n is compact and HLC. Then the singular cochains form a fine R-cover on E_n (see [2, II-13]). By [3, p. 136] the transgression τ_n does not depend on the theory $\tilde{H}^*$ or H^*. Now $\mathcal{G}(B_n)$ is clearly a fine, anticommutative R-cover for B_n. Hence (see Borel [3, Théorèmes 24.1, 25.1]), we have

$$\tilde{H}^*(E_n) \cong \mathcal{H}(\mathcal{G}(B_n) \otimes H^*(F)).$$

Hence, there are algebra isomorphisms

$$g_n : H^*(E_n) \xrightarrow{\cong} \mathcal{H}(\mathcal{G}(B_n) \otimes H^*(F)).$$

Consider the commutative diagram

$$\begin{array}{ccccccccccc} \cdots \to & \pi_i(B_n) & \to & \pi_{i-1}(F) & \to & \pi_{i-1}(E_n) & \to & \pi_{i-1}(B_n) & \to & \pi_{i-2}(F) & \to \cdots \\ & \downarrow i_n^* & & \downarrow \mathrm{Id} & & \downarrow j_n^* & & \downarrow i_n^* & & \downarrow \mathrm{Id} & \\ \cdots \to & \pi_i(B) & \to & \pi_{i-1}(F) & \to & \pi_{i-1}(E) & \to & \pi_{i-1}(B) & \to & \pi_{i-2}(F) & \to \cdots \end{array}$$

in which the rows are exact. i_n^* is an isomorphism in dimensions $< n$. Hence, it follows that $j_n^* : \pi_i(E_n) \to \pi_i(E)$ is an isomorphism in dimensions $< n - 1$. If $n > 2$, Whitehead's theorem [12] tells us that $j_n^* : H^i(E) \to H^i(E_n)$ is an isomorphism for $i < n - 1$. (Recall that $H^*(E)$ and $H_*(E)$ are dual vector spaces.)

As j_n^* is multiplicative, it follows from Lemma 5.4 that

$$\chi_n : H^*(E) \xrightarrow{j_n^*} H^*(E_n) \xrightarrow{g_n^*} \mathcal{H}(\mathcal{G}(B_n) \otimes H^*(F)) \xleftarrow{f_n^*} \mathcal{H}(\mathcal{G}(B) \otimes H^*(F))$$

is a multiplicative isomorphism in dimensions $< n - s$.

The proof will be completed by showing that given two graded algebras,[5] A and B, and a family of isomorphisms

$$\chi_m : \sum_{i<m} A^i \xrightarrow{\cong} \sum_{i<m} B^i, \qquad \text{any} \quad m > 0,$$

if A is finitely-generated as an algebra, then there is an isomorphism[6]

$$\chi : A \xrightarrow{\cong} B.$$

Write $A = F/I$ and where F is a free graded algebra, and I an ideal. F is a tensor product of finitely-generated polynomial and exterior algebras. An ideal of F identifies with a tensor product of an ideal in each of these algebras. It is clear that an ideal in a finitely-generated exterior algebra is finitely-generated, and Hilbert's theorem assures us that an ideal in a finitely-generated polynomial algebra is finitely-generated. Hence, I is finitely-generated.

Choose N greater than the dimension of any generator of F or I. Let $\rho_A : F \to A$ be the projection. Define $\rho_B : F \to B$ by letting ρ_B be $\chi_N \circ \rho_A$ on the generators, and extending by freeness. Notice that ρ_B and $\chi_N \circ \rho_A$ agree in dimensions $< N$.

Now, it is easy to see that A and B have the same number of generators in each dimension. Hence, the image of ρ_B contains the generators of B, so that ρ_B is onto. Set $J = \text{Ker } \rho_B$. It is immediate that $J \supseteq I$. Hence, there is a multiplicative epimorphism

$$\chi : A = F/I \to F/J = B.$$

But clearly A and B are vector spaces, which have the same rank in each dimension. Thus the epimorphism χ must be 1-1.

Remarks. If $H^*(E)$ is not finitely-generated, the theorem asserts that the two algebras are isomorphic in dimensions $< n$, for any n.

The assumption that F is compact is needed twice. (1) E_n must be compact, so that we can find a fine, anticommutative R-cover. (2) F must be compact, so that in the Leray spectral sequence, $E_2 \cong \tilde{H}^*(B) \otimes \tilde{H}^*(F)$.

I conjecture that this theorem is valid for a larger class of fibre spaces; this can be verified additively, in several cases.

Next, we consider cases when we can use Theorem 5.1 to compute effectively. The following notion originated with Borel and Koszul [8]: Let A be an integral domain, $C^* = \sum_{p \geqq 0} C^p$ a graded, additively free A-algebra, with differential d of degree $+1$. Let $H^* = \sum_{p \geqq 0} H^p$ be the derived algebra.

DEFINITION 5.3. A subalgebra $R \subset C^*$ is called a *representative subalgebra* for H^*, if

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[5] Over the real numbers.

[6] *Added in proof.* A. Dold has shown that the algebras need not be multiplicatively isomorphic, if A is not finitely-generated.

(a) R is a graded subalgebra of C, $R^0 = C^0$,
(b) $R \subset \operatorname{Ker} d$,
(c) in each class of H^*, there is exactly one element of R.

Example. Take A to be the reals, and consider the complex projective space $P_n(C)$. Let $C^* = \mathfrak{D}(P_n(C))$ be the algebra of differential forms.

$$H^* = P[u_2]/(u_2^{n+1}),$$

i.e., a truncated polynomial algebra of height n, on a two-dimensional generator, u_2. Choose a representative $u_2' \in u_2$. Let R be the subalgebra of C^* generated by u_2' and 1. It is immediate that R is a representative subalgebra.

From now on A will be the reals.

THEOREM 5.2. *Let X be a countable, simplicial complex. Suppose $H^*(X)$ is a finite tensor product of monogenic polynomial algebras on even-dimensional generators, and monogenic exterior algebras on odd-dimensional generators. Take as C^* the anticommutative cochain algebra $\mathcal{G}(X)$. Then there is a representative subalgebra $R \subset C^*$, for $H^*(X)$.*

Proof. Write

$$H^*(X) = P[x_1] \otimes \cdots \otimes P[x_n] \otimes \Lambda[y_1] \otimes \cdots \otimes \Lambda[y_m],$$

where $\dim x_i$ is even, $\dim y_i$ is odd. For each i, select cocycles $u_i \in x_i$; $v_i \in y_i$; $u_i, v_i \in \mathcal{G}(X)$. Define

$$R = P[u_1] \otimes \cdots \otimes P[u_n] \otimes \Lambda[v_1] \otimes \cdots \otimes \Lambda[v_m] \subset C^*.$$

Clearly, R is a graded subalgebra, $R^0 = C^0$. Since any (formal) polynomial in the u_i and v_j represents that same polynomial in x_i and y_j, we see that there is exactly one element of R in each class of $H^*(X)$.

Examples of spaces satisfying the hypotheses of the preceding theorem are Lie groups and finite products of $K(\pi, n)$ spaces.

The following theorem, adapted directly from Borel [3], shows how representative subalgebras are useful.

THEOREM 5.3. *We take the same hypotheses as in Theorem* 5.1. *Furthermore, suppose $H^*(B)$ has a representative subalgebra $R \subset \mathcal{G}(B)$. Define a differential on $H^*(B) \otimes H^*(F)$ by specifying*

$$d(b \otimes 1) = 0; \qquad d(1 \otimes u_i) = \tau(u_i) \otimes 1.$$

Then, $H^*(E)$ and $\mathcal{H}(H^*(B) \otimes H^*(F))$ are isomorphic.

Proof. By Theorem 5.1, it is sufficient to show that $\mathcal{H}(\mathcal{G}(B) \otimes H^*(F))$ and $\mathcal{H}(H^*(B) \otimes H^*(F))$ are isomorphic algebras. Consider the algebra $R \otimes H^*(F)$ endowed with the differential $d(b \otimes 1) = 0, d(1 \otimes u_i) = \bar{v}_i \otimes 1$, where $\bar{v}_i$ is the unique element of R representing $\tau(u_i)$. There is an obvious map,

$$R \otimes H^*(F) \to H^*(B) \otimes H^*(F)$$

which clearly commutes with the differentials, and induces an isomorphism

$$k : \mathcal{H}(R \otimes H^*(F)) \to \mathcal{H}(H^*(B) \otimes H^*(F)).$$

On the other hand, if we denote the inclusion $R \subset \mathcal{G}(B)$ by i, we have a map

$$(i \otimes \mathrm{Id}) : R \otimes H^*(F) \to \mathcal{G}(B) \otimes H^*(F).$$

Recall that the differential in $\mathcal{G}(B) \otimes H^*(F)$ was specified by choosing $b_i \in v_i = \tau(u_i)$. If we choose $b_i = \bar{v}_i$, then this map commutes with the differentials and induces

$$h : \mathcal{H}(R \otimes H^*(F)) \to \mathcal{H}(\mathcal{G}(B) \otimes H^*(F)).$$

To show that h is an isomorphism, recall the spectral sequence above, $\{\bar{E}_r\}$. Filtering $R \otimes H^*(F)$ in the same way, we get a sequence $\{{}_R E_r\}$, in which ${}_R\bar{E}_2 \cong H^*(B) \otimes H^*(F)$. (See [3, Théorème 25.1].) The map $(i \otimes \mathrm{Id})$ induces an isomorphism

$$(i \otimes \mathrm{Id})^* : {}_R\bar{E}_2 \to \bar{E}_2,$$

so that it follows, as before, that h is an isomorphism. Hence $H^*(E)$ is isomorphic to $\mathcal{H}(H^*(B) \otimes H^*(F))$.

6. Applications

I now give some examples and applications of the theorems of Section 5. First, we consider spaces X which satisfy condition (3) of the previous section, which means that there exists a family of finite subcomplexes, X_i, such that

(a) $b_0 \subset X_1 \subset X_2 \subset \cdots \subset \bigcup_i X_i = X$,

(b) $i_n : X_n \to X$ induce isomorphisms in homotopy in dimensions $< n$.

A simple example is obtained as follows: Consider a sequence of integers, $2 \leqq n_1 < n_2 < \cdots$. Let $X_k = S^{n_1} \times \cdots \times S^{n_k}$. We may choose X_k as a simplicial complex, for each k, with simplicial inclusions $X_k \subset X_{k+1}$. Define $X = \bigcup_k X_k$. Clearly, $i_k : X_k \to X$ induces an isomorphism in integral homology, in dimensions $< k$. The desired property then follows by Whitehead's theorem [12]. It is easy to construct bundles over X which satisfy the other conditions of Section 5.

A second example of spaces satisfying condition (3) is given by

PROPOSITION 6.1. *Let π be a finitely-generated Abelian group, n an integer, say > 1. Then there is a countable simplicial complex K, which is a space of type $K(\pi, n)$, with a sequence of finite subcomplexes satisfying condition* (3).

Proof. R. Thom [10, p. 36] has shown that $K(\pi, n)$ may be realized by a countable complex, all of whose skeletons are finite. Hence, we may choose K_i to be the i^{th} skeleton, $i > 0$.

Remark. With slight modifications, one can prove an analogous proposition for $K(\pi_1, n_1) \times \cdots \times K(\pi_p, n_p)$, with each π_i finitely-generated.

We have already noticed that if the base space, B, has a free cohomology algebra (i.e., product of exterior algebras, odd-dimensional generators, and polynomial algebras, even-dimensional generators), then $H^*(E)$ may be computed from $H^*(B)$, $H^*(F)$, and τ (Theorem 5.3). It was remarked above, examples of such base spaces are Lie groups and Eilenberg-Mac Lane spaces. In fact, using the techniques of Serre [12], one may easily prove the following.

PROPOSITION 6.2. *Let π be a finitely-generated Abelian group, decomposed as*

$$\pi = Z \oplus \cdots \oplus Z \oplus Z_{p_1} \oplus \cdots \oplus Z_{p_k} \qquad (\rho \text{ copies of } Z)$$

where each p_i is some prime power. Then

$$H^*(\pi, n; R) \cong \Lambda[x_1, \cdots, x_p] \quad \text{for } n \text{ odd},$$
$$\cong P[x_1, \cdots, x_p] \quad \text{for } n \text{ even},$$

where $\dim x_i = n$, *all* i.

I now give two applications of Theorem 5.1. The general nature of these theorems is to state what sort of information is necessary to determine the cohomology ring of a bundle, with given base and fibre. The first theorem was discussed by Thom [11] with a proof in the compact case. For compact bundles, see also [9] where much more general results are obtained. *Assume* that conditions (1) and (3) of Section 5 are satisfied ((2) will be automatic), and that $H^*(E)$ is finitely generated.

THEOREM 6.1. *Let $(E, S^n, B; p)$ be a sphere bundle, n odd. (Regard n and B as fixed.) Then $H^*(E)$ is determined up to isomorphism, by $\tau(i)$, i a generator of $H^*(S^n)$.*

THEOREM 6.2. *Let $(E, U(n), B; p)$ be a principal $U(n)$-bundle. (Regard n and B as fixed.) Then $H^*(E)$ is determined by the Chern classes, $c_1, \cdots, c_n$.*

Proof. Theorem 6.1 is immediate from Theorem 5.1. For Theorem 6.2, notice that in the universal bundle, $(E_{U(n)}, U(n), B_{U(n)}; p)$, one can find universally transgressive x_i such that

$$\tau(x_i) = c_i$$

and

$$H^*(U(n); Z) = \Lambda_Z[x_1, \cdots, x_n].$$

(See [4, p. 412].) It is clear that in the bundle $(E, U(n), B; p)$, $c_1, \cdots, c_n$ determine $\tau(x_1), \cdots, \tau(x_n)$ in real cohomology.

Remark. If B admits a representative subalgebra, this information permits effective computation (Theorem 5.3).

BIBLIOGRAPHY

1. C. B. ALLENDOERFER AND J. EELLS, JR., *On the cohomology of smooth manifolds*, Comment. Math. Helv., vol. 32 (1957/58), pp. 165–179.

2. A. BOREL, *Cohomologie des espaces localement compacts, d'après J. Leray*, Zurich, École Polytechnique Fédérale, 1953.

3. ———, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2), vol. 57 (1953), pp. 115–207.

4. ———, *Topology of Lie groups and characteristic classes*, Bull. Amer. Math. Soc., vol. 61 (1955), pp. 397–432.
5. *Séminaire H. Cartan, 1950–51*, Paris, École Normale Supérieure, 1952.
6. J. EELLS, JR., *On the geometry of function spaces*, Symposium Internacional de Topología Algebraica, pp. 303–308, Universidad Nacional Autónoma de México and UNESCO, México, 1958.
7. S. EILENBERG AND N. STEENROD, *Foundations of algebraic topology*, Princeton, Princeton University Press, 1952.
8. J. L. KOSZUL, *Sur une type d'algèbres différentielles en rapport avec la transgression*, Colloque de Topologie (Espaces Fibrés), Bruxelles, 1950, pp. 73–81.
9. W. S. MASSEY, *On the cohomology ring of a sphere bundle*, J. Math. Mech., vol. 7 (1958), pp. 265–289.
10. R. THOM, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv., vol. 28 (1954), pp. 17–86.
11. ———, *Opérations en cohomologie reélle*, Séminaire H. Cartan, 1954/55, Exposé 17, Paris, École Normale Supérieure, 1956.
12. J-P. SERRE, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. (2), vol. 58 (1953), pp. 258–294.

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