AN ISOMORPHISM THEOREM FOR CERTAIN FINITE GROUPS[1]

BY

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Introduction

Let K be a finite field of characteristic p, and let $SL(2, K)$ be the unimodular group of 2 by 2 matrices of determinant one with coefficients in K. We shall be concerned with a finite group G which satisfies a list of axioms which say, roughly speaking, that G is generated by a certain number of subgroups which are homomorphic images of $SL(2, K)$, and that G has p-Sylow subgroups X and Y with certain special properties. We prove that all the finite simple groups G' defined by Chevalley [2] with respect to a finite field K of characteristic $p \geqq 5$, and the variations of them defined by Steinberg [13], satisfy our axioms.

The first main result concerns two finite groups G and $\bar{G}$ satisfying the axioms, and generated by subgroups $\phi_1(SL(2, K_1)), \cdots, \phi_l(SL(2, K_l))$ and $\bar{\phi}_1(SL(2, K_1)), \cdots, \bar{\phi}_l(SL(2, K_l))$, respectively, where the K_i are subfields of K, and the ϕ_i and $\bar{\phi}_i$ are homomorphisms of $SL(2, K_i)$ into G and $\bar{G}$. Let M and $\bar{M}$ be irreducible right ΩG- and $\Omega\bar{G}$-modules respectively, where Ω is an arbitrary extension field of K, and ΩG, $\Omega\bar{G}$ denote the group algebras over Ω of G and $\bar{G}$. A sufficient condition is obtained in order that there exist an Ω-isomorphism $S : M \rightarrow \bar{M}$ such that

$$m\phi_i(g)S = (mS)\bar{\phi}_i(g),$$

for all $m \in M$, $g \in SL(2, K_i)$, and $1 \leqq i \leqq l$. When the hypotheses of this theorem are satisfied, and in addition the modules M and $\bar{M}$ are faithful G- and $\bar{G}$-modules, it follows that $G \cong \bar{G}$, and that the modules M and $\bar{M}$ are isomorphic as ΩG-modules.

The second main theorem again concerns finite groups G and $\bar{G}$ satisfying the axioms, and generated by the same number of homomorphic images of $SL(2, K)$, for a given field K. It is also assumed that the p-Sylow subgroups X and $\bar{X}$ of G and $\bar{G}$ respectively, are isomorphic and satisfy a further condition. It is then proved that both G and $\bar{G}$ satisfy the conditions (1)–(13) of Steinberg's paper [12], and consequently possess irreducible modules over Ω of dimension p^M, where p^M is the order of X. Finally it is shown that if neither G nor $\bar{G}$ has a nontrivial center, then the result of the preceding paragraph can be applied to show that G and $\bar{G}$ are isomorphic. The sufficient condition that $G \cong \bar{G}$ involves only group-theoretic properties of G and $\bar{G}$, and no information about modules over G and $\bar{G}$ is needed in order to apply the theorem.

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A somewhat different application is made to the following problem. Let $\mathfrak{L}$ be a Lie algebra of classical type over an algebraically closed field Ω of characteristic $p \geqq 5$, and let G_0 be the finite group of automorphisms of $\mathfrak{L}$ considered in [6]. Then G_0 is known to satisfy the axioms of the present paper. By the result of Steinberg's paper [12], there exists an irreducible ΩG_0-module M of dimension p^m, where p^m is the order of a p-Sylow subgroup of G_0, and m the number of positive roots of $\mathfrak{L}$ with respect to a Cartan subalgebra. It is proved that one of the irreducible projective representations of G_0 constructed in [6] from an irreducible restricted $\mathfrak{L}$-module, is in fact an ordinary representation of G_0, and is equivalent to the irreducible representation of G_0 afforded by the module M of Steinberg.

1. Axiomatics

This section is written in three parts. In part (a), we give our axioms for G. In part (b) we show that the conditions (1)–(14) of Steinberg's paper [12] are consequences of what has been assumed in (a). In part (c) we prove that the groups defined by Chevalley [2] and Steinberg [13] satisfy our axioms.

First we list a few notations:

$A \triangle B$	A is normal in B
$N_G(A)$	normalizer of A in G
$C_G(A)$	centralizer of A in G
$(a, b) = aba^{-1}b^{-1}$	
(A, B)	the group generated by all commutators (a, b) with $a \in A$, $b \in B$
$[A:B]$	index of a subgroup B in a group A
$[A:1]$	order of the group A
$a^b = bab^{-1}$	
$A^b = bAb^{-1}$	

1a. Throughout the paper, K will denote a finite field of $q = p^f$ elements, where p is a prime and f a positive integer. No other special hypotheses concerning K are needed for §§1a and 1b, and §§2–4. Ω will always denote a field containing K.

Let $SL(2, K)$ denote the group of all 2 by 2 matrices

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \qquad \alpha, \beta, \gamma, \delta \in K, \quad \alpha\delta - \beta\gamma = 1.$$

For all $\xi \in K$, let

$$u(\xi) = \begin{bmatrix} 1 & \xi \\ 0 & 1 \end{bmatrix}, \qquad v(\xi) = \begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix}, \qquad d(\xi) = \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix}, \qquad \xi \neq 0.$$

Let U be the subgroup of $SL(2, K)$ consisting of all $u(\xi)$, $\xi \in K$, V the subgroup consisting of the elements $v(\xi)$, $\xi \in K$, and D the subgroup consisting of all $d(\xi)$, $\xi \neq 0$. Let

$$\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};$$

then computations show that $d^{\omega} = d^{-1}$, $d \in D$, and that

$$\omega = u(1)v(-1)u(1), \quad \omega^2 \in D, \quad \omega U \omega^{-1} = V, \quad U^d = U, \quad V^d = V, \quad d \in D.$$

It is known (see [2, p. 34]) that $U \cup V$ is a set of generators of $SL(2, K)$, and that

$$SL(2, K) = UD \cup UD\omega U.$$

Now we are ready to state our axioms (1.1)–(1.13) concerning a finite group G.

(1.1) *For some positive integer l, there exist subfields $K_1, \cdots, K_l$ of K, and l homomorphisms $\phi_1, \cdots, \phi_l$ of $SL(2, K_i)$ into G such that*

$$\phi_1(SL(2, K_1)) \cup \cdots \cup \phi_l(SL(2, K_l))$$

is a set of generators of G, and $\phi_i(SL(2, K_i)) \neq \{1\}$, $1 \leqq i \leqq l$.

For $1 \leqq i \leqq l$, let $X_i = \phi_i(U)$, $Y_i = \phi_i(V)$, $D_i = \phi_i(D)$, $x_i(\xi) = \phi_i(u(\xi))$, $y_i(\xi) = \phi_i(v(\xi))$, $d_i(\xi) = \phi_i(d(\xi))$, $w_i = \phi_i(\omega)$.

(1.2) *The set $X_1 \cup \cdots \cup X_l$ generates a p-subgroup X of G; the set $Y_1 \cup \cdots \cup Y_l$ generates a p-subgroup Y of G.*

(1.3) *There exists a subgroup H of G such that*

$$D_i \subset H \subset N_G(X_i), \qquad 1 \leqq i \leqq l.$$

(1.4) $(X_i, Y_j) = \{1\}$, $i \neq j$, $1 \leqq i, j \leqq l$.

(1.5) $(Y, Y)^{x_i} \subset Y$ *for all* $x_i \in X_i$, $1 \leqq i \leqq l$; *and*
$(X, X)^{y_i} \subset X$ *for all* $y_i \in Y_i$, $1 \leqq i \leqq l$.

(1.6) $w_i \in N_G(H)$, $1 \leqq i \leqq l$.

We shall see that the axioms (1.1)–(1.6) are sufficient for the first main theorem in §3, and for the application in §5. The remaining axioms are needed in order to prove the conditions (1)–(14) of Steinberg's paper [12].

(1.7) $N_G(X) = XH$, $H \cap X = \{1\}$, *and* $p \nmid [H:1]$.

Let W be the subgroup of G generated by $H \cup \{w_1, \cdots, w_l\}$. Then $H \triangle W$ by (1.6). Let $W^* = W/H$, and denote the coset wH by w^* for all $w \in W$.

(1.8) *There exists an element $w_0 \in W$ such that $X^{w_0} = Y$.*

(1.9) $XH \cap Y = \{1\}$.

From (1.9) it follows that $w_i \notin H$, $1 \leqq i \leqq l$, since $X_i^{w_i} \subset Y$. For the next step we require also the fact that

$$H \subset N_G(Y).$$

To see this, let $h \in H$, and $y \in Y$. By (1.8) it follows that $y = x^{w_0}$ for some $x \in X$. Thus for some $h' \in H$ we have

$$y^h = (x^{w_0})^h = x^{hw_0} = x^{w_0 h'} = (x^{h'})^{w_0} \in Y$$

since $H \subset N_G(X)$. The same argument shows that $H \subset N_G(Y_i)$, $1 \leqq i \leqq l$, since $Y_i = X_i^{w_i}$.

For each coset $w^* = wH$ in W^*, let $X'_{w^*} = \{x \in X : x^w \in X\}$, and let $X''_{w^*} = \{x \in X : x^w \in Y\}$. Since $H \subset N_G(X) \cap N_G(Y)$, it is clear that X'_{w^*} and X''_{w^*} are defined independently of the choice of the coset representatives. It is also clear that X'_{w^*} and X''_{w^*} are subgroups of X such that $X'_{w^*} \cap X''_{w^*} = \{1\}$.

Now let $X \supset X^2 \supset X^3 \supset \cdots$ be the descending central series of the p-group X, where $X^i = (X^{i-1}, X)$, $i = 2, 3, \cdots$, and $X = X^1$.

(1.10) *For each $w^* \in W^*$, each term X^i of the descending central series of X is generated by $X^i \cap X'_{w^*}$ and $X^i \cap X''_{w^*}$.*

(1.11) *For each $w^* \in W^*$, either $X_i \subset X'_{w^*}$ or $X_i \subset X''_{w^*}$, for $1 \leqq i \leqq l$.*

(1.12) *If $X'_{w_1^*}$ and $X'_{w_2^*}$ are conjugate in X, then $w_1^* = w_2^*$.*

(1.13) *There exists a homomorphism $\varepsilon : W^* \to \{1, -1\}$ such that $\varepsilon(w_i^*) = -1$, $1 \leqq i \leqq l$.*

Note that (1.13) is possible in view of the fact that $w_i^* \neq 1$ in W^*, for $1 \leqq i \leqq l$.

1b. For the convenience of the reader we first reproduce the conditions (1)–(14) of Steinberg's paper [12], with some appropriate changes in notation.

(1.14) *There exist two subgroups X and H of G such that $X \cap H = \{1\}$, XH is a group, and $X \triangle XH$.*

(1.15) *There exists a group W^* (the Weyl group) and for each $w^* \in W^*$ an element $w \in G$ such that $\bigcup_{w^* \in W^*} Hw$ is a group W, $H \triangle W$, and $W/H \cong W^*$ under the mapping $w \to Hw = w^*$. The identification $Hw = w^*$ will be made.*

(1.16) *Corresponding to each $w^* \in W^*$, X has two subgroups X'_{w^*} and X''_{w^*} such that:*

(1.17) $X = X'_{w^*} X''_{w^*}$;

(1.18) $wX'_{w^*} w^{-1} \subset X$ *if* $wH = w^*$; *and*

(1.19) $X''_{w_0^*} = X$ *for some* $w_0^* \in W^*$.

(1.20) *Let $\{w_1, \cdots, w_q\}$ be coset representatives of H in W. Then*

$$G = \bigcup_{i=1}^{q} XHw_i X''_{w_i^*},$$

and

$$xhw_i x'' = x_1 h_1 w_j x_1'', \qquad x,\ x_1 \in X, \quad h,\ h_1 \in H, \quad x'' \in X''_{w_i^*}, \quad x_1'' \in X''_{w_j^*}$$

implies $x = x_1,\ h = h_1,\ w_i = w_j,\ x'' = x_1''$.

(1.21) *W^* contains a set of elements $\{w_i^*\}_{1 \leqq i \leqq l}$ such that:*

(1.22) $(w_i^*)^2 = 1, \quad 1 \leqq i \leqq l$;

(1.23) *$\{w_1^*, \cdots, w_l^*\}$ is a set of generators for W^*;*

(1.24) *for each i, $1 \leqq i \leqq l$, $X''_{w_i^*} H \cup X''_{w_i^*} Hw_i X''_{w_i^*}$ is a subgroup of G;*

(1.25) *for each $w^* \in W^*$ and w_i^*, $1 \leqq i \leqq l$, at least one of the inclusions*

$$X''_{w_i^*} \subset X'_{w^*}, \qquad X''_{w_i^*} \subset X'_{w^* w_i^*}$$

is valid; and

(1.26) *there is a homomorphism $\varepsilon : W^* \to \{1, -1\}$ such that $\varepsilon(w_i^*) = -1$, $1 \leqq i \leqq l$.*

The last condition from Steinberg's paper is

(1.27) *There is an element $x \in X$ such that $x \notin X'_{w^*}$ for all $w^* \neq 1$.*

Now we have the task of showing that (1.14)–(1.27) follow from (1.1)–(1.13). Although we do not use any interpretation of the group G in terms of automorphisms of Lie algebras, etc., many of the arguments will be almost identical with those in Chevalley's paper [2].

(1.14) follows from (1.2), (1.3), and (1.7). (1.15) follows from (1.6) and the definition of the group W, if we take for the elements $w \in G$ a set of coset representatives of H in W. The subgroups in (1.16) are those defined after (1.9).

Proof of (1.17). Let $(X'_{w^*})^i$ and $(X''_{w^*})^i$ denote the subgroups $X'_{w^*} \cap X^i$ and $X''_{w^*} \cap X^i$, $i = 1, 2, \cdots$. By (1.10), X^i is generated by $(X'_{w^*})^i$ and $(X''_{w^*})^i$. Since X is a p-group by (1.2), X^i is abelian for sufficiently large i, and in that case $X^i = (X'_{w^*})^i (X''_{w^*})^i$. Now let k be fixed, and suppose that for all $i > k$, $X^i = (X'_{w^*})^i (X''_{w^*})^i$. Then

$$\begin{aligned} X^k &= (X'_{w^*})^k (X''_{w^*})^k (X^k, X^k) = (X'_{w^*})^k (X''_{w^*})^k X^{k+1} \\ &= (X'_{w^*})^k X^{k+1} (X''_{w^*})^k \qquad\qquad (\text{since } X^{k+1} \triangle X^k) \\ &= (X'_{w^*})^k (X'_{w^*})^{k+1} (X''_{w^*})^{k+1} (X''_{w^*})^k = (X'_{w^*})^k (X''_{w^*})^k. \end{aligned}$$

By induction we have $X^i = (X'_{w^*})^i (X''_{w^*})^i$ for all i, and (1.17) is proved.

(1.18) is true by the definition of X'_{w^*}; (1.19) is valid because of (1.8). The proof of (1.20) is the same as the proof of the corresponding result in Chevalley's paper [2, Theorem 2, p. 42], and will be omitted.

The statements (1.21)–(1.23) follow from the definition of the group W, and the fact that for $1 \leqq i \leqq l$, $w_i^2 \in D_i \subset H$ by (1.3).

Proof of (1.24). Because $H \subset N_G(X_i)$ by (1.3), it is sufficient to prove that $X''_{w_i^*} = X_i$, $1 \leqq i \leqq l$. Since $X_i^{w_i} \subset Y$, we have $X_i \subset X''_{w_i^*}$. We next prove that if $j \neq i$, then $X_j \subset X'_{w_i^*}$. Since $w_i = x_i(1)y_i(-1)x_i(1)$, we have for $x_j \in X_j$,

$$\begin{aligned} x_j^{w_i} &= x_i(1)y_i(-1)x_i(1)x_j\, x_i(1)^{-1}y_i(-1)^{-1}x_i(1)^{-1} \\ &= x_i(1)\,(x_i(1), x_j)^{y_i(-1)}\, x_j^{y_i(-1)}\, x_i(-1) \in X \end{aligned}$$

by (1.4) and (1.5). Similarly, if $x \in (X, X)$,

$$x^{w_i} = x_i(1)\ (x_i(1), x)^{y_i(-1)}\, x^{y_i(-1)}\, x_i(1)^{-1} \in X.$$

Since $X_1 \cup \cdots \cup X_l$ generates X, it follows from what has been proved that

$$X = X'_{w_i^*}\, X_i\,, \qquad X'_{w_i^*} \cap X_i = \{1\}.$$

By (1.17) we have also

$$X = X'_{w_i^*}\, X''_{w_i^*}\,, \qquad X'_{w_i^*} \cap X''_{w_i^*} = \{1\}.$$

It follows that $[X_i{:}1] = [X''_{w_i^*}{:}1]$, and since $X_i \subset X''_{w_i^*}$, we have $X_i = X''_{w_i^*}$. As we have remarked, this proves (1.24).

Proof of (1.25). We have already shown that $X_i = X''_{w_i^*}$. Either $X_i \subset X'_{w^*}$ or $X_i \subset X''_{w^*}$, by (1.11). In the latter case, we have $wX_i\, w^{-1} \subset Y$. Setting $w^{-1} = w_i^{-1}w'$, we obtain $(w')^{-1}Y_i\, w' \subset Y$. Then $(w')^{-1}X_i\, w' \subset X$, otherwise $\phi_i(SL(2, K_i))^{(w')^{-1}} \subset Y$, and in particular $D_i^{(w')^{-1}} \subset Y \cap H = \{1\}$ by (1.3), (1.6), and (1.9), which is a contradiction. From $(w')^{-1}X_i\, w' \subset X$ we obtain $X_i \subset X'_{(ww_i^{-1})^*} = X'_{w^*w_i^*}$ since $w_i^{-1} \equiv w_i \pmod{H}$. This completes the proof of (1.25).

We note that (1.26) is included as axiom (1.13). The last condition (1.27) can also be proved from (1.1)–(1.13), but since only (1.14)–(1.26) are needed for the result we shall use from Steinberg's paper [12, Theorem 2, p. 349], we shall not include the proof of (1.27).

1 c. Let G' be the group defined by Chevalley [2, p. 47]. We assume that the characteristic p of K is greater than three. Let $\alpha_1, \cdots, \alpha_l$ be a fundamental set of roots of the Lie algebra $\mathfrak{g}$. We identify X_i with $\mathfrak{X}_{\alpha_i}$, and Y_i with $\mathfrak{X}_{-\alpha_i}$, $1 \leqq i \leqq l$. Then (1.1) is satisfied if we identify K with K_i, and ϕ_i with ϕ_{α_i}, $1 \leqq i \leqq l$, since G' is generated by $\mathfrak{X}_{\alpha_i}$ and $\mathfrak{X}_{-\alpha_i}$, $1 \leqq i \leqq l$ (see [2, p. 48]).

(1.28) LEMMA. *The subgroup $\mathfrak{U}$ of G' (defined in [2, p. 38]) is generated by $\mathfrak{X}_{\alpha_1} \cup \cdots \cup \mathfrak{X}_{\alpha_l}$ if $p \geqq 5$. If $\mathfrak{U}_m$ is the group generated by all subgroups $\mathfrak{X}_\alpha$, where α is a root of height $\geqq m$, then $\mathfrak{U}_m$ coincides with $\mathfrak{U}^m$, where $\mathfrak{U}^i = (\mathfrak{U}^{i-1}, \mathfrak{U})$, $i \geqq 0$, is the i^{th} term in the descending central sum of G'.*

Proof. For each $m \geqq 0$, let $\mathfrak{U}_m$ denote the term generated by the subgroups $\mathfrak{X}_\alpha$, for α a positive root of height $\geqq m$. By [2, p. 39],

$$(\mathfrak{U}_m\,, \mathfrak{U}_{m'}) \subset \mathfrak{U}_{m+m'}\,,$$

where we set $\mathfrak{U}_m = \{1\}$ if all roots of $\mathfrak{g}$ have height $< m$. Let

$$\mathfrak{U} = \mathfrak{U}^1 \supset \mathfrak{U}^2 \supset \cdots$$

be the descending central series of $\mathfrak{U}$. Evidently, $\mathfrak{U}^i \subset \mathfrak{U}_i$, $i \geqq 1$. Suppose for some $i \geqq 1$, $\mathfrak{U}^i = \mathfrak{U}_i$. We shall now prove that $\mathfrak{U}^{i+1} = \mathfrak{U}_{i+1}$, and for this it is sufficient to prove that $\mathfrak{U}_m \subset \mathfrak{U}^{i+1}$ for all $m \geqq i+1$. For sufficiently large m, we have $\mathfrak{U}_m \subset \mathfrak{U}^{i+1}$. Suppose for some $m \geqq i+1$, we have $\mathfrak{U}_{m+j} \subset \mathfrak{U}^{i+1}$ for $j = 1, 2, \cdots$. In order to prove that $\mathfrak{U}_m \subset \mathfrak{U}^{i+1}$, it is sufficient to show that for any positive root α of height m and $\xi \in K$, we have $x_\alpha(\xi) \in \mathfrak{U}^{i+1}$. We can express $\alpha = \beta + \alpha_i$ for some positive root β of height $m-1$ and a fundamental root α_i. Since $p \geqq 5$, the formulas for $N_{\alpha,\beta}$ and $M_{\alpha,\beta,i}$ in [2, p. 36] show that $C_{1,1,\alpha_i,\beta} \neq 0$ in K. Therefore by formula (4) of [2, p. 36], we can find $\xi', \eta' \in K$ such that

$$(x_\beta(\xi'), x_{\alpha_i}(\eta')) = x_\alpha(\xi)x^*$$

where $x^* \in \mathfrak{U}_{m+1} \subset \mathfrak{U}^{i+1}$, and $x_\beta(\xi') \in \mathfrak{U}_{m-1} \subset \mathfrak{U}_i \subset \mathfrak{U}^i$. It follows that $x_\alpha(\xi) \in \mathfrak{U}^{i+1}$, and we have proved that $\mathfrak{U}_m \subset \mathfrak{U}^{i+1}$ for $m \geqq i+1$. Therefore we have

$$\mathfrak{U}^i = \mathfrak{U}_i, \qquad i = 1, 2, \cdots. \tag{1.29}$$

In particular $\mathfrak{U}_2 = (\mathfrak{U}, \mathfrak{U})$, and since $\{\mathfrak{X}_{\alpha_1}, \cdots, \mathfrak{X}_{\alpha_l}\}$ generate $\mathfrak{U}$ modulo $(\mathfrak{U}, \mathfrak{U})$, and $\mathfrak{U}$ is a p-group, it follows from the Burnside basis theorem [8, p. 176] that $\{\mathfrak{X}_{\alpha_1}, \cdots, \mathfrak{X}_{\alpha_l}\}$ generate $\mathfrak{U}$. This completes the proof of Lemma 1.28.

If we identify the subgroup X in (1.2) with $\mathfrak{U}$, then the fact that X is a p-group follows from [2, p. 39, Lemma 6]. Similarly Y is a p-group.

Let H be the subgroup $\mathfrak{H}' = \mathfrak{H} \cap G'$ of G'. Then, remembering that $\mathfrak{U} = X$, we have by [2, Corollary 2, p. 43] that $N_{G'}(U) = \mathfrak{U}\mathfrak{H} \cap G' = \mathfrak{U}\mathfrak{H}'$. The fact $\mathfrak{U} \cap \mathfrak{H}' = \{1\}$ follows from [2, Lemma 13, p. 42]. Finally the inclusions $D_i \subset H \subset N_G(X_i)$, $1 \leqq i \leqq l$, and the fact that $p \nmid [H:1]$ are clear from the definition of H and the formulas (6) and (7) of [2, p. 36]. These remarks prove (1.3) and (1.7). (1.4) follows from formula (4) of [2, p. 36] and the fact that if α and β are fundamental roots, $i\alpha + j\beta$ is a root only if i and j have the same sign.

The second assertion of (1.5) follows from [2, Lemma 8, p. 40] and the fact that for a fundamental root α, $\mathfrak{U}_\alpha \supset \mathfrak{U}_2 = (\mathfrak{U}, \mathfrak{U})$.

If we identify w_i with ω_{α_i} defined in [2, p. 36], then (1.6) follows from [2, Lemma 3, p. 37]. By the argument in the proof of [2, Lemma 4, p. 38], we can identify the group W defined in §1a with the group $\mathfrak{W}$ of [2]. If we select $w_0 \in \mathfrak{W}$ so that its image $\zeta(w_0)$ in the Weyl group is the operation which interchanges positive and negative roots, then (1.8) holds for this choice of w_0. The first statement of (1.5) is also a consequence of what has been shown so far.

(1.9) is an immediate consequence of [2, Lemma 13, p. 42].

To prove (1.10), we begin with the facts that for an element w^* of the Weyl group W/H, $\mathfrak{U}'_{w^*}$ is generated by all $x_\alpha(\xi)$, $\xi \in K$, and α a positive root such that $w^*(\alpha)$ is also positive, while $\mathfrak{U}''_{w^*}$ is generated by all $x_\alpha(\xi)$ with $\alpha > 0$ and $w^*(\alpha) < 0$. It follows from the definition of $\mathfrak{U}_i$, $i \geqq 1$, that $\mathfrak{U}_i$ is generated by $\mathfrak{U}_i \cap \mathfrak{U}'_{w^*}$ and $\mathfrak{U}_i \cap \mathfrak{U}''_{w^*}$. Since $\mathfrak{U}_i = \mathfrak{U}^i$, $i \geqq 1$, by (1.29), we obtain (1.10).

(1.11) is immediate from the definitions of $\mathfrak{U}'_{w^*}$ and $\mathfrak{U}''_{w^*}$.

To prove (1.12), suppose that $\mathfrak{U}'_{w_1^*}$ and $\mathfrak{U}'_{w_2^*}$ are conjugate in $\mathfrak{U}$. By [2, Lemma 12, p. 41], we have $\mathfrak{U}'_{w_1^*} = \mathfrak{U}'_{w_2^*}$, and the roots $\{\alpha > 0 : w_i^*(\alpha) > 0\}$ are the same as the roots $\{\alpha > 0 : w_2^*(\alpha) > 0\}$. Therefore $w_1^*(w_2^*)^{-1}$ maps positive roots onto positive roots, and it follows that $w_1^* = w_2^*$.

Finally, (1.13) is proved by Steinberg's observation [12, p. 350] that for each element w^* of the Weyl group, we can set

$$\varepsilon(w^*) = (-1)^{n(w^*)},$$

where $n(w^*)$ is the number of positive roots α such that $w^*(\alpha) < 0$.

For Lie algebras $\mathfrak{g}$ of types A_l (l odd), D_l ($l \geqq 4$), and E_6, Steinberg has shown in [13] that $\mathfrak{g}$ admits an involution σ, and has defined a certain subgroup $G^{(1)}$ of the set of elements in G' which commute with σ (see [13, pp. 881 and 891]). He proved in [13] that $G^{(1)}$ is a simple group which, in the case of Lie algebras A_l and D_l, can be identified with projective unitary or projective orthogonal groups, respectively (see [13, pp. 882 and 886]). It can be proved using the structure theorems in Steinberg's paper, and arguments similar to those in the first part of this section, that the groups $G^{(1)}$ satisfy the axioms (1.1)–(1.13) of the present paper.[2] The details of this verification will be omitted.

2. Preliminary results on $SL(2, K)$

As in §1a, K denotes an arbitrary finite field. Besides the facts stated in §1a concerning $SL(2, K)$, we require the following formulas:

$$(2.1) \qquad v(\eta)u(\xi) = d(\mu)u(\xi')v(\eta'),$$

where $\mu = (1 + \xi\eta)^{-1}$, $\xi' = \mu^{-1}\xi$, $\eta' = \mu\eta$, if $1 + \xi\eta \neq 0$; and

$$(2.2) \qquad v(\eta)u(\xi) = d(\mu)\omega v(\eta'),$$

where $\mu = \xi$, $\eta' = \mu^{-1}$, in case $1 + \xi\eta = 0$.

These facts may be established by a computation, and we omit the details.

Now let Ω be an arbitrary extension field of K. We let $\Omega(SL(2, K))$ denote the group algebra of $SL(2, K)$ over Ω.

(2.3) Lemma. *Let T be a right $\Omega(SL(2, K))$-module, and let $t_0 \in T$ be such that $t_0 \neq 0$, $t_0 u = t_0$, $u \in U$, and $t_0 d = f(d)t_0$, for all $d \in D$, where $f(d) \in \Omega$. Let $\hat{U} = \sum_{u \in U} u$, and let $\hat{t}_0 = t_0 \omega \hat{U}$. Then $\hat{t}_0 u = \hat{t}_0$, $u \in U$; and $\hat{t}_0 d = \hat{f}(d)t_0$,*

[2] The author is indebted to the referee for this observation, as well as for suggestions which have made important simplifications of the proofs of some of the other theorems in the paper.

$d \in D$, where $\hat{f}(d) \in \Omega$, and

$$t_0\,\omega = \hat{t}_0 + \textstyle\sum_{v \in V} \xi(v)\, t_0\, v,$$

where the coefficients $\xi(v)$ depend only on the function $f : D \to \Omega$.

Proof. For all $u \in U$, we have $\hat{U}u = \hat{U}$ in the group algebra $\Omega(SL(2, K))$; therefore $\hat{t}_0\, u = \hat{t}_0$ for all $u \in U$. If $d \in D$, we have $U^d = U$, and hence $\hat{U}d = d\hat{U}$. Then we have for $d \in D$,

$$\hat{t}_0\, d = t_0\, \omega\hat{U}d = t_0\, \omega d\hat{U} = t_0\, d^{-1}\, \omega\hat{U} = f(d^{-1})\hat{t}_0\,.$$

Finally the properties of ω imply that

$$\begin{aligned} t_0\,\omega\hat{U} &= t_0\, u(1)v(-1)u(1) \textstyle\sum_{\xi \in K} u(\xi) \\ &= t_0\, v(-1) \textstyle\sum_{\xi \in K} u(\xi) \\ &= t_0\, u(1)v(-1)u(1) + \textstyle\sum_{\xi \neq 1} t_0\, v(-1)u(\xi) \\ &= t_0\,\omega + \textstyle\sum_{\xi \neq 1} t_0\, d(1 - \xi)^{-1}\, u((1 - \xi)\xi)v(-(1 - \xi)^{-1}) \\ &= t_0\,\omega + \textstyle\sum_{\xi \neq 1} f(d(1 - \xi)^{-1})t_0\, v(-(1 - \xi)^{-1}), \end{aligned}$$

and the lemma is proved.

3. Equivalence of irreducible ΩG-modules

In this section Ω denotes an arbitrary extension field of K, G a finite group satisfying axioms (1. 1)–(1. 6) of §1a, and M a finite-dimensional right ΩG-module.

(3. 1) DEFINITION. A *maximal vector relative to* G (or in §§3 and 4 simply a maximal vector) is a nonzero element m of M such that $mx = m$ for all $x \in X$, and $mh = f(h)m$ for $h \in H$, where $f(h) \in \Omega$.

Remark. For our purposes it is enough to consider only ΩG-modules which contain maximal vectors. If the group H is abelian (as it is if G is a group G' defined in Chevalley's paper [2]), and Ω is an algebraically closed field, we can prove that any right ΩG-module M contains at least one maximal vector. Indeed, let N be an irreducible $\Omega(XH)$-submodule of M. Since $X \triangle XH$, Clifford's Theorem [3] implies that N is a completely reducible ΩX-module. But X is a p-group and Ω has characteristic p; therefore the action of X on N is trivial. Thus N is in fact an irreducible $\Omega(XH/X)$-module. Since H is abelian and Ω is algebraically closed, it follows that N is one-dimensional, say $N = \Omega n$. From what has been said, we deduce that n is a maximal vector.

Before proceeding, we point out that if M is a right ΩG-module, then each of the homomorphisms $\phi_i : SL(2, K_i) \to G$ gives M the structure of an $\Omega(SL(2, K_i))$-module, the action of $x \in SL(2, K_i)$ on $m \in M$ being given by

$$mx = m\phi_i(x), \qquad 1 \leqq i \leqq l.$$

The results of §2 can of course be applied to each of these $SL(2, K_i)$-modules associated with M. As in that section, we let

$$\hat{X}_i = \sum_{x \in X_i} x, \qquad 1 \leqq i \leqq l.$$

(3.2) LEMMA. *Let m be a maximal vector in a right ΩG-module M. For each i, $1 \leqq i \leqq l$, let $\hat{m}_i = mw_i \hat{X}_i$. Then either $\hat{m}_i = 0$ or $\hat{m}_i$ is a maximal vector, for $1 \leqq i \leqq l$.*

Proof. Suppose $\hat{m}_i \neq 0$, and let $h \in H$. By (1.3), $h \in N_G(X_i)$; hence $\hat{X}_i h = h\hat{X}_i$. Then

$$\hat{m}_i h = mw_i \hat{X}_i h = mw_i h \hat{X}_i = mh^{w_i} w_i \hat{X}_i = f(h^{w_i}) \hat{m}_i,$$

where f is the function on $H \rightarrow \Omega$ associated with m. It is also clear that $\hat{m}_i x = \hat{m}_i$ for all $x \in X_i$. Since X is generated by $\{X_1, \cdots, X_l\}$, it is sufficient to prove that for $x \in X_j$, $j \neq i$, we have $\hat{m}_i x = \hat{m}_i$. Since

$$w_i = x_i(1) y_i(-1) x_i(1),$$

we have, for $x \in X_j$,

$$(3.3) \qquad \begin{aligned} \hat{m}_i x &= mw_i \hat{X}_i x = mx_i(1) y_i(-1) x_i(1) \big(\textstyle\sum_{\xi \in K_i} x_i(\xi)\big) x \\ &= my_i(-1) \big(\textstyle\sum_{\xi \in K_i} x_i(\xi)\big) x = my_i(-1) x^{-1} \big(\textstyle\sum_{\xi \in K_i} x_i(\xi)\big) x, \end{aligned}$$

because m is a maximal vector and

$$my_i(-1) = mx^{-1} y_i(-1) = my_i(-1) x^{-1}$$

by (1.4). Continuing we have, by (3.3),

$$\begin{aligned} \hat{m}_i x &= my_i(-1) \big(\textstyle\sum_{\xi \in K_i} x_i(\xi)\big) + \textstyle\sum_{\xi \in K_i} my_i(-1) [x^{-1} x_i(\xi) x - x_i(\xi)] \\ &= \hat{m}_i + \textstyle\sum_{\xi \in K_i} my_i(-1) [(x^{-1}, x_i(\xi)) - 1] x_i(\xi) \\ &= \hat{m}_i + \textstyle\sum_{\xi \in K_i} m[(x^{-1}, x_i(\xi))^{y_i(-1)} - 1] y_i(-1) x_i(\xi) \\ &= \hat{m}_i, \end{aligned}$$

since $(x^{-1}, x_i(\xi))^{y_i(-1)} \in X$ by (1.5), and $m(g - 1) = 0$ for $g \in X$ because m is a maximal vector. This completes the proof of the lemma.

For any right ΩG-module M the set of maximal vectors in M generate an Ω-subspace of M which we shall denote by M_+.

(3.4) LEMMA. *Let M be an irreducible right ΩG-module such that $M_+ = \Omega m_0$, $m_0 \neq 0$. Then $M = m_0 \Omega Y = \Omega m_0 \oplus m_0$ rad ΩY, where* rad ΩY *is the radical of the group algebra ΩY.*

Proof. The set $X_1 \cup \cdots \cup X_l \cup Y$ is a set of generators for G such that $m_0 x_i = m_0$, $x_i \in X_i$, $1 \leqq i \leqq l$. Since M is irreducible, in order to prove that $M = m_0 \Omega Y$, it is sufficient to prove that if $y \in Y$ and $x_i \in X_i$, $1 \leqq i \leqq l$, then

$$m_0 y x_i \in m_0 \Omega Y.$$

First suppose that $y = y_i \in Y_i$. Then $y_i = y_i(\eta)$, $x_i = x_i(\xi)$ for some $\xi, \eta \in K_i$. By (2.1) and (2.2) we have either

$$y_i x_i = d_i(\mu)x_i(\xi')y_i(\eta') \quad \text{if} \quad 1 + \xi\eta \neq 0,$$

and $m_0 y_i x_i = m_0 d_i(\mu)x_i(\xi')y_i(\eta') = f(d_i(\mu))m_0 y_i(\eta') \in m_0 \Omega Y$, or

$$y_i x_i = d_i(\mu)w_i y_i(\eta') \quad \text{if} \quad 1 + \xi\eta = 0.$$

In the latter case we have by Lemma 2.3

$$m_0 y_i x_i = f(d_i(\mu))[m_0 w_i \hat{X}_i y_i(\eta') + \textstyle\sum_{\lambda \in K_i} \xi_\lambda m_0 y_i(\lambda + \eta')], \quad \xi_\lambda \in \Omega.$$

Since $M_+ = \Omega m_0$, Lemma 3.2 implies that $m_0 w_i \hat{X}_i \in \Omega m_0$, and hence $m_0 y_i x_i \in m_0 \Omega Y$ as required.

Now let $y \in Y$ be arbitrary. Then we can write $y = y_i (\prod_{j \neq i} y_j)\bar{y}$, where $y_j \in Y_j$, $1 \leqq j \leqq l$, and $\bar{y} \in (Y, Y)$. Then we have

$$m_0 yx_i = m_0 y_i x_i (\textstyle\prod_{j \neq i} y_j) x_i^{-1}\bar{y}x_i \in m_0 \Omega Y$$

by what has been proved together with the facts that $x_i^{-1}y_j x_i = y_j$, $i \neq j$, by (1.4), and $x_i^{-1}\bar{y}x_i \in Y$, by (1.5).

For the last statement of the lemma, we use the fact that since Y is a p-group, and Ω has characteristic p, $\Omega Y = \Omega \cdot 1 \oplus \operatorname{rad} \Omega Y$, and $\operatorname{rad} \Omega Y$ has a basis over Ω consisting of the elements $y - 1$, $y \in Y$, $y \neq 1$. From these remarks, together with the first part of the lemma, it is clear that $M = \Omega m_0 \oplus m_0 \operatorname{rad} \Omega Y$, and the lemma is proved.

Let M be an irreducible ΩG-module satisfying the hypotheses of the preceding lemma. Then there exists a function $f : G \to \Omega$ such that for all $x \in G$,

$$(3.5) \qquad m_0 x \equiv f(x)m_0 \pmod{m_0 \operatorname{rad} \Omega Y}.$$

The main theorem of this section asserts that this function determines the module M up to isomorphism. For our purposes, it is necessary to prove a more general theorem. Let $\bar{G}$ be another finite group satisfying the axioms (1.1)–(1.6), where the field K, the subfields K_i, and the number l are the same as for G. Let $\bar{\phi}_1, \cdots, \bar{\phi}_l$ be the given homomorphisms of $SL(2, K_i) \to \bar{G}$, and let $\bar{x}_i(\xi), \bar{y}_i(\xi), \bar{d}_i(\xi), \bar{w}_i$ be defined as the corresponding elements were defined for G.

(3.6) THEOREM. *Let G and $\bar{G}$ be finite groups satisfying the axioms (1.1)–(1.6), with respect to the same field K, the same subfields K_i, and with the same number of homomorphisms in (1.1) of $SL(2, K_i)$ into G or $\bar{G}$. Suppose there exists an isomorphism θ of Y onto $\bar{Y}$ such that $\theta(y_i(\xi)) = \bar{y}_i(\xi)$, $1 \leqq i \leqq l$, $\xi \in K_i$, and such that for $x_i(\xi) \in X_i$, and $y \in (Y, Y)$, $\theta(y^{x_i(\xi)}) = \theta(y)^{\bar{x}_i(\xi)}$. Let Ω be an extension field of K, and let M and $\bar{M}$ be irreducible ΩG- and $\Omega\bar{G}$-modules such that $M_+ = \Omega m_0$, $\bar{M}_+ = \Omega\bar{m}_0$. Let f and $\bar{f}$ be the functions on G and $\bar{G}$*

to Ω defined by (3.5), *and suppose that*

$$f(d_i(\mu)) = \bar{f}(\bar{d}_i(\mu)), \qquad 1 \leqq i \leqq l, \quad \mu \in K_i,$$

and

$$f(w_i) = \bar{f}(\bar{w}_i), \qquad 1 \leqq i \leqq l.$$

Then there exists an Ω-isomorphism S of M onto $\bar{M}$ such that for all $m \in M$,

$$(mx_i(\xi))S = (mS)\bar{x}_i(\xi), \qquad (my_i(\xi))S = (mS)\bar{y}_i(\xi),$$

for $\xi \in K_i$ and $1 \leqq i \leqq l$.

Proof. Let $Z = X_i \cup \cdots \cup X_l \cup Y_1 \cup \cdots \cup Y_l$. Then Z is a set of generators of G, and $\bar{Z} = \bar{X}_1 \cup \cdots \cup \bar{X}_l \cup \bar{Y}_1 \cup \cdots \cup \bar{Y}_l$ is a set of generators for $\bar{G}$. Every element of M can be expressed as a linear combination of elements of the form

$$m_0 z_1 \cdots z_s, \qquad z_i \in Z, \quad s \geqq 0,$$

which is irredundant in the sense that if two adjacent elements $z_i z_{i+1}$ belong to the same X_j or Y_j, then $z_i z_{i+1}$ is replaced by $z = z_i z_{i+1}$. Corresponding to each such expression we have an irredundant expression $\bar{m}_0 \bar{z}_1 \cdots \bar{z}_s$ in $\bar{M}$, where if $z_j = x_j(\xi)$ or $y_j(\xi)$, $\bar{z}_j = \bar{x}_j(\xi)$ or $\bar{y}_j(\xi)$ respectively. Define $f(z_1, \cdots, z_s) \in \Omega$ by the formula

$$m_0 z_1 \cdots z_s \equiv f(z_1, \cdots, z_s) m_0 \pmod{m_0 \operatorname{rad} \Omega Y}$$

and $\bar{f}(\bar{z}_1, \cdots, \bar{z}_s) \in \Omega$ by setting

$$\bar{m}_0 \bar{z}_1 \cdots \bar{z}_s \equiv \bar{f}(\bar{z}_1, \cdots, \bar{z}_s) m_0 \pmod{m_0 \operatorname{rad} \Omega \bar{Y}}.$$

We shall prove first that

$$f(z_1, \cdots, z_s) = \bar{f}(\bar{z}_1, \cdots, \bar{z}_s) \tag{3.7}$$

for all irredundant expressions $m_0 z_1 \cdots z_s$ and $\bar{m}_0 \bar{z}_1 \cdots \bar{z}_s$.

We use induction on the number t of factors z_i which belong to $X_1 \cup \cdots \cup X_l$, the result being obvious if $t = 0$, since in that case $f(z_1, \cdots, z_s) = \bar{f}(\bar{z}_1, \cdots, \bar{z}_s) = 0$. We may assume $t > 0$, and that the result is valid for expressions with less than t factors from $X_1 \cup \cdots \cup X_l$. If $z_1 \in X_j$, then by the induction hypothesis and the fact that $m_0 z_1 = m_0$,

$$f(z_1, \cdots, z_s) = f(z_2, \cdots, z_s) = \bar{f}(\bar{z}_2, \cdots, \bar{z}_s) = \bar{f}(\bar{z}_1, \cdots, \bar{z}_s).$$

For fixed t, we may now assume that (3.7) holds for expressions $m_0 z_1' \cdots z_q'$ in which the index of the first $z_j' \in X_1 \cup \cdots \cup X_l$ is less than the index of the first $z_k \in X_1 \cup \cdots \cup X_l$ in a given expression $m_0 z_1 \cdots z_s$. Next suppose $z_1 \in Y_j$, $z_2 \in X_k$. Then $f(z_1, \cdots, z_s) = f(z_2, z_1, \cdots, z_s)$ if $j \neq k$, and we are back to the first case. If $j = k$, let $z_1 = y_j(\eta)$ and $z_2 = x_j(\xi)$, $\xi, \eta \in K_j$. Then by (2.1) and (2.2) we have either of two cases. Suppose first that

$$z_1 z_2 = d_j(\mu) x_j(\xi') y_j(\eta');$$

then

$$\begin{aligned} f(z_1, \cdots, z_s) &= f(d_j(\mu))f(y_j(\eta'), \cdots, z_s) = \bar{f}(\bar{d}_j(\mu))\bar{f}(\bar{y}_j(\eta'), \cdots, \bar{z}_s) \\ &= \bar{f}(\bar{z}_1, \cdots, \bar{z}_s) \end{aligned}$$

by the induction hypothesis. Now let

$$z_1 z_2 = d_j(\mu) w_j y_j(\eta').$$

Then by the proof of Lemma 2.3 and the hypothesis of the theorem we have

$$\begin{aligned} m_0 w_j &= m_0 w_j \hat{X}_j + \textstyle\sum_{\lambda \in K_j, \lambda \neq 0} \xi_\lambda m_0 y_j(\lambda) \\ &\equiv (\xi^* + \textstyle\sum \xi_\lambda) m_0 \qquad\qquad (\text{mod } m_0 \text{ rad } \Omega Y), \end{aligned}$$

where $m_0 w_j \hat{X}_j = \xi^* m_0$ by Lemma 3.2, and $\xi_\lambda = f(d_j(-\lambda))$, $\lambda \in K_j$. Since $f(w_j) = \bar{f}(\bar{w}_j)$, it follows that if $\bar{m}_0 \bar{w}_j \hat{\bar{X}}_j = \bar{\xi}^* \bar{m}_0$, then $\xi^* = \bar{\xi}^*$. Then by Lemma 2.3,

$$\begin{aligned} m_0 z_1 \cdots z_s = f_j(d_j(\mu))[&\xi^* m_0 y_j(\eta') z_3 \cdots z_s \\ &+ \textstyle\sum_{\lambda \neq 0} f(d_j(-\lambda)) m_0 y_j(\eta' + \lambda) z_3 \cdots z_s)], \end{aligned}$$

and a similar expression holds for $\bar{m}_0 z_1 \cdots z_s$, with $\xi^* = \bar{\xi}^*$. Then

$$\begin{aligned} f(z_1, \cdots, z_s) &= f(d_j(\mu))[\xi^* f(y_j(\eta'), z_3, \cdots, z_s) \\ &\qquad + \textstyle\sum_{\lambda \neq 0} f(d_j(-\lambda)) f(y_j(\eta' + \lambda), z_3, \cdots, z_s)] \\ &= \bar{f}(\bar{d}_j(\mu))[\bar{\xi}^* \bar{f}(\bar{y}_j(\eta'), \bar{z}_3, \cdots, \bar{z}_s) \\ &\qquad + \textstyle\sum_{\lambda \neq 0} \bar{f}(\bar{d}_j(-\lambda)) \bar{f}(\bar{y}_j(\eta' + \lambda), \bar{z}_3, \cdots, \bar{z}_s)] \\ &= \bar{f}(\bar{z}_1, \cdots, \bar{z}_s) \end{aligned}$$

by the induction hypothesis. Finally suppose that $z_q \in Y_1 \cup \cdots \cup Y_l$ for $1 \leqq q \leqq i-1$ for some $i \geqq 3$, and let $z_i \in X_j$. If $z_{i-1} \in Y_k$ for $k \neq j$, then

$$f(z_1, \cdots, z_s) = f(z_1, \cdots, z_{i-2}, z_i, z_{i-1}, \cdots, z_s) = \bar{f}(\bar{z}_1, \cdots, \bar{z}_s)$$

by the induction hypothesis. If on the other hand, $z_i \in X_j$, $z_{i-1} \in Y_j$, and $z_{i-2} \in Y_k$, $k \neq j$, then by (1.5),

$$(z_{i-i}^{-1}, z_{i-2})^{z_i^{-1}} = z^{(1)} \cdots z^{(t)}, \qquad z^{(r)} \in Y_1 \cup \cdots \cup Y_l, \quad 1 \leqq r \leqq t,$$

and by the hypothesis of the theorem

$$(\bar{z}_{i-1}^{-1}, \bar{z}_{i-2})^{\bar{z}_i^{-1}} = \bar{z}^{(1)} \cdots \bar{z}^{(t)}.$$

Then since $z_{i-2} z_{i-1} z_i = z_{i-1} z_i (z_{i-1}^{-1}, z_{i-2})^{z_i^{-1}} z_{i-2}^{z_i^{-1}}$, we have

$$\begin{aligned} f(z_1, \cdots, z_s) &= f(z_1, \cdots, z_{i-3}, z_{i-1}, z_i, z^{(1)}, \cdots, z^{(t)}, z_{i-2}, \cdots) \\ &= \bar{f}(\bar{z}_1, \cdots, \bar{z}_{i-3}, \bar{z}_{i-1}, \bar{z}_i, \bar{z}^{(1)}, \cdots, \bar{z}^{(t)}, \bar{z}_{i-2}, \cdots) \\ &= \bar{f}(\bar{z}_1, \cdots, \bar{z}_s) \end{aligned}$$

by the induction hypothesis. This completes the proof of (3.7).

Now consider the set $\bar{N}$ of all elements in $\bar{M}$,

$$\sum \alpha(i_1, \cdots, i_s)\bar{m}_0 \bar{z}_{i_1} \cdots \bar{z}_{i_s}, \qquad \alpha(i_1, \cdots, i_s) \in \Omega,$$

for which the corresponding expression

$$\sum \alpha(i_1, \cdots, i_s) m_0 z_{i_1} \cdots z_{i_s} = 0$$

in M. Then $\bar{N}$ is a submodule of $\bar{M}$, and since $\bar{M}$ is irreducible, $\bar{N}$ is either zero or $\bar{N} = M$. We prove[3] $\bar{N} = \{0\}$ by showing that $\bar{m}_0 \notin \bar{N}$. If on the contrary $\bar{m}_0 \in \bar{N}$, then we have

$$\bar{m}_0 = \sum \alpha(i_1, \cdots, i_s)\bar{m}_0 \bar{z}_{i_1} \cdots \bar{z}_{i_s},$$

while

$$\sum \alpha(i_1, \cdots, i_s) m_0 z_{i_1} \cdots z_{i_s} = 0$$

in M. By (3.7), upon taking congruences mod $\bar{m}_0$ rad $\Omega\bar{Y}$ and m_0 rad ΩY, we obtain the contradiction

$$\begin{aligned} 1 &= \sum \alpha(i_1, \cdots, i_s)\bar{f}(\bar{z}_{i_1}, \cdots, \bar{z}_{i_s}) \\ &= \sum \alpha(i_1, \cdots, i_s) f(z_{i_1}, \cdots, z_{i_s}) = 0. \end{aligned}$$

Thus $\bar{N} = 0$. It follows that the mapping

$$S : \sum \alpha(i_1, \cdots, i_s) m_0 z_{i_1} \cdots z_{i_s} \rightarrow \sum \alpha(i_1, \cdots, i_s)\bar{m}_0 \bar{z}_{i_1} \cdots \bar{z}_{i_s}$$

is an Ω-homomorphism for M onto $\bar{M}$. Since M is irreducible, the kernel is zero, and S is an Ω-isomorphism. The fact that S intertwines the generators of G and $\bar{G}$ in the required way is clear from the definition of S. This completes the proof of the theorem.

(3.8) COROLLARY. *Let G, $\bar{G}$, M, $\bar{M}$ satisfy the hypotheses of the previous theorem. If M and $\bar{M}$ are faithful G- and $\bar{G}$-modules, respectively, then G and $\bar{G}$ are isomorphic.*

Proof. Consider the mapping

$$\rho : z_1 \cdots z_s \rightarrow \bar{z}_1 \cdots \bar{z}_s$$

of G onto $\bar{G}$. If $z_1 \cdots z_s = 1$, it follows from the theorem that $\bar{m}\bar{z}_1 \cdots \bar{z}_s = \bar{m}$ for all $\bar{m} \in \bar{M}$, and since $\bar{M}$ is a faithful $\bar{G}$-module, we have $\bar{z}_1 \cdots \bar{z}_s = 1$. Similarly, $\bar{z}_1 \cdots \bar{z}_s = 1$ implies $z_1 \cdots z_s = 1$. From these remarks it is clear that ρ is an isomorphism of G onto $\bar{G}$.

4. The isomorphism theorem

We shall combine the theorem of the preceding section with Steinberg's result [12] on the construction of irreducible modules for finite groups satisfying (1.14)–(1.26) to obtain an isomorphism theorem for finite groups satisfying (1.1)–(1.13). As in §3, Ω denotes an arbitrary extension field of K.

[3] This argument is similar to the well-known proof of E. Cartan and H. Weyl that an irreducible representation of a semisimple Lie algebra is determined by its highest weight.

(4.1) LEMMA. *Let G be a finite group satisfying the axioms* (1.1)–(1.13). *Then there exists an irreducible ΩG-module M with the following properties.*

(i) $M_+ = \Omega m_0$ *for some* $m_0 \neq 0$ *in* M.

(ii) $m_0 h = m_0$ *for all* $h \in H$.

(iii) $m_0 w_i \in m_0 \operatorname{rad} \Omega Y$.

(iv) *M is a faithful G-module if* $C_G(X) \cap H = \{1\}$.

Proof. By §1b, G satisfies (1.14)–(1.26). Moreover, we shall prove that X is a p-Sylow subgroup of G. If X is not a p-Sylow subgroup, then there exists a p-group $X' \subset N_G(X)$ such that X' properly contains X, and this contradicts (1.7). By Theorem 2 of Steinberg [12], there exists an irreducible right ΩG-module M constructed in the following way. In the group algebra ΩG, let

$$\hat{X} = \textstyle\sum_{x \in X} x, \qquad \hat{H} = \textstyle\sum_{h \in H} h,$$

and let $\{w\}$ be a complete set of coset representatives of H in W. In ΩG form the element

$$e = \hat{X}\hat{H} \textstyle\sum \varepsilon(w^*) w,$$

the summation being over all coset representatives of H in W. Then let $M = e\Omega X$.

In [12, Theorem 2(i)], it is shown that M viewed as a right ΩX-module is isomorphic to the right ΩX-module ΩX itself. Since X is a p-group, ΩX is indecomposable, and has a unique minimal submodule, which is a one-dimensional space on which X acts trivially. Therefore the space M_+ is at most one-dimensional.

From what has been said we have $e\hat{X} \neq 0$, and clearly $e\hat{X}x = e\hat{X}$ for all $x \in X$. Now let $h \in H$. Since $H \subset N_G(X)$, we have $\hat{X}h = h\hat{X}$, and

$$\begin{aligned} e\hat{X}h &= eh\hat{X} = \hat{X}\hat{H}(\textstyle\sum \varepsilon(w^*) wh)\hat{X} \\ &= \hat{X}\hat{H}(\textstyle\sum h^w \varepsilon(w^*) w)\hat{X} \\ &= \hat{X}\hat{H}(\textstyle\sum \varepsilon(w^*) w)\hat{X} = e\hat{X} \end{aligned}$$

since $\hat{H}h^w = \hat{H}$. This proves that $m_0 = e\hat{X}$ is a maximal vector, and we have established parts (i) and (ii) of the lemma.

Now consider w_i, $1 \leqq i \leqq l$. Since $X_i \subset X''_{w_i^*}$, we have $X''_{w_i^*} \neq \{1\}$. By (1.17) and the fact that $X'_{w_i^*} \cap X''_{w_i^*} = \{1\}$ we have

$$\hat{X} = \hat{X}'_{w_i^*} \hat{X}''_{w_i^*}.$$

Then by (i) of Lemma 1, [12, p. 348], $ew_i^{-1} = -e$, and

$$\begin{aligned} m_0 w_i^{-1} &= e\hat{X}w_i^{-1} = e\hat{X}'_{w_i^*} \hat{X}''_{w_i^*} w_i^{-1} \\ &= ew_i^{-1}(\hat{X}'_{w_i^*})^{w_i}(\hat{X}''_{w_i^*})^{w_i} \\ &= -e(\hat{X}'_{w_i^*})^{w_i}(\hat{X}''_{w_i^*})^{w_i}. \end{aligned}$$

We have

$$(\hat{X}''_{w_i*})^{w_i} = \sum_x x^{w_i} = \sum_x [1 + (x^{w_i} - 1)] \qquad (x \in X''_{w_i*})$$
$$= \sum_x (x^{w_i} - 1) \in \operatorname{rad} \Omega Y,$$

since $x^{w_i} \in Y$ for $x \in X''_{w_i*}$, and since $X''_{w_i*} \neq \{1\}$, so that

$$\sum_x 1 = 0 \qquad (x \in X''_{w_i*}).$$

We have shown that

$$m_0 w_i^{-1} \in M \operatorname{rad} \Omega Y \subset m_0 \operatorname{rad} \Omega Y,$$

since $M = \Omega m_0 \oplus m_0 \operatorname{rad} \Omega Y$ by Lemma 3.4. Since $hw_i = w_i^{-1}$ for some $h \in H$, we have

$$m_0 w_i^{-1} = m_0 hw_i = m_0 w_i \in m_0 \operatorname{rad} \Omega Y,$$

and (iii) is proved.

It remains to prove (iv). Let $g \in G$ be expressed uniquely according to (1.20) as

$$g = hxwx', \qquad x \in X, \quad h \in H, \quad x' \in X''_{w*}, \quad g \neq 1.$$

If $m_0 g = m_0$, then from what has been proved we have $m_0 w = m_0$, but if $w \notin H$, we have $w \notin N_G(X)$, and hence $X''_{w*} \neq \{1\}$. By the proof of part (iii) we obtain also $m_0 \in m_0 \operatorname{rad} \Omega Y$, which is a contradiction. Therefore $m_0(g - 1) \neq 0$ if $w \notin H$. If $w \in H$, we may assume that $w = 1$; then $X''_{w*} = \{1\}$, and we have $g = xh$. If $x \neq 1$, then $xh = hx'$ for some $x' \in X$, $x' \neq 1$, and we have

$$exh = ehx' = ex' \neq e$$

since $eh = e$ and because the elements ex, $x \in X$, are linearly independent. It remains to consider the case $g = h \in H$. If $C_G(X) \cap H = \{1\}$, then for some $x \in X$, $xh = hx'$ for $x' \in X$, $x' \neq x$. Then $exh = ehx' = ex' \neq ex$. This completes the proof of the lemma.

Finally we can state our main theorem.

(4.2) THEOREM. *Let G and $\bar{G}$ be finite groups satisfying the axioms (1.1)–(1.13). Suppose that both G and $\bar{G}$ satisfy the condition $C_G(X) \cap H = \{1\}$. Suppose that the field K and the subfields K_i, are the same in both cases, and that $\{\phi_1, \cdots, \phi_l\}$ and $\{\bar{\phi}_1, \cdots, \bar{\phi}_l\}$ are the given homomorphisms of $SL(2, K_i)$ into G and $\bar{G}$ respectively. Let*

$$x_i(\xi) = \phi_i(u(\xi)), \qquad \bar{x}_i(\xi) = \bar{\phi}_i(u(\xi)), \quad \textit{etc.}$$

Finally suppose there exists an isomorphism θ of the p-Sylow subgroup Y of G onto the p-Sylow subgroup $\bar{Y}$ of $\bar{G}$ such that $\theta(y_i(\xi)) = \bar{y}_i(\xi)$, and for all $y \in (Y, Y)$, $x_i(\xi) \in X_i$, $1 \leqq i \leqq l$, we have $\theta(y^{x_i(\xi)}) = \theta(y)^{\bar{x}_i(\xi)}$. Then the mapping

$$x_i(\xi) \to \bar{x}_i(\xi), \qquad y_i(\xi) \to \bar{y}_i(\xi), \qquad 1 \leqq i \leqq l,$$

can be extended to an isomorphism of G onto $\bar{G}$.

The proof is immediate by Lemma 4.1, Theorem 3.6, and Corollary 3.8.

We prove finally that for a group G satisfying (1.1)–(1.13), $C_G(X) \cap H$ is contained in the center of G, so that the hypothesis of Theorem 4.2 is satisfied for the simple groups constructed by Chevalley [2] and Steinberg [13]. Let $h \in C_G(X) \cap H$. Then by the proof of part (iv) of Lemma 4.1, $M(h - 1) = 0$. Since the set of all $g \in G$ such that $M(g - 1) = 0$ is a normal subgroup of G contained in $C_G(X) \cap H$, we have $w_0^{-1}hw_0 \in C_G(X)$, and hence $h \in C_G(w_0 X w_0^{-1}) = C_G(Y)$. Since $X \cup Y$ is a set of generators for G, it follows that h belongs to the center of G, and our assertion is proved.

5. Irreducible modules of dimension p^m for Lie algebras of classical type

We shall prove first that if G is the subgroup defined in [6] of the group of invariant automorphisms of a Lie algebra of classical type $\mathfrak{L}$ associated with a complex semisimple Lie algebra $\mathfrak{L}^c$, then G satisfies the axioms (1.1)–(1.13). Therefore, by Steinberg's result [12], G has an irreducible module M of dimension p^m, where m is the number of positive roots of $\mathfrak{L}$ with respect to a Cartan subalgebra. The purpose of this section is to prove, as an application of Theorem 3.6, that this module is isomorphic to an ΩG-module constructed from an irreducible restricted $\mathfrak{L}$-module by the methods of [5] and [6].

Changing the notation of [2, p. 32] slightly, we let $\mathfrak{L}^c$ be a complex semisimple Lie algebra, and $(X_1, \cdots, X_\nu)$ the basis of $\mathfrak{L}^c$ defined in [2, p. 32], containing the root elements X_α of $\mathfrak{L}^c$ relative to a Cartan subalgebra $\mathfrak{H}^c$. Let Ω be an algebraically closed field of characteristic $p \geqq 5$, and let K be the prime field in Ω. Let $\mathfrak{L}_Z$ be the Lie algebra over the integers with basis $(X_1, \cdots, X_\nu)$, and let $\mathfrak{L} = \Omega \otimes \mathfrak{L}_Z$. Then $\mathfrak{L}$ is a Lie algebra over Ω with basis elements $(X_1^*, \cdots, X_\nu^*)$, where $X_i^* = 1 \otimes X_i$, $1 \leqq i \leqq \nu$, and the constants of structure of $\mathfrak{L}$ relative to this basis all belong to K. Among the X_i^* appear the elements E_α corresponding to the root elements X_α of $\mathfrak{L}$, and the remaining basis elements generate an abelian subalgebra $\mathfrak{H}$ of $\mathfrak{L}$ which is easily seen to be a Cartan subalgebra of $\mathfrak{L}$. Then $\mathfrak{L}$ has a Cartan decomposition

$$\mathfrak{L} = \mathfrak{H} + \sum \Omega E_\alpha,$$

where we may view each element E_α as a root element belonging to a nonzero root α of $\mathfrak{L}$ with respect to $\mathfrak{H}$. We shall assume in this section that $\mathfrak{L}$ satisfies the axioms (i)–(v) of Mills and Seligman [9, p. 520]. The question of which Lie algebras of classical type can be obtained from complex semisimple Lie algebras by reduction modulo p has been settled by Seligman [10]. (See also [4] for the case of Lie algebras with nondegenerate Killing forms.)

There is a one-to-one mapping of the set of roots of $\mathfrak{L}^c$ onto the roots of $\mathfrak{L}$ which preserves additive relations in the sense that if a sum of two nonzero roots is a nonzero root of $\mathfrak{L}^c$, the same holds for the corresponding roots in $\mathfrak{L}$. Let $\alpha_1, \cdots, \alpha_l$ be the roots of $\mathfrak{L}$ corresponding to a fundamental system

(= maximal simple system) of roots of $\mathfrak{L}^c$. Then $\alpha_i - \alpha_j$, $i \neq j$, is not a root of $\mathfrak{L}$, otherwise $[E_{\alpha_i}, E_{\alpha_j}] \neq 0$ by [9, (xiii), p. 524], and this is impossible since $[X_{\alpha_i}, X_{\alpha_j}] = 0$ in $\mathfrak{L}^c$. Therefore $\{\alpha_1, \cdots, \alpha_l\}$ is a simple system of roots, and it is clear that $\{\alpha_1, \cdots, \alpha_l\}$ is a maximal simple system of roots in the sense of [4]. Moreover the roots $\{\alpha_1, \cdots, \alpha_l\}$ are linearly independent, for if they were not, there would exist $H \in \mathfrak{H}$, $H \neq 0$, such that $\alpha(H) = 0$ for all roots α, and H would belong to the center of $\mathfrak{L}$, contrary to [9, axiom (ii)]. Letting H_α be a generator of the one-dimensional space $[\mathfrak{L}_{-\alpha}, \mathfrak{L}_\alpha]$ for a root $\alpha \neq 0$, a computation shows easily that if α, β, $\alpha + \beta$ are nonzero roots, then $H_{\alpha+\beta}$ is a linear combination of H_α and H_β. Therefore every H_α is a linear combination of the elements $H_i \in [\mathfrak{L}_{-\alpha_i}, \mathfrak{L}_{\alpha_i}]$, $1 \leqq i \leqq l$, such that $\alpha_i(H_i) = 2$. Since the elements H_α generate $\mathfrak{H}$ by [9, (viii)], and since $\mathfrak{H}$ has dimension l, it follows that $H_1, \cdots, H_l$ is a basis of $\mathfrak{H}$ over Ω.

Now let G be the group of automorphisms of $\mathfrak{L}$ generated by the automorphisms

$$x_\alpha(\xi) = \exp \operatorname{ad} \xi E_\alpha, \qquad \xi \in K,$$

where α is a root $\neq 0$. From the discussion in [2, pp. 32–36] it follows that G is isomorphic to the group G' defined in [2] relative to the complex semisimple Lie algebra $\mathfrak{L}^c$ and the field K. If we let

$$x_i(\xi) = \exp \operatorname{ad} \xi E_{\alpha_i}, \qquad y_i(\xi) = \exp \operatorname{ad} \xi E_{-\alpha_i}, \quad 1 \leqq i \leqq l, \qquad \xi \in K,$$

then the mapping $\phi_i : SL(2, K) \to G$ given by

$$\phi_i(u(\xi)) = x_i(\xi), \qquad \phi_i(v(\xi)) = y_i(\xi), \qquad \xi \in K,$$

defines a homomorphism of $SL(2, K)$ into G for $1 \leqq i \leqq l$. With this interpretation of the homomorphisms $\{\phi_1, \cdots, \phi_l\}$, the results of §1c imply that G satisfies the axioms (1.1)–(1.13) of §1a.

Let $\bar{M}$ be the irreducible restricted right $\mathfrak{L}$-module whose maximal weight λ satisfies $\lambda(H_i) = p - 1$, $1 \leqq i \leqq l$ (see [5, Theorem 2, p. 315]). We summarize some of the properties of $\bar{M}$ in the following lemma.

(5.1) Lemma. *The irreducible restricted right $\mathfrak{L}$-module $\bar{M}$ whose maximal weight is $\lambda : \lambda(H_i) = p - 1$, $1 \leqq i \leqq l$, has the following properties.*

(i) *If $\bar{m}_0$ is a maximal vector (see [5, p. 312]) in $\bar{M}$, then for $1 \leqq i \leqq l$, the elements $\{\bar{m}_0, \bar{m}_0 E_{-\alpha_i}, \cdots, \bar{m}_0 E_{-\alpha_i}^{p-1}\}$ are linearly independent, and span an irreducible $\mathfrak{L}_i$-submodule V_i of M, where $\mathfrak{L}_i$ is the three-dimensional simple subalgebra of $\mathfrak{L}$ with basis $\{E_{-\alpha_i}, E_{\alpha_i}, H_i\}$.*

(ii) *There exists an irreducible projective representation $F : G \to GL(M)$ of G by linear transformations $F(g)$, $g \in G$, such that for all $m \in \bar{M}, A \in \mathfrak{L}$, and $g \in G$, we have*

$$(mA)F(g) = mF(g)A^g,$$

where $A \to A^g$ is the automorphism g of $\mathfrak{L}$.

(iii) *The restrictions $F \mid X$ and $F \mid Y$ of F to the subgroups X and Y of G are ordinary representations of these subgroups.*

(iv) *A vector* $m \in \bar{M}$ *satisfies* $mF(x) = m$ *for all* $x \in X$ *if and only if* $m \in \Omega \bar{m}_0$.

Proof. (i) We may assume that $[E_{-\alpha_i}, E_{\alpha_i}] = H_i$, and $\alpha_i(H_i) = 2$. Let $\bar{m}_0 E_{-\alpha_i}^v \neq 0$ for $0 \leqq v \leqq k-1$, and $\bar{m}_0 E_{-\alpha_i}^k = 0$. In order to prove (i) it is sufficient to prove that $k = p$, because of the well known classification of the irreducible restricted modules for the three-dimensional simple Lie algebra.

The subspace $V_i = \sum_{\nu=0}^{k-1} \Omega \bar{m}_0 E_{-\alpha_i}^{\nu}$ is invariant relative to $\mathfrak{L}_i$, and we have

$$\bar{m}_0 E_{-\alpha_i}^{\nu} H_i = (-1 - 2\nu) \bar{m}_0 E_{-\alpha_i}^{\nu}, \qquad 0 \leqq \nu \leqq k-1,$$

since $\bar{m}_0 H_i = -\bar{m}_0$, and $[E_{-\alpha_i}, H_i] = -2E_{-\alpha_i}$. Then computing the trace of H_i on the space V_i we have, since $H_i = [E_{-\alpha_i}, E_{\alpha_i}]$,

$$0 = \sum_{\nu=0}^{k-1} (-1 - 2\nu) = -k - 2(k(k-1)/2) = -k^2 .$$

Therefore $k = p$, and (i) is proved.

(ii) follows from the definition of the projective representation F given in [5, §II.2], and the theorem of [6, p. 856].

(iii) We prove first that $F \mid X$ is an ordinary representation. We refer to the construction of the representation F in [5, pp. 317 and 318]. By the discussion there, it follows that for any $x \in X$ (not necessarily a generator), we may define $F(x)$ by

$$(5.2) \qquad F(x) : \bar{m}_0 E_{\gamma_1} \cdots E_{\gamma_r} \rightarrow \bar{m}_0 E_{\gamma_1}^{x} \cdots E_{\gamma_r}^{x},$$

and obtain an invertible linear transformation of $\bar{M}$ such that (5) of [5, p. 318] is satisfied, namely

$$(5.3) \qquad (mA)F(x) = mF(x)A^{x}, \qquad A \in \mathfrak{L}, \quad m \in M.$$

Because of (5.3) and the fact that $\bar{M}$ is an irreducible $\mathfrak{L}$-module, any two determinations of $F(x)$ satisfying (5.3) differ by a scalar factor, so that the definition (5.2) is consistent with the rest of the discussion in [5] and [6]. Since F is a projective representation we have

$$(5.4) \qquad F(x_1 x_2) = F(x_1)F(x_2)\alpha(x_1, x_2), \qquad \alpha(x_1, x_2) \in \Omega.$$

Since $m_0 F(x) = m_0$ for all $x \in X$, (5.4) implies that $\alpha(x_1, x_2) = 1$ for all $x_1, x_2 \in X$, and $F \mid X$ is an ordinary representation. A similar discussion applies to $F \mid Y$.

Finally (iv) follows from [6, Lemma 1.7, p. 856], and Lemma 5.1 is proved.

(5.5) LEMMA. *Let* $\bar{G}$ *be the group of linear transformations on* $\bar{M}$ *generated by*

$$\bar{x}_i(\xi) = F(x_i(\xi)), \qquad \bar{y}_i(\xi) = F(y_i(\xi)), \qquad \xi \in K, \quad 1 \leqq i \leqq l.$$

Then the following statements hold.

(i) *For* $1 \leqq i \leqq l$, $\bar{m}_0 \bar{y}_i(\xi) \in V_i = \sum_{\nu=0}^{p-1} \Omega \bar{m}_0 E_{-\alpha_i}^{\nu}$.

(ii) *The mapping*

$$\bar{\phi}_i : u(\xi) \to \bar{x}_i(\xi), \quad v(\xi) \to \bar{y}_i(\xi)$$

can be extended to a homomorphism $\bar{\phi}_i$ *of* $SL(2, K)$ *into* $\bar{G}$.

(iii) *Letting* $\bar{d}_i(\xi) = \bar{\phi}_i(d(\xi))$, $\bar{w}_i = \bar{\phi}_i(w_i)$, *we have*

$$\bar{m}_0 \bar{d}_i(\xi) = \bar{m}_0, \qquad \xi \in K, \quad 1 \leqq i \leqq l,$$

and

$$\bar{m}_0 \bar{w}_i \in \bar{m}_0 \operatorname{rad} \Omega \bar{Y}_i, \qquad 1 \leqq i \leqq l.$$

Proof. By the proof of Lemma 1.6 in [6, p. 855], we have

$$\bar{m}_0 \bar{y}_i(\xi) = \bar{m}_0 + \xi \bar{m}_1 + \xi^2 \bar{m}_2 + \cdots, \tag{5.6}$$

where if ρ_0 is the rank of $\bar{m}_0$, $\bar{m}_1$ has rank $\rho_0 + \varepsilon_i$, $\bar{m}_2$ has rank $p_0 + 2\varepsilon_i$, etc. Since $\bar{M}$ has a basis consisting of rank vectors of the form

$$\bar{m}_0 E_{-\alpha_{i_1}} E_{-\alpha_{i_2}} \cdots,$$

which has rank $\rho_0 + \varepsilon_{i_1} + \varepsilon_{i_2} + \cdots$, and since vectors of different ranks are linearly independent, it follows that $\bar{m}_\nu$ is a multiple of $\bar{m}_0 E^{\nu}_{-\alpha_i}$ for $\nu = 1, 2, \cdots$, and (5.6) implies (i).

(ii) Consider the space V_i of $\bar{M}$ defined in (i). By the definition of $F(x_i(\xi))$, and by (i) of Lemma 5.5, we have $\bar{m}_0 \bar{x}_i(\xi) \in V_i$, and $\bar{m}_0 \bar{y}_i(\xi) \in V_i$, $\xi \in K$. Then we have by (5.3),

$$\bar{m}_0 E^{\nu}_{-\alpha_i} \bar{x}_i(\xi) = \bar{m}_0 (E^{x_i(\xi)}_{-\alpha_i})^{\nu}, \qquad \bar{m}_0 E^{\nu}_{-\alpha_i} \bar{y}_i(\xi) = \bar{m}_0 \bar{y}_i(\xi) (E^{y_i(\xi)}_{-\alpha_i})^{\nu}, \tag{5.7}$$

for $\nu = 0, 1, 2, \cdots$. From these formulas it is clear that V_i is invariant relative to $\bar{x}_i(\xi)$ and $\bar{y}_i(\xi)$, $\xi \in K$.

Since the elements $u(\xi)$ and $v(\xi)$, $\xi \in K$, generate $SL(2, K)$, it follows by (ii) of Lemma 5.1 that the mappings

$$u(\xi) \to x_i(\xi) \to F(x_i(\xi)) = \bar{x}_i(\xi), \qquad v(\xi) \to \bar{y}_i(\xi)$$

define a projective representation $\bar{\phi}_i$ of $SL(2, K)$ on M, with V_i as an invariant subspace. (For later use we remark that since V_i is an irreducible $\mathfrak{L}_i$-module, and because of (5.7), it follows by the theorem of [6, p. 856] that V_i is an irreducible invariant subspace.) We have

$$\bar{\phi}_i(gg') = \bar{\phi}_i(g)\bar{\phi}_i(g')\alpha(g, g'), \qquad \alpha(g, g') \in \Omega, \quad g, g' \in SL(2, K). \tag{5.8}$$

Since $\det_{V_i} \bar{\phi}_i(g) = 1$ for all $g \in SL(2, K)$ by (iii) of Lemma 5.1, (5.8) implies that $\alpha(g, g')^p = 1$, and hence $\alpha(g, g') = 1$. Therefore $\bar{\phi}_i$ is an ordinary representation of $SL(2, K)$, and (ii) is proved.

(iii) We have shown that $\bar{\phi}_i \mid V_i$ is an irreducible representation of $SL(2, K)$ on the space V_i of dimension p. From the classification of the irreducible modular representations of $SL(2, K)$ (see [1, p. 588]), it follows that $\bar{\phi}_i \mid V_i$ is equivalent to the representation of $SL(2, K)$ afforded by the space W of homogeneous polynomials of degree $p - 1$ in two variables x, y such that

$$\begin{aligned} xu(\xi) &= x + \xi y, & yu(\xi) &= y, \\ xv(\xi) &= x, & yv(\xi) &= \xi x + y, \\ xd(\xi) &= \xi x, & yd(\xi) &= \xi^{-1} y. \end{aligned}$$

The maximal vector in W relative to $SL(2, K)$ is y^{p-1}, and we have

$$y^{p-1} d(\xi) = (\xi^{-1})^{p-1} y^{p-1} = y^{p-1},$$

since $\xi^{p-1} = 1$ for all $\xi \in K$, $\xi \neq 0$. Also since $\omega = u(1)v(-1)u(1)$, we obtain

$$y^{p-1}\omega = y^{p-1}v(-1)u(1) = (-x+y)^{p-1}u(1) = (-x-y+y)^{p-1} = x^{p-1},$$

which belongs to $y^{p-1}\operatorname{rad} \Omega V$. Transferring these results to V_i, we obtain (iii), and Lemma 5.5 is proved.

Now let ρ be the mapping of $\bar{G} \to G$ defined by

$$(5.9) \qquad \rho(\bar{x}_i(\xi)) = x_i(\xi), \qquad \rho(\bar{y}_i(\xi)) = y_i(\xi), \qquad \xi \in K, \quad 1 \leqq i \leqq l.$$

Let $\bar{z}_{i_1}, \cdots, \bar{z}_{i_s}$ be generators of $\bar{G}$ (as in §3) such that $\bar{z}_{i_1} \cdots \bar{z}_{i_s} = 1$. For all $m \in \bar{M}$, $A \in \mathfrak{L}$, we have by (5.3), letting $\langle z \rangle = z_{i_1} \cdots z_{i_s}$,

$$mA\bar{z}_{i_1} \cdots \bar{z}_{i_s} = mA = m\bar{z}_{i_1} \cdots \bar{z}_{i_s} A^{\langle z \rangle} = mA^{\langle z \rangle}.$$

Therefore $\bar{M}(A - A^{\langle z \rangle}) = 0$, and since $\bar{M}$ is a faithful $\mathfrak{L}$-module, we have $\langle z \rangle = z_{i_1} \cdots z_{i_s} = 1$. Therefore ρ is a well-defined mapping of $\bar{G}$ onto G, and is clearly a homomorphism. The kernel of ρ is $\Omega \cdot 1^- \cap \bar{G}$, by (5.3) and Schur's Lemma.

(5.10) LEMMA. *The group $\bar{G}$ satisfies the axioms* (1.1)–(1.6) *of* §1a, *with the definition of $\bar{\phi}_1, \cdots, \bar{\phi}_l$ given in Lemma* 5.5. *Moreover the homomorphism ρ defined by* (5.9) *is an isomorphism of $\bar{Y}$ onto Y such that $\rho(\bar{y}_i(\xi)) = y_i(\xi)$, and if $\bar{y} \in (\bar{Y}, \bar{Y})$,*

$$(5.11) \qquad \rho(\bar{y}^{\bar{x}_i(\xi)}) = \rho(\bar{y})^{x_i(\xi)}, \qquad 1 \leqq i \leqq l, \quad \xi \in K.$$

Proof. (1.1) follows from (ii) of Lemma 5.5, and the definition of $\bar{G}$. By the definition of ρ, we have

$$(5.12) \qquad \rho(F(x)) = x, \qquad x \in X.$$

Since $F \mid X$ is an ordinary representation by (iii) of Lemma 5.1, (5.12) implies that ρ is an isomorphism of $\bar{X}$ upon X, and hence $\bar{X}$ is a p-group. Similarly $\rho \mid \bar{Y}$ is an isomorphism of $\bar{Y}$ upon Y, and hence $\bar{Y}$ is a p-group. Moreover (5.11) follows from the definition of ρ, as soon as the fact that $(\bar{Y}, \bar{Y})^{\bar{x}_i} \subset \bar{Y}$ is established.

Let $\bar{H} = \rho^{-1}(H)$; then $\bar{D}_i \subset \bar{H}$ for $1 \leqq i \leqq l$. By the argument of [2, p. 48], H is generated by the set $D_1 \cup \cdots \cup D_l$. Therefore $\bar{H}$ is generated by $\bar{D}_1 \cup \cdots \cup \bar{D}_l \cup \rho^{-1}(1)$, where $\rho^{-1}(1)$ is the kernel of ρ. In order to prove that $\bar{H} \subset N_{\bar{G}}(\bar{X}_i)$, it is sufficient to prove that $\bar{D}_j \subset N_{\bar{G}}(\bar{X}_i)$. Let $\bar{d}_j \in \bar{D}_j$,

$\bar{x} \in \bar{X}_i$. Then there exists an element $\bar{x}_1 \in \bar{X}_i$ and $\xi \in \rho^{-1}(1)$ such that

$$\bar{d}_j \bar{x} \bar{d}_j^{-1} = \bar{x}_1 \xi.$$

Applying both sides of this relation to the maximal vector $\bar{m}_0$, and using (iii) of Lemma 5.5, we obtain $\xi = 1$, and hence $\bar{d}_j \in N_{\bar{G}}(\bar{X}_i)$ as required. This completes the proof of axiom (1.3).

By a similar argument, if $\bar{x}_i \in \bar{X}_i$, $\bar{y}_j \in \bar{Y}_j$, $i \neq j$, then

$$\bar{x}_i \bar{y}_j = \bar{y}_j \bar{x}_i \mu, \qquad \mu \in \Omega. \tag{5.13}$$

Now consider the actions of $\bar{x}_i$ and $\bar{y}_j$ on $\bar{m}_0$. We have

$$\bar{m}_0 \bar{x}_i \bar{y}_j = \bar{m}_0 \bar{y}_j \in V_j.$$

If $\bar{m}_0 \bar{y}_j = \sum a_\nu \bar{m}_0 E_{-\alpha_j}^\nu$, $a_\nu \in \Omega$, then by (5.3)

$$\bar{m}_0 \bar{y}_j \bar{x}_i = \sum a_\nu \bar{m}_0 (E_{-\alpha_j}^{x_i})^\nu.$$

But

$$E_{-\alpha_j}^{x_i} = E_{-\alpha_j} \exp \operatorname{ad} \xi E_{\alpha_i} = E_{-\alpha_j}$$

since $-\alpha_j + \alpha_i$ is not a root if $i \neq j$. Therefore

$$\bar{m}_0 \bar{y}_j \bar{x}_i = \bar{m}_0 \bar{y}_j = \bar{m}_0 \bar{x}_i \bar{y}_j.$$

Comparing this equation with (5.13), we obtain $\mu = 1$ in (5.13), and (1.4) is proved.

We have already shown that ρ is an isomorphism of $\bar{Y}$ onto Y such that $\rho(\bar{y}_i(\xi)) = y_i(\xi)$, $1 \leqq i \leqq l$, $\xi \in K$. From §1c, it follows that (Y, Y) is generated by the elements $x_{-\alpha}(\xi)$, $\xi \in K$, where α is a positive root $\neq \alpha_1, \cdots, \alpha_l$. By letting

$$\bar{x}_{-\alpha}(\xi) = F(x_{-\alpha}(\xi)),$$

the facts that F is a homomorphism of Y onto $\bar{Y}$ and $\rho(F(y)) = y$ for $y \in Y$, imply that $\rho(\bar{x}_{-\alpha}(\xi)) = x_{-\alpha}(\xi)$. Then for $x_i \in X_i$, we have

$$\bar{x}_{-\alpha}(\xi)^{\bar{x}_i} = \bar{y}\eta$$

for some $\eta \in \Omega$ and $\bar{y} \in \bar{Y}$. In order to show that $\eta = 1$, it is sufficient to prove that if m_- is a minimal vector in $\bar{M}$, then $m_- \bar{x}_{-\alpha}(\xi)^{\bar{x}_i} = m_-$. As we have pointed out before, the methods of [6] show that

$$m_- \bar{x}_i = \sum \xi_\nu m_- E_{\alpha_i}^\nu.$$

Then

$$m_- \bar{x}_i \bar{x}_{-\alpha}(\xi) = \sum \xi_\nu m_-(E_{\alpha_i}^\nu \exp \operatorname{ad} \xi E_{-\alpha}).$$

We shall now prove by induction that

$$m_-(E_{\alpha_i}^\nu \exp \operatorname{ad} \xi E_{-\alpha}) = m_- E_{\alpha_i}^\nu.$$

Suppose the result is valid for ν. Then by the induction hypothesis,

$$m_-(E_{\alpha_i}^{\nu+1} \exp \operatorname{ad} \xi E_{-\alpha}) = m_- E_{\alpha_i}^\nu (E_{\alpha_i} + \xi[E_{\alpha_i}, E_{-\alpha}] + \tfrac{1}{2}\xi^2[[E_{\alpha_i}, E_{-\alpha}]E_{-\alpha}] \cdots),$$

The commutators $[[E_{\alpha_i}, E_{-\alpha}] \cdots]$ are multiples of $E_{\alpha_i - j\alpha}$, and the roots $\alpha_i - j\alpha$ are all < 0, since α_i is a fundamental root and α is not. For $j > 0$, we have for some $a, b, \cdots$ in Ω,

$$m_- E_{\alpha_i}^{\nu} E_{\alpha_i - j\alpha} = am_- E_{\alpha_i - j\alpha} E_{\alpha_i}^{\nu} + bm_- E_{2\alpha_i - j\alpha} E_{\alpha_i}^{\nu-1} + \cdots = 0$$

since all the roots $k\alpha_i - j\alpha$ are < 0, and m_- is a minimal vector. Returning to our original formula we have

$$m_- \bar{x}_i \bar{x}_{-\alpha}(\xi) \bar{x}_i^{-1} = m_-,$$

and $\bar{x}_{-\alpha}(\xi)^{\bar{x}_i} = \bar{y}\eta$ implies $\eta = 1$. We have now proved the first half of axiom (1.5); a similar argument proves the second half.

Finally let $\bar{h} \in \bar{H}$. Then for $1 \leqq i \leqq l$, $\rho(\bar{w}_i) = w_i$, and

$$\rho(\bar{h}^{\bar{w}_i}) = \rho(\bar{h})^{w_i} \in H$$

by (1.6) for G. Since $\bar{H} = \rho^{-1}(H)$, $\bar{h}^{\bar{w}_i} \in H$, and (1.6) holds for $\bar{G}$. We have already proved (5.11), so that Lemma 5.10 is established.

We come now to the main theorem of this section. Because G satisfies the conditions (1.14)–(1.26), either by the results of §1b or by the argument in Steinberg's paper [12], the construction of Steinberg can be applied to G to construct an irreducible ΩG-module of dimension p^m, where p^m is the order of X.

(5.14) THEOREM. *Let $\mathfrak{L}$ be a Lie algebra of classical type, which is obtained from a complex semisimple Lie algebra by reduction modulo p, over Ω of characteristic $p \geqq 5$. Let $\bar{M}$ be the irreducible restricted $\mathfrak{L}$-module whose maximal weight λ satisfies $\lambda(H_i) = p - 1$, $1 \leqq i \leqq l$. Let G be the group of automorphisms of $\mathfrak{L}$ generated by $x_i(\xi)$ and $y_i(\xi)$ for $\xi \in K$ and $1 \leqq i \leqq l$, where K is the prime field in Ω. Let F be the irreducible projective representation of G on $\bar{M}$. Let M be the irreducible right ΩG-module of dimension p^m defined by Steinberg. Then F is an ordinary representation of G, and is equivalent to the representation of G afforded by the module M of Steinberg.*

Proof. Because of Lemma 5.10, it is possible to apply Theorem 3.6 to G and $\bar{G}$, M and $\bar{M}$. The homomorphism $\rho : \bar{Y} \to Y$ has the properties required of the isomorphism θ in Theorem 3.6. By Lemma 4.1, M has a one-dimensional space of maximal vectors relative to G. By (iv) of Lemma 5.1, the space of maximal vectors of $\bar{M}$ is at most one-dimensional. We shall prove that $\bar{m}_0$ is a maximal vector in $\bar{M}$. It is sufficient to prove that $\bar{m}_0 \bar{h} \in \Omega\bar{m}_0$ for all $\bar{h} \in \bar{H}$. Since $\bar{H}$ is generated by $\bar{D}_1 \cup \cdots \cup \bar{D}_l \cup \rho^{-1}(1)$, this result is clear by (iii) of Lemma 5.5. By (iii) of Lemma 5.5 and (ii) and (iii) of Lemma 4.1, the functions f and $\bar{f}$ associated with m_0 and $\bar{m}_0$ satisfy the conditions

$$\bar{f}(\bar{w}_i) = f(w_i), \qquad \bar{f}(\bar{d}_i(\xi)) = f(d_i(\xi)),$$

for $1 \leqq i \leqq l$ and $\xi \in K$. By Theorem 3.6, there exists a vector-space isomorphism S of $\bar{M}$ onto M such that for all $\bar{m} \in \bar{M}$ and generators $\{z_i\}$, $\{\bar{z}_i\}$ of

G and $\bar{G}$ respectively, we have

$$(5.15) \qquad \bar{m}F(z_1) \cdots F(z_s) S = (\bar{m}S) z_1 \cdots z_s .$$

From this it follows that $\dim \bar{M} = p^m$. Moreover by (iii) of Lemma 5.1, we have

$$\det F(g) = 1, \qquad g \in G.$$

Therefore, as we saw earlier in the case of $SL(2, K)$,

$$F(gg') = F(g)F(g')\alpha(g, g'), \qquad \alpha(g, g') \in \Omega,$$

implies $\alpha(g, g')^{p^m} = 1$, and hence $\alpha(g, g') = 1$. Therefore F is an ordinary representation of G, $G \cong \bar{G}$, and (5.15) asserts that F is equivalent to the representation afforded by the module M of Steinberg. This completes the proof of the theorem.

Remark 1. Theorem 5.14 asserts that $\mathfrak{L}$ has an irreducible restricted module $\bar{M}$ of dimension p^m, where m is the number of positive roots of $\mathfrak{L}$ with respect to $\mathfrak{H}$. This result complements Theorem 2 of [7], in which it was was proved that $\dim M \leqq p^m$ for all irreducible restricted $\mathfrak{L}$-modules.

Remark 2. If we assume that $\mathfrak{L}$ has a nondegenerate Killing form, then with $\bar{M}$ we have an associated $\mathfrak{L}^c$-module V in the sense of [7, p. 137], where $\mathfrak{L}^c$ is the complex semisimple Lie algebra belonging to $\mathfrak{L}$. The maximal weight Λ of V satisfies

$$\Lambda(H_i) = p - 1, \qquad 1 \leqq i \leqq l.$$

Applying the Weyl formula for the dimension of V we obtain

$$(5.16) \qquad \dim V = \prod_{\alpha' > 0} \frac{(\Lambda + \rho)(H_{\alpha'})}{\rho(H_{\alpha'})} = p^m.$$

To prove (5.16), let $H_{\alpha'} = \sum \mu_i H_i$, $\mu_i \in Q$. Then

$$(\Lambda + \rho)(H_{\alpha'}) = (\textstyle\sum \mu_i)p,$$

since $\rho(H_i) = 1$, $1 \leqq i \leqq l$. Therefore each factor

$$\frac{(\Lambda + \rho)(H_{\alpha'})}{\rho(H_{\alpha'})} = p,$$

and $\dim V = p^m$. From the results of [7] it follows that the $\mathfrak{L}$-module $\bar{V}$ obtained from V by reduction modulo p is irreducible, and isomorphic to $\bar{M}$.

Remark 3. We shall apply Theorem 5.14 to construct a minimal right ideal of dimension p^m in the u-algebra $\mathfrak{U}$ of $\mathfrak{L}$. The u-algebra of $\mathfrak{L}$ has a basis over Ω consisting of the standard monomials

$$u(P, Q, R) = E_{\alpha_1}^{p_1} \cdots E_{\alpha_m}^{p_m} H_1^{q_1} \cdots H_l^{q_l} E_{-\alpha_1}^{r_1} \cdots E_{-\alpha_m}^{r_m}$$

where $0 \leqq p_i, q_i, r_i \leqq p - 1$. For a vector exponent P, we write $|P| = \sum p_i$. Let $\mathfrak{U}_+$ be the nilpotent subalgebra of $\mathfrak{U}$ consisting of all standard monomials $u(P, 0, 0)$, $|P| \neq 0$, $\mathfrak{U}_-$ the nilpotent subalgebra consisting of all $u(0, 0, R)$, $|R| \neq 0$, and $\mathfrak{C}$ the subalgebra of $\mathfrak{U}$ generated by 1 and $\{H_1, \cdots, H_l\}$.

(5.17) LEMMA. *There exists an element $u_0 \in \mathfrak{U}_+ \mathfrak{C}$ such that $u_0 \neq 0$, $u_0 E_\alpha = 0$, $\alpha > 0$, and $u_0 H_i = -u_0$ for $1 \leqq i \leqq l$.*

Proof. Since $\mathfrak{U}_+$ is a nilpotent algebra, there exists an element $u_0' \neq 0$ in $\mathfrak{U}_+$ such that $u_0' E_\alpha = 0$ for all $\alpha > 0$. Let H_i^R denote the right multiplication $x \to xH_i$ in the subalgebra $\Omega[H_i]$ of $\mathfrak{C}$. Since the powers H_i^s, $0 \leqq s \leqq p - 1$, are linearly independent and $H_i^p = H_i$, the minimum polynomial of H_i^R is $\lambda^p - \lambda$, which has $\lambda + 1$ as a factor. Therefore there is an element $c_i \in \Omega[H_i]$ such that $c_i H_i = -c_i$, $1 \leqq i \leqq l$. Since $\mathfrak{C} \cong \Omega[H_1] \otimes \cdots \otimes \Omega[H_l]$, $c = \prod_{i=1}^{l} c_i \neq 0$. Moreover $cH_i = -c$ for $1 \leqq i \leqq l$. Now let $u_0 = u_0' c$. Because the standard monomials $u(P, Q, 0)$ are linearly independent, it follows that $u_0 \neq 0$. Moreover, $u_0 H_i = u_0'(cH_i) = -u_0$, $1 \leqq i \leqq l$. For all roots $\alpha > 0$, $u_0 E_\alpha = u_0' c E_\alpha = u_0' E_\alpha c' = 0$, for some $c' \in \mathfrak{C}$ depending on c and E_α. This completes the proof of the lemma.

(5.18) LEMMA. *$\mathfrak{J} = \Omega u_0 + u_0 \mathfrak{U}_-$ is a right ideal in $\mathfrak{U}$.*

Proof. $\mathfrak{J}$ is clearly a subspace of $\mathfrak{U}$ such that $\mathfrak{J}\mathfrak{U}_- \subset \mathfrak{J}$. If $H \in \mathfrak{H}$, then

$$u_0 u(0, 0, R)H = [-1 - \sum r_i \alpha_i(H)]u_0 u(0, 0, R) \in \mathfrak{J}.$$

For $\alpha > 0$, we show that $u_0 u(0, 0, R)E_\alpha \in \mathfrak{J}$ by induction on $|R|$. If $|R| > 0$, write

$$u(0, 0, R) = u(0, 0, R')E_{-\beta}, \quad |R'| < |R|, \quad \beta > 0,$$

and obtain

$$u_0 u(0, 0, R)E_\alpha = u_0 u(0, 0, R')E_\alpha E_{-\beta} + u_0 u(0, 0, R')[E_{-\beta}, E_\alpha],$$

where $[E_{-\beta}, E_\alpha]$ is either 0, in $\mathfrak{H}$, or a multiple of $E_{-\beta+\alpha}$ for a root $-\beta + \alpha \neq 0$. In all cases both summands are in $\mathfrak{J}$ by the induction hypothesis, and the lemma is proved.

(5.19) LEMMA. *The elements u_0 and $u_0 u(0, 0, R)$, $|R| \neq 0$, form a basis of $\mathfrak{J}$ over Ω.*

Proof. By Lemma 5.18, the indicated elements generate $\mathfrak{J}$ over Ω. Since $u_0 \in \mathfrak{U}_+\mathfrak{C}$, u_0 is a linear combination of standard monomials $u(P, Q, 0)$. We have $u(P, Q, 0)u(0, 0, R) = u(P, Q, R)$ for all R, and since the standard monomials $u(P, Q, R)$ are linearly independent, the conclusion of the lemma follows.

(5.20) COROLLARY. *The dimension of $\mathfrak{J}$ over Ω is p^m, where m is the number of positive roots of $\mathfrak{L}$ with respect to $\mathfrak{H}$.*

(5.21) THEOREM. *The right ideal $\mathfrak{J}$ constructed in Lemma 5.18 is a minimal right ideal in $\mathfrak{U}$.*

Proof. By Lemma 5.17, u_0 is a maximal vector in $\mathfrak{J}$ of weight λ such that $\lambda(H_i) = -1$, $1 \leqq i \leqq l$. Then $\mathfrak{J}$ has an irreducible homomorphic image $\mathfrak{J}_1 = \mathfrak{J}/\mathfrak{N}$ such that the maximal weight of $\mathfrak{J}_1$ is λ. By [5, Theorem 1, p. 312], $\mathfrak{J}_1 \cong \bar{M}$ where $\bar{M}$ is the irreducible $\mathfrak{L}$-module appearing in Theorem 5.14. By Theorem 5.14, $\dim \mathfrak{J}_1 = p^m$, hence $\mathfrak{N} = \{0\}$, and $\mathfrak{J}$ is an irreducible right $\mathfrak{U}$-module.

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